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Article

Product Convolution of Generalized Subexponential Distributions

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Abstract: Assume that ξ and η are two independent random variables with distribution functions F_ξ and F_η, respectively. The distribution of a random variable $\xi\eta$, denoted by $F_\xi \otimes F_\eta$, is called the product-convolution of F_ξ and F_η. It is proved that $F_\xi \otimes F_\eta$ is a generalized subexponential distribution if F_ξ belongs to the class of generalized subexponential distributions and η is nonnegative and not degenerated at zero.

Keywords: tail function; closure property; product-convolution; generalized subexponential distribution; heavy-tailed distribution

MSC: 60E05; 60G70; 91G10; 26A21

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1. Introduction

The distribution of the product of two independent random variables (r.v.s) is considered in this paper. If ξ and η are two real-valued independent r.v.s with distribution functions (d.f.s) $F_\xi(x) = \mathbb{P}(\xi \leqslant x)$ and $F_\eta(x) = \mathbb{P}(\eta \leqslant x)$, then the d.f. of the product $\xi\eta$ is

$$F_\xi \otimes F_\eta(x) := \mathbb{P}(\xi\eta \leqslant x) = \int\limits_{(-\infty,0)} \left(1 - F_\xi\left(\frac{x}{y}-\right)\right)\mathrm{d}F_\eta(y)$$
$$+ \int\limits_{(0,\infty)} F_\xi\left(\frac{x}{y}\right)\mathrm{d}F_\eta(y) + \big(F_\eta(0) - F_\eta(0-)\big)\mathbb{1}_{[0,\infty)}(x),$$

see, e.g., Section 1.2 of [1]. The d.f. $F_\xi \otimes F_\eta$ is called the *product-convolution* of d.f.s F_ξ and F_η. In the case of a nonnegative r.v. η, we have $F_\eta(0-) = 0$ implying that

$$F_\xi \otimes F_\eta(x) = \int\limits_{(0,\infty)} F_\xi\left(\frac{x}{y}\right)\mathrm{d}F_\eta(y) + F_\eta(0)\mathbb{1}_{[0,\infty)}(x).$$

Our interest lies in the closure properties under multiplication of independent r.v.s. More exactly, we focus on the closure under multiplication of generalized subexponential distributions.

A d.f. F is said to be generalized subexponential or O-subexponential, denoted by $F \in \mathcal{OS}$, if

$$\limsup_{x\to\infty} \frac{\overline{F*F}(x)}{\overline{F}(x)} < \infty.$$

Here and further, the notation $\overline{F}(x) = 1 - F(x), x \in \mathbb{R}$, denotes the tail of the d.f. F, and for any two d.f.s F_1 and F_2, the symbol $F_1 * F_2$ denotes their convolution:

$$F_1 * F_2(x) = \int\limits_{-\infty}^{\infty} F_1(x-y)\mathrm{d}F_2(y),\ x \in \mathbb{R}.$$

If F_ξ and F_η are d.f.s of two independent r.v.s ξ and η, then

$$\mathbb{P}(\xi + \eta \leqslant x) = F_\xi * F_\eta(x),\ x \in \mathbb{R}.$$

Generalized subexponential distributions were firstly mentioned by Klüppelberg [2] as weakly idempotent distributions. Later, class $\mathcal{OS}$ was studied in [3–8]. Class of d.f.s $\mathcal{OS}$ is the generalization of the standard class of subexponential distributions.

A d.f. F, satisfying $F(0-) = 0$, is said to be subexponential, denoted $F \in \mathcal{S}$, if

$$\lim_{x\to\infty} \frac{\overline{F*F}(x)}{\overline{F}(x)} = 2.$$

In the general case, F is said to be subexponential if F^+ is subexponential, where

$$F^+(x) := F(x)\mathbb{1}_{[0,\infty)}(x)$$

is the positive part of d.f. F.

The concept of subexponentiality was introduced by Chistyakov [9]. Later, subexponential distributions, together with O-subexponential distributions, found numerous applications in applied probability including financial mathematics, risk theory, actuarial mathematics, branching processes, queuing theory, etc., see, for instance, [10–25]. It is well known that class $\mathcal{S}$ represents a subset of the class of *long-tailed* d.f.s $\mathcal{L}$, see [9] or Section 3 of [26] for details.

A d.f. F is said to be long-tailed, denoted $F \in \mathcal{L}$, if for any positive y,

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1.$$

Similarly, as in the case of classes $\mathcal{S} \subset \mathcal{OS}$, one can introduce the O-version of class $\mathcal{L}$ according to [3].

A d.f. F is said to belong to the class $\mathcal{OL}$ of generalized long-tailed distributions if for any positive y

$$\limsup_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} < \infty.$$

Similarly to the inclusion $\mathcal{S} \subset \mathcal{L}$, it holds that $\mathcal{OS} \subset \mathcal{OL}$, see, e.g., Proposition 2.1 in [3]. Examples of d.f.s $F \in \mathcal{OL} \setminus \mathcal{OS}$ can be found in [27,28]. Some useful characterizations of class $\mathcal{OL}$ are given in [29]. For instance, according to results by Albin and Sundén [29], an absolutely continuous d.f. F belongs to the class $\mathcal{OL}$ if and only if

$$\overline{F}(x) = \exp\left\{ -\int_{-\infty}^{x} \big(a(y) + b(y)\big)\mathrm{d}y \right\}$$

for some measurable functions $a = a(x)$ and $b = b(x)$ with $a(x) + b(x) \geqslant 0$, $x \in \mathbb{R}$, such that

$$\limsup_{x\to\infty} |a(x)| < \infty, \quad \lim_{x\to\infty} \int_{-\infty}^{x} a(y)\mathrm{d}y = \infty, \quad \limsup_{x\to\infty} \left| \int_{-\infty}^{x} b(y)\mathrm{d}y \right| < \infty.$$

In this work, the closure problem of the class $\mathcal{OS}$ with respect to the product-convolution is actually solved. Any class of d.f.s $\mathcal{K}$ is called closed with respect to some operation, if for any element from the set $\mathcal{K}$, the result remains in the same class after the operation. The classes of d.f.s $\mathcal{S}, \mathcal{L}, \mathcal{OL}, \mathcal{OS}$ defined in this section, together with other classes of d.f.s not defined in this paper, are closed with respect to certain operations. Let us limit ourselves to the d.f. class $\mathcal{OS}$, the main class considered in this paper.

Watanabe and Yamamuro ([6] Lemma 3.1) proved that the class $\mathcal{OS}$ is closed with respect to the weak tail equivalence (see also [12] and [30] (Proposition A1)) for more detailed proof). The closure of class $\mathcal{OS}$ with respect to the weak tail equivalence means the following statement:

- *If a d.f. $F \in \mathcal{OS}$ and $\overline{F} \underset{x\to\infty}{\asymp} \overline{G}(x)$ for a d.f. G, then $G \in \mathcal{OS}$.*

In the same Lemma 3.1 of [6], it is proved that the class $\mathcal{OS}$ is closed with respect to the convolution. The closure of O-subexponential distributions with respect to the convolution means the following statement:

- *If d.f.s F_ξ and F_η of two independent r.v.s ξ and η belong to the class $\mathcal{OS}$, then the convolution $F_\xi * F_\eta$ is O-subexponential as well.*

The closure with respect to the minimum was established by Lin and Wang in [8] (Lemma 3.1) by proving the following statement:

- *If independent r.v.s ξ and η are O-subexponentially distributed, then d.f. $F_{\xi\wedge\eta}$ of minimum $\xi \wedge \eta$ is also O-subexponential.*

One of the closure properties is the closure with respect to the product-convolution. In such a case, the goal is to find minimal conditions for the r.v. η so that the distribution of the product $\xi\eta$ is O-subexponential if the distribution of the first multiplier is O-subexponential. Theorem 5 presented in Section 3 below is the last result in such a direction. In that theorem, there is an additional technical requirement for the r.v. η, which is not necessary. In this paper, we improve the result of Theorem 5 by removing the additional requirement for the second random multiplier η.

The rest of the paper is organized as follows. In Section 2, the main result of the paper is formulated. In Section 3 several related results are reviewed. In Section 4, the proof of the main result is given. Section 5 provides several examples to demonstrate the theoretical meaning of the obtained results. Finally, in Section 6, possible applications of the obtained results to insurance and financial models are discussed.

2. Main Result

As mentioned above, the main result of the paper is on the product-convolution for generalized subexponential distributions.

Theorem 1. *Let ξ and $\eta \geqslant 0$ be two independent r.v.s with d.f.s F_ξ and F_η. If d.f. F_ξ belongs to the class $\mathcal{OS}$ and r.v. η is not degenerated at zero, then the d.f. of the product $F_\xi \otimes F_\eta$ belongs to the class $\mathcal{OS}$ as well.*

If we consider only positive random variables belonging to the class $\mathcal{OS}$, then Theorem 1 shows that the class $\mathcal{OS}$ is closed under the product-convolution. That is, by multiplying two independent r.v.s having generalised subexponential d.f.s with at least one of them being positive, we will always get an r.v. with a generalised subexponential d.f. Among other things this property gives the ability to generate a lot of new r.v.s with d.f.s from class $\mathcal{OS}$.

3. Related Results

In this section, a brief review is given of the related results found in the literature, regarding the product-convolution of distributions belonging to classes close to generalized subexponential distributions. The following conditions for the product-convolution closure for d.f.s from class $\mathcal{L}$ was obtained by Tang [31] (Theorem 1.1).

Theorem 2. *Let ξ and η be two independent r.v.s with d.f.s $F_\xi \in \mathcal{L}$ and F_η. Let η be nonnegative and not degenerated at zero. Then,*

(i) $F_\xi \in \mathcal{L} \otimes F_\eta \in \mathcal{L}$ *if and only if either the set* $\mathcal{D}(F_\xi)$ *of all positive points of discontinuity of d.f.* F_ξ *is empty, or* $\mathcal{D}(F_\xi) \neq \emptyset$ *and*

$$\overline{F}_\eta\Big(\frac{x}{a}\Big) - \overline{F}_\eta\Big(\frac{x+1}{a}\Big) = o\Big(\overline{F_\xi \otimes F_\eta}(x)\Big) \ \ for\ all \ \ a \in \mathcal{D}(F_\xi),$$

(ii) *If* $F_\eta \in \mathcal{L}$, *then* $F_\xi \otimes F_\eta \in \mathcal{L}$.

A similar assertion holds for class $\mathcal{S}$. The following results was proved by Xu et al. [32] (Theorem 1.3).

Theorem 3. *Let* ξ *and* η *be two independent r.v.s with d.f.s* $F_\xi \in \mathcal{S}$ *and* F_η. *Let, in addition,* $\eta \geqslant 0$ *and* $\overline{F}_\eta(0) > 0$. *Then,*

(i) $F_\xi \otimes F_\eta \in \mathcal{S}$ *if and only if either* $\mathcal{D}(F_\xi) = \emptyset$, *or* $\mathcal{D}(F_\xi) \neq \emptyset$ *and*

$$\overline{F}_\eta\Big(\frac{x}{a}\Big) - \overline{F}_\eta\Big(\frac{x+1}{a}\Big) = o\Big(\overline{F_\xi \otimes F_\eta}(x)\Big) \ \ for\ all \ \ a \in \mathcal{D}(F_\xi),$$

(ii) *If* $F_\eta \in \mathcal{L}$, *then* $F_\xi \otimes F_\eta \in \mathcal{S}$.

The assertion below on the class $\mathcal{OL}$ was recently proved by Cui and Wang [27] (Theorem 1).

Theorem 4. *Let* ξ *and* η *be two independent nonnegative r.v.s with d.f.s* F_ξ *and* F_η. *If* $F_\xi \in \mathcal{OL}$ *and* η *is not degenerate at zero, then* $F_\xi \otimes F_\eta \in \mathcal{OL}$.

To our knowledge, the assertion below is the latest known result on the product-convolution closure of d.f.s from class $\mathcal{OS}$. The proof of the following theorem can be found in [33] (Theorem 3).

Theorem 5. *Let* ξ *and* $\eta \geqslant 0$ *be two independent r.v.s with d.f.s* F_ξ *and* F_η. *If* $F_\xi \in \mathcal{OS}$, η *is not degenerated at zero and*

$$\sup_{y>0} \limsup_{x\to\infty} \frac{\overline{F}_\eta(yx)}{\overline{F_\xi \otimes F_\eta}(x)} < \infty,$$

then $F_\xi \otimes F_\eta \in \mathcal{OS}$.

The main theorem of this paper improves on the last statement. Our theorem asserts that the d.f. of the product of two r.v.s remains in the class $\mathcal{OS}$ if the d.f. of the first r.v. belongs to the class $\mathcal{OS}$. The second r.v. should satisfy only the natural requirements.

4. Proofs

This section provides a detailed proof of the main result. Our proof is related with cutting off the second random multiplier. A similar approach was used by Cui and Wang [27] in the proof of Theorem 4. Before the direct proof, we present two auxiliary lemmas.

Lemma 1. *Let* F *and* G *be two d.f.s. If* $F \in \mathcal{OS}$ *and* $\overline{F}(x) \underset{x\to\infty}{\asymp} \overline{G}(x)$, *then* $G \in \mathcal{OS}$.

Proof of Lemma 1. In fact, the statement of the lemma was proved by Watanabe and Yamamuro in [6] (Lemma 3.1). For the sake of completeness, we give here a short proof of the lemma with the additional comments useful for the future.

By definition of the class $\mathcal{OS}$, we get

$$F \in \mathcal{OS} \ \ \Leftrightarrow \ \ c_F := \sup_{x\in\mathbb{R}} \frac{\overline{F*F}(x)}{\overline{F}(x)} < \infty. \tag{1}$$

In addition,

$$\overline{F}(x) \underset{x\to\infty}{\asymp} \overline{G}(x) \quad \Leftrightarrow \quad 0 < \liminf_{x\to\infty} \frac{\overline{G}(x)}{\overline{F}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{G}(x)}{\overline{F}(x)} < \infty,$$

implying that

$$\overline{F}(x) \underset{x\to\infty}{\asymp} \overline{G}(x) \quad \Leftrightarrow \quad c_* := \inf_{x\in\mathbb{R}} \frac{\overline{G}(x)}{\overline{F}(x)} > 0,\ \ c^* := \sup_{x\in\mathbb{R}} \frac{\overline{G}(x)}{\overline{F}(x)} < \infty.$$

For all $x \in \mathbb{R}$,

$$\begin{aligned} \overline{G*G}(x) &= \int\limits_{\mathbb{R}} \overline{G}(x-y)\mathrm{d}G(y) \leqslant c^* \int\limits_{\mathbb{R}} \overline{F}(x-y)\mathrm{d}G(y) \\ &= c^* \int\limits_{\mathbb{R}} \overline{G}(x-y)\mathrm{d}F(y) \leqslant (c^*)^2 \int\limits_{\mathbb{R}} \overline{F}(x-y)\mathrm{d}F(y) \\ &= (c^*)^2\, \overline{F*F}(x). \end{aligned}$$

Therefore,

$$\frac{\overline{G*G}(x)}{\overline{G}(x)} \leqslant \frac{(c^*)^2}{c_*} \frac{\overline{F*F}(x)}{\overline{F}(x)} \leqslant \frac{(c^*)^2}{c_*} c_F.$$

According to relation (1), $G \in \mathcal{OS}$. The lemma is proved. □

Lemma 2. *Let ξ and η be two independent r.v.s with d.f.s F_ξ and F_η. In addition, let $F_\eta(0-) = 0$ and $\overline{F}_\eta(d) > 0$ for some $d > 0$. Then, $F_\xi \otimes F_\eta \in \mathcal{OS}$ if and only if $(F_\xi \otimes F_\eta)_d \in \mathcal{OS}$, where*

$$(F_\xi \otimes F_\eta)_d(x) = \mathbb{P}(\xi \max(\eta, d) \leqslant x),\ x \in \mathbb{R}.$$

Proof of Lemma 2. According to Lemma 1, it is sufficient to prove that

$$\overline{(F_\xi \otimes F_\eta)_d}(x) \underset{x\to\infty}{\asymp} \overline{F_\xi \otimes F_\eta}(x). \tag{2}$$

The simple estimate

$$\overline{F_\xi \otimes F_\eta}(x) = \mathbb{P}(\xi\eta > x) \leqslant \mathbb{P}\big(\xi \max(\eta, d) > x\big) = \overline{(F_\xi \otimes F_\eta)_d}(x),\ x > 0,$$

gives that

$$\limsup_{x\to\infty} \frac{\overline{F_\xi \otimes F_\eta}(x)}{\overline{(F_\xi \otimes F_\eta)_d}(x)} \leqslant 1. \tag{3}$$

On the other hand, for positive x

$$\begin{aligned} \overline{F_\xi \otimes F_\eta}(x) &= \mathbb{P}(\xi\eta > x) \geqslant \mathbb{P}(\xi\eta > x, \eta > d) \\ &= \mathbb{P}(\xi \max\{\eta, d\} > x, \eta > d) \\ &= \mathbb{P}(\xi \max\{\eta, d\} > x) - \mathbb{P}(\xi \max\{\eta, d\} > x, \eta \leqslant d) \\ &= \overline{(F_\xi \otimes F_\eta)_d}(x) - \mathbb{P}(\xi d > x, \eta \leqslant d) \\ &\geqslant \overline{(F_\xi \otimes F_\eta)_d}(x) - F_\eta(d)\mathbb{P}(\xi \max\{\eta, d\} > x) \\ &= \overline{F}_\eta(d)\overline{(F_\xi \otimes F_\eta)_d}(x). \end{aligned}$$

Therefore,

$$\liminf_{x\to\infty} \frac{\overline{F_\xi \otimes F_\eta}(x)}{\overline{\left(F_\xi \otimes F_\eta\right)_d}(x)} \geqslant \overline{F}_\eta(d) > 0. \tag{4}$$

Estimates (3) and (4) imply relation (2). The lemma is proved. □

Proof of Theorem 1. R.v. η is nonnegative and not degenerated at zero. Hence, there exists $d > 0$ such that $\overline{F}_\eta(d) > 0$. By means of Lemma 2, it is sufficient to prove that $\left(F_\xi \otimes F_\eta\right)_d \in \mathcal{OS}$ where

$$\left(F_\xi \otimes F_\eta\right)_d(x) = \mathbb{P}(\xi\eta_d \leqslant x)$$

with $\eta_d = \max\{\eta, d\}$. It is clear that

$$\overline{\left(\left(F_\xi \otimes F_\eta\right)_d\right)^{*2}}(x) = \overline{\left(F_\xi \otimes F_\eta\right)_d * \left(F_\xi \otimes F_\eta\right)_d}(x) = \mathbb{P}(\xi_1\eta_{d1} + \xi_2\eta_{d2} > x),$$

where $\eta_{d1} = \max\{\eta_1, d\}$, $\eta_{d2} = \max\{\eta_2, d\}$ and random vectors (ξ_1, η_1), (ξ_2, η_2) are supposed to be independent copies of the vector (ξ, η). Temporally denote

$$\xi_1^+ = \max\{\xi_1, 0\} \text{ and } \xi_2^+ = \max\{\xi_2, 0\}.$$

For a positive x, we have

$$\begin{aligned}
\overline{\left(\left(F_\xi \otimes F_\eta\right)_d\right)^{*2}}(x) &\leqslant \mathbb{P}(\xi_1^+\eta_{d1} + \xi_2^+\eta_{d2} > x) \\
&= \mathbb{P}(\xi_1^+\eta_{d1} + \xi_2^+\eta_{d2} > x, \eta_{d1} \leqslant \eta_{d2}) \\
&\quad + \mathbb{P}(\xi_1^+\eta_{d1} + \xi_2^+\eta_{d2} > x, \eta_{d2} < \eta_{d1}) \\
&\leqslant \mathbb{P}((\xi_1^+ + \xi_2^+)\eta_{d2} > x, \eta_{d1} \leqslant \eta_{d2}) \\
&\quad + \mathbb{P}((\xi_1^+ + \xi_2^+)\eta_{d1} > x, \eta_{d2} \leqslant \eta_{d1}) \\
&= 2\,\mathbb{P}((\xi_1^+ + \xi_2^+)\eta_{d2} > x, \eta_{d1} \leqslant \eta_{d2}) \\
&\leqslant 2\,\mathbb{P}((\xi_1^+ + \xi_2^+)\eta_{d2} > x) \\
&= 2 \int\limits_{[0,\infty)} \overline{F_{\xi^+}^{*2}}\left(\frac{x}{y}\right) \mathrm{d}F_{\eta_d}(y) \\
&\leqslant 2 \sup_{d\leqslant y<\infty} \frac{\overline{F_{\xi^+}^{*2}}\left(\frac{x}{y}\right)}{\overline{F}_{\xi^+}\left(\frac{x}{y}\right)} \int\limits_{[d,\infty)} \overline{F}_{\xi^+}\left(\frac{x}{y}\right) \mathrm{d}F_{\eta_d}(y) \\
&= 2 \sup_{d\leqslant y<\infty} \frac{\overline{F_{\xi^+}^{*2}}\left(\frac{x}{y}\right)}{\overline{F}_{\xi^+}\left(\frac{x}{y}\right)} \overline{F_{\xi^+} \otimes F_{\eta_d}}(x),
\end{aligned} \tag{5}$$

where F_{ξ^+} denotes the d.f. of r.v. $\xi^+ = \max\{\xi, 0\}$, and F_{η_d} denotes the d.f. of r.v. η_d.

It is clear that for a positive x

$$\begin{aligned}
\overline{F_{\xi^+} \otimes F_{\eta_d}}(x) &= \mathbb{P}(\xi^+\eta_d > x, \xi > 0) + \mathbb{P}(\xi^+\eta_d > x, \xi \leqslant 0) \\
&= \mathbb{P}(\xi\eta_d > x, \xi > 0) \leqslant \mathbb{P}(\xi\eta_d > x) \\
&= \overline{\left(F_\xi \otimes F_\eta\right)_d}(x).
\end{aligned}$$

Hence, the estimate (5) implies that

$$\frac{\overline{\left(\left(F_\xi \otimes F_\eta\right)_d\right)^{*2}}(x)}{\overline{\left(F_\xi \otimes F_\eta\right)_d}(x)} \leqslant \sup_{z>0} \frac{\overline{F_{\xi^+}^{*2}}(z)}{\overline{F}_{\xi^+}(z)} \tag{6}$$

for all positive x's.

If $z > 0$, then

$$\begin{aligned}
\overline{F^{*2}_{\xi^+}}(z) &= \mathbb{P}\big(\xi_1^+ + \xi_2^+ > z,\big) \\
&= \mathbb{P}\big(\xi_1^+ + \xi_2^+ > z, \xi_1 \geqslant 0, \xi_2 \geqslant 0\big) + \mathbb{P}\big(\xi_1^+ + \xi_2^+ > z, \xi_1 \geqslant 0, \xi_2 < 0\big) \\
&\quad + \mathbb{P}\big(\xi_1^+ + \xi_2^+ > z, \xi_1 < 0, \xi_2 \geqslant 0\big) \\
&= \mathbb{P}\big(\xi_1 + \xi_2 > z, \xi_1 \geqslant 0, \xi_2 \geqslant 0\big) + \mathbb{P}\big(\xi_1 > z, \xi_1 \geqslant 0, \xi_2 < 0\big) \\
&\quad + \mathbb{P}\big(\xi_2 > z, \xi_1 < 0, \xi_2 \geqslant 0\big) \\
&\leqslant \mathbb{P}\big(\xi_1 + \xi_2 > z\big) + 2\mathbb{P}\big(\xi_1 > z\big) \\
&= \overline{F^{*2}_{\xi}}(z) + 2\overline{F}_{\xi}(z),
\end{aligned}$$

and

$$\begin{aligned}
\overline{F}_{\xi^+}(z) &= \mathbb{P}(\xi^+ > z, \xi \geqslant 0) + \mathbb{P}(\xi^+ > z, \xi < 0) \\
&= \mathbb{P}(\xi > z, \xi \geqslant 0) = \mathbb{P}(\xi > z) \\
&= \overline{F}_{\xi}(z).
\end{aligned}$$

Hence,

$$\sup_{z>0} \frac{\overline{F^{*2}_{\xi^+}}(z)}{\overline{F}_{\xi^+}(z)} \leqslant 2 + \sup_{z>0} \frac{\overline{F^{*2}_{\xi}}(z)}{\overline{F}_{\xi}(z)} < \infty \tag{7}$$

by (1) because of $F_\xi \in \mathcal{OS}$. The inequality (6) and the last estimate (7) imply that

$$\limsup_{x\to\infty} \frac{\overline{\left(\left(F_\xi \otimes F_\eta\right)_d\right)^{*2}}(x)}{\overline{\left(F_\xi \otimes F_\eta\right)_d}(x)} < \infty.$$

Therefore, $\left(F_\xi \otimes F_\eta\right)_d \in \mathcal{OS}$ as required. The theorem is proved. □

5. Examples

In Section 2, it was mentioned that with the help of Theorem 1, the new d.f.s belonging to the class $\mathcal{OS}$ can be constructed using the product-convolution. In this section, we present two examples that demonstrate this procedure.

Example 1. *Let ξ be the classical Peter and Paul r.v., i.e.,*

$$\mathbb{P}(\xi = 2^k) = 2^{-k},\ k \in \mathbb{N}.$$

For this r.v., the tail of the d.f. is

$$\overline{F}_\xi(x) = \mathbb{1}_{(-\infty,2)}(x) + 2^{-\lfloor \log_2 x \rfloor}\mathbb{1}_{[2,\infty)}(x),$$

where the symbol $\lfloor a \rfloor$ denotes the integer part of the real number a. It follows from this (for details see [34]) that

$$\limsup_{x\to\infty} \frac{\overline{F}_\xi\left(\frac{x}{2}\right)}{\overline{F}_\xi(x)} < \infty$$

implying $F_\xi \in \mathcal{OS}$, because for a positive x

$$\frac{\overline{F_\xi * F_\xi}(x)}{\overline{F}_\xi(x)} \leqslant 2\frac{\overline{F}_\xi\left(\frac{x}{2}\right)}{\overline{F}_\xi(x)}.$$

It follows from Theorem 1 that the d.f. of the product $\xi\eta$ belongs to the class of O-subexponential distributions for each r.v. $\eta \geqslant 0$ with condition $\mathbb{P}(\eta = 0) < 1$.

In particular, if η_1 is an independent copy of ξ, then the d.f. $F_\xi \otimes F_{\eta_1}$ of r.v. $\xi\eta_1$ with local probabilities

$$\mathbb{P}(\xi\eta_1 = 2^{n+1}) = \frac{n}{2^{n+1}},\ n \in \mathbb{N},$$

and the tail function

$$\begin{aligned}\overline{F_\xi \otimes F_{\eta_1}}(x) &= \mathbb{1}_{(-\infty,4)}(x) + \left(\lfloor \log_2 x \rfloor + 1\right) 2^{-\lfloor \log_2 x \rfloor} \mathbb{1}_{[4,\infty)]}(x) \\ &= \mathbb{1}_{(-\infty,4)}(x) = \sum_{k=2}^{\infty} \frac{k+1}{2^k} \mathbb{1}_{[2^k,2^{k+1})]}(x)\end{aligned}$$

belongs to the class $\mathcal{OS}$.

If r.v. $\eta_2 = \mathcal{U}$ is uniformly distributed in the interval $[0,1]$, then the d.f. $F_\xi \otimes F_{\mathcal{U}}$ with the tail function

$$\begin{aligned}\overline{F_\xi \otimes F_{\mathcal{U}}}(x) &= \int\limits_0^{\min\{1,x/2\}} 2^{-\left\lfloor \log_2\left(\frac{x}{u}\right)\right\rfloor} \mathrm{d}u + \int\limits_{[0,1]\cap(x/2,\infty)} \mathrm{d}u \\ &= \mathbb{1}_{(-\infty,0)}(x) + \left(1 - \frac{x}{3}\right)\mathbb{1}_{[0,2)}(x) + x\mathbb{1}_{[2,\infty)]}(x) \int\limits_x^{\infty} 2^{-\lfloor \log_2 y \rfloor} \frac{\mathrm{d}y}{y^2} \\ &= \mathbb{1}_{(-\infty,0)}(x) + \left(1 - \frac{x}{3}\right)\mathbb{1}_{[0,2)}(x) + \sum_{k=1}^{\infty} \frac{1}{2^k}\left(1 - \frac{x}{3 \cdot 2^k}\right)\mathbb{1}_{[2^k,2^{k+1}]}(x)\end{aligned}$$

belongs to the class $\mathcal{OS}$ as well.

Example 2. *Let ξ be an r.v. with tail function*

$$\overline{F}_\xi(x) = \mathbb{1}_{(-\infty,1)}(x) + \frac{\mathrm{e}}{x^2}\mathrm{e}^{-x}\mathbb{1}_{[1,\infty)}(x).$$

According to the results presented in [12,35,36], the limit

$$\lim_{x\to\infty} \frac{\overline{F_\xi * F_\xi}(x)}{\overline{F}_\xi(x)}$$

exists, implying that $F_\xi \in \mathcal{OS}$.

It follows from Theorem 1 that the d.f. of the product $\xi\eta$ is O-subexponential if $\eta \geqslant 0$ and $\mathbb{P}(\eta = 0) < 1$.

If η_1 is an independent copy of ξ, then the d.f. of $\xi\eta_1$ belongs to the class $\mathcal{OS}$. In this case, the tail function is the following:

$$\begin{aligned}\mathbb{P}(\xi\eta_1 > x) = \overline{F_\xi \otimes F_{\eta_1}}(x) &= \int\limits_{(1,\infty)\cap(x,\infty)} \frac{\mathrm{e}}{y^2}\,\mathrm{e}^{-y}\left(1 + \frac{2}{y}\right)\mathrm{d}y + \frac{\mathrm{e}^2}{x^2}\int\limits_1^x \mathrm{e}^{-\left(y + \frac{x}{y}\right))}\left(1 + \frac{2}{y}\right)\mathrm{d}y \\ &= \mathbb{1}_{(-\infty,1)}(x) + \frac{\mathrm{e}}{x^2}\left(\mathrm{e}^{-x} + \int\limits_1^x \left(1 + \frac{2}{y}\right)\mathrm{e}^{1-y-x/y}\mathrm{d}y\right)\mathbb{1}_{[1,\infty)]}(x)\end{aligned}$$

If η_2 is a discrete uniform r.v. with parameter three, i.e.,

$$\mathbb{P}(\eta_2 = 0) = \mathbb{P}(\eta_2 = 1) = \mathbb{P}(\eta_2 = 2) = \frac{1}{3},$$

then the d.f. $F_\xi \otimes F_{\eta_2}$ is also O-subexponential with the tail function

$$\overline{F_\xi \otimes F_{\eta_2}}(x) = \mathbb{1}_{(-\infty,0)}(x) + \frac{2}{3}\mathbb{1}_{[0,1)}(x) + \frac{1}{3}\Big(1 + \frac{\mathrm{e}}{x^2}\,\mathrm{e}^{-x}\Big)\mathbb{1}_{[1,2)}(x) + \frac{\mathrm{e}}{3x^2}\big(\mathrm{e}^{-x} + \mathrm{e}^{-x/2}\big)\mathbb{1}_{[2,\infty)}(x).$$

6. Conclusions

In this work, we established that O-subexponential distributions satisfied the closure property with respect to the product convolution. This means that for any independent random variables ξ and η with a d.f. $F_\xi \in \mathcal{OS}$, the product d.f. $F_{\xi\eta} = F_\xi \otimes F_\eta$ also belongs to the class $\mathcal{OS}$, only if the r.v. η is nonnegative and not degenerated at zero. Since the class $\mathcal{OS}$ is also closed with respect to the usual convolution, see [6] (Lemma 3.1), it follows from the obtained results that the distribution function of the sum

$$S_n^{\theta\xi} := \theta_1\xi_1 + \theta_2\xi_2 + \ldots \theta_n\xi_n \tag{8}$$

remains in the class $\mathcal{OS}$ for any fixed n, if r.v.s $\{\theta_1, \theta_2, \ldots, \theta_n, \xi_1, \xi_2 \ldots, \xi_n\}$ are independent, $F_{\xi_k} \in \mathcal{OS}$ for all $k \in \{1, 2, \ldots, n\}$, and the r.v.s $\{\theta_1, \theta_2, \ldots, \theta_n\}$ are nonnegative and not degenerated at zero.

The sums of random variables (8) are usually applied in risk theory. From the point of view of insurance risk theory, the sum (8) describes the so-called discrete-time stochastic risk model with insurance and financial risks. In such a model, each ξ_k is interpreted as the net loss (the total claim amount minus the total premium income) of an insurance company during period k, θ_k is the corresponding stochastic discount factor to the origin and the sum $S_n^{\theta\xi}$ represents the stochastic present value of the aggregate net losses. For details, see [37–43].

From a financial point of view, the sum (8) describes the behaviour of an investment portfolio consisting of n distinct asset classes or lines of business. In such a case, r.v. ξ_k, $k \in \{1, 2, \ldots, n\}$, could correspond to the loss incurred from the kth instrument. As for the role of random weights, there could be different viewpoints: θ_k, $k \in \{1, 2, \ldots, n\}$, could be treated as a stochastic discount factor of the kth asset class or, for instance, as a weight corresponding to the kth instrument in the portfolio. Then, the random sum $S_n^{\theta\xi}$ would correspond to the present value of the total loss of a portfolio at the present moment in the former case, and the total weighted portfolio loss in the later case. For details, see [34,44–50].

It should be noted that for both actuarial and financial models, it is not enough to know which regularity class the d.f. of $S_n^{\theta\xi}$ belongs to. We still need to find asymptotic formulas for distributions of large values of such a sum. The results obtained in this paper simplify the research on the behaviour of the large values of sum $S_n^{\theta\xi}$.

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