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Static Spherically Symmetric Perfect Fluid Solutions in Teleparallel $F(T)$ Gravity

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Abstract: In this paper, we investigate static spherically symmetric teleparallel $F(T)$ gravity containing a perfect isotropic fluid. We first write the field equations and proceed to find new teleparallel $F(T)$ solutions for perfect isotropic and linear fluids. By using a power-law ansatz for the coframe components, we find several classes of new non-trivial teleparallel $F(T)$ solutions. We also find a new class of teleparallel $F(T)$ solutions for a matter dust fluid. After, we solve the field equations for a non-linear perfect fluid. Once again, there are several new exact teleparallel $F(T)$ solutions and also some approximated teleparallel $F(T)$ solutions. All these classes of new solutions may be relevant for future cosmological and astrophysical applications.

Keywords: teleparallel gravity; static spherically symmetric; perfect fluid; teleparallel $F(T)$ solutions; frame approach; orthonormal frames

MSC: 83D05; 83F05; 83-10

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1. Introduction

There are a number of alternative theories of gravity to General Relativity (GR). The $F(T)$-type teleparallel theories of gravity are very promising [1–3]. In these theories, the geometry is characterized by the torsion which is a function of the coframe, $\mathbf{h}^a$, derivatives of the coframe, and a zero curvature and metric compatible spin-connection one-form $\omega^a{}_b$. Hence, in teleparallel gravity, it is necessary to work with a frame basis instead of a metric tensor. In such theories the role of symmetry is no longer as clearly defined as in pseudo-Riemannian geometry, where symmetry is defined in terms of an isometry of the metric or Killing Vectors (KVs). In GR, the Riemannian geometry is completely defined by the curvature of a Levi-Civita connection and calculated from the metric. But this is not really the case for some alternative theories, in particular for teleparallel $F(T)$-type gravity.

The development of a frame-based approach for determining the symmetries of a spacetime has been explored [4–6]. A possible complication arises due to the possible existence of a non-trivial linear isotropy group: a Lie group of Lorentz frame transformations keeping the associated tensors of the geometry invariant. If a given spacetime has a non-trivial linear isotropy group, determining the group of symmetries requires solving a set of inhomogeneous differential equations [7]:

$$\mathcal{L}_{\mathbf{X}}\mathbf{h}^a = \lambda^a{}_b\mathbf{h}^b \text{ and } \mathcal{L}_{\mathbf{X}}\omega^a{}_{bc} = 0, \tag{1}$$

where $\mathbf{h}^a$ is the orthonormal coframe basis, $\lambda^a{}_b$ is a Lie algebra generator of Lorentz transformations and $\omega^a{}_{bc}$ are the components of the spin connection.

In ref. [8], Coley et al. introduced a new approach to determine the symmetries of any geometry based on an independent frame and connection which admits the torsion tensor and the curvature tensor as geometric objects. In these theories, the connection is an independent object. They call any geometry where the non-metricity and curvature tensors vanish a *teleparallel geometry*. The approach relies on the existence of a particular class of invariantly defined frames known as symmetry frames, which facilitates solving the differential equations arising from Equation (1), by fixing the $\lambda^a{}_b$ in an invariant way.

This assumes an orthonormal frame where the gauge metric is $g_{ab} = diag[-1, 1, 1, 1]$. The spin connection, $\omega^a{}_{bc}$, is defined in terms of an arbitrary Lorentz transformation, $\Lambda^a{}_b$, through the equation

$$\omega^a{}_{bc} = \Lambda^a{}_d \mathbf{h}_c((\Lambda^{-1})^d{}_b). \tag{2}$$

A particular subclass of teleparallel gravitational theories is dynamically equivalent to GR and is called the Teleparallel Equivalent to General Relativity (TEGR), which is based on a torsion scalar T constructed from the torsion tensor [1]. The most common generalization of TEGR is $F(T)$-type teleparallel gravity, where F is an arbitrary function of the torsion scalar T [9–11]. In the *covariant* approach to $F(T)$-type gravity, the teleparallel geometry is defined in a gauge-invariant manner as a geometry where the spin connection has zero curvature and zero non-metricity. The spin connection will vanish in the special class of proper frames, and will be non-zero in all other frames [1,3,12]. Therefore, the resulting teleparallel gravity theory has Lorentz covariant FEs and is locally Lorentz invariant [13]. A proper frame is not invariantly defined since it is defined in terms of the connection, which is not a tensorial quantity, which leads to a number of problems when using such a frame to determine symmetries.

There are several papers in the literature about static and non-static spherically symmetric solutions in teleparallel $F(T)$ gravity [14–29]. There are several perturbative solutions in TEGR (Teleparallel Equivalent of General Relativity) and there are some power-law $F(T)$ solutions with power-law frame components (see [14–17] and references within). These papers essentially use the Weitzenback gauge (leading to proper frames) because the antisymmetric FEs are trivially satisfied, but there are arising some extra degrees of freedom (DoFs) by imposing the zero spin connection. This requirement leads to only symmetric parts of FEs and the presented solutions are essentially limited to power-law in $F(T)$ and frame components by using a complex coframe. Beyond these considerations, even if the symmetric parts of FEs and their solutions are similar between the different gauges, the fact remains that the extra DoF potential issue associated with the proper frame should be resolved by a frame changing. For this requirement, it is necessary to go towards a frame where the spin connections can be found by solving the non-trivial antisymmetric parts of FEs. From there, all the DoFs will be covered by all the FEs and the solutions will be found by a non-trivial approach for the spin-connection and coframe components and then for the $F(T)$ solutions.

For rectifying this extra DoF potential issue and going further than power-law $F(T)$ solutions, there is a paper on general teleparallel spherically symmetric geometries with an emphasis on vacuum solutions and possible additional symmetry structures [18]. They found the general FEs in an orthonormal gauge assuming a diagonal frame and a non-trivial spin connection, leading to specific antisymmetric parts of FEs and then to well-determined symmetric parts of FEs without extra DoFs. There are some specific symmetry structures such as static (radial coordinate dependent), Kantowski–Sacks (KS) (time coordinate dependent) and an additional affine symmetry called X_4. For static geometries, the study is restricted to find the $F(T)$ solutions in the vacuum. They found more power-law solutions, but also more general $F(T)$ solutions such as products, quotient, exponential and/or a mix of these type of functions. In this case, the X_4 symmetry will be defined by the time-coordinate derivative ∂_t leading to radial coordinate dependence for coframes, spin connections and FEs.

For non-vacuum spherically symmetric $F(T)$ solutions, there are in principle several possible types of energy-momentum sources. The most interesting are the perfect isotropic cosmological and astrophysical fluids, and there are some teleparallel $F(T)$ solutions such as Bahamonde and Camci's [20]. In this paper, there are some specific power-law $F(T)$ solutions leading to some specific types of fluid where they find specific expressions for P and ρ. But this type of approach is restrictive because this supposes first a power-law $F(T)$ solution and then they look for possible ρ and P. Alternatively, for finding new solutions, a different approach would assume an energy-momentum source with an equation of state (EoS) (relation between P and ρ as $P = P(\rho)$) and then find all possible $F(T)$ solutions satisfying the FEs with these EoS relations.

For this paper, we assume a static (r-coordinate dependent) spherically symmetric teleparallel geometry in an orthonormal gauge as defined in ref. [18]. We will focus on finding non-vacuum static spherically symmetric teleparallel $F(T)$ solutions. After a brief summary of the static spherically symmetric teleparallel geometry and FEs in Section 2, we will find in Section 3 several possible $F(T)$ solutions for the linear and isotropic perfect fluids. In Section 4, we will do the same with a dust fluid, because this special case arises from the conservation laws. In Section 5, we will solve FEs and find some $F(T)$ solutions for a non-linear perfect fluid.

We will use the notation as follows: the coordinate indices are $\mu, \nu, \ldots$ and the tangent space indices are $a, b, \ldots$ as in ref. [8]. The spacetime coordinates will be x^μ. The frame fields are denoted as $\mathbf{h}_a$ and the dual coframe one-forms are $\mathbf{h}^a$. The vierbein components are $h_a^{\ \mu}$ or $h^a_{\ \mu}$. The spacetime metric is $g_{\mu\nu}$ and the Minkowski tangent space metric is η_{ab}. For a local Lorentz transformation leaving η_{ab} unchanged, we write $\Lambda_a^{\ b}(x^\mu)$. The spin-connection one-form $\omega^a_{\ b}$ is defined as $\omega^a_{\ b} = \omega^a_{\ bc}\mathbf{h}^c$. The curvature and torsion tensors are, respectively, $R^a_{\ bcd}$ and $T^a_{\ bc}$. Covariant derivatives with respect to a metric-compatible connection are denoted using a semi-colon, $T_{abc;e}$.

2. Summary of Teleparallel Spherically Symmetric Spacetimes and Field Equations

2.1. Teleparallel Static Spherical Symmetry

The teleparallel spherically symmetric spacetimes were defined and discussed in detail with all necessary justifications in ref. [18]. From this last paper, there is a new coordinate system to "diagonalize" the frame described by a static spherically symmetric vierbein satisfying Equation (1), as follows:

$$h^a_{\ \mu} = \mathrm{Diag}[A_1(r),\, A_2(r),\, A_3(r),\, A_3(r)\sin(\theta)]. \tag{3}$$

This frame choice in Equation (3) is an invariant symmetry frame, and the most general static spherically symmetric spin connection is as follows [18]:

$$\begin{aligned}
&\omega_{341} = W_1(r),\ \omega_{342} = W_2(r),\ \omega_{233} = \omega_{244} = W_3(r),\ \omega_{234} = -\omega_{243} = W_4(r),\\
&\omega_{121} = W_5(r),\ \omega_{122} = W_6(r),\ \omega_{133} = \omega_{144} = W_7(r),\\
&\omega_{134} = -\omega_{143} = W_8(r),\ \omega_{344} = -\frac{\cos(\theta)}{A_3\sin(\theta)}.
\end{aligned} \tag{4}$$

We determine the most general connection by imposing the flatness condition on the geometry. The resulting eqns can be solved so that any spherically symmetric teleparallel geometry is defined by the three arbitrary functions in the vierbein described by Equation (3) and by the following spin-connection components [18]:

$$\begin{aligned}
W_1 = 0,\ W_2 = -\frac{\chi'}{A_2},\ W_3 &= \frac{\cosh(\psi)\cos(\chi)}{A_3}, W_4 = \frac{\cosh(\psi)\sin(\chi)}{A_3},\\
W_5 = 0, W_6 = -\frac{\psi'}{A_2},\ W_7 &= \frac{\sinh(\psi)\cos(\chi)}{A_3}, W_8 = \frac{\sinh(\psi)\sin(\chi)}{A_3},
\end{aligned} \tag{5}$$

where $\chi(r)$ and $\psi(r)$ are arbitrary functions of the radial coordinate r and then $\chi' = \partial_r\chi(r)$ and $\psi' = \partial_r\psi(r)$. We will subsequently find via the antisymmetric parts of the FEs the exact expressions for $\chi(r)$ and $\psi(r)$, and therefore the components of Equations (4) and (5).

Any choice of the arbitrary functions, ψ and χ, picks out a unique teleparallel geometry, as any change in the form of the spin connection which could affect the form of ψ or χ leads to a change in the form of the vierbein. For a given pair of functions, the invariantly defined frame up to the linear isotropy group $\bar{H}_q$ arising from the Cartan–Karlhede (CK) algorithm could be computed to provide further sub-classification. We note that there are only five arbitrary functions required to specify a geometry: A_1, A_2, A_3, ψ and χ [18]. We note that special subclasses of these teleparallel geometries have been studied earlier in teleparallel gravity using the Killing equations for an arbitrary spherically symmetric metric and using the non-invariant proper frame approach [25,30–32].

2.2. Summary of Teleparallel Field Equations

The teleparallel action integral is as follows [1–3,18]:

$$S_{F(T)} = \int d^4\,x \left[\frac{h}{2\kappa}\,F(T) + \mathcal{L}_{Matter}\right] \tag{6}$$

By applying the least-action principle to Equation (6), we obtain the symmetric and anti-symmetric parts of FEs [18]:

$$\kappa\,\Theta_{(ab)} \;=\; F'(T)\overset{\circ}{G}_{ab} + F''(T)\,S_{(ab)}^{\;\;\;\;\mu}\,\partial_\mu T + \frac{g_{ab}}{2}\left[F(T) - T\,F'(T)\right], \tag{7a}$$

$$0 \;=\; F''(T)\,S_{[ab]}^{\;\;\;\;\mu}\,\partial_\mu T, \tag{7b}$$

where $\overset{\circ}{G}_{ab}$ is the Einstein tensor, $\Theta_{(ab)}$ the energy momentum, T the torsion scalar, g_{ab} the gauge metric, $S_{ab}^{\;\;\;\mu}$ the superpotential (torsion dependent) and κ the coupling constant. The canonical energy momentum is obtained from the $\mathcal{L}_{Matter}$ term of Equation (6) and defined as follows:

$$\Theta_a^{\;\;\mu} = \frac{1}{h}\frac{\mathcal{L}_{Matter}}{\delta h^a_{\;\;\mu}} \tag{8}$$

The antisymmetric and symmetric parts of Equation (8) are, respectively [18]:

$$\Theta_{[ab]} = 0, \qquad \Theta_{(ab)} = T_{ab} \tag{9}$$

where T_{ab} is the symmetric part of the energy-momentum tensor. In Equation (9), we see that Θ_{ab} is a symmetric physical quantity. Equation (9) is valid especially for the case where the matter field interacts with the metric $g_{\mu\nu}$ associated with the coframe $h^a_{\;\;\mu}$ and the gauge g_{ab}, and is not intricately coupled to the $F(T)$ gravity. This consideration is valid in the case of this paper, because there is an absence of hypermomentum (i.e., $\mathfrak{T}^{\mu\nu} = 0$ as defined in ref. [16]). The conservation of energy momentum in teleparallel gravity states that $\Theta_a^{\;\;\mu}$ must satisfy the following relation [1,2]:

$$\nabla_\nu\left(\Theta^{\mu\nu}\right) = 0, \tag{10}$$

where ∇_ν is th covariant derivative and $\Theta^{\mu\nu}$ is the conserved energy-momentum tensor. Equation (10) is the same conservation of energy-momentum expression as GR. Satisfying Equation (10) is automatically required by the previous equations in cases of null hypermomentum ($\mathfrak{T}^{\mu\nu} = 0$ case). In non-zero hypermomentum situations ($\mathfrak{T}^{\mu\nu} \neq 0$ case), we will need to satisfy more complex conservation equations than Equation (10) as shown in ref. [16].

For a perfect and isotropic fluid with any EoS (linear or not), the T_{ab} tensor is defined as follows [33,34]:

$$T_{ab} = \left(P(\rho(r)) + \rho(r)\right)u_a\,u_b + g_{ab}\,P(\rho(r)), \tag{11}$$

where $P(\rho(r))$ is the EoS in terms of the static fluid density $\rho(r)$ and $u_a = (1, 0, 0, 0)$ for a stationary fluid. In some astrophysical applications, the pressure is sometimes modeled with radial and tangential components, especially for stellar modeling [33,34]. But this is not the specific purpose of this paper.

2.3. General Static Spherically Symmetric Perfect Fluid Field Equations

For the static spherically symmetric case, the antisymmetric FEs in terms of Equations (4) and (5) components are as follows [18]:

$$0 = \frac{F''(T)\,\partial_r T}{\kappa\, A_2\, A_3}\,[\cos\chi\,\sinh\psi] \quad \text{and} \quad 0 = \frac{F''(T)\,\partial_r T}{\kappa\, A_2\, A_3}\,[\sin\chi\,\cosh\psi]. \tag{12}$$

Assuming $T \neq$ constant, Equation (12) admits only the solution $\sin\chi = 0$ and $\sinh\psi = 0$, with $\chi = n\,\pi$ and $\psi = 0$ where $n \in \mathbb{Z}$ is an integer and $\cos\chi = \cos(n\,\pi) = \pm 1 = \delta$. Substituting these expressions for χ and ψ into Equation (5), we obtain that $W_3(r) = \frac{\delta}{A_3}$ is the only non-zero component (i.e., $W_i = 0$ for all $i \neq 3$) and find as Equation (4) for the non-vanishing spin-connection components:

$$\omega_{233} = \omega_{244} = \frac{\delta}{A_3},\ \omega_{344} = -\frac{\cos(\theta)}{A_3\sin(\theta)}. \tag{13}$$

Equation (13), for non-vanishing spin-connections components, goes in the same direction and improves the expressions obtained recently in refs. [3,16].
By using the solution for Equation (12), we find the three equations for the symmetric FEs [18]:

$$\partial_r\left[\ln\, F'(T(r))\right] = \frac{g_1(r)}{k_1(r)}, \tag{14a}$$

$$\kappa\,[\rho + P] = -\,2\,F''(T)\,(\partial_r\,T)\,k_2(r) + 2\,F'(T)\,g_2(r), \tag{14b}$$

$$\kappa\,\rho = -\,\frac{F(T)}{2} - 2\,F''(T)\,(\partial_r\,T)\,k_3(r) + 2\,F'(T)\,g_3(r), \tag{14c}$$

where the g_i and k_i components are expressed in Appendix A. In addition, because we are not in a vacuum, we need to satisfy the static conservation law [18]

$$[\rho + P]\,\partial_r(\ln\,A_1) + \partial_r\,P = 0, \tag{15}$$

where $A_1' = \partial_r\,A_1(r)$ and $P' = \partial_r\,P(r)$ are r radial coordinate derivatives. For $P = -\rho$, we obtain from Equation (15) that $P = P_0 = -\rho_0 =$ constant. In this case, we will obtain with Equations (14a)–(14c) the vacuum solutions, but with a $-2\kappa\,\rho_0$ shifting inside the $F(T)$ solutions [18,19]. From the FE components in Appendix A, **we have** $k_2(r) = k_3(r)$ **for all** A_i**,** $i = 1, 2, 3$. By also substituting Equation (14a) into Equations (14b) and (14c), we find that the FEs become the following:

$$F'(T(r)) = F'(T(0))\,\exp\left[\int dr'\,\frac{g_1(r')}{k_1(r')}\right], \tag{16a}$$

$$\kappa\,[\rho + P] = 2\,F'(T)\left[-\left(\frac{g_1(r)}{k_1(r)}\right)k_2(r) + g_2(r)\right], \tag{16b}$$

$$\kappa\,\rho = -\,\frac{F(T)}{2} + 2\,F'(T)\left[-\left(\frac{g_1(r)}{k_1(r)}\right)k_2(r) + g_3(r)\right]. \tag{16c}$$

From there, we have to solve Equations (15)–(16c) for non-vacuum solutions by using the torsion scalar expression and the g_i and k_i.

3. Perfect Linear Fluid Solutions

As the first case of a non-vacuum solution with an isotropic fluid having a linear EoS, we have $P(r) = \alpha\,\rho(r)$ with $-1 < \alpha < 0$ and $0 < \alpha \leq 1$ (i.e., $\alpha \neq 0$), the static perfect cosmological fluid case. First, Equation (15) will simplify as follows [18]:

$$(1+\alpha)\,(\ln\,A_1)' + \alpha\,(\ln\,\rho)' = 0, \tag{17}$$

where $\rho'(r) = \partial_r \rho(r)$. By integration, we find as a solution for Equation (17) the following:

$$\rho(r) = \rho_0 \left[A_1(r)\right]^{-\frac{(1+\alpha)}{\alpha}}. \tag{18}$$

In a such case, the density of the fluid $\rho(r)$ is directly dependent on $A_1(r)$ for $\alpha \neq 0$ and the energy condition constraints to satisfy $\rho(r) \geq 0$ for positive mass density. For $\alpha = 0$ (dust fluid), we will need to solve this case separately to avoid the singular solution for Equation (18). If we set an ansatz for the A_i, $\rho(r)$ will depend directly on this same ansatz according to the conservation laws. But since $\rho(r)$ depends only on $A_1(r)$, one can in principle perform a coordinate change $dt' \rightarrow A_1(r)\, dt$ for going to a frame where we have a constant and positive fluid density $\rho = \rho_0$ [18].

Then, although Equation (14a) remains unchanged, Equations (16b) and (16c) will simplify as follows:

$$\kappa\,\rho = \frac{2\,F'(T)}{(1+\alpha)}\left[-\frac{g_1(r)\,k_2(r)}{k_1(r)} + g_2(r)\right], \tag{19a}$$

$$\kappa\,\rho = -\frac{F(T)}{2} + 2\,F'(T)\left[-\frac{g_1(r)\,k_2(r)}{k_1(r)} + g_3(r)\right]. \tag{19b}$$

With Equations (19a) and (19b), we can put them together eliminating ρ to finally have a relation linking $F(T)$ and $F'(T)$:

$$F(T) = 4\,F'(T)\left[-\frac{\alpha}{1+\alpha}\,\frac{g_1(r)\,k_2(r)}{k_1(r)} + \left(g_3(r) - \frac{g_2(r)}{1+\alpha}\right)\right]. \tag{20}$$

The torsion scalar is as follows:

$$T(r) = -2\left(\frac{\delta}{A_3} + \frac{A_3'}{A_2\,A_3}\right)\left(\frac{\delta}{A_3} + \frac{A_3'}{A_2\,A_3} + \frac{2\,A_1'}{A_1\,A_2}\right). \tag{21}$$

There are a number of possible approaches for solutions to the FEs described by Equations (14a), (19a), (19b) and (21) added by the conservation law solution described by Equation (18). The main goal is to find several possible $F(T)$ solutions from these previous equations. For this purpose, we will solve for $A_3 =$ constant and $A_3 = r$ as in ref. [18]. We can do this because there is a set of coordinates where $A_3 = r$ is valuable without any loss of generality and the constant A_3 system is the exception to this rule. This consideration is only for a local coordinate definition. The constant A_3 case is an exception because we cannot perform a local transformation allowing this to change into a non-constant term. All other non-constant A_3 can be changed by a local transformation into an $A_3 = r$ system.

3.1. Constant A_3 Field Equation Solutions

By setting $A_3 = c_0 =$ constant in our FEs, Equations (14a), (19a) and (19b) become the following with Equation (A2) components:

$$F'(T) = F'(0)\,\exp\left[-\int dr\,\frac{\left(A_1'' - \frac{A_1'\,A_2'}{A_2} + \frac{A_1\,A_2^2}{c_0^2}\right)}{\left(A_1' + \frac{\delta\,A_1\,A_2}{c_0}\right)}\right], \tag{22a}$$

$$\kappa\,(1+\alpha)\,\rho = 2\,F'(T)\left[\frac{\left(A_1'' - \frac{A_1'\,A_2'}{A_2} + \frac{A_1\,A_2^2}{c_0^2}\right)}{\left(A_1' + \frac{\delta\,A_1\,A_2}{c_0}\right)}\,\frac{\delta}{A_2\,c_0}\right], \tag{22b}$$

$$\kappa\,\rho = -\frac{F(T)}{2} + 2\,F'(T)\left[\frac{\left(A_1'' - \frac{A_1'\,A_2'}{A_2} + \frac{A_1\,A_2^2}{c_0^2}\right)}{\left(A_1' + \frac{\delta\,A_1\,A_2}{c_0}\right)}\,\frac{\delta}{A_2\,c_0} - \frac{\delta\,A_1'}{A_1\,A_2\,c_0}\right]. \tag{22c}$$

Equation (21) for torsion scalar becomes the following:

$$T(r) = -\frac{2}{c_0^2} - \frac{4\delta\, A_1'}{c_0\, A_1\, A_2}. \tag{23}$$

3.1.1. Power-Law Solutions

We will solve Equations (22a)–(22c) by using the following ansatz:

$$A_1(r) = a_0\, r^a, \qquad A_2(r) = b_0\, r^b. \tag{24}$$

In the supplement, Equation (18) from the conservation laws becomes the following:

$$\rho(r) = \rho_1\, r^{-\frac{a(1+\alpha)}{\alpha}}. \tag{25}$$

where $\rho_1 = \rho_0\, a_0^{-\frac{a(1+\alpha)}{\alpha}}$ and $\alpha \neq 0$. Equations (22a)–(23) become the following:

$$F'(T) = F'(0)\, \exp\left[\int dr\, \frac{\left[a(1-a+b)\, r^{-2(b+1)} - \left(\frac{b_0}{c_0}\right)^2\right]}{r\left[a\, r^{-2(b+1)} + \delta\left(\frac{b_0}{c_0}\right) r^{-(b+1)}\right]}\right] \tag{26a}$$

$$\kappa\,\rho(r) = -\frac{2\delta\, F'(T(r))}{(1+\alpha)\, b_0 c_0}\left[\frac{\left[a(1-a+b)\, r^{-2(b+1)} - \left(\frac{b_0}{c_0}\right)^2\right]}{\left[a\, r^{-(b+1)} + \delta\left(\frac{b_0}{c_0}\right)\right]}\right], \tag{26b}$$

$$\kappa\,\rho(r) = -\frac{F(T(r))}{2} - \frac{2\delta\, F'(T(r))}{b_0 c_0}\left[a\, r^{-(b+1)} + \frac{\left[a(1-a+b)\, r^{-2(b+1)} - \left(\frac{b_0}{c_0}\right)^2\right]}{\left[a\, r^{-(b+1)} + \delta\left(\frac{b_0}{c_0}\right)\right]}\right]. \tag{26c}$$

$$T(r) = -\frac{2}{c_0^2} - \frac{4\delta a}{b_0\, c_0}\, r^{-(b+1)} \tag{26d}$$

For setting Equations (26a)–(26c) in terms of torsion scalar T, we isolate $r(T)$ from Equation (26d):

$$r^{-(b+1)}(T) = -\frac{\delta\, b_0\, c_0}{4a}\left(T + \frac{2}{c_0^2}\right) \tag{27}$$

By substituting Equation (27) into Equations (26a)–(26c) and by simplifying Equation (26a), we obtain the following:

$$F'(T) = F'(0)\left(T - \frac{2}{c_0^2}\right)^{\frac{2a}{(1+b)}-1}\left(T + \frac{2}{c_0^2}\right)^{-\frac{a}{(b+1)}} \exp\left[\frac{4a}{c_0^2\,(b+1)\left(T + \frac{2}{c_0^2}\right)}\right] \tag{28a}$$

$$\kappa\,\rho = \frac{2\,F'(T)}{(1+\alpha)}\left[\frac{\frac{(1+b-a)}{4a}\left(T + \frac{2}{c_0^2}\right)^2 - \frac{4}{c_0^4}}{\left(T - \frac{2}{c_0^2}\right)}\right], \tag{28b}$$

$$\kappa\,\rho = -\frac{F(T)}{2} + 2\,F'(T)\left[\frac{1}{4}\left(T + \frac{2}{c_0^2}\right) + \frac{\frac{(1+b-a)}{4a}\left(T + \frac{2}{c_0^2}\right)^2 - \frac{4}{c_0^4}}{\left(T - \frac{2}{c_0^2}\right)}\right]. \tag{28c}$$

By putting Equations (28b) and (28c) and then by substituting Equation (28a), we find the following as a solution for $F(T)$:

$$F(T) = F'(0)\left(T-\frac{2}{c_0^2}\right)^{\frac{2a}{(1+b)}-2}\left(T+\frac{2}{c_0^2}\right)^{-\frac{a}{(b+1)}}\exp\left[\frac{4a}{c_0^2\,(b+1)\left(T+\frac{2}{c_0^2}\right)}\right]$$
$$\times\left[\left(T+\frac{2}{c_0^2}\right)\left(T-\frac{2}{c_0^2}\right)+\frac{\alpha}{(1+\alpha)}\left(\frac{(1+b-a)}{a}\left(T+\frac{2}{c_0^2}\right)^2-\frac{16}{c_0^4}\right)\right], \quad (29)$$

where $a \neq 0$ and $b \neq -1$. Equation (29) is a new non-trivial $F(T)$ teleparallel solution arising from the $A_3 =$ constant case. Then, Equation (25) for the fluid density in terms of T will be expressed as follows:

$$\rho(T) = \rho_2\left(T+\frac{2}{c_0^2}\right)^{-\frac{a(1+\alpha)}{(1+b)\alpha}}. \quad (30)$$

where $\rho_2 = \rho_0\, a_0^{-\frac{a(1+\alpha)}{\alpha}}\left(-\frac{4\delta\, a}{b_0\, c_0}\right)^{\frac{a(1+\alpha)}{(1+b)\alpha}}$. Therefore, Equation (29) has two possible singularities:

- $T = -\frac{2}{c_0^2}$: This singularity appears in two terms of Equation (29) leading to an undefined $\lim_{T\to-\frac{2}{c_0^2}} F(T)$ and then $F(T)$ is undefined in all situations. For fluid density, Equation (30) will lead to the following situations:
  - $\frac{a}{(1+b)\alpha} > 0$ subcase: $\rho(T)$ is undefined.
  - $\frac{a}{(1+b)\alpha} < 0$ subcase: $\rho(T) = 0$, the vacuum situation.

  Then, Equation (27) will lead to the following situations:
  - $b > -1$ subcase: $r(T)$ is undefined.
  - $b < -1$ subcase: $r(T) \to 0$: a point-like singularity.
- $T = +\frac{2}{c_0^2}$: This singularity only occurs for $b \neq -1$ and $a < 1+b$. For Equations (27) and (30), there are no real consequences because we obtain definite values of $r(T)$ and $\rho(T)$. This is only that $\lim_{T\to+\frac{2}{c_0^2}} F(T) = \infty$.

For $a = b+1$, Equation (29) becomes the following:

$$F(T) = F'(0)\,\exp\left[\frac{4}{c_0^2\,T+2}\right]\left[\left(T-\frac{2}{c_0^2}\right)-\frac{16\alpha}{c_0^4(1+\alpha)}\left(T+\frac{2}{c_0^2}\right)^{-1}\right], \quad (31)$$

and then Equation (30) will simplify as follows:

$$\rho(T) = \rho_2\left(T+\frac{2}{c_0^2}\right)^{-\frac{(1+\alpha)}{\alpha}}, \quad (32)$$

where $\rho_2 = \rho_0\, a_0^{-\frac{a(1+\alpha)}{\alpha}}\left(-\frac{4\delta\, a}{b_0\, c_0}\right)^{\frac{(1+\alpha)}{\alpha}}$. The $T = -\frac{2}{c_0^2}$ singularity is now the remaining one inside Equation (31) and leads to an undefined $F(T)$. We obtain from Equation (32) that the fluid density is as follows:

- $\alpha > 0$ subcase: $\rho(T)$ is undefined.
- $\alpha < 0$ subcase: $\rho(T) = 0$, the vacuum situation.

For Equation (27), we find the following:

- $b > -1$ subcase: $r(T)$ is undefined.

- $b < -1$ subcase: $r(T) \rightarrow 0$, a point-like singularity.

For $a = 0$ and/or $b = -1$, these are constant torsion scalar spacetime cases according to Equation (26d) and are GR solutions.

3.1.2. Constant A_2 and Exponential A_1 Solutions

Another possible ansatz for $F(T)$ solutions is $A_1(r) = a_0(1 - e^{-ar})$ and $A_2(r) = b_0 = 1$. We replace the component A_1 of the simple power-law ansatz expressed in Equation (24) by an infinite series of power laws leading to an exponential ansatz defined as $A_1(r) = a_0 \sum_{k\geq 0} \frac{(-1)^k}{(k+1)!} (a\,r)^{k+1}$. We then set $b = 0$ for the component A_2 of this same ansatz thus generalizing the power-law ansatz as expressed in Equation (24). Then, Equation (23) becomes the following:

$$T(r) = -\frac{2}{c_0^2} - \frac{4\delta\,a}{c_0\,(e^{ar}-1)}$$
$$\Rightarrow\; e^{-a\,r(T)} = \frac{\left(T + \frac{2}{c_0^2}\right)}{\left(T + \frac{2}{c_0^2}(1 - 2\delta\,a\,c_0)\right)} \tag{33}$$

Equations (22a)–(22c) become the following:

$$F'(T) = F'(0)\left(T - \frac{2}{c_0^2}\right)^{\frac{(\delta + a\,c_0)}{(\delta - a\,c_0)}} \left(T + \frac{2}{c_0^2}\right)^{\frac{\delta}{a\,c_0}} \left(T + \frac{2}{c_0^2}(1 - 2\delta\,a\,c_0)\right)^{-\frac{(1 + a^2\,c_0^2)}{a\,c_0(\delta - a\,c_0)}}, \tag{34a}$$

$$-\kappa\,\rho = \frac{F'(T)}{(1+\alpha)}\left[\frac{\frac{8}{c_0^4} + \frac{2\delta a}{c_0}\left(T + \frac{2}{c_0^2}\right)}{\left(T - \frac{2}{c_0^2}\right)}\right], \tag{34b}$$

$$-\kappa\,\rho = \frac{F(T)}{2} + F'(T)\left[\frac{\left(\frac{8}{c_0^4} + \frac{2\delta a}{c_0}\left(T + \frac{2}{c_0^2}\right)\right)}{\left(T - \frac{2}{c_0^2}\right)} - \frac{\left(T + \frac{2}{c_0^2}\right)}{2}\right]. \tag{34c}$$

By putting Equations (34b) and (34c) together and then by substituting Equation (34a), we find the following:

$$F(T) = F'(0)\left(T - \frac{2}{c_0^2}\right)^{\frac{(\delta + a\,c_0)}{(\delta - a\,c_0)}} \left(T + \frac{2}{c_0^2}\right)^{\frac{\delta}{a\,c_0}} \left(T + \frac{2}{c_0^2}(1 - 2\delta\,a\,c_0)\right)^{-\frac{(1 + a^2\,c_0^2)}{a\,c_0(\delta - a\,c_0)}}$$
$$\times\left[-\frac{2\alpha}{(1+\alpha)}\frac{\left(\frac{8}{c_0^4} + \frac{2\delta a}{c_0}\left(T + \frac{2}{c_0^2}\right)\right)}{\left(T - \frac{2}{c_0^2}\right)} + \left(T + \frac{2}{c_0^2}\right)\right], \tag{35}$$

where $a \neq \left\{0, \frac{\delta}{c_0}\right\}$. Equation (35) is another new non-trivial $F(T)$ teleparallel solution with $A_3 =$ constant. Then, Equation (18) for the fluid density in terms of T will be as follows:

$$\rho(T) = \rho_3\left(T + \frac{2}{c_0^2}(1 - 2\delta\,a\,c_0)\right)^{\frac{(1+\alpha)}{\alpha}}, \tag{36}$$

where $\rho_3 = \rho_0\left(-\frac{4\delta\,a_0\,a}{c_0}\right)^{-\frac{(1+\alpha)}{\alpha}}$ and $\alpha \neq 0$. From Equation (35), we find three singularities:

- $T = -\frac{2}{c_0^2}$: This singularity arises when $\frac{\delta}{a\,c_0} < 0$ and $\lim_{T\to -\frac{2}{c_0^2}} F(T)$ is undefined. There is no impact on $\rho(T)$, only a limit of $\rho(T) = \rho_0$. We have that $r(T) \to \infty$ for $a > 0$ and $r(T)$ is undefined for $a < 0$ according to Equation (33).
- $T = \frac{2}{c_0^2}$: This singularity arises when $ac_0 < 0$ only and leads to an undefined $\lim_{T\to \frac{2}{c_0^2}} F(T)$. There are no real consequences on $\rho(T)$ and $r(T)$: these quantities will be constant.
- $T = \frac{2}{c_0^2}(2\delta\,a\,c_0 - 1)$: This case arises when $0 < a\,c_0 < 1$ for $\delta = +1$ and $-1 < a\,c_0 < 0$ for $\delta = -1$ only, all leading to an undefined $\lim_{T\to \frac{2}{c_0^2}(2\delta\,a\,c_0-1)} F(T)$, an undefined $\rho(T)$ for $\alpha < 0$ and to $\rho(T) = 0$ (vacuum case) for $\alpha > 0$ according to Equation (36). For Equation (33), we find that $r(T) \to \infty$ if $a < 0$ (point-like singularity) and $r(T)$ is undefined if $a > 0$.

For $a = \frac{\delta}{c_0}$, Equation (33) will be as follows:

$$e^{-\frac{\delta}{c_0}\,r(T)} = \frac{\left(T + \frac{2}{c_0^2}\right)}{\left(T - \frac{2}{c_0^2}\right)}, \tag{37}$$

where $T \neq \frac{2}{c_0^2}$. Then, Equations (34a)–(34c) will become the following:

$$F'(T) = F'(0)\,\frac{\left(T + \frac{2}{c_0^2}\right)}{\left(T - \frac{2}{c_0^2}\right)}\,\exp\left[-2\,\frac{\left(T + \frac{2}{c_0^2}\right)}{\left(T - \frac{2}{c_0^2}\right)}\right], \tag{38a}$$

$$-\kappa\,\rho = \frac{2F'(T)}{c_0^2(1+\alpha)}\,\frac{\left(T + \frac{6}{c_0^2}\right)}{\left(T - \frac{2}{c_0^2}\right)}, \tag{38b}$$

$$-\kappa\,\rho = \frac{F(T)}{2} + \frac{2F'(T)}{c_0^2}\left[\frac{\left(T + \frac{6}{c_0^2}\right)}{\left(T - \frac{2}{c_0^2}\right)} - \frac{c_0^2}{4}\left(T + \frac{2}{c_0^2}\right)\right]. \tag{38c}$$

By putting Equations (38b) and (38c) and then by substituting Equation (38a) inside, we find the following:

$$F(T) = \frac{F'(0)}{c_0^2}\,\frac{\left(T + \frac{2}{c_0^2}\right)}{\left(T - \frac{2}{c_0^2}\right)^2}\exp\left[-2\,\frac{\left(T + \frac{2}{c_0^2}\right)}{\left(T - \frac{2}{c_0^2}\right)}\right]\left[c_0^2\left(T + \frac{2}{c_0^2}\right)\left(T - \frac{2}{c_0^2}\right) - \frac{4\alpha}{(1+\alpha)}\left(T + \frac{6}{c_0^2}\right)\right]. \tag{39}$$

Then, Equation (36) for fluid density will be as follows:

$$\rho(T) = \rho_3\left(T - \frac{2}{c_0^2}\right)^{\frac{(1+\alpha)}{\alpha}}, \tag{40}$$

where $\rho_3 = \rho_0\left(-\frac{4\,a_0}{c_0^2}\right)^{-\frac{(1+\alpha)}{\alpha}}$ and $\alpha \neq 0$. In the case of Equation (39), the only and remaining singularity is $T = \frac{2}{c_0^2}$ leading to an undefined $F(T)$, $\rho(T) = 0$ for $\alpha > 0$ (vacuum) and an undefined $\rho(T)$ for $\alpha < 0$ according to Equation (40), all with $r \to \infty$ and $\frac{\delta}{c_0} < 0$ from Equation (37).

For $a = 0$, we obtain from Equation (33) that the torsion scalar is constant (i.e., $T = -\frac{2}{c_0^2}$), $A_1 = a_0$ and $\rho(T) = \rho_0\,a_0^{-\frac{(1+\alpha)}{\alpha}}$ = constant leading to GR solutions.

In comparison with ref. [18], we obtain as a result for the pure vacuum case a linear $F(T)$, which is a GR solution. But for a perfect fluid with $\alpha \neq 0$, we find some new and non-trivial teleparallel $F(T)$ specific solutions. These are all new teleparallel fluid non-vacuum solutions for the $A_3 =$ constant class.

3.2. $A_3 = r$ Field Equation Solutions

For $A_3 = r$ FEs, Equations (14a), (19a) and (19b) become the following with Equation (A3) components:

$$\begin{aligned}
F'(T) =& F'(0)\, \exp\left[\int \frac{dr}{A_2\, r}\, \frac{\left[-A_2\, r^2\, A_1'' + A_1\, A_2 + (A_1\, A_2)'\, r + r^2\, A_1'\, A_2' - A_1\, A_2^3\right]}{\left[A_1 + r\, A_1' + \delta\, A_1\, A_2\right]}\right], && \text{(41a)}\\
\kappa\,(1+\alpha)\,\rho =& 2\, F'(T)\left[-\frac{\left[-A_2\, r^2\, A_1'' + A_1\, A_2 + (A_1\, A_2)'\, r + r^2\, A_1'\, A_2' - A_1\, A_2^3\right]}{\left[A_1 + r\, A_1' + \delta\, A_1\, A_2\right]}\, \frac{(1+\delta\, A_2)}{A_2^3\, r^2}\right.\\
&\left. + \frac{(A_1\, A_2)'}{A_1\, A_2^3\, r}\right], && \text{(41b)}\\
\kappa\,\rho + \frac{F(T)}{2} =& 2\, F'(T)\left[-\frac{\left[-A_2\, r^2\, A_1'' + A_1\, A_2 + (A_1\, A_2)'\, r + r^2\, A_1'\, A_2' - A_1\, A_2^3\right]}{\left[A_1 + r\, A_1' + \delta\, A_1\, A_2\right]}\, \frac{(1+\delta\, A_2)}{A_2^3\, r^2}\right.\\
&\left. + \frac{1}{A_1\, A_2^3\, r^2}\left[-A_1\, A_2 - A_2\, r\, A_1' - \delta\, A_1\, A_2^2 + A_1\, r\, A_2' - \delta\, A_2^2\, r\, A_1'\right]\right]. && \text{(41c)}
\end{aligned}$$

Equation (21) for torsion scalar becomes the following:

$$T(r) = \quad -\frac{2}{r^2\, A_2^2}\left[(\delta\, A_2 + 1)^2 + \frac{2\, r\, A_1'}{A_1}\, (\delta\, A_2 + 1)\right]. \tag{42}$$

There are a number of approach for solving Equations (41a)–(42) to find specific pure $F(T)$ new solutions in the general perfect fluid case with $\alpha \neq 0$. For conservation laws, $\rho(r)$ is still described by Equation (18), because $\rho(r)$ depends only on the $A_1(r)$ component.

3.2.1. General Power-Law Field Equations

For FEs and conservation law in terms of power-law solutions, we will use the Equation (24) ansatz in Equations (25) and (41a)–(42). From there, we obtain the following:

$$\begin{aligned}
F'(T(r)) =& F'(T(0))\, \exp\left[\int dr\, \frac{\left[(2a - a^2 + ab + b + 1)\, r^{-2b} - b_0^2\right]}{\left[(a+1)\, r^{1-2b} + \delta\, b_0\, r^{1-b}\right]}\right] && \text{(43a)}\\
\kappa\,(1+\alpha)\,\rho =& \frac{2\, F'(T)}{b_0^2}\left[-\frac{\left[(2a - a^2 + ab + b + 1)\, r^{-2b} - b_0^2\right]}{\left[(a+1)\, r^{1-2b} + \delta\, b_0\, r^{1-b}\right]}\left[r^{-2b-1} + \delta\, b_0\, r^{-b-1}\right] + (a+b)\, r^{-2b-2}\right], && \text{(43b)}\\
\kappa\,\rho =& -\frac{F(T)}{2} + \frac{2\, F'(T)}{b_0^2}\left[-\frac{\left[(2a - a^2 + ab + b + 1)\, r^{-2b} - b_0^2\right]}{\left[(a+1)\, r^{2(1-b)-1} + \delta\, b_0\, r^{(1-b)}\right]}\left[r^{-2b-1} + \delta\, b_0\, r^{-b-1}\right]\right.\\
&\left. + (-a + b - 1)\, r^{-2b-2} - \delta\, b_0\, (a+1)\, r^{-b-2}\right], && \text{(43c)}\\
T(r) =& -\frac{2}{b_0^2}\left[b_0^2\, r^{-2} + 2\delta\, b_0\, (1+a)\, r^{-2-b} + (2\,a+1)\, r^{-2-2b}\right], && \text{(43d)}\\
\rho(r) =& \rho_1\, r^{-\frac{a\,(1+\alpha)}{\alpha}}, && \text{(43e)}
\end{aligned}$$

where $\rho_1 = a_0^{-\frac{a(1+\alpha)}{\alpha}} =$ constant and $\alpha \neq 0$. From Equation (43d), we find the following characteristic eqn for $r(T)$:

$$0 = \frac{b_0^2\, T}{2} + b_0^2\, r^{-2} + 2\delta\, b_0\, (1+a)\, r^{-2-b} + (2\,a+1)\, r^{-2-2b}. \tag{44}$$

From Equation (44), we can in principle isolate for each value of a and b a relation $r(T)$ for finding a specific solution $F(T)$, which is the main aim of this rigorous work.

3.2.2. Simple Spacetime Solutions

Before going to more complex solutions, it is important to consider the simplest case of pure flat cosmological spacetimes where $\mathbf{a} = \mathbf{b} = \mathbf{0}$. In this case, Equation (44) becomes the following:

$$\begin{aligned} 0 =& \frac{b_0^2\, T}{2} + (1+\delta\, b_0)^2\, r^{-2}, \\ \Rightarrow\; r^{-2}(T) =& \frac{b_0^2}{2(1+\delta\, b_0)^2}(-T). \end{aligned} \tag{45}$$

Then, Equations (43a)–(43c) and Equation (25) become the following:

$$F'(T) = F'(0)\left(\frac{b_0}{\sqrt{2}(1+\delta\, b_0)}\right)^{\delta b_0 - 1}(-T)^{\frac{\delta b_0 - 1}{2}} \tag{46a}$$

$$\kappa\,\rho = \frac{(1-\delta b_0)}{(1+\delta\, b_0)(1+\alpha)}\, T\, F'(T), \tag{46b}$$

$$\kappa\,\rho = -\frac{F(T)}{2} + \frac{(2-\delta b_0)}{(1+\delta\, b_0)}\, T\, F'(T), \tag{46c}$$

$$\rho = \rho_1 = \rho_0\, a_0^{-\frac{a(1+\alpha)}{\alpha}} = \text{const.} \tag{46d}$$

Equations (46a)–(46c) are expressed in terms of T, $F(T)$ and $F'(T)$ only. By putting Equations (46b) and (46c) together, we find the following:

$$F(T) = F(0)\, T^{\frac{(1+\delta\, b_0)(1+\alpha)}{2(1+\alpha(2-\delta b_0))}}, \tag{47}$$

where $F(0)$ is an integration constant. Equation (47) is a pure power-law solution for static simple cosmological spacetimes where $\alpha \neq 0$, which is similar to Bahamonde–Camci solutions [20]. If $b_0 = \delta$, we find that Equation (47) will be reduced to the TEGR-like solution $F(T) = F(0)\, T$. For a pure flat null torsion spacetime, we require that $b_0 = -\delta$ in Equations (45)–(46c), which leads to $T = 0$ and $F(T) = F(0) = \text{constant}$ without any other condition. If then $F(0) = 0$, we obtain the pure Minkowski spacetime [35].

3.2.3. General Case Solutions

We will consider different cases according to the value of b for the general case (i.e., $a \neq \left\{-1, -\frac{1}{2}\right\}$):

1. $\mathbf{b} = \mathbf{0}$ case: Equation (44) will be as follows:

$$\begin{aligned} 0 =& \frac{b_0^2\, T}{2} + \left(b_0^2 + 2\delta\, b_0\,(1+a) + 2\,a + 1\right) r^{-2}, \\ \Rightarrow\; r^{-1}(T) =& \pm\frac{b_0}{\sqrt{2}\sqrt{b_0^2 + 2\delta\, b_0\,(1+a) + 2\,a + 1}}(-T)^{1/2}, \end{aligned} \tag{48}$$

where $T \leq 0$. As for the simple case presented in Section 3.2.2, we substitute Equation (48) into Equations (43a)–(43c) by setting $b = 0$. After that, by putting Equations (43b) and (43c) together, and then by substituting Equation (43a), we find the following:

$$\begin{aligned} F(T) =& 4F'(0)\left(2\left(b_0^2 + 2\delta\, b_0\,(1+a) + 2\,a + 1\right)\right)^{-\frac{\left(a^2+(1+\delta\, b_0)^2\right]}{2(a+1+\delta\, b_0)}}\, b_0^{-\frac{\left[2a-a^2+1-b_0^2\right]}{(a+1+\delta\, b_0)}} \\ &\times\left[\frac{\alpha\,\left[a^2-2a-1+b_0^2\right]}{(1+\alpha)(a+1+\delta\, b_0)}(1+\delta\, b_0) - \frac{a}{(1+\alpha)} - (a+1)(1+\delta\, b_0)\right] \\ &\times (-T)^{\frac{\left(a^2+(1+\delta\, b_0)^2\right]}{2(a+1+\delta\, b_0)}}, \\ =&\; F_1\,(-T)^{\frac{\left(a^2+(1+\delta\, b_0)^2\right]}{2(a+1+\delta\, b_0)}}, \end{aligned} \tag{49}$$

where F_1 is a constant. Here, we have a power-law solution similar to the Bahamonde–Camci solution where $\rho(r)$ is described by Equation (25) [20]. In terms of torsion scalar, this Equation (25) becomes the following:

$$\rho(T) = \rho_1 \left(\frac{b_0^2}{2\left(b_0^2 + 2\delta\, b_0\,(1+a) + 2\,a + 1\right)} \right)^{\frac{a(1+\alpha)}{2\alpha}} (-T)^{\frac{a(1+\alpha)}{2\alpha}}, \tag{50}$$

where $\alpha \neq 0$. We have a direct density-linked $F(T)$ solution in this case.

2. $\mathbf{b = 1}$ case: Equation (44) will be as follows:

$$0 = \frac{b_0^2\,T}{2} + b_0^2\,r^{-2} + 2\delta\,b_0\,(1+a)\,r^{-3} + (2\,a+1)\,r^{-4}, \tag{51}$$

The solutions are as follows:

$$\begin{aligned}
&r^{-1}(T) = \\
&\frac{\delta_1\,b_0(a+1)}{2(2a+1)} - \frac{\delta_1}{2}\Bigg[\Bigg(-\frac{2b_0^2}{3(2a+1)} + \frac{b_0^2(a+1)^2}{(2a+1)^2} \\
&+ \frac{1}{6\sqrt[3]{2}(2a+1)}\Bigg(\Bigg(16b_0^6 + \Big(432(a+1)^2 - 288\,(2a+1)\Big)\,b_0^4\,T \\
&+ \sqrt{\left(\left(16b_0^6 + (432(a+1)^2 - 288(2a+1))\,b_0^4\,T\right)^2 - 4b_0^6\left(4b_0^2 + 24(2a+1)\,T\right)^3\right)}\Bigg)^{1/3}\Bigg) \\
&+ \frac{2^{4/3}b_0^2(b_0^2 + 6\,(2a+1)\,T)}{3(2a+1)}\Bigg(16b_0^6 + \Big(432(a+1)^2 - 288\,(2a+1)\Big)\,b_0^4\,T \\
&+ \sqrt{\left(\left(16b_0^6 + (432(a+1)^2 - 288(2a+1))\,b_0^4\,T\right)^2 - 4b_0^6\left(4b_0^2 + 24(2a+1)\,T\right)^3\right)}\Bigg)^{-1/3}\Bigg)\Bigg]^{1/2} \\
&- \frac{\delta_2}{2}\Bigg[\Bigg(-\frac{4b_0^2}{3(2a+1)} + \frac{2b_0^2(a+1)^2}{(2a+1)^2} \\
&- \frac{1}{6\sqrt[3]{2}(2a+1)}\Bigg(\Bigg(16b_0^6 + \Big(432(a+1)^2 - 288\,(2a+1)\Big)\,b_0^4\,T \\
&+ \sqrt{\left(\left(16b_0^6 + (432(a+1)^2 - 288(2a+1))\,b_0^4\,T\right)^2 - 4b_0^6\left(4b_0^2 + 24(2a+1)\,T\right)^3\right)}\Bigg)^{1/3}\Bigg) \\
&+ \frac{2^{4/3}b_0^2(b_0^2 + 6\,(2a+1)\,T)}{3(2a+1)}\Bigg(16b_0^6 + \Big(432(a+1)^2 - 288\,(2a+1)\Big)\,b_0^4\,T \\
&+ \sqrt{\left(\left(16b_0^6 + (432(a+1)^2 - 288(2a+1))\,b_0^4\,T\right)^2 - 4b_0^6\left(4b_0^2 + 24(2a+1)\,T\right)^3\right)}\Bigg)^{-1/3}\Bigg) \\
&+ \delta_1\,\frac{8\delta\,b_0^3\,a^2\,(a+1)}{(2a+1)^3}\Bigg[4\Bigg[\Bigg(-\frac{2b_0^2}{3(2a+1)} + \frac{b_0^2(a+1)^2}{(2a+1)^2} \\
&+ \frac{1}{6\sqrt[3]{2}(2a+1)}\Bigg(\Bigg(16b_0^6 + \Big(432(a+1)^2 - 288\,(2a+1)\Big)\,b_0^4\,T \\
&+ \sqrt{\left(\left(16b_0^6 + (432(a+1)^2 - 288(2a+1))\,b_0^4\,T\right)^2 - 4b_0^6\left(4b_0^2 + 24(2a+1)\,T\right)^3\right)}\Bigg)^{1/3}\Bigg) \\
&+ \frac{2^{4/3}b_0^2(b_0^2 + 6\,(2a+1)\,T)}{3(2a+1)}\Bigg(16b_0^6 + \Big(432(a+1)^2 - 288\,(2a+1)\Big)\,b_0^4\,T \\
&+ \sqrt{\left(\left(16b_0^6 + (432(a+1)^2 - 288(2a+1))\,b_0^4\,T\right)^2 - 4b_0^6\left(4b_0^2 + 24(2a+1)\,T\right)^3\right)}\Bigg)^{-1/3}\Bigg)\Bigg]^{1/2}\Bigg]^{-1}\Bigg]^{1/2},
\end{aligned} \tag{52}$$

where $(\delta_1, \delta_2) = (\pm 1, \pm 1)$. By putting Equations (43b) and (43c) together and then by substituting Equations (43a) and (52), we find as a solution:

$$\begin{aligned} F(T) =& \frac{4\,F'(0)}{b_0^2}\, e^{-\delta\, b_0\, r(T)}\, \left(a+1+\delta\, b_0\, r(T)\right)^{\frac{2a^2}{a+1}-1} \left[r(T)\right]^{\frac{2+3a-a^2}{a+1}} \\ &\times \Bigg[-\frac{\alpha\left[\left(3a-a^2+2\right) r^{-2}(T) - b_0^2\right]}{(1+\alpha)\left[(a+1)\, r^{-1}(T)+\delta\, b_0\right]} \left(r^{-1}(T)+\delta\, b_0\right) r^{-2}(T) \\ &- \frac{(a+1)}{(1+\alpha)}\, r^{-4}(T) - a\, r^{-4}(T) - \delta\, b_0\,(a+1)\, r^{-3}(T)\Bigg], \end{aligned} \tag{53}$$

where $r(T)$ is Equation (52).

3. $\mathbf{b = -1}$ case: Equation (44) will be as follows:

$$\begin{aligned} 0 =& \left(\frac{b_0^2\, T}{2} + 2\,a + 1\right) + 2\delta\, b_0\,(1+a)\, r^{-1} + b_0^2\, r^{-2}, \\ &\Rightarrow\ r^{-1}(T) = -\frac{\delta\,(a+1)}{b_0} \pm \sqrt{\frac{a^2}{b_0^2} - \frac{T}{2}} \end{aligned} \tag{54}$$

By putting Equations (43b) and (43c) together and then by substituting Equations (43a) and (54), we find the following:

$$\begin{aligned} F(T) =& \frac{4\,F'(0)}{b_0^2}\, \left[r(T)\right]^{a+1} \left[(a+1)\, r(T) + \delta\, b_0\right]^{-\frac{2a^2+a+1}{a+1}} \exp\left(\frac{\delta\, b_0}{r(T)}\right) \\ &\times \Bigg[-\frac{\alpha\left[a(1-a)\, r^2(T) - b_0^2\right]\left(r(T)+\delta\, b_0\right)}{(1+\alpha)\left[(a+1)\, r^3(T) + \delta\, b_0\, r^2(T)\right]} + \frac{(1-a)}{(1+\alpha)} - (a+2) \\ &- \delta\, b_0\,(a+1)\, r^{-1}(T)\Bigg], \end{aligned} \tag{55}$$

where $r(T)$ is Equation (54).

4. $\mathbf{b = -2}$ case: Equation (44) will be as follows:

$$\begin{aligned} 0 =& b_0^2 + \left(\frac{b_0^2\, T}{2} + 2\delta\, b_0\,(1+a)\right) r^2 + (2\,a+1)\, r^4, \\ &\Rightarrow\ r(T) = \pm\frac{1}{2\sqrt{2a+1}}\, \sqrt{\delta_2\, b_0 \sqrt{16a^2 + 8\delta\,(a+1)\, b_0\, T + b_0^2\, T^2} - 4\delta\, b_0 (a+1) - b_0^2\, T}. \end{aligned} \tag{56}$$

Then, we will set the positive $r(T)$ case and $\delta_2 = \pm 1$. By putting Equations (43b) and (43c) together and then by substituting Equations (43a) and (56), we find the following:

$$\begin{aligned} F(T) =& \frac{4\,F'(0)}{b_0^2} \Bigg[\frac{\alpha\left[\left(a^2+1\right) r^4(T) + b_0^2\right]\left[r^2(T)+\delta\, b_0\right]}{(1+\alpha)\, r^2(T)\left[(a+1)\, r^2(T)+\delta\, b_0\right]} - \frac{(a+2\alpha)}{(1+\alpha)}\, r^2(T) \\ &- (a+1)\left[r^2(T)+\delta\, b_0\right]\Bigg] \left[(a+1) r^2(T) + b_0 + \delta\right]^{-\frac{(a^2+a+1)}{(a+1)}} r^{(a+1)}(T)\, \exp\left(\frac{\delta\, b_0}{2\, r^2(T)}\right), \end{aligned} \tag{57}$$

where $r(T)$ is Equation (56).

3.2.4. $a = -1$ Case Solutions

For solving $\mathbf{a} = -\mathbf{1}$ specific cases, Equation (44) will simplify as follows:

$$0 = \frac{b_0^2\,T}{2} + b_0^2\,r^{-2} - r^{-2-2b}. \tag{58}$$

Then, Equations (43a)–(43e) and (43e) will be simplified as follows:

$$F'(T) = F'(0)\,\exp\left[-\frac{\delta}{b_0}\int dr\left(2\,r^{-b-1} + b_0^2 r^{b-1}\right)\right] \tag{59a}$$

$$\kappa\,\rho = \frac{2\,F'(T)}{b_0^2\,(1+\alpha)}\left[\frac{\delta}{b_0}\left[2\,r^{-b-1} + b_0^2\,r^{b-1}\right]\left[r^{-2b-1} + \delta\,b_0\,r^{-b-1}\right] + (b-1)\,r^{-2b-2}\right], \tag{59b}$$

$$\kappa\,\rho = -\frac{F(T)}{2} + \frac{2\,F'(T)}{b_0^2}\left[\frac{\delta}{b_0}\left[2\,r^{-b-1} + b_0^2\,r^{b-1}\right]\left[r^{-2b-1} + \delta\,b_0\,r^{-b-1}\right] + b\,r^{-2b-2}\right], \tag{59c}$$

$$\rho = \rho_1\,r^{\frac{(1+\alpha)}{\alpha}}, \tag{59d}$$

where $\rho_1 = \rho_0\,a_0^{\frac{(1+\alpha)}{\alpha}}$ is a constant.

From Equation (58), there are several new subcases arising and leading to new $F(T)$ solutions for Equations (59a)–(59c). These subcases are as follows:

1. $\mathbf{b} = \mathbf{0}$: Equation (58) becomes the following:

$$0 = \frac{b_0^2\,T}{2} + \left(b_0^2 - 1\right) r^{-2}.$$

$$r^{-2}(T) = \left(\frac{b_0^2}{2(1-b_0^2)}\right) T \tag{60}$$

By putting together Equations (59b) and (59c) and then by substituting Equations (59a) and (60), we obtain a power-law $F(T)$ solution:

$$F(T) = \left[\frac{4F'(0)}{b_0^2(1+\alpha)}\left(\frac{b_0^2}{2(1-b_0^2)}\right)^{1+\frac{\delta}{b_0}\left(1+\frac{b_0^2}{2}\right)}\left[\delta\alpha\left(2+b_0^2\right)(1+\delta\,b_0) + b_0\right]\right] T^{1+\frac{\delta}{b_0}\left(1+\frac{b_0^2}{2}\right)}$$
$$= F_2\,T^{1+\frac{\delta}{b_0}\left(1+\frac{b_0^2}{2}\right)}, \tag{61}$$

where F_2 is a constant. Equation (61) is a pure power-law solution and this is similar to the Bahamonde–Camci solution [20].

2. $\mathbf{b} = \frac{\mathbf{1}}{\mathbf{2}}$: Equation (58) becomes the following:

$$0 = \frac{b_0^2\,T}{2} + b_0^2\,r^{-2} - r^{-3},$$

$$\Rightarrow\; r^{-1}(T) = \frac{1}{3}\left[b_0^2 + \frac{2^{2/3}\,b_0^4}{\left[3^{3/2}\sqrt{27b_0^4\,T^2 + 8b_0^8\,T + 27b_0^2\,T + 4b_0^6}\right]^{1/3}} + \frac{\left[3^{3/2}\sqrt{27b_0^4\,T^2 + 8b_0^8\,T + 27b_0^2\,T + 4b_0^6}\right]^{1/3}}{2^{2/3}}\right]. \tag{62}$$

By putting Equations (59b) and (59c) together, and then by substituting Equations (59a) and (62), we find the following:

$$F(T) = \frac{4\,F'(0)}{b_0^2}\left[\frac{\delta\,\alpha}{b_0(1+\alpha)}\left(2\,r^{-1}(T)+b_0^2\right)\left(r^{-\frac{1}{2}}(T)+\delta\,b_0\right)r^{-2}(T)+\frac{(2+\alpha)}{2(1+\alpha)}\,r^{-3}(T)\right] \times \exp\left[\frac{2\delta}{b_0}\left(2\,r^{-\frac{1}{2}}(T)-b_0^2 r^{\frac{1}{2}}(T)\right)\right], \tag{63}$$

where $r^{-1}(T)$ is Equation (62).

3. $\mathbf{b} = -\frac{\mathbf{1}}{\mathbf{2}}$: Equation (58) becomes the following:

$$0 = r^{-2} - \frac{1}{b_0^2}\,r^{-1} + \frac{T}{2} \Rightarrow\; r^{-1}(T) = \frac{1}{2b_0^2}\left[1+\delta_1\sqrt{1-2\,b_0^4\,T}\right] \tag{64}$$

where $\delta_1 = \pm 1$. By putting Equations (59b) and (59c) together, and then by substituting Equations (59a) and (64), we obtain the following:

$$F(T) = \frac{F'(0)}{(1+\alpha)\,b_0^4}\,\exp\left[\frac{\sqrt{2}\left[-3+\delta_1\sqrt{1-2\,b_0^4\,T}\right]}{\left[1+\delta_1\sqrt{1-2\,b_0^4\,T}\right]^{1/2}}\right]\left[1+\delta_1\sqrt{1-2\,b_0^4\,T}\right]^{1/2} \times\left[\left(2+\alpha\left[4+\delta_1\sqrt{1-2\,b_0^4\,T}\right]\right)\left[1+\delta_1\sqrt{1-2\,b_0^4\,T}\right]^{1/2}+\sqrt{2}\alpha\left[5+\delta_1\sqrt{1-2\,b_0^4\,T}\right]\right]. \tag{65}$$

4. $\mathbf{b} = \mathbf{1}$, Equation (58) becomes the following:

$$0 = r^{-4} - b_0^2\,r^{-2} - \frac{b_0^2\,T}{2} \Rightarrow\; r^{-2}(T) = \frac{b_0^2}{2}\left[1+\delta_1\sqrt{1+\frac{2\,T}{b_0^2}}\right]. \tag{66}$$

where $\delta_1 = \pm 1$. By putting Equations (59b) and (59c) together, and then by substituting Equations (59a) and (66), we obtain the following:

$$F(T) = F'(0)\,b_0^2\exp\left[\frac{+\delta_1\sqrt{2}\sqrt{1+\frac{2\,T}{b_0^2}}}{\left[1+\delta_1\sqrt{1+\frac{2\,T}{b_0^2}}\right]^{1/2}}\right]\left[1+\delta_1\sqrt{1+\frac{2\,T}{b_0^2}}\right] \times\left[1+\delta_1\sqrt{1+\frac{2\,T}{b_0^2}}+\frac{\alpha\left[2+\delta_1\sqrt{1+\frac{2\,T}{b_0^2}}\right]}{(1+\alpha)}\left[\sqrt{2}\left[1+\delta_1\sqrt{1+\frac{2\,T}{b_0^2}}\right]^{1/2}+2\right]\right]. \tag{67}$$

5. $\mathbf{b} = -\mathbf{1}$: Equation (58) becomes the following:

$$0 = \frac{b_0^2\,T}{2} + b_0^2\,r^{-2} - 1, \Rightarrow\; r^{-1}(T) = \delta_1\sqrt{\frac{1}{b_0^2}-\frac{T}{2}}, \tag{68}$$

where $T \leq \frac{2}{b_0^2}$. By putting Equations (59b) and (59c) together, and then by substituting Equations (59a) and (68), we obtain the following:

$$\begin{aligned} F(T) =& \frac{4\,F'(0)}{b_0^2}\left[\frac{\delta\,\alpha}{b_0(1+\alpha)}\left(2+b_0^2\,r^{-2}(T)\right)(r(T)+\delta\,b_0)+\frac{(1-\alpha)}{(1+\alpha)}\right] \\ & \times\,\exp\left[-\frac{\delta}{b_0}\left(2\,r(T)-b_0^2 r^{-1}(T)\right)\right], \end{aligned} \tag{69}$$

where $r(T)$ is Equation (68).

6. $\mathbf{b} = -\frac{3}{2}$: Equation (58) becomes the following:

$$\begin{aligned} 0 =& r^{-3}+\frac{T}{2}\,r^{-1}-\frac{1}{b_0^2}, \\ \Rightarrow r^{-1}(T) =& \frac{1}{6^{2/3}b_0^2}\left[\sqrt{6}b_0^4\sqrt{b_0^4\,T^3+54}+18b_0^4\right]^{1/3} \\ & -\frac{b_0^2\,T}{6^{1/3}}\left[\sqrt{6}b_0^4\sqrt{b_0^4\,T^3+54}+18b_0^4\right]^{-1/3}. \end{aligned} \tag{70}$$

By putting Equations (59b) and (59c) together, and then by substituting Equations (59a) and (70), we obtain the following:

$$\begin{aligned} F(T) =& \frac{4\,F'(0)}{b_0^2}\left[\frac{\delta\,\alpha}{b_0(1+\alpha)}\left(2+b_0^2\,r^{-3}(T)\right)\left(r^{\frac{3}{2}}(T)+\delta\,b_0\right)r(T)+\frac{r(T)(2-3\alpha)}{2(1+\alpha)}\right] \\ & \times\,\exp\left[-\frac{2\delta}{3b_0}\left(2\,r^{\frac{3}{2}}(T)-b_0^2 r^{-\frac{3}{2}}(T)\right)\right], \end{aligned} \tag{71}$$

where $r(T)$ is Equation (70).

7. $\mathbf{b} = \mathbf{2}$: Equation (58) becomes the following:

$$\begin{aligned} 0 =& \frac{b_0^2\,T}{2}+b_0^2\,r^{-2}-r^{-6}, \\ \Rightarrow r^{-1}(T) =& \delta_1\left[\frac{2^{2/3}\,b_0^2}{3^{1/3}}\left[\sqrt{3}b_0^2\sqrt{27\,T^2-16b_0^2}+9b_0^2\,T\right]^{-\frac{1}{3}}\right. \\ & \left.+\frac{1}{6^{2/3}}\left[\sqrt{3}b_0^2\sqrt{27\,T^2-16b_0^2}+9b_0^2\,T\right]^{\frac{1}{3}}\right]^{\frac{1}{2}}, \end{aligned} \tag{72}$$

where $\delta_1 = \pm 1$. By putting Equations (59b) and (59c) together, and then by substituting Equations (59a) and (72), we obtain the following:

$$\begin{aligned} F(T) =& \frac{4\,F'(0)}{b_0^2}\left[\frac{\alpha\,\delta}{b_0(1+\alpha)}\left(2+b_0^2\,r^4(T)\right)\left(r^{-2}(T)+\delta\,b_0\right)r^{-6}(T)+\frac{(1+2\alpha)}{(1+\alpha)}r^{-6}(T)\right] \\ & \times\,\exp\left[-\frac{\delta}{2b_0}\left(-2\,r^{-2}(T)+b_0^2 r^2(T)\right)\right], \end{aligned} \tag{73}$$

where $r(T)$ is Equation (72).

8. $\mathbf{b} = -\mathbf{2}$: Equation (58) becomes the following:

$$\begin{aligned} 0 =& \frac{b_0^2\,T}{2}\,r^2+b_0^2-r^4, \\ \Rightarrow\ r(T) =& \frac{\delta_2}{2}\sqrt{b_0^2\,T+\delta_1\,b_0\sqrt{b_0^2\,T^2+16}}. \end{aligned} \tag{74}$$

By putting Equations (59b) and (59c) together, and then by substituting Equations (59a) and (74), we obtain the following:

$$F(T) = \frac{4\,F'(0)}{b_0^2}\left[\frac{\alpha\,\delta}{b_0(1+\alpha)}\left(2+b_0^2\,r^{-4}(T)\right)\left(r^2(T)+\delta\,b_0\right)r^2(T)+\frac{(1-2\alpha)}{(1+\alpha)}\,r^2(T)\right]$$
$$\times\,\exp\left[-\frac{\delta}{2b_0}\left(2\,r^2(T)-b_0^2 r^{-2}(T)\right)\right], \tag{75}$$

where $r(T)$ is Equation (74).

9. $\mathbf{b}=\mathbf{3}$: Equation (58) becomes the following:

$$\begin{aligned}
&0=\frac{b_0^2\,T}{2}+b_0^2\,r^{-2}-r^{-8},\\
&\Rightarrow\; r^{-2}(T)=\\
&\frac{\delta_1}{2}\left[\frac{b_0}{\sqrt[3]{2}\,3^{2/3}}\left(\sqrt{3}\sqrt{32T^3+27b_0^2}+9b_0\right)^{1/3}-2b_0\sqrt[3]{\frac{2}{3}}\,T\left(\sqrt{3}\sqrt{32T^3+27b_0^2}+9b_0\right)^{-1/3}\right]^{1/2}\\
&+\frac{\delta_2}{2}\left[2b_0\sqrt[3]{\frac{2}{3}}\,T\left(\sqrt{3}\sqrt{32T^3+27b_0^2}+9b_0\right)^{-1/3}+2\delta_1\,b_0^2\left[\frac{b_0}{\sqrt[3]{2}\,3^{2/3}}\left(\sqrt{3}\sqrt{32T^3+27b_0^2}+9b_0\right)^{1/3}\right.\right.\\
&\left.\left.-2b_0\sqrt[3]{\frac{2}{3}}\,T\left(\sqrt{3}\sqrt{32T^3+27b_0^2}+9b_0\right)^{-1/3}\right]^{-1/2}-\frac{b_0}{\sqrt[3]{2}\,3^{2/3}}\left(\sqrt{3}\sqrt{32T^3+27b_0^2}+9b_0\right)^{1/3}\right]^{1/2},
\end{aligned}\tag{76}$$

where $(\delta_1,\delta_2)=(\pm1,\,\pm1)$. By putting Equations (59b) and (59c) together, and then by substituting Equations (59a) and (76), we obtain the following:

$$F(T) = \frac{4\,F'(0)}{b_0^2}\left[\frac{\alpha\,\delta}{b_0(1+\alpha)}\left(2+b_0^2\,r^{6}(T)\right)\left(r^{-3}(T)+\delta\,b_0\right)r^{-8}(T)+\frac{(1+3\alpha)}{(1+\alpha)}\,r^{-8}(T)\right]$$
$$\times\,\exp\left[-\frac{\delta}{3b_0}\left(-2\,r^{-3}(T)+b_0^2 r^{3}(T)\right)\right], \tag{77}$$

where $r(T)$ is Equation (76).

3.2.5. $a=-\frac{1}{2}$ Case Solutions

For solving $\mathbf{a}=-\frac{\mathbf{1}}{\mathbf{2}}$ specific cases, Equation (44) will simplify as follows:

$$0=\frac{b_0\,T}{2}+b_0\,r^{-2}+\delta\,r^{-2-b}. \tag{78}$$

Equations (43a)–(43c) and (43e) become the following:

$$F'(T) = F'(0)\,\exp\left[\int dr\,\frac{\left[\left(\frac{2b-1}{4}\right)r^{-2b}-b_0^2\right]}{\left[\frac{1}{2}\,r^{1-2b}+\delta\,b_0\,r^{1-b}\right]}\right] \tag{79a}$$

$$\kappa\,\rho = \frac{2\,F'(T)}{b_0^2\,(1+\alpha)}\left[-\frac{\left[\left(\frac{2b-1}{4}\right)r^{-2b}-b_0^2\right]}{\left[\frac{1}{2}\,r^{1-2b}+\delta\,b_0\,r^{1-b}\right]}\left[r^{-2b-1}+\delta\,b_0\,r^{-b-1}\right]+(b-\frac{1}{2})\,r^{-2b-2}\right], \tag{79b}$$

$$\kappa\,\rho = -\frac{F(T)}{2}+\frac{2\,F'(T)}{b_0^2}\left[-\frac{\left[\left(\frac{2b-1}{4}\right)r^{-2b}-b_0^2\right]}{\left[\frac{1}{2}\,r^{1-2b}+\delta\,b_0\,r^{(1-b)}\right]}\left[r^{-2b-1}+\delta\,b_0\,r^{-b-1}\right]+(b-\frac{1}{2})\,r^{-2b-2}\right.$$
$$\left.-\frac{\delta\,b_0}{2}\,r^{-b-2}\right], \tag{79c}$$

$$\rho = \rho_1\,r^{\frac{(1+\alpha)}{2\alpha}} \tag{79d}$$

where $\rho_1=\rho_0\,a_0^{\frac{(1+\alpha)}{2\alpha}}=\text{constant}$.

The possible cases are:

1. $\mathbf{b} = \mathbf{0}$ case: Equation (78) becomes the following:

$$\begin{aligned} 0 =& \frac{b_0\, T}{2} + (b_0 + \delta)\, r^{-2}, \\ \Rightarrow\ r^{-1}(T) =& \sqrt{\frac{b_0}{2(b_0+\delta)}}\sqrt{-T}. \end{aligned} \tag{80}$$

By putting Equations (79b) and (79c) together, and then substituting Equations (79a) and (80) inside, we obtain the following:

$$\begin{aligned} F(T) =& \frac{4\,F'(0)}{b_0^2}\left(\frac{b_0}{2(b_0+\delta)}\right)^{1+\frac{\left(\frac{1}{4}+b_0^2\right)}{(1+2\delta\, b_0)}}\left[\frac{\alpha\left(\frac{1}{4}+b_0^2\right)(1+\delta\, b_0)}{(1+\alpha)\left(\frac{1}{2}+\delta\, b_0\right)} - \frac{\alpha}{2(1+\alpha)} - \frac{\delta\, b_0}{2}\right] \\ & \times\ (-T)^{1+\frac{\left(\frac{1}{4}+b_0^2\right)}{(1+2\delta\, b_0)}}, \\ =& F_3\,(-T)^{1+\frac{\left(\frac{1}{4}+b_0^2\right)}{(1+2\delta\, b_0)}}, \end{aligned} \tag{81}$$

where F_3 is a constant. Once again, we have a pure power-law solution as in ref. [20].

2. $\mathbf{b} = \mathbf{1}$ case: Equation (78) becomes more simple as follows:

$$\begin{aligned} 0 =& \frac{\delta\, b_0\, T}{2} + \delta\, b_0\, r^{-2} + r^{-3}, \\ \Rightarrow\ r^{-1}(T) =& \frac{1}{3}\left[-\,\delta\, b_0 + \frac{2^{2/3}\, b_0^2}{\sqrt[3]{-4\delta\, b_0^3 + 3\sqrt{3}\sqrt{27b_0^2\, T^2 + 8b_0^4\, T} - 27\delta\, b_0\, T}}\right. \\ & \left. + \frac{1}{2^{2/3}}\sqrt[3]{-4\delta\, b_0^3 + 3\sqrt{3}\sqrt{27b_0^2\, T^2 + 8b_0^4\, T} - 27\delta\, b_0\, T}\right]. \end{aligned} \tag{82}$$

Equation (82) leads to only one real solution for $r^{-1}(T)$. By putting Equations (79b) and (79c) together, and then substituting Equations (79a) and (82) inside, we obtain the following:

$$\begin{aligned} F(T) =& \frac{4\,F'(0)}{b_0^2}\, \exp[-\delta\, b_0\, r(T)]\left[-\frac{\alpha}{(1+\alpha)}\left(\frac{r^{-1}(T)}{2} - \delta\, b_0\right)\left(r^{-1}(T) + \delta\, b_0\right) r^{-3/2}(T)\right. \\ & \left. + \frac{\alpha}{(1+\alpha)}\frac{r^{-7/2}(T)}{2} - \delta\, b_0\, \frac{r^{-5/2}(T)}{2}\right], \end{aligned} \tag{83}$$

where $r(T)$ is described by Equation (82).

3. $\mathbf{b} = \mathbf{-1}$ case: Equation (78) becomes the following:

$$\begin{aligned} 0 =& \frac{b_0\, T}{2} + b_0\, r^{-2} + \delta\, r^{-1}, \\ \Rightarrow\ r^{-1}(T) =& -\frac{\delta}{2b_0} \pm \sqrt{\frac{1}{4b_0^2} - \frac{T}{2}}. \end{aligned} \tag{84}$$

By putting Equations (79b) and (79c) together, and then substituting Equations (79a) and (84) inside, we obtain the following:

$$F(T) = \frac{4\,F'(0)}{b_0^2}\left[\frac{\alpha\left(\frac{3}{4}\,r^2(T)+b_0^2\right)(r(T)+\delta\,b_0)}{r^2(T)(1+\alpha)\left(\frac{r(T)}{2}+\delta\,b_0\right)} - \frac{3\alpha}{2(1+\alpha)} - \frac{\delta\,b_0}{2\,r(T)}\right]\frac{\sqrt{r(T)}\,\exp\left(\frac{\delta\,b_0}{r(T)}\right)}{(2\delta\,b_0+r(T))^2}, \tag{85}$$

where $r(T)$ is described by Equation (84).

4. $\mathbf{b} = \mathbf{2}$ case: Equation (78) becomes the following:

$$\begin{aligned} 0 =& \frac{\delta\,b_0\,T}{2} + \delta\,b_0\,r^{-2} + r^{-4}, \\ \Rightarrow\ r^{-1}(T) =& \ \delta_2\sqrt{-\frac{\delta\,b_0}{2} + \delta_1\sqrt{\frac{b_0^2}{4} - \frac{\delta\,b_0\,T}{2}}}. \end{aligned} \tag{86}$$

By putting Equations (79b) and (79c) together, and then substituting Equations (79a) and (86) inside, we obtain the following:

$$\begin{aligned} F(T) =& \frac{4\,F'(0)}{b_0^2}\left[-\frac{\alpha\left(\frac{3}{4}-b_0^2\,r^4(T)\right)\left(1+\delta\,b_0 r^2(T)\right)}{(1+\alpha)\left(\frac{1}{2}+\delta\,b_0\,r^2(T)\right)} + \frac{3\alpha}{2(1+\alpha)} - \frac{\delta\,b_0 r^2(T)}{2}\right] \\ &\times \frac{r^{-\frac{9}{2}}(T)\,\exp\left(-\frac{\delta\,b_0\,r^2(T)}{2}\right)}{\sqrt{1+2\delta\,b_0\,r^2(T)}}, \end{aligned} \tag{87}$$

where $r(T)$ is described by Equation (86).

5. $\mathbf{b} = -\mathbf{2}$ case: Equation (78) will simplify as follows:

$$\begin{aligned} 0 =& b_0 + \left(\frac{b_0\,T}{2} + \delta\right) r^2, \\ \Rightarrow\ r(T) =& \pm\frac{\sqrt{2b_0}}{\sqrt{-b_0\,T - 2\delta}}, \end{aligned} \tag{88}$$

where $b_0\,T + 2\delta < 0$. By putting Equations (79b) and (79c) together, and then substituting Equations (79a) and (88) inside, we obtain the following:

$$\begin{aligned} F(T) =& \frac{4\,F'(0)\,\exp\left(\frac{\delta\,b_0}{2\,r^2(T)}\right)}{b_0^2\,(r^2(T)+2\delta\,b_0)^{3/2}}\left[\frac{\alpha\left[\frac{5}{4}\,r^4(T)+b_0^2\right]\left[r^2(T)+\delta\,b_0\right]}{(1+\alpha)r^{3/2}(T)\left[\frac{1}{2}\,r^2(T)+\delta\,b_0\right]} - \frac{5\alpha\,r^{5/2}(T)}{2(1+\alpha)}\right. \\ &\left. - \frac{\delta\,b_0}{2}r^{1/2}(T)\right], \end{aligned} \tag{89}$$

where $r(T)$ is Equation (88).

6. $\mathbf{b} = -\mathbf{3}$ case: Equation (78) becomes the following:

$$\begin{aligned} 0 =& r^3 + \frac{\delta b_0\,T}{2}\,r^2 + \delta b_0, \\ \Rightarrow\ r(T) =& \frac{1}{6}\left[\left[-108\delta\,b_0 - \delta\,b_0^3\,T^3 + 6^{3/2}\,b_0\sqrt{54+b_0^2\,T^3}\right]^{1/3}\right. \\ &\left. + b_0^2\,T^2\left[-108\delta\,b_0 - \delta\,b_0^3\,T^3 + 6^{3/2}\,b_0\sqrt{54+b_0^2\,T^3}\right]^{-1/3} - \delta\,b_0\,T\right]. \end{aligned} \tag{90}$$

By putting Equations (79b) and (79c) together, and then substituting Equations (79a) and (90) inside, we obtain the following:

$$F(T) = \frac{4\,F'(0)}{b_0^2}\left[\frac{\alpha\left(\frac{7}{4}\,r^6(T)+b_0^2\right)\left(r^3(T)+\delta\,b_0\right)}{(1+\alpha)r^3(T)\left(\frac{r^3(T)}{2}+\delta\,b_0\right)} - \frac{7\alpha\,r^3(T)}{2(1+\alpha)} - \frac{\delta\,b_0}{2}\right]\frac{r^{\frac{3}{2}}(T)\,\exp\left(\frac{\delta\,b_0}{3\,r^3(T)}\right)}{\left(2\delta\,b_0+r^3(T)\right)^{\frac{4}{3}}}, \tag{91}$$

where $r(T)$ is described by Equation (90).

7. $\mathbf{b} = \mathbf{4}$ case: Equation (78) becomes the following:

$$\begin{aligned} 0 =& \frac{b_0\,T}{2} + b_0\,r^{-2} + \delta\,r^{-6}, \\ \Rightarrow\ r^{-1}(T) =& \delta_1\left[\frac{1}{6^{\frac{2}{3}}}\left[\sqrt{3}\,b_0\sqrt{16\delta b_0+27\,T^2} - 9\delta\,b_0\,T\right]^{\frac{1}{3}}\right. \\ & \left. - \frac{\delta\,b_0\,2^{\frac{2}{3}}}{3^{\frac{1}{3}}}\left[\sqrt{3}\,b_0\sqrt{16\delta b_0+27\,T^2} - 9\delta\,b_0\,T\right]^{-\frac{1}{3}}\right]^{\frac{1}{2}}, \end{aligned} \tag{92}$$

where $\delta_1 = \pm 1$. By putting Equations (79b) and (79c) together, and then substituting Equations (79a) and (92) inside, we obtain the following:

$$\begin{aligned} F(T) =& \frac{4\,F'(0)}{b_0^2}\left[-\frac{\alpha\left(\frac{7}{4}-b_0^2\,r^8(T)\right)\left(1+\delta\,b_0 r^4(T)\right)}{(1+\alpha)r^4(T)\left(\frac{1}{2}+\delta\,b_0\,r^4(T)\right)} + \frac{7\alpha}{2(1+\alpha)\,r^4(T)} - \frac{\delta\,b_0}{2}\right] \\ & \times \frac{r^{-\frac{5}{2}}(T)\,\exp\left(-\frac{\delta\,b_0\,r^4(T)}{4}\right)}{\left(1+2\delta\,b_0\,r^4(T)\right)^{\frac{3}{4}}}, \end{aligned} \tag{93}$$

where $r(T)$ is described by Equation (92).

8. $\mathbf{b} = -\mathbf{4}$ case: Equation (78) becomes the following:

$$\begin{aligned} 0 =& r^4 + \frac{\delta\,b_0\,T}{2}\,r^2 + \delta\,b_0, \\ \Rightarrow\ r(T) =& \delta_2\sqrt{-\frac{\delta\,b_0\,T}{4} + \delta_1\sqrt{\frac{b_0^2\,T^2}{16}} - \delta\,b_0}. \end{aligned} \tag{94}$$

By putting Equations (79b) and (79c) together, and then substituting Equations (79a) and (94) inside, we obtain the following:

$$F(T) = \frac{4\,F'(0)}{b_0^2}\left[\frac{\alpha\left(\frac{9}{4}\,r^8(T)+b_0^2\right)\left(r^4(T)+\delta\,b_0\right)}{(1+\alpha)r^5(T)\left(\frac{r^4(T)}{2}+\delta\,b_0\right)} - \frac{9\alpha\,r^3(T)}{2(1+\alpha)} - \frac{\delta\,b_0}{2}\right]\frac{r^{\frac{7}{2}}(T)\,\exp\left(\frac{\delta\,b_0}{4\,r^4(T)}\right)}{\left(2\delta\,b_0+r^4(T)\right)^{\frac{5}{4}}}, \tag{95}$$

where $r(T)$ is described by Equation (94).

9. $\mathbf{b} = \mathbf{6}$ case: Equation (78) becomes the following:

$$
\begin{aligned}
0 =& \frac{b_0\,T}{2} + b_0\,r^{-2} + \delta\,r^{-8},\\
\Rightarrow r^{-2}(T) =& \frac{\delta_1}{2}\Bigg[2\sqrt[3]{\frac{2}{3}}b_0\,T\left(9\delta\,b_0^2+\sqrt{3}\,b_0\sqrt{27b_0^2-32\delta\,b_0\,T^3}\right)^{-1/3}\\
&+\frac{\delta}{\sqrt[3]{2}\,3^{2/3}}\left(9\delta\,b_0^2+\sqrt{3}\,b_0\sqrt{27b_0^2-32\delta\,b_0\,T^3}\right)^{1/3}\Bigg]^{1/2}\\
&+\frac{\delta_2}{2}\Bigg[-2\sqrt[3]{\frac{2}{3}}b_0\,T\left(9\delta\,b_0^2+\sqrt{3}\,b_0\sqrt{27b_0^2-32\delta\,b_0\,T^3}\right)^{-1/3}\\
&-\frac{\delta\,\delta_2}{2b_0}\Bigg[2\sqrt[3]{\frac{2}{3}}b_0\,T\left(9\delta\,b_0^2+\sqrt{3}\,b_0\sqrt{27b_0^2-32\delta\,b_0\,T^3}\right)^{-1/3}\\
&+\frac{\delta}{\sqrt[3]{2}\,3^{2/3}}\left(9\delta\,b_0^2+\sqrt{3}\,b_0\sqrt{27b_0^2-32\delta\,b_0\,T^3}\right)^{1/3}\Bigg]^{-1/2}\\
&-\frac{\delta}{\sqrt[3]{2}3^{2/3}}\left(9\delta\,b_0^2+\sqrt{3}\,b_0\sqrt{27b_0^2-32\delta\,b_0\,T^3}\right)^{1/3}\Bigg]^{1/2},
\end{aligned}
\tag{96}
$$

where the possible solutions are $(\delta_1, \delta_2) = (\pm 1, \pm 1)$. By putting Equations (79b) and (79c) together, and then substituting Equations (79a) and (96) inside, we obtain the following:

$$
\begin{aligned}
F(T) =& \frac{4\,F'(0)}{b_0^2}\left[-\frac{\alpha\left(\frac{11}{4}\,r^{-12}(T)-b_0^2\right)}{(1+\alpha)\left(\frac{1}{2}\,r^{-6}(T)+\delta\,b_0\right)}\left(r^{-6}(T)+\delta\,b_0\right)+\frac{11\alpha}{2(1+\alpha)}\,r^{-12}(T)-\frac{\delta\,b_0}{2}\,r^{-6}(T)\right]\\
&\times\frac{r^{\frac{7}{2}}(T)}{(2\delta\,b_0\,r^6(T)+1)^{\frac{5}{6}}}\,\exp\left(-\frac{\delta\,b_0}{6}\,r^6(T)\right),
\end{aligned}
\tag{97}
$$

where $r(T)$ is described by Equation (96).

10. $\mathbf{b} = -\mathbf{6}$ case: Equation (78) becomes the following:

$$
\begin{aligned}
0 =& r^6 + \frac{\delta\,b_0\,T}{2}\,r^2 + \delta\,b_0,\\
\Rightarrow\ r^2(T) =& \frac{\left[\sqrt{6}\,b_0\sqrt{\delta\,b_0\,T^3+54}-18\delta\,b_0\right]^{2/3}-\sqrt[3]{6}\,\delta\,b_0\,T}{6^{2/3}\left[\sqrt{6}\,b_0\sqrt{\delta\,b_0\,T^3+54}-18\delta\,b_0\right]^{1/3}}
\end{aligned}
\tag{98}
$$

By putting Equations (79b) and (79c) together, and then substituting Equations (79a) and (98) inside, we obtain the following:

$$
\begin{aligned}
F(T) =& \frac{4\,F'(0)}{b_0^2}\left[\frac{\alpha\left(\frac{13}{4}\,r^{12}(T)+b_0^2\right)}{r^6(T)(1+\alpha)\left(\frac{1}{2}\,r^6(T)+\delta\,b_0\right)}\left(r^6(T)+\delta\,b_0\right)-\frac{13\alpha}{2(1+\alpha)}\,r^6(T)-\frac{\delta\,b_0}{2}\right]\\
&\times\exp\left(\frac{\delta\,b_0}{6\,r^6(T)}\right)\frac{r^{9/2}(T)}{(2\delta\,b_0+r^6(T))^{\frac{7}{6}}},
\end{aligned}
\tag{99}
$$

where $r(T)$ is described by Equation (98).

11. **b** $= -8$ case: Equation (78) becomes the following:

$$
\begin{aligned}
0 =& r^8 + \frac{\delta\, b_0\, T}{2}\, r^2 + \delta\, b_0, \\
\Rightarrow r^2(T) =& \frac{\delta_1}{2}\left[\frac{8\delta\, b_0}{\sqrt[3]{3}}\left(\sqrt{3}b_0\sqrt{27b_0^2\, T^4 - 4096\delta\, b_0} + 9b_0^2\, T^2\right)^{-1/3}\right. \\
&\left. + \frac{1}{2\cdot 3^{2/3}}\left(\sqrt{3}b_0\sqrt{27b_0^2\, T^4 - 4096\delta\, b_0} + 9b_0^2\, T^2\right)^{1/3}\right]^{1/2} \\
&+ \frac{\delta_2}{2}\left[-\frac{8\delta\, b_0}{\sqrt[3]{3}}\left(\sqrt{3}b_0\sqrt{27b_0^2\, T^4 - 4096\delta\, b_0} + 9b_0^2\, T^2\right)^{-1/3}\right. \\
&- \delta_1\delta\, b_0\, T\left[\frac{8\delta\, b_0}{\sqrt[3]{3}}\left(\sqrt{3}b_0\sqrt{27b_0^2\, T^4 - 4096\delta\, b_0} + 9b_0^2\, T^2\right)^{-1/3}\right. \\
&\left. + \frac{1}{2\cdot 3^{2/3}}\left(\sqrt{3}b_0\sqrt{27b_0^2\, T^4 - 4096\delta\, b_0} + 9b_0^2\, T^2\right)^{1/3}\right]^{-1/2} \\
&\left. - \frac{1}{2\cdot 3^{2/3}}\left(\sqrt{3}b_0\sqrt{27b_0^2\, T^4 - 4096\delta\, b_0} + 9b_0^2\, T^2\right)^{1/3}\right]^{1/2}
\end{aligned} \tag{100}
$$

By putting Equations (79b) and (79c) together, and then substituting Equations (79a) and (100) inside, we obtain the following:

$$
\begin{aligned}
F(T) =& \frac{4\, F'(0)}{b_0^2}\left[\frac{\alpha\left(\frac{17}{4}\, r^{16}(T) + b_0^2\right)}{r^8(T)(1+\alpha)\left(\frac{1}{2}\, r^8(T) + \delta\, b_0\right)}\left(r^8(T) + \delta\, b_0\right) - \frac{17\alpha}{2(1+\alpha)}\, r^8(T) - \frac{\delta\, b_0}{2}\right] \\
&\times \exp\left(\frac{\delta\, b_0}{8\, r^8(T)}\right)\frac{r^{13/2}(T)}{(2\delta\, b_0 + r^8(T))^{\frac{9}{8}}},
\end{aligned} \tag{101}
$$

where $r(T)$ is described by Equation (100).

In this section, all these previous non-power-law teleparallel $F(T)$ solutions are new. We may also use several different coframe ansatz leading to additional new $F(T)$ solutions. Equation (24) power-law ansatz based $F(T)$ solutions are sufficient for the current paper's aims and purposes. We may study several specific cases such as radiation fluids $\alpha = \frac{1}{3}$ to name an example [33,34]. We are also able to study the physical properties of possible singularities arising from each new previous $F(T)$ solution. Even if there are numerous new and more complex singularities in these previous $F(T)$ solutions, they may lead to some possible black hole solutions (point-like or not) and/or matter absorbing points. This task is beyond the aims of the paper and might be for potential future works.

4. Dust Perfect Fluid Solutions ($\alpha = 0$)

This specific case arises from $P(r) = 0$ and $\rho(r) \neq 0$ consideration. By setting $\alpha = 0$ inside Equation (17), the conservation law becomes the following:

$$
A_1'(r) = 0. \tag{102}
$$

We require that $A_1(r) = a_0 =$ constant. Then, Equation (14a) remains unchanged, but Equations (19a) and (19b) will be simplified:

$$\kappa\,\rho =2\,F'(T)\left[-\left(\frac{g_1(r)}{k_1(r)}\right)k_2(r)+g_2(r)\right], \tag{103a}$$

$$\kappa\,\rho =-\frac{F(T)}{2}+2\,F'(T)\left[-\left(\frac{g_1(r)}{k_1(r)}\right)k_2(r)+g_3(r)\right]. \tag{103b}$$

By combining Equations (103a) and (103b) and substituting Equations (A1b) and (A1c) FE components, we will obtain a simplified relation for $F(T(r))$ as follows:

$$\begin{aligned} F(T(r)) =&4\,F'(T(r))\,[g_3(r)-g_2(r)],\\ =&-\frac{4\,F'(T(r))\,A_3'}{A_2\,A_3^2}\left(A_3'+\delta\,A_2\right). \end{aligned} \tag{104}$$

Equation (14a) becomes the following:

$$F'(T) =F'(0)\,\exp\left[\int dr\,\frac{\left[-A_2\,A_3\,A_3''+A_2\,A_3'^2+A_2'\,A_3\,A_3'-A_2^3\right]}{A_2\,A_3\,(A_3'+\delta\,A_2)}\right], \tag{105}$$

Equation (21) for the torsion scalar becomes the following:

$$T(r) = -2\left(\frac{\delta}{A_3}+\frac{A_3'}{A_2\,A_3}\right)^2. \tag{106}$$

As for previous cases, we will apply the $A_3 = r$ coordinate set. The $A_3 = c_0 =$ constant coordinate leads to constant torsion scalar and GR solutions, which is not relevant for the current purpose.

For the $A_3 = r$ coordinate system, Equations (104)–(106) become the following:

$$F(T) = -\frac{4\,F'(T)}{A_2^2\,r^2}\left(1+\delta\,A_2\right), \tag{107a}$$

$$F'(T) =F'(0)\,\frac{r\,A_2}{(1+\delta\,A_2)}\,\exp\left[-\delta\int dr\,\frac{A_2}{r}\right], \tag{107b}$$

$$T =T(r) = -\frac{2}{A_2^2\,r^2}\left(1+\delta\,A_2\right)^2. \tag{107c}$$

By substituting Equation (107b) into Equation (107a), we find the following:

$$F(T) = -\frac{4\,F'(0)}{r\,A_2}\,\exp\left[-\delta\int dr\,\frac{A_2}{r}\right], \tag{108}$$

The best way for solving Equations (107b)–(108) is by a power-law solution ansatz as $A_2(r) = b_0\,r^b$. Note also that, by setting $A_2(r) = \left(1-k\,r^2\right)^{-1/2}$ for static Robertson–Walker spacetimes, we obtain that $F(T)$ will be linear and this is a GR solution. Equations (107b)–(108) become the following:

$$F(T) = -\frac{4\,F'(0)}{b_0\,r^{b+1}}\,\exp\left[-\frac{\delta\,b_0}{b}\,r^b\right], \tag{109a}$$

$$F'(T) = \frac{F'(0)\,b_0\,r^{b+1}}{\left(1+\delta\,b_0\,r^b\right)}\,\exp\left[-\frac{\delta\,b_0}{b}\,r^b\right] = -\frac{b_0^2\,r^{2(b+1)}}{4\left(1+\delta\,b_0\,r^b\right)}\,F(T), \tag{109b}$$

$$T(r) = -\frac{2}{b_0^2\,r^{2(b+1)}}\left(1+\delta\,b_0\,r^b\right)^2, \tag{109c}$$

where $b \neq 0$. The case $b = 0$ is the simple static cosmological spacetime and this case may be considered as a special case. By substituting Equations (109a) and (109c) into Equation (109b), we obtain the simplified DE to solve for $F(T)$ in a cosmological dust fluid where $b \neq 0$:

$$T\,F'(T) = \frac{\left(1+\delta\,b_0\,r^b(T)\right)}{2}\,F(T). \tag{110}$$

By using Equation (103a) and then substituting Equations (110) and (A3), we find the fluid density:

$$\rho(T) = \frac{F(T)}{\kappa\,b_0^2\,(-T)}\,r^{-2b-2}(T)\left(1+\delta\,b_0\,r^b(T)\right)^2\left(1-\delta\,b_0\,r^b(T)\right), \tag{111}$$

where $F(T)$ is given by Equation (110) solutions. We will solve Equation (110) for some values of b. For pure $F(T)$ solutions, we need to find from Equation (109c) the characteristic equation and then solve for $r(T)$:

$$0 = r^{b+1} - \sqrt{-\frac{2}{T}}\,r^b - \frac{\delta}{b_0}\sqrt{-\frac{2}{T}}. \tag{112}$$

There are some specific values of b leading to an analytic $r(T)$ function and then to an $F(T)$ solution:

1. $\mathbf{b = 0}$: For this simple case of cosmological spacetime, Equation (112) becomes the following:

$$r^2(T) = -\left(1+\frac{\delta}{b_0}\right)^2\frac{2}{T}. \tag{113}$$

Equations (107a)–(107c) for $A_2 = b_0$ will be summarized by Equation (110):

$$T\,F'(T) = \frac{(1+\delta\,b_0)}{2}\,F(T) \tag{114}$$

We solve Equation (114) and obtain as a solution for a flat dust fluid:

$$F(T) = F_0\,T^{\frac{1+\delta\,b_0}{2}}, \tag{115}$$

where $b_0 \neq \pm\delta$ for a teleparallel solution (i.e., $b_0 = \delta$ leads to GR solutions). Once again, we obtain a pure power-law solution as in ref. [18] for general X_4 similarity (here $\rho = \rho(r)$ without any other constraint). By using Equation (111), setting $b = 0$ and substituting Equation (115), the fluid density $\rho(T)$ is as follows:

$$\rho(T) = \frac{F_0}{2\kappa}\,(1-\delta\,b_0)\,T^{\frac{\delta\,b_0+1}{2}}. \tag{116}$$

Equation (116) is again a power-law function of T as usual. If $b_0 = \delta$, we find that $\rho(T) = 0$ for $F(T) = F_0\, T$.

2. $\mathbf{b=1}$: Equation (112) becomes the following:

$$0 = r^2 - \sqrt{-\frac{2}{T}}\, r - \frac{\delta}{b_0}\sqrt{-\frac{2}{T}},$$
$$\Rightarrow\ r(T) = \frac{1}{\sqrt{(-2T)}}\left[1+\delta_1\sqrt{1+\frac{2\delta}{b_0}\sqrt{(-2T)}}\right], \tag{117}$$

where $\delta_1 = \pm 1$. Then, Equation (110) becomes the following:

$$T\,F'(T) = \frac{F(T)}{2}\left[1+\frac{\delta\, b_0}{\sqrt{(-2T)}}\left[1+\delta_1\sqrt{1+\frac{2\delta}{b_0}\sqrt{(-2T)}}\right]\right]. \tag{118}$$

The solution of Equation (118) is as follows:

$$F(T) = F_0\,\sqrt{-T}\left[\frac{1-\sqrt{1+\frac{2\sqrt{2}\delta}{b_0}\sqrt{-T}}}{1+\sqrt{1+\frac{2\sqrt{2}\delta}{b_0}\sqrt{-T}}}\right]^{\delta_1}\exp\left[-\frac{\delta\, b_0}{\sqrt{2}\,\sqrt{-T}}\left(1+\delta_1\sqrt{1+\frac{2\delta}{b_0}\sqrt{(-2T)}}\right)\right], \tag{119}$$

where $T \leq 0$. The fluid density $\rho(T)$ will be the following from Equation (111):

$$\begin{aligned}\rho(T) =& \frac{4F_0\,(-T)^{3/2}}{b_0^2\kappa\left[1+\delta_1\sqrt{1+\frac{2\delta}{b_0}\sqrt{(-2T)}}\right]^4}\left[1+\frac{\delta\, b_0}{\sqrt{(-2T)}}\left[1+\delta_1\sqrt{1+\frac{2\delta}{b_0}\sqrt{(-2T)}}\right]\right]^2\\ &\times\left[1-\frac{\delta\, b_0}{\sqrt{(-2T)}}\left[1+\delta_1\sqrt{1+\frac{2\delta}{b_0}\sqrt{(-2T)}}\right]\right]\left[\frac{1-\sqrt{1+\frac{2\sqrt{2}\delta}{b_0}\sqrt{-T}}}{1+\sqrt{1+\frac{2\sqrt{2}\delta}{b_0}\sqrt{-T}}}\right]^{\delta_1}\\ &\times\exp\left[-\frac{\delta\, b_0}{\sqrt{2}\,\sqrt{-T}}\left[1+\delta_1\sqrt{1+\frac{2\delta}{b_0}\sqrt{(-2T)}}\right]\right].\end{aligned} \tag{120}$$

3. $\mathbf{b=-1}$: Equation (112) becomes the following:

$$0 = 1 - \sqrt{-\frac{2}{T}}\, r^{-1} - \frac{\delta}{b_0}\sqrt{-\frac{2}{T}}.$$
$$\Rightarrow\ r^{-1}(T) = \sqrt{-\frac{T}{2}} - \frac{\delta}{b_0}. \tag{121}$$

Equations (110) becomes a simple DE:

$$\frac{dF}{d(-T)} = \frac{\delta b_0}{2\sqrt{2}}\,(-T)^{-1/2}\,F(T). \tag{122}$$

The solution of Equation (122) is as follows:

$$F(T) = F_1\,\exp\left[\frac{\delta b_0}{\sqrt{2}}\sqrt{-T}\right], \tag{123}$$

where $T \leq 0$ and F_1 is an integration constant. The fluid density $\rho(T)$ will be the following from Equation (111):

$$\rho(T) = \frac{F_1}{\kappa}\left[1-\frac{\delta b_0\sqrt{-T}}{2\sqrt{2}}\right]\exp\left[\frac{\delta b_0}{\sqrt{2}}\sqrt{-T}\right]. \tag{124}$$

4. $\mathbf{b} = -\mathbf{2}$: Equation (112) becomes the following:

$$0 = r^{-1} - \sqrt{-\frac{2}{T}}\, r^{-2} - \frac{\delta}{b_0}\sqrt{-\frac{2}{T}},$$
$$\Rightarrow\; r^{-1}(T) = \sqrt{-\frac{T}{8}} + \delta_1\sqrt{-\frac{T}{8} - \frac{\delta}{b_0}}, \tag{125}$$

where $T \leq 0$ and $\delta_1 = \pm 1$. Equation (110) will be a DE and the solution is as follows:

$$F(T) = F_2\,\exp\left[-\frac{\delta\, b_0}{8}\, T\left(1 + \delta_1\sqrt{1 + \frac{8\delta}{b_0\, T}}\right)\right]\left[\frac{1 - \sqrt{1 + \frac{8\delta}{b_0\, T}}}{1 + \sqrt{1 + \frac{8\delta}{b_0\, T}}}\right]^{\frac{\delta_1}{2}}, \tag{126}$$

where F_2 is an integration constant. The fluid density $\rho(T)$ will be the following from Equation (111):

$$\rho(T) = \frac{8F_2}{\kappa\, b_0^2\, T^2}\left(1 + \delta_1\sqrt{1 + \frac{8\delta}{b_0\, T}}\right)^{-2}\left[1 - \frac{\delta\, b_0}{8}\, T\left(1 + \delta_1\sqrt{1 + \frac{8\delta}{b_0\, T}}\right)^2\right]^2\left[\frac{1 - \sqrt{1 + \frac{8\delta}{b_0\, T}}}{1 + \sqrt{1 + \frac{8\delta}{b_0\, T}}}\right]^{\frac{\delta_1}{2}}$$
$$\times\left[1 + \frac{\delta\, b_0}{8}\, T\left(1 + \delta_1\sqrt{1 + \frac{8\delta}{b_0\, T}}\right)^2\right]\exp\left[-\frac{\delta\, b_0}{8}\, T\left(1 + \delta_1\sqrt{1 + \frac{8\delta}{b_0\, T}}\right)\right]. \tag{127}$$

5. $\mathbf{b} = \{\mathbf{2},\, \mathbf{3},\, -\mathbf{3},\, -\mathbf{4}\}$: We can in principle find analytic $r(T)$ solutions to the Equation (112) characteristic equation. However, these $r(T)$ cannot lead to solvable and well-defined $F(T)$ solutions and this explains the limited number of possible power-law ansatz analytical $F(T)$ solutions for dust fluids.

All these teleparallel $F(T)$ solutions found in this section are all new. We may also use several other possible ansatz for finding further new $F(T)$ solutions as for Section 3. However, we only used in this section power-law ansatz as defined by Equation (24) with $a = 0$ (because of Equation (102)) and solved for several new and interesting $F(T)$ solutions all useful for many types of astrophysical or cosmological dust fluids. We may still study and look in detail for singularities and their related physical characteristics in potential future works as for Section 3 solutions. We can also find some point-like singularity solutions and/or matter absorbing singularities in these new $F(T)$ solutions.

5. Non-Linear Perfect Fluid Solutions

Another class of non-vacuum solutions assumes an EoS $P(r) = \alpha\,\rho(r) + \beta\,\rho^w(r)$ with $-1 < \alpha \leq 1$, $w > 1$ where it is often assumed that $\beta\,\rho^{w-1}(r) \ll \alpha$. The second term of this EoS can describe non-linear dissipating terms. Equation (15) will simplify as follows:

$$(\ln\, A_1)' + \frac{\left[\alpha + \beta\, w\,\rho^{w-1}\right]}{\left[(1+\alpha)\rho + \beta\,\rho^w\right]}\rho' = 0. \tag{128}$$

The general solution is as follows:

$$A_1(r) = A_1(0)\,\left[(1+\alpha)\rho^{1-w} + \beta\right]^{\left[\frac{\alpha}{(1+\alpha)(w-1)} - \frac{w}{w-1}\right]}\rho^{-w} \tag{129}$$

We need to set the FEs for a power-w fluid density EoS and then solve for new $F(T)$ solutions. We will have that Equations (14a) and (21) remain unchanged and then Equations (16b) and (16c) will be as follows:

$$(1+\alpha)\,(\kappa\rho)+\kappa^{1-w}\,\beta\,(\kappa\rho)^{w} = 2\,F'(T)\left[-\frac{g_1(r)}{k_1(r)}\,k_2(r)+g_2(r)\right], \tag{130a}$$

$$\kappa\,\rho = -\frac{F(T)}{2}+2\,F'(T)\left[-\frac{g_1(r)}{k_1(r)}\,k_2(r)+g_3(r)\right]. \tag{130b}$$

By putting Equations (130a) and (130b) together, we find the unified equation linking $F(T)$ and $F'(T)$:

$$\begin{aligned}(1+\alpha)&\left[-\frac{F(T)}{2}+2\,F'(T)\left[-\frac{g_1(r)}{k_1(r)}\,k_2(r)+g_3(r)\right]\right]\\ &+\kappa^{1-w}\,\beta\left[-\frac{F(T)}{2}+2\,F'(T)\left[-\frac{g_1(r)}{k_1(r)}\,k_2(r)+g_3(r)\right]\right]^{w}=2\,F'(T)\left[-\frac{g_1(r)}{k_1(r)}\,k_2(r)+g_2(r)\right].\end{aligned} \tag{131}$$

There are in principle several possible ansatz for solving the system governed by Equations (14a), (21), (130b) and (131), completed by the Equation (129) conservation law solution. We will present some possible solvable solutions.

5.1. A_3 = Constant Power-Law Solutions

By using Equation (24) ansatz and setting $A_3 = c_0 =$ constant, we will solve Equations (16a), (26d), (129), (130a) and (130b):

$$F'(T)=F'(0)\,\exp\left[\int dr\,\frac{\left[(a(1-a+b))\,r^{-2b-2}-\left(\frac{b_0}{c_0}\right)^2\right]}{\left[a\,r^{-2b-1}+\delta\left(\frac{b_0}{c_0}\right)r^{-b}\right]}\right], \tag{132a}$$

$$(1+\alpha)\,(\kappa\rho)+\kappa^{1-w}\,\beta\,(\kappa\rho)^{w} = -\frac{2\delta}{b_0\,c_0}\,F'(T)\left[\frac{\left[(a(1-a+b))\,r^{-2b-2}-\left(\frac{b_0}{c_0}\right)^2\right]}{\left[a\,r^{-b-1}+\delta\left(\frac{b_0}{c_0}\right)\right]}\right], \tag{132b}$$

$$\kappa\,\rho = -\frac{F(T)}{2}-\frac{2\delta}{b_0\,c_0}\,F'(T)\left[\frac{\left[(a(1-a+b))\,r^{-2b-2}-\left(\frac{b_0}{c_0}\right)^2\right]}{\left[a\,r^{-b-1}+\delta\left(\frac{b_0}{c_0}\right)\right]}+a\,r^{-b-1}\right], \tag{132c}$$

$$T(r)=-\frac{2}{c_0^2}-\frac{4\delta a}{b_0\,c_0}\,r^{-(b+1)}, \tag{132d}$$

$$\tilde{a}_0\,r^{a}=\left[(1+\alpha)\rho^{1-w}+\beta\right]^{\left[\frac{\alpha}{(1+\alpha)(w-1)}-\frac{w}{w-1}\right]}\rho^{-w}, \tag{132e}$$

where $\tilde{a}_0$ is a constant from conservation laws. From Equation (132d), $a=0$ and/or $b=-1$ lead to GR solutions (because of the constant torsion scalar). For all other cases, we isolate $r(T)$ from Equation (132d):

$$\begin{aligned}r(T)&=\left(-\frac{4\delta a}{b_0\,c_0}\right)^{\frac{1}{b+1}}\left(T+\frac{2}{c_0^2}\right)^{-\frac{1}{b+1}},\\ dr&=-\frac{r(T)}{(b+1)}\,\frac{dT}{\left(T+\frac{2}{c_0^2}\right)},\end{aligned} \tag{133}$$

where $b\neq -1$. Then, Equations (132a)–(132c) become the following:

$$F'(T) = F'(0)\left(-c_0^2\right)^{1-\frac{a}{(b+1)}}\left(2-c_0^2\,T\right)^{\frac{2a}{(b+1)}-1}\left(2+c_0^2\,T\right)^{-\frac{a}{(b+1)}}\exp\left[\frac{4a}{(b+1)(2+c_0^2\,T)}\right] \tag{134a}$$

$$(1+\alpha)\,(\kappa\rho)+\kappa^{1-w}\,\beta\,(\kappa\rho)^w = \frac{F'(T)}{2}\left[\frac{\left(\frac{b+1}{a}-1\right)\left(T+\frac{2}{c_0^2}\right)^2-\frac{16}{c_0^4}}{\left(T-\frac{2}{c_0^2}\right)}\right], \tag{134b}$$

$$\kappa\,\rho = -\frac{F(T)}{2}+\frac{F'(T)}{2}\left[\frac{\left(\frac{b+1}{a}-1\right)\left(T+\frac{2}{c_0^2}\right)^2-\frac{16}{c_0^4}}{\left(T-\frac{2}{c_0^2}\right)}+\left(T+\frac{2}{c_0^2}\right)\right]. \tag{134c}$$

By putting Equations (134b) and (134c) together, we obtain the following:

$$\left[-F(T)+F'(T)\left(T+\frac{2}{c_0^2}\right)\right]+\alpha\left[-F(T)+F'(T)\left[\frac{\left(\frac{b+1}{a}-1\right)\left(T+\frac{2}{c_0^2}\right)^2-\frac{16}{c_0^4}}{\left(T-\frac{2}{c_0^2}\right)}+\left(T+\frac{2}{c_0^2}\right)\right]\right]$$
$$+\frac{\beta}{(2\kappa)^{w-1}}\left[-F(T)+F'(T)\left[\frac{\left(\frac{b+1}{a}-1\right)\left(T+\frac{2}{c_0^2}\right)^2-\frac{16}{c_0^4}}{\left(T-\frac{2}{c_0^2}\right)}+\left(T+\frac{2}{c_0^2}\right)\right]\right]^w = 0, \tag{135}$$

where $F(T) \neq F'(T)\left(T+\frac{2}{c_0^2}\right)$. Otherwise, we obtain a linear $F(T)$ leading to GR solutions. Equation (135) can also be written in the following form:

$$\left[-F(T)+F'(T)\left(T+\frac{2}{c_0^2}\right)\right]+\alpha\,G_1(T,\,F(T),\,F'(T))+\frac{\beta}{(2\kappa)^{w-1}}\left[G_1(T,\,F(T),\,F'(T))\right]^w = 0, \tag{136}$$

where the function $G_1(T,\,F(T),\,F'(T))$ is

$$G_1(T,\,F(T),\,F'(T)) = -F(T)+F'(T)\left[\frac{\left(\frac{b+1}{a}-1\right)\left(T+\frac{2}{c_0^2}\right)^2-\frac{16}{c_0^4}}{\left(T-\frac{2}{c_0^2}\right)}+\left(T+\frac{2}{c_0^2}\right)\right]. \tag{137}$$

We may solve Equation (136) describing a polynomial of degree w only for $w=2$, 3 and 4, because G_1 is linear in $F(T)$ and $F'(T)$. For $w=2$, we obtain as a solution to Equation (136) the following:

$$F(T)-F'(T)\left[\frac{\left(\frac{b+1}{a}-1\right)\left(T+\frac{2}{c_0^2}\right)^2-\frac{16}{c_0^4}}{\left(T-\frac{2}{c_0^2}\right)}+\left(T+\frac{2}{c_0^2}\right)\right]$$
$$=\frac{\alpha\,\kappa}{\beta}\left[1-\delta_2\sqrt{1+\frac{2\beta}{\alpha^2\kappa}\left(F(T)-F'(T)\left(T+\frac{2}{c_0^2}\right)\right)}\right] \tag{138}$$

Equation (138) is difficult to solve for an exact solution because of the square root to the r.h.s. But if we use the approximation $\beta \ll \alpha$ for a slightly non-linear fluid approximation, then Equation (138) will simplify and becomes a linear DE:

$$\frac{(\alpha+\delta_2)\left(T-\frac{2}{c_0^2}\right)}{\left[\left(\alpha\left(\frac{b+1}{a}\right)+\delta_2\right)T^2+\frac{4\alpha}{c_0^2}\left(\frac{b+1}{a}-1\right)T+\frac{4}{c_0^4}\left(\alpha\left(\frac{b+1}{a}-6\right)-\delta_2\right)\right]}=\frac{F'(T)}{\left[F(T)-\frac{\alpha^2\,\kappa(1-\delta_2)}{\beta(\alpha+\delta_2)}\right]}. \tag{139}$$

The general solution of Equation (139) is as follows:

$$\begin{aligned}F(T)=&\frac{\alpha^2\,\kappa(1-\delta_2)}{\beta(\alpha+\delta_2)}+\left[F(0)-\frac{\alpha^2\,\kappa(1-\delta_2)}{\beta(\alpha+\delta_2)}\right]\\&\times\left[(4\alpha(b+1-6a)-4a\alpha_2)+\left(4\alpha\,c_0^2(b+1-a)\right)T+\left(c_0^4\,(a\,\delta_2+\alpha\,(b+1))\right)T^2\right]^{\frac{(\alpha+\delta_2)\,a}{2(\delta_2\,a+\alpha(1+b))}}\\&\times\exp\left[\frac{(\alpha+\delta_2)\sqrt{a}(\alpha(a-2(b+1))-a\,\delta_2)}{(\delta_2\,a+\alpha(1+b))\sqrt{6a\alpha\,\delta_2+a+\alpha^2(a+4(b+1))}}\right.\\&\left.\times\tanh^{-1}\left(\frac{2\alpha(a-b-1)-c_0^2(a\delta_2+\alpha(b+1))\,T}{2\sqrt{a}\sqrt{6a\alpha\,\delta_2+a+\alpha^2(a+4(b+1))}}\right)\right].\end{aligned} \tag{140}$$

For $\delta_2=+1$, the solution will be the same as a linear perfect fluid which is solved in Section 3. The most interesting case is $\delta_2=-1$ where Equation (140) becomes the following:

$$\begin{aligned}F(T)=&\frac{2\alpha^2\,\kappa}{\beta(\alpha-1)}+\left[F(0)-\frac{2\alpha^2\,\kappa}{\beta(\alpha-1)}\right]\\&\times\left[(4\alpha(b+1-6a)+4a)+\left(4\alpha\,c_0^2(b+1-a)\right)T+\left(c_0^4\,(-a+\alpha\,(b+1))\right)T^2\right]^{\frac{(\alpha-1)\,a}{2(-a+\alpha(1+b))}}\\&\times\exp\left[\frac{(\alpha-1)\sqrt{a}(\alpha(a-2(b+1))+a)}{(-a+\alpha(1+b))\sqrt{-6a\alpha+a+\alpha^2(a+4(b+1))}}\right.\\&\left.\times\tanh^{-1}\left(\frac{2\alpha(a-b-1)-c_0^2(-a+\alpha(b+1))\,T}{2\sqrt{a}\sqrt{-6a\alpha+a+\alpha^2(a+4(b+1))}}\right)\right]\end{aligned} \tag{141}$$

where $\alpha\neq 1$. We found in Equation (141) a real quadratic new $F(T)$ solution for a weak correction in ρ^2 to the linear and isotropic perfect fluid. We can proceed to the same exercise for $w=3$ and $w=4$ corrections; we will just obtain a slightly different $F(T)$ solution in both cases.

5.2. $A_3=r$ Power-Law Solutions

By using the Equation (24) ansatz and setting $A_3=r$ as the coordinate choice, we will solve (16a), (42), (129), (130a) and (130b):

$$F'(T)=F'(0)\,\exp\left[\int dr\,\frac{\left[(2a-a^2+ab+b+1)\,r^{-2b}-b_0^2\right]}{\left[(a+1)\,r^{2(1-b)-1}+\delta\,b_0\,r^{(1-b)}\right]}\right], \tag{142a}$$

$$\begin{aligned}(1+\alpha)\,(\kappa\rho)+\kappa^{1-w}\,\beta\,(\kappa\rho)^w=&\frac{2}{b_0^2}\,F'(T)\left[-\frac{\left[(2a-a^2+ab+b+1)\,r^{-2b}-b_0^2\right]}{\left[(a+1)\,r^{2(1-b)-1}+\delta\,b_0\,r^{(1-b)}\right]}\left[r^{-2b-1}+\delta\,b_0\,r^{-b-1}\right]\right.\\&\left.+(a+b)\,r^{-2b-2}\right],\end{aligned} \tag{142b}$$

$$\begin{aligned}\kappa\,\rho=&-\frac{F(T)}{2}+\frac{2}{b_0^2}\,F'(T)\left[-\frac{\left[(2a-a^2+ab+b+1)\,r^{-2b}-b_0^2\right]}{\left[(a+1)\,r^{2(1-b)-1}+\delta\,b_0\,r^{(1-b)}\right]}\left[r^{-2b-1}+\delta\,b_0\,r^{-b-1}\right.\right.\\&\left.\left.+\left[(-a+b-1)\,r^{-2b-2}-\delta\,b_0\,(a+1)\,r^{-b-2}\right]\right]\right],\end{aligned} \tag{142c}$$

$$T(r)=-\frac{2}{b_0^2}\left[b_0^2\,r^{-2}+2\delta\,b_0\,(1+a)\,r^{-2-b}+(2\,a+1)\,r^{-2-2b}\right]. \tag{142d}$$

$$\tilde{a}_0\,r^a=\left[(1+\alpha)\rho^{1-w}+\beta\right]^{\left[\frac{\alpha}{(1+\alpha)(w-1)}-\frac{w}{w-1}\right]}\rho^{-w} \tag{142e}$$

Because we are still looking for new $F(T)$ solutions, we need as in all previous cases to isolate $r(T)$ from Equation (142d) leading to the following characteristic equation:

$$0 = \frac{b_0^2\,T}{2} + b_0^2\,r^{-2} + 2\delta\,b_0\,(1+a)\,r^{-2-b} + (2\,a+1)\,r^{-2-2b}. \tag{143}$$

As in the Section 3.2, there are three classes of solution according to a:

1. **General case ($\mathbf{a \neq \left\{-1, -\frac{1}{2}\right\}}$):** We can solve Equation (143) for $b = \{0, 1, -1, -2\}$ leading to an $r(T)$ solution in each case. Then, we have to solve for each of the four values of b Equations (142a)–(142c) (the FEs) with respect to Equation (129) (conservation laws). As a relevant case, we will solve the $b = 0$ subcase for comparison with the perfect and dust fluid solutions. First, Equation (142e) for the conservation law remains invariant and Equation (143) will be as follows:

$$\begin{aligned} 0 &= \frac{b_0^2\,T}{2}\,r^2 + \left(b_0^2 + 2\delta\,b_0\,(1+a) + 2\,a + 1\right), \\ \Rightarrow r^2(T) &= \frac{2\left(b_0^2 + 2\delta\,b_0\,(1+a) + 2\,a+1\right)}{b_0^2\,(-T)}, \end{aligned} \tag{144}$$

where $T \leq 0$ and $\delta_1 = \pm 1$. Equations (142a)–(142c) will simplify as follows:

$$F'(-T) = -F'(0)\left[\frac{2\left(b_0^2 + 2\delta\,b_0\,(1+a) + 2\,a+1\right)}{b_0^2}\right]^{\frac{\left(2a-a^2+1-b_0^2\right)}{2(a+1+\delta\,b_0)}}(-T)^{-\frac{\left(2a-a^2+1-b_0^2\right)}{2(a+1+\delta\,b_0)}}, \tag{145a}$$

$$(1+\alpha)(\kappa\rho) + \kappa^{1-w}\,\beta\,(\kappa\rho)^w = \frac{(-T)\,F'(T)}{\left(b_0^2 + 2\delta\,b_0\,(1+a) + 2\,a+1\right)}\left[-\frac{\left(2a-a^2+1-b_0^2\right)}{(a+1+\delta\,b_0)}(1+\delta\,b_0) + a\right], \tag{145b}$$

$$\kappa\,\rho = -\frac{F(T)}{2} + \frac{(1+\delta\,b_0)\,(-T)\,F'(T)}{\left(b_0^2 + 2\delta\,b_0\,(1+a) + 2\,a+1\right)}\left[-\frac{\left(2a-a^2+1-b_0^2\right)}{(a+1+\delta\,b_0)} - (a+1)\right]. \tag{145c}$$

By substituting Equation (145c) into Equation (145b) and by expressing F and F' in terms of $(-T)$, we obtain a DE for a pure $F(T)$ solution as a characteristic algebraic equation with $F'(-T)$ expressed by Equation (145a). This expression for $w = 2$ and its solution will be expressed as follows:

$$\begin{aligned} 0 =& G^2(F(-T),-T) + \frac{\kappa(1+\alpha)}{\beta}\,G(F(-T),-T) - C(-T), \\ \Rightarrow\ & G(F(-T),-T) = -\frac{\kappa(1+\alpha)}{2\beta} - \delta_1\sqrt{\left(\frac{\kappa(1+\alpha)}{2\beta}\right)^2 + C(-T)} \end{aligned} \tag{146}$$

where $\delta_1 = \pm 1$ and

$$\begin{aligned} G(F(-T),-T) =& -\frac{F(-T)}{2} - \frac{(1+\delta\,b_0)F'(0)}{\left(b_0^2 + 2\delta\,b_0\,(1+a) + 2\,a+1\right)}\left[\frac{\left(2a-a^2+1-b_0^2\right)}{(a+1+\delta\,b_0)} + (a+1)\right] \\ & \times \left[\frac{2\left(b_0^2 + 2\delta\,b_0\,(1+a) + 2\,a+1\right)}{b_0^2}\right]^{\frac{\left(2a-a^2+1-b_0^2\right)}{2(a+1+\delta\,b_0)}}(-T)^{1-\frac{\left(2a-a^2+1-b_0^2\right)}{2(a+1+\delta\,b_0)}}, \end{aligned} \tag{147a}$$

$$\begin{aligned} C(-T) =& -\frac{\kappa F'(0)}{\beta\left(b_0^2 + 2\delta\,b_0\,(1+a) + 2\,a+1\right)}\left[\frac{\left(2a-a^2+1-b_0^2\right)}{(a+1+\delta\,b_0)}(1+\delta\,b_0) - a\right] \\ & \times \left[\frac{2\left(b_0^2 + 2\delta\,b_0\,(1+a) + 2\,a+1\right)}{b_0^2}\right]^{\frac{\left(2a-a^2+1-b_0^2\right)}{2(a+1+\delta\,b_0)}}(-T)^{1-\frac{\left(2a-a^2+1-b_0^2\right)}{2(a+1+\delta\,b_0)}} \end{aligned} \tag{147b}$$

The Equation (146) solution will be as follows for $F(-T)$:

$$\begin{aligned}F(-T) =& \frac{\kappa(1+\alpha)}{\beta} - \frac{2(1+\delta\, b_0)F'(0)}{\left(b_0^2+2\delta\, b_0\,(1+a)+2\,a+1\right)}\left[\frac{\left(2a-a^2+1-b_0^2\right)}{(a+1+\delta\, b_0)}+(a+1)\right] \\ &\times\left[\frac{2\left(b_0^2+2\delta\, b_0\,(1+a)+2\,a+1\right)}{b_0^2}\right]^{\frac{\left(2a-a^2+1-b_0^2\right)}{2(a+1+\delta\, b_0)}}(-T)^{1-\frac{\left(2a-a^2+1-b_0^2\right)}{2(a+1+\delta\, b_0)}} \\ &+\delta_1\Bigg[\left(\frac{\kappa(1+\alpha)}{\beta}\right)^2-\frac{4\kappa F'(0)}{\beta\left(b_0^2+2\delta\, b_0\,(1+a)+2\,a+1\right)} \\ &\times\left[\frac{\left(2a-a^2+1-b_0^2\right)}{(a+1+\delta\, b_0)}(1+\delta\, b_0)-a\right] \\ &\times\left[\frac{2\left(b_0^2+2\delta\, b_0\,(1+a)+2\,a+1\right)}{b_0^2}\right]^{\frac{\left(2a-a^2+1-b_0^2\right)}{2(a+1+\delta\, b_0)}}(-T)^{1-\frac{\left(2a-a^2+1-b_0^2\right)}{2(a+1+\delta\, b_0)}}\Bigg]^{1/2}\end{aligned} \tag{148}$$

As expected, Equation (148) describes a complex non-linear perfect cosmological fluid teleparallel $F(T)$ solution. Here is a proof that non-linear fluids can lead to relevant solutions.

2. $\mathbf{a}=\mathbf{b}=\mathbf{0}$ special case: Equation (143) becomes the following:

$$\begin{aligned}0 =& \frac{b_0^2\,T}{2}+(1+\delta\, b_0)^2\,r^{-2}, \\ &\Rightarrow\; r^{-2}(T) = -\frac{b_0^2\,T}{2(1+\delta\, b_0)^2}\end{aligned} \tag{149}$$

By substituting Equation (142c) into Equation (142b) and then by putting Equation (142a) inside, we find an algebraic equation:

$$\begin{aligned}0 =& \left(-\frac{F(T)}{2}+F'(0)\left[\frac{(-2)^{\frac{1-\delta\, b_0}{2}}(\alpha\,(2-\delta\, b_0)+1)}{(1+\alpha)b_0^{1-\delta\, b_0}(1+\delta\, b_0)^{\delta\, b_0}}\right]T^{\frac{(1+\delta\, b_0)}{2}}\right) \\ &+\frac{\kappa^{1-w}\,\beta}{(1+\alpha)}\left(-\frac{F(T)}{2}+F'(0)\left[\frac{(-2)^{\frac{1-\delta\, b_0}{2}}(2-\delta\, b_0)}{b_0^{1-\delta\, b_0}(1+\delta\, b_0)^{\delta\, b_0}}\right]T^{\frac{(1+\delta\, b_0)}{2}}\right)^w,\end{aligned} \tag{150}$$

Equation (150) is a degree w polynomial equation in terms of $F(T)$ and it is solvable for $w=2$, 3 and 4. For $w=2$, Equation (150) is the following with simplifications:

$$\begin{aligned}0 =& \left(\frac{\beta}{2\kappa(1+\alpha)}\right)F(T)^2+\left(\frac{\beta\, F'(0)}{\kappa(1+\alpha)}\left[\frac{(-2)^{\frac{3-\delta\, b_0}{2}}(2-\delta\, b_0)}{b_0^{1-\delta\, b_0}(1+\delta\, b_0)^{\delta\, b_0}}\right]T^{\frac{(1+\delta\, b_0)}{2}}-1\right)F(T) \\ &+\left(\frac{\beta\, F'(0)^2}{2\kappa(1+\alpha)}\left[\frac{(-2)^{\frac{3-\delta\, b_0}{2}}(2-\delta\, b_0)}{b_0^{1-\delta\, b_0}(1+\delta\, b_0)^{\delta\, b_0}}\right]^2 T^{1+\delta\, b_0}-F'(0)\left[\frac{(-2)^{\frac{3-\delta\, b_0}{2}}(\alpha\,(2-\delta\, b_0)+1)}{(1+\alpha)b_0^{1-\delta\, b_0}(1+\delta\, b_0)^{\delta\, b_0}}\right]T^{\frac{(1+\delta\, b_0)}{2}}\right),\end{aligned} \tag{151}$$

The general solution is as follows:

$$F(T) = \frac{\kappa(1+\alpha)}{\beta}\left[1 - \left(\frac{\beta\,F'(0)(-2)^{\frac{3-\delta\,b_0}{2}}(2-\delta\,b_0)}{\kappa(1+\alpha)b_0^{1-\delta\,b_0}(1+\delta\,b_0)^{\delta\,b_0}}\right)T^{\frac{(1+\delta\,b_0)}{2}}\right.$$
$$\left. + \delta_2\sqrt{1 + \left(\frac{\beta\,F'(0)(-2)^{\frac{5-\delta\,b_0}{2}}(1-\delta\,b_0)}{\kappa(1+\alpha)^2 b_0^{1-\delta\,b_0}(1+\delta\,b_0)^{\delta\,b_0}}\right)T^{\frac{(1+\delta\,b_0)}{2}}}\right]. \quad (152)$$

Equation (152) is a new teleparallel $F(T)$ solution and this function is not an approximated form. Hence, for a flat teleparallel spacetime, there is a non-trivial $F(T)$ solution which is not a pure power-law solution.

3. $\mathbf{a} = -\mathbf{1}$ case: Equation (143) will simplify as follows:

$$0 = \frac{b_0^2\,T}{2} + b_0^2\,r^{-2} - r^{-2-2b}. \quad (153)$$

Then, we can solve Equation (153) for $b = \left\{0, \frac{1}{2}, -\frac{1}{2}, 1, -1, -\frac{3}{2}, 2, -2, 3\right\}$ as in Section 3.2. After that, Equations (142a)–(142e) become the following for $a = -1$:

$$F'(T) = F'(0)\exp\left[-\frac{\delta}{b_0\,b}\left(-2\,r^{-b}(T) + b_0^2\,r^b(T)\right)\right] \qquad b \neq 0, \quad (154a)$$

$$= F'(0)\,[r(T)]^{-\frac{\delta\left(2+b_0^2\right)}{b_0}} \qquad b = 0, \quad (154b)$$

$$(1+\alpha)\,(\kappa\rho) + \kappa^{1-w}\,\beta\,(\kappa\rho)^w = \frac{2}{b_0^2}\,F'(T)\left[\frac{\left(2\,r^{-2b} + b_0^2\right)}{\delta\,b_0}\left(r^{-b} + \delta\,b_0\right)r^{-2} + (b-1)\,r^{-2b-2}\right], \quad (154c)$$

$$\kappa\,\rho = -\frac{F(T)}{2} + \frac{2}{b_0^2}\,F'(T)\left[\frac{\left(2\,r^{-2b} + b_0^2\right)}{\delta\,b_0}\left(r^{-b} + \delta\,b_0\right)r^{-2} + b\,r^{-2b-2}\right], \quad (154d)$$

$$\frac{\tilde{a}_0}{r} = \left[(1+\alpha)\rho^{1-w} + \beta\right]^{\left[\frac{\alpha}{(1+\alpha)(w-1)} - \frac{w}{w-1}\right]}\rho^{-w} \quad (154e)$$

We will only solve Equations (154b)–(154e) for the $b = 0$ subcase solution. Equation (154e) remains unchanged and Equation (153) becomes the following:

$$r^{-2}(T) = \frac{b_0^2}{2(1-b_0^2)}\,T. \quad (155)$$

By substituting Equation (154d) and then Equation (154b) into Equation (154c) for $b = 0$ and $w = 2$, we will obtain the algebraic equation in the Equation (146) form. (We only change terms in $-T$ for T terms!) The solution of this new Equation (146) will be as follows:

$$F(T) = \frac{2F'(0)\,(2+b_0^2)}{\delta\,b_0(1-\delta b_0)}\left[\frac{b_0^2}{2(1-b_0^2)}\right]^{\frac{\delta\left(2+b_0^2\right)}{2b_0}} T^{1+\frac{\delta\left(2+b_0^2\right)}{2b_0}} + \frac{\kappa(1+\alpha)}{\beta}$$
$$+ \delta_1\left[\left(\frac{\kappa(1+\alpha)}{\beta}\right)^2 - \frac{4\kappa\,F'(0)}{\beta}\left[\frac{b_0^2}{2(1-b_0^2)}\right]^{\frac{\delta\left(2+b_0^2\right)}{2b_0}}\left[\frac{(2+b_0^2)}{\delta\,b_0(1-\delta\,b_0)} - \frac{1}{(1-b_0^2)}\right]T^{1+\frac{\delta\left(2+b_0^2\right)}{2b_0}}\right]^{1/2} \quad (156)$$

4. $\mathbf{a} = -\frac{1}{2}$ case: Equation (143) will simplify as follows:

$$0 = \frac{b_0\,T}{2} + b_0\,r^{-2} + \delta\,r^{-2-b}. \tag{157}$$

We can solve Equation (157) for $b = \{0, 1, -1, 2, -2, -3, 4, -4, 6, -6, -8\}$ as in Section 3.2. Then, Equations (142a)–(142e) become the following for $a = -\frac{1}{2}$:

$$F'(T) = F'(0)\,\exp\left[\int dr\,\frac{\left[\left(-\frac{1}{4}+\frac{b}{2}\right)r^{-2b} - b_0^2\right]}{\left[\frac{1}{2}\,r^{1-2b} + \delta\,b_0\,r^{1-b}\right]}\right], \tag{158a}$$

$$(1+\alpha)\,(\kappa\rho) + \kappa^{1-w}\,\beta\,(\kappa\rho)^w = \frac{2}{b_0^2}\,F'(T)\left[\frac{\left[\left(\frac{1}{4}-\frac{b}{2}\right)r^{-2b} + b_0^2\right]}{r^2\left(\frac{r^{-b}}{2} + \delta\,b_0\right)}\left(r^{-b} + \delta\,b_0\right) + \left(b - \frac{1}{2}\right)r^{-2b-2}\right], \tag{158b}$$

$$\kappa\,\rho = -\frac{F(T)}{2} + \frac{2}{b_0^2}\,F'(T)\left[\frac{\left[\left(\frac{1}{4}-\frac{b}{2}\right)r^{-2b} + b_0^2\right]}{r^2\left(\frac{r^{-b}}{2} + \delta\,b_0\right)}\left(r^{-b} + \delta\,b_0\right) + \left(b - \frac{1}{2}\right)r^{-2b-2} - \frac{\delta\,b_0}{2}\,r^{-b-2}\right], \tag{158c}$$

$$\frac{\tilde{a}_0}{\sqrt{r}} = \left[(1+\alpha)\rho^{1-w} + \beta\right]^{\left[\frac{\alpha}{(1+\alpha)(w-1)} - \frac{w}{w-1}\right]}\rho^{-w} \tag{158d}$$

We will again solve Equations (158a)–(158d) only for the $b = 0$ subcase solution. Equation (158d) remains unchanged and Equation (157) becomes the following:

$$r^{-2}(T) = \frac{\delta\,b_0}{2(1+\delta\,b_0)}\,(-T). \tag{159}$$

By substituting Equation (158c) and then Equation (158a) into Equation (158b) for $b = 0$ and by setting $w = 2$, we obtain another quadratic equation in Equation (146) form. The solution of this relation will be as follows:

$$\begin{aligned} F(T) =& \frac{2\delta\,F'(0)}{b_0}\left[\frac{\left(\frac{1}{4}+b_0^2\right)}{\left(\frac{1}{2}+\delta\,b_0\right)} - \frac{1}{2}\right]\left[\frac{\delta\,b_0}{2(1+\delta\,b_0)}\right]^{\frac{\left(\frac{1}{4}+b_0^2\right)}{(1+2\delta\,b_0)}}(-T)^{1+\frac{\left(\frac{1}{4}+b_0^2\right)}{(1+2\delta\,b_0)}} + \frac{\kappa(1+\alpha)}{\beta} \\ & -\delta_1\left[\left(\frac{\kappa(1+\alpha)}{\beta}\right)^2 + \frac{4\kappa\,\delta\,F'(0)}{\beta\,b_0}\left[\frac{\left(\frac{1}{4}+b_0^2\right)}{\left(\frac{1}{2}+\delta\,b_0\right)} - \frac{1}{2(1+\delta\,b_0)}\right]\left[\frac{\delta\,b_0}{2(1+\delta\,b_0)}\right]^{\frac{\left(\frac{1}{4}+b_0^2\right)}{(1+2\delta\,b_0)}}\right. \\ & \left.\times\,(-T)^{1+\frac{\left(\frac{1}{4}+b_0^2\right)}{(1+2\delta\,b_0)}}\right]^{1/2} \end{aligned} \tag{160}$$

All these previous $b = 0$ teleparallel $F(T)$ solutions are expressing some possible cosmological spacetime geometries. In the recent literature, there are some simple pure power-law $F(T) \sim (-T)^k$, logarithmic $F(T) \sim \ln(-T)$ or $F(T) \sim (-T)^k \ln(-T)$ leading to some stable solutions [2,19,36]. In addition, we can carry out the same type of development for all $b \neq 0$ subcases and we may find some more complex non-linear fluid $F(T)$ solutions. As for Sections 3 and 4, some of these new solutions may lead to some black hole and/or matter absorbing singularity solutions by the end of this process. But in this current section, all necessary FEs and conservation laws are there for further investigation in this way.

6. Discussion and Conclusions

In this paper, we first solved conservation laws and FEs and then found in Sections 3–5 dozens of new teleparallel $F(T)$ solutions in static spherically symmetric spacetimes for perfect fluids. These new $F(T)$ solutions are products of exponential, power, quotients and some mixtures of these types of expression. In some of these new $F(T)$ solutions, we found some new singularities which arise to point-like discontinuity or undefined $F(T)$ functions. In Section 3.1, we found new teleparallel $F(T)$ solutions for the constant A_3 where we used a power-law ansatz in Section 3.1.1 and a special ansatz defined by an $A_2 =$ constant (i.e., $b = 0$ set as in ref. [18]) and an exponential A_1 component in Section 3.1.2. This A_1 component generalizes the power-law ansatz by a summation of an infinite number of integer power-law terms. By this approach, we found the same singularities as in Section 3.1.1 and an additional singularity arising from the new ansatz.

For the rest of the paper (Sections 3.2–5.2), we used a power-law ansatz approach to find new $F(T)$ solutions by choosing an $A_3 = r$ coordinate system. If $A_3 \equiv$ constant, then we found for slightly quadratic perfect fluid approximation ($\beta \ll \alpha$ and $w = 2$) some new approximated $F(T)$ solutions as shown by Equations (140) and (141) in Section 5.1. The solutions found in Section 5.2 for non-linear perfect fluids (in particular $w = 2$) are usually generalizing the power-law $F(T)$ found in Sections 3.2.2–3.2.5 for $b = 0$ and are exact. We can easily make the same assumptions for $b \neq 0$ cases for generalizing Section 3.2 new solutions. In addition, the new $F(T)$ solutions in Section 4 for cosmological dust fluids ($\alpha = 0$) should be useful for studying some cosmological models with baryonic matter [34].

Then, we look for non-perfect fluid $F(T)$ solutions, but we will have at least to add supplementary terms to the Equation (11) definition of energy momentum. We will at least have to add some factors such as viscosity and any fluid imperfections. Equation (11) characterizes an ideal fluid without any viscosity or imperfection where the pressure and the density are directly linked by an EoS. But this assumption of Equation (11) cannot necessarily be performed for non-perfect fluids because of these additional physical factors. Several works may be carried out in the future, but we can expect more complex $F(T)$ solutions than those found in this paper.

For astrophysical and cosmological applications, a detailed analysis for each $F(T)$ solution obtained will be necessary for determining the stability conditions and their physical processes. There are several recent works on this type of study (see [34,36–43] and references within). They sometimes replace the fluid by a scalar field source in some of these studies [37,39]. In addition, we should also study the physical processes around the singularities for each $F(T)$ solution in some future works. We can also work with electromagnetic energy-momentum sources for new classes of $F(T)$ solutions and for possible "electromagnetic" BH horizons, but new $F(T)$ solutions will be necessary [21,24–27]. The teleparallel $F(T)$ solutions obtained in this paper can also be used as conditions for dynamical cosmological models. These solutions can be used for (r, t)-coordinates-based $F(T)$ solutions in some astrophysical and cosmological problems. In addition, there are in this paper many teleparallel $F(T)$ solutions for solving these physical problems and there are necessary ingredients for a complete cosmological analysis.

To proceed further in this approach, there are some ongoing developments concerning Kantowski–Sachs spacetime solutions in teleparallel $F(T)$ gravity where we look for general, fluid and other solutions (see [18] and references within). There are some possible works on axially symmetric teleparallel $F(T)$ geometries allowing solving more astrophysical problems with teleparallel gravity [44,45]. Another possible work is looking for teleparallel $F(T, B)$-type geometries. All these possibilities deserve serious and tactful considerations.

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Abbreviations

The following abbreviations are used in this manuscript:

FEs	Field Equations
EoS	Equation of State
BH	Black Hole
Eqn	Equation
AL	Alexandre Landry
GR	General Relativity
TEGR	Teleparallel Equivalent of General Relativity

Appendix A. Field Equation Components

This appendix is for presenting the exact FE components found in ref. [18]. There are general, constant A_3 and $A_3 = r$ power-law ansatz FE components for the current paper's purposes.

Appendix A.1. General Components

$$\frac{g_1}{k_1} = \frac{\left[- A_2\, A_3^2\, A_1'' - A_1\, A_2\, A_3\, A_3'' + A_1\, A_2\, A_3'^2 + (A_1\, A_2)'\, A_3\, A_3' + A_3^2\, A_1'\, A_2' - A_1\, A_2^3 \right]}{\left[A_1\, A_2\, A_3\, A_3' + A_2\, A_3^2\, A_1' + \delta\, A_1\, A_2^2\, A_3\right]} \tag{A1a}$$

$$g_2 = \frac{1}{A_1\, A_2^3\, A_3} \left[- A_1\, A_2\, A_3'' + (A_1\, A_2)'\, A_3' \right] \tag{A1b}$$

$$g_3 = \frac{1}{A_1\, A_2^3\, A_3^2} \left[- A_1\, A_2\, A_3\, A_3'' - A_1\, A_2\, A_3'^2 - A_2\, A_3\, A_3'\, A_1' - \delta\, A_1\, A_2^2\, A_3' + A_1\, A_3\, A_2'\, A_3' - \delta\, A_2^2\, A_3\, A_1' \right], \tag{A1c}$$

$$k_2 = k_3 = \frac{1}{A_2^2\, A_3} \left[A_3' + \delta\, A_2\right] \tag{A1d}$$

Appendix A.2. $A_3 = c_0 =$ Constant Power-Law Components

Equations (A1a)–(A1d) with Equations (24) power-laws ansatz are as follows:

$$\frac{g_1}{k_1} = \frac{\left[(a(1-a+b))\, r^{-2b-2} - \left(\frac{b_0}{c_0}\right)^2 \right]}{\left[a\, r^{-2b-1} + \delta \left(\frac{b_0}{c_0}\right) r^{-b} \right]}, \quad g_2 = 0, \quad g_3 = -\left(\frac{\delta\, a}{b_0\, c_0}\right) r^{-b-1}, \quad k_2 = k_3 = \left(\frac{\delta}{b_0\, c_0}\right) r^{-b}. \tag{A2}$$

Appendix A.3. $A_3 = r$ Power-Law Components

Equations (A1a)–(A1d) with Equations (24) power-laws ansatz are as follows:

$$\begin{aligned} \frac{g_1}{k_1} &= \frac{\left[\left(2a - a^2 + ab + b + 1\right) r^{-2b} - b_0^2 \right]}{\left[(a+1)\, r^{2(1-b)-1} + \delta\, b_0\, r^{(1-b)} \right]}, & g_2 &= \frac{(a+b)}{b_0^2}\, r^{-2b-2}, \\ g_3 &= \frac{1}{b_0^2} \left[(-a+b-1)\, r^{-2b-2} - \delta\, b_0\, (a+1)\, r^{-b-2} \right], & k_2 = k_3 &= \frac{1}{b_0^2} \left[r^{-2b-1} + \delta\, b_0\, r^{-b-1} \right]. \end{aligned} \tag{A3}$$

References

1. Aldrovandi, R.; Pereira, J.G. *Teleparallel Gravity, An Introduction*; Springer: Berlin/Heidelberg, Germany, 2013. Available online: https://link.springer.com/book/10.1007/978-94-007-5143-9 (accessed on 16 May 2024).
2. Bahamonde, S.; Dialektopoulos, K.; Escamilla-Rivera, C.; Farrugia, G.; Gakis, V.; Hendry, M.; Hohmann, M.; Said, J.L.; Mifsud, J.; Di Valentino, E.; Teleparallel Gravity: From Theory to Cosmology. *Rep. Prog. Phys.* **2023**, *86*, 026901. [CrossRef]
3. Krssak, M.; van den Hoogen, R.; Pereira, J.; Boehmer, C.; Coley, A. Teleparallel Theories of Gravity: Illuminating a Fully Invariant Approach. *Class. Quantum Gravity* **2019**, *36*, 183001. [CrossRef]

4. Chinea, F. Symmetries in tetrad theories. *Class. Quantum Gravity* **1988**, *5*, 135. [CrossRef]
5. Estabrook, F.; Wahlquist, H. Moving frame formulations of 4-geometries having isometries. *Class. Quantum Gravity* **1996**, *13*, 1333. [CrossRef]
6. Papadopoulos, G.; Grammenos, T. Locally homogeneous spaces, induced Killing vector fields and applications to Bianchi prototypes. *J. Math. Phys.* **2012**, *53*, 072502. [CrossRef]
7. Olver, P. *Equivalence, Invariants and Symmetry*; Cambridge University Press: Cambridge, UK, 1995. Available online: https://books.google.ca/books/about/Equivalence_Invariants_and_Symmetry.html?id=YuTzf61HILAC&redir_esc=y (accessed on 16 May 2024).
8. McNutt, D.D.; Coley, A.A.; van den Hoogen, R.J. A frame based approach to computing symmetries with non-trivial isotropy groups. *J. Math. Phys.* **2023**, *64*, 032503. [CrossRef]
9. Ferraro, R.; Fiorini, F. Modified teleparallel gravity: Inflation without an inflation. *Phys. Rev.* **2007**, *75*, 084031. [CrossRef]
10. Ferraro, R.; Fiorini, F. On Born-Infeld Gravity in Weitzenbock spacetime. *Phys. Rev.* **2008**, *78*, 124019. [CrossRef]
11. Linder, E. Einstein's Other Gravity and the Acceleration of the Universe. *Phys. Rev.* **2010**, *81*, 127301; Erratum: *Phys. Rev.* **2010**, *82*, 109902. [CrossRef]
12. Lucas, T.G.; Obukhov, Y.; Pereira, J.G. Regularizing role of teleparallelism. *Phys. Rev.* **2009**, *80*, 064043. [CrossRef]
13. Krssak, M.; Pereira, J.G. Spin Connection and Renormalization of Teleparallel Action. *Eur. Phys. J.* **2015**, *75*, 519. [CrossRef]
14. Golovnev, A.; Guzman, M.-J. Approaches to spherically symmetric solutions in $f(T)$-gravity. *Universe* **2021**, *7*, 121. [CrossRef]
15. Golovnev, A. Issues of Lorentz-invariance in $f(T)$-gravity and calculations for spherically symmetric solutions. *Class. Quantum Gravity* **2021**, *38*, 197001. [CrossRef]
16. Golovnev, A.; Guzman, M.-J. Bianchi identities in $f(T)$-gravity: Paving the way to confrontation with astrophysics. *Phys. Lett.* **2020**, *810*, 135806. [CrossRef]
17. DeBenedictis, A.; Ilijić, S.; Sossich, M. On spherically symmetric vacuum solutions and horizons in covariant $f(T)$ gravity theory. *Phys. Rev.* **2022**, *105*, 084020. [CrossRef]
18. Coley, A.A.; Landry, A.; van den Hoogen, R.J.; McNutt, D.D. Spherically symmetric teleparallel geometries. *Eur. Phys. J.* **2024**, *84*, 334. [CrossRef]
19. Coley, A.A.; Landry, A.; van den Hoogen, R.J.; McNutt, D.D. Generalized Teleparallel de Sitter geometries. *Eur. Phys. J.* **2023**, *83*, 977. [CrossRef]
20. Bahamonde, S.; Camci, U. Exact Spherically Symmetric Solutions in Modified Teleparallel gravity. *Symmetry* **2019**, *11*, 1462. [CrossRef]
21. Awad, A.; Golovnev, A.; Guzman, M.-J.; El Hanafy, W. Revisiting diagonal tetrads: New Black Hole solutions in $f(T)$-gravity. *Eur. Phys. J.* **2022**, *82*, 972. [CrossRef]
22. Bahamonde, S.; Golovnev, A.; Guzmán, M.-J.; Said, J.L.; Pfeifer, C. Black Holes in $f(T,B)$ Gravity: Exact and Perturbed Solutions. *J. Cosmol. Astropart. Phys.* **2022**, *01* 037. [CrossRef]
23. Bahamonde, S.; Faraji, S.; Hackmann, E.; Pfeifer, C. Thick accretion disk configurations in the Born-Infeld teleparallel gravity. *Phys. Rev.* **2022**, *106*, 084046. [CrossRef]
24. Nashed, G.G.L. Quadratic and cubic spherically symmetric black holes in the modified teleparallel equivalent of general relativity: Energy and thermodynamics. *Class. Quantum Gravity* **2021**, *38*, 125004. [CrossRef]
25. Pfeifer, C.; Schuster, S. Static spherically symmetric black holes in weak $f(T)$-gravity. *Universe* **2021**, *7*, 153. [CrossRef]
26. El Hanafy, W.; Nashed, G.G.L. Exact Teleparallel Gravity of Binary Black Holes. *Astrophys. Space Sci.* **2016**, *361*, 68. [CrossRef]
27. Aftergood, J.; DeBenedictis, A. Matter Conditions for Regular Black Holes in $f(T)$ Gravity. *Phys. Rev.* **2014**, *90*, 124006. [CrossRef]
28. Bahamonde, S.; Doneva, D.D.; Ducobu, L.; Pfeifer, C.; Yazadjiev, S.S. Spontaneous Scalarization of Black Holes in Gauss-Bonnet Teleparallel Gravity. *Phys. Rev.* **2023**, *107* 10, 104013. [CrossRef]
29. Bahamonde, S.; Ducobu, L.; Pfeifer, C. Scalarized Black Holes in Teleparallel Gravity. *J. Cosmol. Astropart. Phys.* **2022**, *04*, 018. [CrossRef]
30. Sharif, M.; Majeed, B. Teleparallel Killing Vectors of Spherically Symmetric Spacetimes. *Commun. Theor. Phys.* **2009**, *52*, 435. [CrossRef]
31. Hohmann, M.; J arv, L.; Krssak, M.; Pfeifer, C. Modified teleparallel theories of gravity in symmetric spacetimes. *Phys. Rev.* **2019**, *100*, 084002. [CrossRef]
32. Pfeifer, C. A quick guide to spacetime symmetry and symmetric solutions in teleparallel gravity. *arXiv* **2022**, arXiv:2201.04691.
33. Hawking, S.W.; Ellis, G.F.R. *The Large Scale Structure of Space-Time*; Cambridge University Press: Cambridge, UK, 2010. Available online: https://www.cambridge.org/core/books/large-scale-structure-of-spacetime/1E6B961EC9878EDDBBD6AC0AF031CC93 (accessed on 16 May 2024).
34. Coley, A.A. *Dynamical Systems and Cosmology*; Kluwer Academic: Dordrecht, The Netherlands, 2003. Available online: https://link.springer.com/book/10.1007/978-94-017-0327-7 (accessed on 16 May 2024).
35. Landry, A.; van den Hoogen, R.J. Teleparallel Minkowski Spacetime with Perturbative Approach for Teleparallel Theories on the Proper Frame. *Universe* **2013**, *9*, 232. [CrossRef]
36. Duchaniya, L.K.; Gandhi, K.; Mishra, B. Attractor behavior of $f(T)$ modified gravity and the cosmic acceleration. *Phys. Dark Universe* **2024**, *44*, 101464. [CrossRef]

37. Paliathanasis, A.; Leon, G. $f(T,B)$ gravity in a Friedmann-Lemaitre-Robertson-Walker universe with nonzero spatial curvature. *Math. Methods Appl. Sci. arXiv* **2022**, arXiv:2201.12189. [CrossRef]
38. Kofinas, G.; Leon, G.; Saridakis, E.N. Dynamical behavior in $f(T,T_G)$ cosmology. *Class. Quantum Gravity* **2014**, *31*, 175011. [CrossRef]
39. Leon, G.; Paliathanasis, A. Anisotropic spacetimes in $f(T,B)$ theory III: LRS Bianchi III Universe. *Eur. Phys. J. Plus* **2022**, *137*, 927. [CrossRef]
40. Bahamonde, S.; Bohmer, C.G.; Carloni, S.; Copeland, E.J.; Fang,W.; Tamanini, N. Dynamical systems applied to cosmology: Dark energy and modified gravity. *Phys. Rep.* **2018**, *775–777*, 1–122. [CrossRef]
41. Ruggiero, M.L.; Radicella, N. Weak-Field Spherically Symmetric Solutions in $f(T)$ gravity. *Phys. Rev.* **2015**, *91*, 104014. [CrossRef]
42. Iorio, L.; Radicella, N.; Ruggiero, M.L. Constraining $f(T)$ gravity in the Solar System. *J. Cosmol. Astropart. Phys.* **2015**, *8*, 021. [CrossRef]
43. Coley, A.A.; Landry, A.; Gholami, F. Teleparallel Robertson-Walker Geometries and Applications. *Universe* **2023**, *9*, 454. [CrossRef]
44. Bahamonde, S.; Valcarcel, J.; J arv, L.; Pfeifer, C. Exploring axial symmetry in modified teleparallel gravity. *Phys. Rev.* **2021**, *103*, 044058. [CrossRef]
45. Nurbaki, A.N.; Capozziello, S.; Deliduman, C. Spherical and cylindrical solutions in $f(T)$ gravity by Noether Symmetry Approach. *Eur. Phys. J.* **2020**, *80*, 108. [CrossRef]