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Article

Hybrid System of Proportional Hilfer-Type Fractional Differential Equations and Nonlocal Conditions with Respect to Another Function

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Abstract: In this paper, a new class of coupled hybrid systems of proportional sequential ψ-Hilfer fractional differential equations, subjected to nonlocal boundary conditions were investigated. Based on a generalization of the Krasnosel'skiĭ's fixed point theorem due to Burton, sufficient conditions were established for the existence of solutions. A numerical example was constructed illustrating the main theoretical result. For special cases of the parameters involved in the system many new results were covered. The obtained result is new and significantly contributes to existing results in the literature on coupled systems of proportional sequential ψ-Hilfer fractional differential equations.

Keywords: coupled system; Hilfer fractional proportional derivative; nonlocal conditions; fixed-point theorem

MSC: 26A33; 34A08; 34B15

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1. Introduction

Fractional calculus (differentiation and integration of arbitrary order) has proved to be an important tool in describing many mathematical models in science and engineering. In fact, this branch of calculus has found its application in physics, mechanics, control theory, economics, biology, signal and image precessing, etc. Fractional differential equations describe many real world processes more accurately than classical differential equations and have been addressed by many researchers. For theoretical and application details of fractional differential equations, we refer the reader to the books [1–6], while an extensive study of fractional boundary value problems can be found in the monograph [7]. Usually, fractional derivative operators are defined via fractional integral operators and in the literature one can find a variety of such operators, such as Riemann-Liouville, Caputo, Hadamard, Erdélyi-Kober, Hilfer fractional derivatives, etc., to name some of them. In [8], with the help of Euler's k-gamma function, the k-Riemann-Liouville fractional integral operator was introduced, generalizing the concept of Riemann-Liouville fractional integral operator, which was used to define the k-Riemann-Liouville fractional derivative in [9]. The Hilfer fractional derivative [10] extends both Riemann-Liouville and Caputo fractional derivatives. For applications of Hilfer fractional derivatives in mathematics, physics, etc., see [11–16]. For recent results on boundary value problems for fractional differential equations and inclusions with Hilfer fractional derivative, see the survey paper by Ntouyas [17]. The ψ-Riemann-Liouville fractional integral and derivative operators, which are fractional calculus with respect to a function ψ, are discussed in [1], while the ψ-Hilfer fractional derivative is discussed in [18].

Recently, the notion of generalized proportional fractional derivative was introduced by Jarad et al. [19–21]. For some recent results on fractional differential equations with generalized proportional derivatives, see [22,23].

In [24], an initial value problem of the form

$$\begin{cases} D^{\alpha}\left[\dfrac{D^{\omega}\mathfrak{u}(\mathfrak{t})-\sum\limits_{i=1}^{n} I^{\delta_i}h_i(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))}{\mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))}\right]=\Upsilon(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),I^{\gamma}\mathfrak{u}(t)), \quad \mathfrak{t}\in[0,T], \\ \mathfrak{u}(0)=0, \quad D^{\omega}\mathfrak{u}(0)=0, \end{cases} \tag{1}$$

was studied, where D^{v} indicates the Caputo fractional derivative of order $v \in \{\alpha,\omega\}$ with $0<\alpha,\omega\leq 1, 1<\alpha+\omega\leq 2$, I^{δ_i}, I^{γ} are the fractional integrals of Riemann–Liouville type of order $\delta_i>0, \gamma>0$, $h_i\in C([0,T]\times\mathbb{R},\mathbb{R})$, for $i=1,2,\ldots,n$, $\mathfrak{f}\in C([0,T]\times\mathbb{R},\mathbb{R}\setminus\{0\})$ and $\Upsilon\in C([0,T]\times\mathbb{R},\mathbb{R})$. A three-point boundary value problem of the form (1) was studied in [25], by replacing the initial conditions with $\mathfrak{u}(0)=0, \quad D^{\omega}\mathfrak{u}(0)=0, \quad \mathfrak{u}(1)=\delta\mathfrak{u}(\eta)$, $\delta\in\mathbb{R}, \ 0<\eta<1$, where $0<\alpha\leq 1, 1<\omega\leq 2$, and using a generalized Krasnosel'skiĭ's fixed-point theorem.

Fractional coupled systems are also important, as such systems appear in the mathematical models associated with fractional dynamics [26], bio-engineering [27], financial economics [28], etc. In [29], the authors studied the existence and Ulam-Hyers stability results of a coupled system of ψ-Hilfer sequential fractional differential equations. In [30], by using Krasnosel'skiĭ's fixed point theorem, the existence of solutions are established for the following nonlinear system involving generalized Hilfer fractional operators

$$\begin{cases} {}^{H}D^{\delta,\vartheta,\psi}\left[\dfrac{{}^{H}D^{\nu,\kappa,\psi}\mathfrak{u}(\mathfrak{t})-\sum\limits_{i=1}^{m} I^{q_i,\psi}H_i(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}{\mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}\right]=P(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})), \ \mathfrak{t}\in[0,\mathfrak{b}_0], \\ {}^{H}D^{\delta,\vartheta,\psi}\left[\dfrac{{}^{H}D^{\nu,\kappa,\psi}\mathfrak{s}(\mathfrak{t})-\sum\limits_{i=1}^{m} I^{q_i,\psi}G_i(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}{\mathfrak{g}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}\right]=Q(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})), \ \mathfrak{t}\in[0,\mathfrak{b}_0], \\ I^{1-\gamma,\psi}\mathfrak{u}(0)=0, \ I^{1-\gamma,\psi}\mathfrak{s}(0)=0, \ {}^{H}D^{\nu,\kappa,\psi}\mathfrak{u}(0)=0, \ {}^{H}D^{\nu,\kappa,\psi}\mathfrak{s}(0)=0, \end{cases} \tag{2}$$

where ${}^{H}D^{A,B,\psi}$ is the ψ-Hilfer fractional derivative of order $A\in\{\delta,\nu\}$ with $\delta,\nu\in(0,1)$ and types $B\in\{\vartheta,\kappa\}$, $\vartheta,\kappa\in[0,1]$, $I^{1-\gamma,\psi}, I^{q_i,\psi}$ are the ψ-Riemann-Liouville fractional integrals of order $1-\gamma>0, q_i>0$, $i=1,2,\ldots,m$ and $P,Q\in C([0,\mathfrak{b}_0]\times\mathbb{R}^2,\mathbb{R})$. In [31], the existence and uniqueness results are derived for a coupled system of Hilfer–Hadamard fractional differential equations with fractional integral boundary conditions. Recently, in [32] a coupled system of nonlinear fractional differential equations involving the (k,ψ)-Hilfer fractional derivative operators complemented with multi-point nonlocal boundary conditions was discussed. Moreover, Samadi et al. [33] have considered a coupled system of Hilfer-type generalized proportional fractional differential equations.

In this article, motivated by the above works, we study a hybrid system of proportional Hilfer-type fractional differential equations of the form:

$$\begin{cases} {}^{H}D^{\delta_1,\vartheta_1,\rho,\psi}\left[\dfrac{{}^{H}D^{\delta_2,\vartheta_2,\rho,\psi}\mathfrak{u}(\mathfrak{t})-\sum\limits_{i=1}^{n}{}^{p}I^{\eta_i,\rho,\psi}H_i(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}{\mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}\right]=P(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})),\ \mathfrak{t}\in[\mathfrak{a}_0,\mathfrak{b}_0],\\ {}^{H}D^{\delta_3,\vartheta_3,\rho,\psi}\left[\dfrac{{}^{H}D^{\delta_4,\vartheta_4,\rho,\psi}\mathfrak{s}(\mathfrak{t})-\sum\limits_{j=1}^{m}{}^{p}I^{\overline{\eta}_j,\rho,\psi}G_j(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}{\mathfrak{g}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}\right]=Q(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})),\ \mathfrak{t}\in[\mathfrak{a}_0,\mathfrak{b}_0], \end{cases} \tag{3}$$

subject to coupled nonlocal boundary conditions

$$\begin{cases} \mathfrak{u}(\mathfrak{a}_0)={}^{H}D^{\delta_2,\vartheta_2,\rho,\psi}\mathfrak{u}(\mathfrak{a}_0)=0,\ \ \mathfrak{u}(\mathfrak{b}_0)=\theta_1\mathfrak{s}(\xi_1),\\ \mathfrak{s}(\mathfrak{a}_0)={}^{H}D^{\delta_4,\vartheta_4,\rho,\psi}\mathfrak{s}(\mathfrak{a}_0)=0,\ \ \mathfrak{s}(\mathfrak{b}_0)=\theta_2\mathfrak{u}(\xi_2), \end{cases} \tag{4}$$

where ${}^{H}D^{\delta,\vartheta_1,\rho,\psi}$ denotes the ψ-Hilfer generalized proportional derivatives of order $\delta\in\{\delta_1,\delta_2,\delta_3,\delta_4\}$, with parameters ϑ_l, $0\le\vartheta_l\le 1$, $l\in\{1,2,3,4\}$, $I^{\eta,\rho,\psi}$ is the generalized proportional integral of order $\eta>0$, $\eta\in\{\eta_i,\overline{\eta}_j\}$, $\theta_1,\theta_2\in\mathbb{R}$, $\xi_1,\xi_2\in[\mathfrak{a}_0,\mathfrak{b}_0]$, $\mathfrak{f},\mathfrak{g}\in C([\mathfrak{a}_0,\mathfrak{b}_0]\times\mathbb{R}^2,\mathbb{R}\setminus\{0\})$ and $H_i,G_j,P,Q\in C([\mathfrak{a}_0,\mathfrak{b}_0]\times\mathbb{R}^2,\mathbb{R})$, for $i=1,2,\dots,n$ and $j=1,2,\dots,m$.

To establish our main existence result, we first transform the problem (3) and (4) into a fixed-point problem by using a linear variant of the problem (3) and (4), and then apply a generalization of the Krasnosel'skiĭ's fixed-point theorem due to Burton.

Our problem enriches the literature on hybrid sequential coupled systems of proportional ψ-Hilfer differential equations of fractional order with nonlocal boundary conditions. The nonlocal boundary conditions can be applied in physics, thermodynamics, wave propagation, etc., and are more general than classical boundary conditions. For some applications see [34,35] and the references cited therein. For applications of Hilfer fractional derivative operators in applied sciences (such as physics, filtration processes, cobweb economics model, stochastic equations etc.), we refer the reader to [36–41] and their references.

Comparing our problem with the problem studied in [30], we note that:

- We study a system involving ψ-Hilfer proportional fractional derivatives.
- Our equations are more general as the contained fractional derivatives have different orders.
- Our system contains nonlocal coupled boundary conditions.
- Our system covers many special cases by fixing the parameters involved in the problem. For example, by taking $\mathfrak{f},\mathfrak{g}=1$ in the problem (3), we have the following new nonlocal coupled system of sequential Hilfer-type proportional fractional differential equations

$$\begin{cases} {}^{H}D^{\delta_1,\vartheta_1,\rho,\psi}\left[{}^{H}D^{\delta_2,\vartheta_2,\rho,\psi}\mathfrak{u}(\mathfrak{t})-\sum\limits_{i=1}^{n}{}^{p}I^{\eta_i,\rho,\psi}H_i(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))\right]=P(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})),\ \mathfrak{t}\in[\mathfrak{a}_0,\mathfrak{b}_0],\\ {}^{H}D^{\delta_3,\vartheta_3,\rho,\psi}\left[{}^{H}D^{\delta_4,\vartheta_4,\rho,\psi}\mathfrak{u}(\mathfrak{t})-\sum\limits_{j=1}^{m}{}^{p}I^{\overline{\eta}_j,\rho,\psi}G_j(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))\right]=Q(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})),\ \mathfrak{t}\in[\mathfrak{a}_0,\mathfrak{b}_0],\\ \mathfrak{u}(\mathfrak{a}_0)={}^{H}D^{\delta_2,\vartheta_2,\rho,\psi}\mathfrak{u}(\mathfrak{a}_0)=0,\ \ \mathfrak{u}(\mathfrak{b}_0)=\theta_1\mathfrak{s}(\xi_1),\\ \mathfrak{s}(\mathfrak{a}_0)={}^{H}D^{\delta_4,\vartheta_4,\rho,\psi}\mathfrak{s}(\mathfrak{a}_0)=0,\ \ \mathfrak{s}(\mathfrak{b}_0)=\theta_2\mathfrak{u}(\xi_2). \end{cases}$$

- Note that if $n = m = 1$, $H_i(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), \mathfrak{s}(\mathfrak{t})) = -\lambda\mathfrak{u}(\mathfrak{t})$, $G_j(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), \mathfrak{s}(\mathfrak{t})) = -\mu\mathfrak{s}(\mathfrak{t})$, λ, μ constants, then we have a nonlocal coupled system of sequential Hilfer-type proportional fractional differential equations of Langevin-type

$$\begin{cases} {}^H D^{\delta_1,\vartheta_1,\rho,\psi}\left[\dfrac{({}^H D^{\delta_2,\vartheta_2,\rho,\psi}+\lambda)\mathfrak{u}(\mathfrak{t})}{\mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}\right] = P(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})), \ \mathfrak{t}\in[\mathfrak{a}_0,\mathfrak{b}_0], \\ {}^H D^{\delta_3,\vartheta_3,\rho,\psi}\left[\dfrac{({}^H D^{\delta_4,\vartheta_4,\rho,\psi}+\mu)\mathfrak{s}(\mathfrak{t})}{\mathfrak{g}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}\right] = Q(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})), \ \mathfrak{t}\in[\mathfrak{a}_0,\mathfrak{b}_0], \\ \mathfrak{u}(\mathfrak{a}_0) = {}^H D^{\delta_2,\vartheta_2,\rho,\psi}\mathfrak{u}(\mathfrak{a}_0) = 0, \ \ \mathfrak{u}(\mathfrak{b}_0) = \theta_1\mathfrak{s}(\xi_1), \\ \mathfrak{s}(\mathfrak{a}_0) = {}^H D^{\delta_4,\vartheta_4,\rho,\psi}\mathfrak{s}(\mathfrak{a}_0) = 0, \ \ \mathfrak{s}(\mathfrak{b}_0) = \theta_2\mathfrak{u}(\xi_2), \end{cases}$$

  which is a generalization of the well-known classical results in [42].

The structure of this article has been organized as follows: In Section 2, some necessary concepts and basic results concerning our problem are presented. The main result for the problem (3) and (4) is proved in Section 3, while Section 4 contains an example illustrating the obtained result.

2. Preliminaries

In this section, we summarize some known definitions and lemmas needed in our results.

Definition 1 ([22,23])**.** *Let $\rho \in (0,1]$ and $\delta > 0$. The fractional proportional integral of order δ of the function $\mathfrak{F}$ is defined by*

$${}^p I^{\delta,\rho,\psi}\mathfrak{F}(\mathfrak{t}) = \frac{1}{\rho^\delta\Gamma(\delta)}\int_{\mathfrak{a}_0}^{\mathfrak{t}} e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(s))}(\psi(\mathfrak{t})-\psi(s))^{\delta-1}\mathfrak{F}(s)\psi'(s)ds.$$

Definition 1 unifies several known definitions of fractional integrals for $\rho = 1$, for example, for $\psi(\mathfrak{t}) = \mathfrak{t}$, it corresponds to Riemann-Liouville fractional integral, for $\psi(\mathfrak{t}) = \log\mathfrak{t}$, to Hadamard fractional integral, while for $\psi(\mathfrak{t}) = \frac{\mathfrak{t}^\alpha}{\alpha}, \alpha > 0$, to Katugampola fractional integral.

Definition 2 ([22,23])**.** *Let $\rho \in (0,1]$, $\delta > 0$, and $\psi(\mathfrak{t})$ is a continuous function on $[\mathfrak{a}_0, \mathfrak{b}_0]$, $\psi'(\mathfrak{t}) > 0$. The generalized proportional fractional derivative of order δ of the continuous function $\mathfrak{F}$ is defined by*

$$({}^p D^{\delta,\rho,\psi}\mathfrak{F})(\mathfrak{t}) = \frac{({}^p D^{n,\rho,\psi})}{\rho^{n-\delta}\Gamma(n-\delta)}\int_{\mathfrak{a}_0}^{\mathfrak{t}} e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(s))}(\psi(\mathfrak{t})-\psi(s))^{\delta-1}\mathfrak{F}(s)\psi'(s)ds,$$

where $n = [\delta] + 1$ and $[\delta]$ denotes the integer part of the real number δ and $D^{n,\rho,\psi} = \underbrace{D^{\rho,\psi}\dots D^{\rho,\psi}}_{n-times}$.

Now the generalized Hilfer proportional fractional derivative of order δ of function $\mathfrak{F}$ with respect to another function ψ is introduced.

Definition 3 ([43])**.** *Let $\rho \in (0,1]$, $\mathfrak{F}, \psi \in C^m([\mathfrak{a}_0, \mathfrak{b}_0], \mathbb{R})$ in which ψ is positive and strictly increasing with $\psi'(\mathfrak{t}) \neq 0$ for all $\mathfrak{t} \in [\mathfrak{a}_0, \mathfrak{b}_0]$. The ψ-Hilfer generalized proportional fractional derivative of order δ and type ϑ for $\mathfrak{F}$ with respect to another function ψ is defined by*

$$\left({}^H D^{\delta,\vartheta,\rho,\psi}\mathfrak{F}\right)(\mathfrak{t}) = {}^p I^{\vartheta(n-\delta),\rho,\psi}[{}^p D^{n,\rho,\psi}({}^p I^{(1-\vartheta)(n-\delta),\rho,\psi}\mathfrak{F})](\mathfrak{t}),$$

where $n-1 < \delta < n$ and $0 \leq \vartheta \leq 1$.

Remark 1 ([43]). *It is assumed that the parameters δ, ϑ and γ (involved in the above definitions) satisfy the relations:*

$$\gamma = \delta + \vartheta(n-\delta), \;\; n-1<\delta, \;\; \gamma \le n, \;\; 0 \le \vartheta \le 1,$$

and

$$\gamma \ge \delta, \;\; \gamma > \vartheta, \;\; n-\gamma < n - \vartheta(n-\delta).$$

Lemma 1 ([43]). *Let $n-1<\delta<n \in \mathbb{N}$, $0<\rho\le 1$, $0 \le \vartheta \le 1$ and $n-1<\gamma<n$ such that $\gamma = \delta + n\vartheta - \delta\vartheta$. If $\mathfrak{F} \in C([\mathfrak{a}_0,\mathfrak{b}_0],\mathbb{R})$ and ${}^{p}I^{(n-\gamma,\rho,\psi)}\mathfrak{F} \in C^n([\mathfrak{a}_0,\mathfrak{b}_0],\mathbb{R})$, then*

$$\left({}^{p}I^{\delta,\rho,\psi\,H}D^{\delta,\vartheta,\rho,\psi}\mathfrak{F}\right)(\mathfrak{t}) = \mathfrak{F}(\mathfrak{t}) - \sum_{j=1}^{n} \frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma-j}}{\rho^{\gamma-j}\Gamma(\gamma-j+1)}\left({}^{p}I^{j-\gamma,\rho,\psi}\mathfrak{F}\right)(\mathfrak{a}_0).$$

To prove the main result we need the following lemma, which concerns a linear variant of the ψ-Hilfer proportional coupled system (3) and (4), and is used to convert the nonlinear problem in system (3) and (4) into a fixed-point problem.

Lemma 2. *Let $0<\delta_1,\delta_3 \le 1$, $1<\delta_2,\delta_4 \le 2$, $0 \le \vartheta_i \le 1$, $\gamma_i = \delta_i + \vartheta_i(1-\delta_i)$, $\gamma_j = \delta_j + \vartheta_j(2-\delta_j)$, $i=1,3$, $j=2,4$, $\Theta = M_1N_2 - M_2N_1 \neq 0$ and $U, S \in C([\mathfrak{a}_0,\mathfrak{b}_0],\mathbb{R})$, $\mathfrak{f},\mathfrak{g} \in C([\mathfrak{a}_0,\mathfrak{b}_0]\times\mathbb{R}^2,\mathbb{R}\setminus\{0\})$, $H_i, G_j \in C([\mathfrak{a}_0,\mathfrak{b}_0]\times\mathbb{R}^2,\mathbb{R})$, for $i=1,2,\ldots,n$ and $j=1,2,\ldots,m$ and ${}^{p}I^{(n-\gamma_i,\rho,\psi)}U \in C^n([\mathfrak{a}_0,\mathfrak{b}_0],\mathbb{R}),{}^{p}I^{(n-\gamma_i,\rho,\psi)}S \in C^n([\mathfrak{a}_0,\mathfrak{b}_0],\mathbb{R}), i=1,2,3,4$. Then the pair $(\mathfrak{u},\mathfrak{s})$ is a solution of the system*

$$\begin{cases} {}^{H}D^{\delta_1,\vartheta_1,\rho,\psi}\left[\dfrac{{}^{H}D^{\delta_2,\vartheta_2,\rho,\psi}\mathfrak{u}(\mathfrak{t}) - \sum_{i=1}^{n}{}^{p}I^{\eta_i,\rho,\psi}H_i(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}{\mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}\right] = U(\mathfrak{t}), \;\; \mathfrak{t}\in[\mathfrak{a}_0,\mathfrak{b}_0], \\ {}^{H}D^{\delta_3,\vartheta_3,\rho,\psi}\left[\dfrac{{}^{H}D^{\delta_4,\vartheta_4,\rho,\psi}\mathfrak{s}(\mathfrak{t}) - \sum_{j=1}^{m}{}^{p}I^{\overline{\eta}_j,\rho,\psi}G_j(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}{\mathfrak{g}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}\right] = S(\mathfrak{t}), \;\; \mathfrak{t}\in[\mathfrak{a}_0,\mathfrak{b}_0], \\ \mathfrak{u}(\mathfrak{a}_0) = {}^{H}D^{\delta_2,\vartheta_2,\rho,\psi}\mathfrak{u}(\mathfrak{a}_0) = 0, \;\; \mathfrak{u}(\mathfrak{b}_0) = \theta_1\mathfrak{s}(\xi_1), \\ \mathfrak{s}(\mathfrak{a}_0) = {}^{H}D^{\delta_4,\vartheta_4,\rho,\psi}\mathfrak{s}(\mathfrak{a}_0) = 0, \;\; \mathfrak{s}(\mathfrak{b}_0) = \theta_2\mathfrak{u}(\xi_2), \end{cases} \tag{5}$$

if and only if

$$\begin{aligned} \mathfrak{u}(\mathfrak{t}) \;=\; & \sum_{i=1}^{n}{}^{p}I^{\eta_i+\delta_2,\rho,\psi}H_i(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})) + {}^{p}I^{\delta_2,\rho,\psi}\mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))^{p}I^{\delta_1,\rho,\psi}U(\mathfrak{t}) \\ & + \frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma_2-1}}{\Theta\rho^{\gamma_2-1}\Gamma(\gamma_2)}\Bigg\{N_2\Bigg[\theta_1\sum_{j=1}^{m}{}^{p}I^{\overline{\eta}_j+\delta_4,\rho,\psi}G_j(\xi_1,\mathfrak{u}(\xi_1),\mathfrak{s}(\xi_1)) \\ & + \theta_1{}^{p}I^{\delta_4,\rho,\psi}\mathfrak{g}(\xi_1,\mathfrak{u}(\xi_1),\mathfrak{s}(\xi_1))^{p}I^{\delta_3,\rho,\psi}S(\xi_1) \\ & - \sum_{i=1}^{n}{}^{p}I^{\eta_i+\delta_2,\rho,\psi}H_i(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0)) - {}^{p}I^{\delta_2,\rho,\psi}\mathfrak{f}(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0))^{p}I^{\delta_1,\rho,\psi}U(\mathfrak{b}_0)\Bigg] \\ & + M_2\Bigg[\theta_2\sum_{i=1}^{n}{}^{p}I^{\eta_i+\delta_2,\rho,\psi}H_i(\xi_2,\mathfrak{u}(\xi_2),\mathfrak{s}(\xi_2)) \\ & + \theta_2{}^{p}I^{\delta_2,\rho,\psi}\mathfrak{f}(\xi_2,\mathfrak{u}(\xi_2),\mathfrak{s}(\xi_2))^{p}I^{\delta_1,\rho,\psi}U(\xi_2) - \sum_{j=1}^{m}{}^{p}I^{\overline{\eta}_j+\delta_4,\rho,\psi}G_j(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0)) \\ & - {}^{p}I^{\delta_4,\rho,\psi}\mathfrak{g}(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0))^{p}I^{\delta_3,\rho,\psi}S(\mathfrak{b}_0)\Bigg]\Bigg\} \end{aligned} \tag{6}$$

and

$$\begin{aligned}\mathfrak{s}(\mathfrak{t}) &= \sum_{j=1}^{m} {}^{p}I^{\overline{\eta}_j+\delta_4,\rho,\psi}G_j(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})) + {}^{p}I^{\delta_4,\rho,\psi}\mathfrak{g}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))^{p}I^{\delta_3,\rho,\psi}S(\mathfrak{t}) \\ &+ \frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma_4-1}}{\Theta\rho^{\gamma_4-1}\Gamma(\gamma_4)}\Bigg\{N_1\Bigg[\theta_1\sum_{j=1}^{m} {}^{p}I^{\overline{\eta}_j+\delta_4,\rho,\psi}G_j(\xi_1,\mathfrak{u}(\xi_1),\mathfrak{s}(\xi_1)) \\ &+\theta_1{}^{p}I^{\delta_4,\rho,\psi}\mathfrak{g}(\xi_1,\mathfrak{u}(\xi_1),\mathfrak{s}(\xi_1))^{p}I^{\delta_3,\rho,\psi}S(\xi_1) \\ &-\sum_{i=1}^{n} {}^{p}I^{\eta_i+\delta_2,\rho,\psi}H_i(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0)) - {}^{p}I^{\delta_2,\rho,\psi}\mathfrak{f}(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0))^{p}I^{\delta_1,\rho,\psi}U(\mathfrak{b}_0)\Bigg] \\ &+M_1\Bigg[\theta_2\sum_{i=1}^{n} {}^{p}I^{\eta_i+\delta_2,\rho,\psi}H_i(\xi_2,\mathfrak{u}(\xi_2),\mathfrak{s}(\xi_2)) \\ &+\theta_2{}^{p}I^{\delta_2,\rho,\psi}\mathfrak{f}(\xi_2,\mathfrak{u}(\xi_2),\mathfrak{s}(\xi_2))^{p}I^{\delta_1,\rho,\psi}U(\xi_2) - \sum_{j=1}^{m} {}^{p}I^{\overline{\eta}_j+\delta_4,\rho,\psi}G_j(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0)) \\ &-{}^{p}I^{\delta_4,\rho,\psi}\mathfrak{g}(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0))^{p}I^{\delta_3,\rho,\psi}S(\mathfrak{b}_0)\Bigg]\Bigg\}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} M_1 &= \frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{b}_0)-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{b}_0)-\psi(\mathfrak{a}_0))^{\gamma_2-1}}{\rho^{\gamma_2-1}\Gamma(\gamma_2)}, \\ M_2 &= \frac{\theta_1 e^{\frac{\rho-1}{\rho}(\psi(\xi_1)-\psi(\mathfrak{a}_0))}(\psi(\xi_1)-\psi(\mathfrak{a}_0))^{\gamma_4-1}}{\rho^{\gamma_4-1}\Gamma(\gamma_4)}, \\ N_1 &= \theta_2\frac{e^{\frac{\rho-1}{\rho}(\psi(\xi_2)-\psi(\mathfrak{a}_0))}(\psi(\xi_2)-\psi(\mathfrak{a}_0))^{\gamma_2-1}}{\rho^{\gamma_2-1}\Gamma(\gamma_2)}, \\ N_2 &= \frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{b}_0)-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{b}_0)-\psi(\mathfrak{a}_0))^{\gamma_4-1}}{\rho^{\gamma_4-1}\Gamma(\gamma_4)}. \end{aligned} \tag{8}$$

Proof. Due to Lemma 1 with $n = 1$, we obtain

$$\begin{aligned} &\frac{{}^{H}D^{\delta_2,\vartheta_2,\rho,\psi}\mathfrak{u}(\mathfrak{t}) - \sum_{i=1}^{n} {}^{p}I^{\eta_i,\rho,\psi}H_i(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}{\mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))} = {}^{p}I^{\delta_1,\rho,\psi}U(\mathfrak{t}) \\ &+c_0\frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma_1-1}}{\rho^{\gamma_1-1}\Gamma(\gamma_1)}, \\ &\frac{{}^{H}D^{\delta_4,\vartheta_4,\rho,\psi}\mathfrak{s}(\mathfrak{t}) - \sum_{j=1}^{m} {}^{p}I^{\overline{\eta}_j,\rho,\psi}G_j(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}{\mathfrak{g}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))} = {}^{p}I^{\delta_3,\rho,\psi}S(\mathfrak{t}) \\ &+d_0\frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma_3-1}}{\rho^{\gamma_3-1}\Gamma(\gamma_3)}, \end{aligned} \tag{9}$$

where $c_0, d_0 \in \mathbb{R}$. Now applying the boundary conditions

$${}^{H}D^{\delta_2,\vartheta_2,\rho,\psi}\mathfrak{u}(\mathfrak{a}_0) = {}^{H}D^{\delta_4,\vartheta_4,\rho,\psi}\mathfrak{s}(\mathfrak{a}_0) = 0,$$

we obtain $c_0 = d_0 = 0$. Hence

$$\begin{aligned} {}^{H}D^{\delta_2,\vartheta_2,\rho,\psi}\mathfrak{u}(\mathfrak{t}) &= \sum_{i=1}^{n} {}^{p}I^{\eta_i,\rho,\psi}H_i(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})) + \mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))\,{}^{p}I^{\delta_1,\rho,\psi}U(\mathfrak{t}), \\ {}^{H}D^{\delta_4,\vartheta_4,\rho,\psi}\mathfrak{s}(\mathfrak{t}) &= \sum_{j=1}^{m} {}^{p}I^{\overline{\eta}_j,\rho,\psi}G_j(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})) + \mathfrak{g}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))\,{}^{p}I^{\delta_3,\rho,\psi}S(\mathfrak{t}). \end{aligned} \tag{10}$$

Now, by taking the operators ${}^{p}I^{\delta_2,\rho,\psi}$ and ${}^{p}I^{\delta_4,\rho,\psi}$ into both sides of (10) and using Lemma 1, we obtain

$$\begin{aligned} \mathfrak{u}(\mathfrak{t}) &= \sum_{i=1}^{n} {}^{p}I^{\eta_i+\delta_2,\rho,\psi}H_i(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})) + {}^{p}I^{\delta_2,\rho,\psi}\mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))\,{}^{p}I^{\delta_1,\rho,\psi}U(\mathfrak{t}) \\ &\quad + c_1\frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma_2-1}}{\rho^{\gamma_2-1}\Gamma(\gamma_2)} \\ &\quad + c_2\frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma_2-2}}{\rho^{\gamma_2-2}\Gamma(\gamma_2-1)}, \\ \mathfrak{s}(\mathfrak{t}) &= \sum_{j=1}^{m} {}^{p}I^{\overline{\eta}_j+\delta_4,\rho,\psi}G_j(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})) + {}^{p}I^{\delta_4,\rho,\psi}\mathfrak{g}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))\,{}^{p}I^{\delta_3,\rho,\psi}S(\mathfrak{t}) \\ &\quad + d_1\frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma_4-1}}{\rho^{\gamma_4-1}\Gamma(\gamma_4)} \\ &\quad + d_2\frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma_4-2}}{\rho^{\gamma_4-2}\Gamma(\gamma_4-1)}. \end{aligned} \tag{11}$$

Applying in (11) the conditions $\mathfrak{u}(\mathfrak{a}_0) = \mathfrak{s}(\mathfrak{a}_0) = 0$, we obtain $c_2 = d_2 = 0$ since $\gamma_2 \in [\delta_2, 2]$ and $\gamma_4 \in [\delta_4, 2]$ (Remark 1). Thus we have

$$\begin{aligned} \mathfrak{u}(\mathfrak{t}) &= \sum_{i=1}^{n} {}^{p}I^{\eta_i+\delta_2,\rho,\psi}H_i(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})) + {}^{p}I^{\delta_2,\rho,\psi}\mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))\,{}^{p}I^{\delta_1,\rho,\psi}U(\mathfrak{t}) \\ &\quad + c_1\frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma_2-1}}{\rho^{\gamma_2-1}\Gamma(\gamma_2)}, \\ \mathfrak{s}(\mathfrak{t}) &= \sum_{j=1}^{m} {}^{p}I^{\overline{\eta}_j+\delta_4,\rho,\psi}G_j(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})) + {}^{p}I^{\delta_4,\rho,\psi}\mathfrak{g}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))\,{}^{p}I^{\delta_3,\rho,\psi}S(\mathfrak{t}) \\ &\quad + d_1\frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma_4-1}}{\rho^{\gamma_4-1}\Gamma(\gamma_4)}. \end{aligned} \tag{12}$$

In view of (12) and the conditions $\mathfrak{u}(\mathfrak{b}_0) = \theta_1\mathfrak{s}(\xi_1)$ and $\mathfrak{s}(\mathfrak{b}_0) = \theta_2\mathfrak{u}(\xi_2)$, we obtain

$$\begin{aligned} &\sum_{i=1}^{n} {}^{p}I^{\eta_i+\delta_2,\rho,\psi}H_i(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0)) + {}^{p}I^{\delta_2,\rho,\psi}\mathfrak{f}(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0))\,{}^{p}I^{\delta_1,\rho,\psi}U(\mathfrak{b}_0) \\ &+ c_1\frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{b}_0)-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{b}_0)-\psi(\mathfrak{a}_0))^{\gamma_2-1}}{\rho^{\gamma_2-1}\Gamma(\gamma_2)} \\ &= \theta_1\sum_{j=1}^{m} {}^{p}I^{\overline{\eta}_j+\delta_4,\rho,\psi}G_j(\xi_1,\mathfrak{u}(\xi_1),\mathfrak{s}(\xi_1)) + \theta_1\,{}^{p}I^{\delta_4,\rho,\psi}\mathfrak{g}(\xi_1,\mathfrak{u}(\xi_1),\mathfrak{s}(\xi_1))\,{}^{p}I^{\delta_3,\rho,\psi}S(\xi_1) \end{aligned}$$

$$+d_1\frac{\theta_1 e^{\frac{\rho-1}{\rho}(\psi(\xi_1)-\psi(\mathfrak{a}_0))}(\psi(\xi_1)-\psi(\mathfrak{a}_0))^{\gamma_4-1}}{\rho^{\gamma_4-1}\Gamma(\gamma_4)}, \tag{13}$$

and

$$\begin{aligned}
&\sum_{j=1}^{m} {}^{p}I^{\overline{\eta}_j+\delta_4,\rho,\psi}G_j(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0)) + {}^{p}I^{\delta_4,\rho,\psi}\mathfrak{g}(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0))^{p}I^{\delta_3,\rho,\psi}S(\mathfrak{b}_0)\\
&+d_1\frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{b}_0)-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{b}_0)-\psi(\mathfrak{a}_0))^{\gamma_4-1}}{\rho^{\gamma_2-1}\Gamma(\gamma_2)}\\
&=\theta_2\sum_{i=1}^{n} {}^{p}I^{\eta_i+\delta_2,\rho,\psi}H_i(\xi_2,\mathfrak{u}(\xi_2),\mathfrak{s}(\xi_2)) + \theta_2{}^{p}I^{\delta_2,\rho,\psi}\mathfrak{f}(\xi_2,\mathfrak{u}(\xi_2),\mathfrak{s}(\xi_2))^{p}I^{\delta_1,\rho,\psi}U(\xi_2)\\
&+c_1\frac{\theta_2 e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\xi_2)-\psi(\mathfrak{a}_0))^{\gamma_2-1}}{\rho^{\gamma_2-1}\Gamma(\gamma_2)}.
\end{aligned} \tag{14}$$

Due to (8) and (13), (14), we have

$$\begin{aligned}
c_1M_1 - d_1M_2 &= M,\\
-c_1N_1 + d_1N_2 &= N,
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
M &= \theta_1\sum_{j=1}^{m} {}^{p}I^{\overline{\eta}_j+\delta_4,\rho,\psi}G_j(\xi_1,\mathfrak{u}(\xi_1),\mathfrak{s}(\xi_1)) + \theta_1{}^{p}I^{\delta_4,\rho,\psi}\mathfrak{g}(\xi_1,\mathfrak{u}(\xi_1),\mathfrak{s}(\xi_1))^{p}I^{\delta_3,\rho,\psi}S(\xi_1)\\
&\quad-\sum_{i=1}^{n} {}^{p}I^{\eta_i+\delta_2,\rho,\psi}H_i(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0)) - {}^{p}I^{\delta_2,\rho,\psi}\mathfrak{f}(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0))^{p}I^{\delta_1,\rho,\psi}U(\mathfrak{b}_0),\\
N &= \theta_2\sum_{i=1}^{n} {}^{p}I^{\eta_i+\delta_2,\rho,\psi}H_i(\xi_2,\mathfrak{u}(\xi_2),\mathfrak{s}(\xi_2)) + \theta_2{}^{p}I^{\delta_2,\rho,\psi}\mathfrak{f}(\xi_2,\mathfrak{u}(\xi_2),\mathfrak{s}(\xi_2))^{p}I^{\delta_1,\rho,\psi}U(\xi_2)\\
&\quad-\sum_{j=1}^{m} {}^{p}I^{\overline{\eta}_j+\delta_4,\rho,\psi}G_j(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0)) - {}^{p}I^{\delta_4,\rho,\psi}\mathfrak{g}(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0))^{p}I^{\delta_3,\rho,\psi}S(\mathfrak{b}_0).
\end{aligned}$$

By solving the above system, we conclude that

$$c_1 = \frac{1}{\Theta}[N_2M + M_2N],\ \ d_1 = \frac{1}{\Theta}[M_1N + N_1M].$$

Replacing the values c_1 and d_1 in the Equation (12) we obtain the solutions (6) and (7). The converse follows by direct computation. The proof is completed. □

The following version of Krasnosel'skiĭ's fixed point theorem due to Burton is the basic tool in proving our main existence result.

Lemma 3 ([44])**.** *Let S be a nonempty, convex, closed, and bounded set such that $\mathbb{S} \subset \mathbb{X}$, and let $\mathcal{A} : \mathbb{X} \to \mathbb{X}$ and $\mathcal{B} : \mathbb{S} \to \mathbb{X}$ be two operators which satisfy the following:*

(i) *$\mathcal{A}$ is a contraction;*
(ii) *$\mathcal{B}$ is completely continuous; and*
(iii) *$x = \mathcal{A}x + \mathcal{B}y, \forall y \in S \Rightarrow x \in \mathbb{S}$.*

Then there exists a solution of the operator equation $x = \mathcal{A}x + \mathcal{B}x$.

3. An Existence Result

Let $\mathbb{Y} = C([\mathfrak{a}_0,\mathfrak{b}_0],\mathbb{R}) = \{\mathfrak{u} : [\mathfrak{a}_0,\mathfrak{b}_0] \longrightarrow \mathbb{R} \text{ is continuous}\}$. The space $\mathbb{Y}$ is a Banach space with the norm $\|\mathfrak{u}\| = \sup_{\mathfrak{t}\in[\mathfrak{a}_0,\mathfrak{b}_0]}|\mathfrak{u}(\mathfrak{t})|$. Obviously, the space $(\mathbb{Y}\times\mathbb{Y}, \|(\mathfrak{u},\mathfrak{s})\|)$ is also a Banach space with the norm $\|(\mathfrak{u},\mathfrak{s})\| = \|\mathfrak{u}\| + \|\mathfrak{s}\|$.

Definition 4. *A function $(\mathfrak{u},\mathfrak{s}) \in \mathbb{Y}\times\mathbb{Y}$ is said to be a solution of nonlocal ψ-Hilfer sequential proportional coupled system (3) and (4) if* $\mathfrak{u} \to \frac{{}^{H}D^{\delta_2,\vartheta_2,\rho,\psi}\mathfrak{u}(\mathfrak{t})-\sum_{i=1}^{n}{}^{p}I^{\eta_i,\rho,\psi}H_i(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}{\mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}$ *and* $\mathfrak{s} \to \frac{{}^{H}D^{\delta_4,\vartheta_4,\rho,\psi}\mathfrak{s}(\mathfrak{t})-\sum_{j=1}^{m}{}^{p}I^{\overline{\eta}_j,\rho,\psi}G_j(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}{\mathfrak{g}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}$ *are continuous for each $(\mathfrak{u},\mathfrak{s}) \in \mathbb{Y}\times\mathbb{Y}$ and satisfies nonlocal ψ-Hilfer sequential proportional coupled system (3) and the boundary conditions in (4).*

By Lemma 2, we define an operator $\mathbb{U} : \mathbb{Y}\times\mathbb{Y} \to \mathbb{Y}\times\mathbb{Y}$ by

$$\mathbb{U}(\mathfrak{u},\mathfrak{s})(\mathfrak{t}) = \begin{pmatrix} \mathbb{U}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{t}) \\ \mathbb{U}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{t}) \end{pmatrix}, \tag{16}$$

where

$$\begin{aligned}
&\mathbb{U}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{t})\\
= \;& \sum_{i=1}^{n}{}^{p}I^{\eta_i+\delta_2,\rho,\psi}H_i(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))\\
&+{}^{p}I^{\delta_2,\rho,\psi}\mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))^{p}I^{\delta_1,\rho,\psi}P(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))\\
&+\frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma_2-1}}{\Theta\rho^{\gamma_2-1}\Gamma(\gamma_2)}\Bigg\{N_2\Bigg[\theta_1\sum_{j=1}^{m}{}^{p}I^{\overline{\eta}_j+\delta_4,\rho,\psi}G_j(\xi_1,\mathfrak{u}(\xi_1),\mathfrak{s}(\xi_1))\\
&+\theta_1{}^{p}I^{\delta_4,\rho,\psi}\mathfrak{g}(\xi_1,\mathfrak{u}(\xi_1),\mathfrak{s}(\xi_1))^{p}I^{\delta_3,\rho,\psi}Q(\xi_1,\mathfrak{u}(\xi_1),\mathfrak{s}(\xi_1))\\
&-\sum_{i=1}^{n}{}^{p}I^{\eta_i+\delta_2,\rho,\psi}H_i(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0))\\
&-{}^{p}I^{\delta_2,\rho,\psi}\mathfrak{f}(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0))^{p}I^{\delta_1,\rho,\psi}P(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0))\Bigg]\\
&+M_2\Bigg[\theta_2\sum_{i=1}^{n}{}^{p}I^{\eta_i+\delta_2,\rho,\psi}H_i(\xi_2,\mathfrak{u}(\xi_2),\mathfrak{s}(\xi_2))\\
&+\theta_2{}^{p}I^{\delta_2,\rho,\psi}\mathfrak{f}(\xi_2,\mathfrak{u}(\xi_2),\mathfrak{s}(\xi_2))^{p}I^{\delta_1,\rho,\psi}P(\xi_2,\mathfrak{u}(\xi_2),\mathfrak{s}(\xi_2))\\
&-\sum_{j=1}^{m}{}^{p}I^{\overline{\eta}_j+\delta_4,\rho,\psi}G_j(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0))\\
&-{}^{p}I^{\delta_4,\rho,\psi}\mathfrak{g}(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0))^{p}I^{\delta_3,\rho,\psi}Q(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0))\Bigg]\Bigg\},
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
&\mathbb{U}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{t})\\
= \;& \sum_{j=1}^{m}{}^{p}I^{\overline{\eta}_j+\delta_4,\rho,\psi}G_j(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))\\
&+{}^{p}I^{\delta_4,\rho,\psi}\mathfrak{g}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))^{p}I^{\delta_3,\rho,\psi}Q(\mathfrak{t},\mathfrak{u}w),\mathfrak{s}(\mathfrak{t}))\\
&+\frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma_4-1}}{\Theta\rho^{\gamma_4-1}\Gamma(\gamma_4)}\Bigg\{N_1\Bigg[\theta_1\sum_{j=1}^{m}{}^{p}I^{\overline{\eta}_j+\delta_4,\rho,\psi}G_j(\xi_1,\mathfrak{u}(\xi_1),\mathfrak{s}(\xi_1))\\
&+\theta_1{}^{p}I^{\delta_4,\rho,\psi}\mathfrak{g}(\xi_1,\mathfrak{u}(\xi_1),\mathfrak{s}(\xi_1))^{p}I^{\delta_3,\rho,\psi}Q(\xi_1,\mathfrak{u}(\xi_1),\mathfrak{s}(\xi_1))\\
&-\sum_{i=1}^{n}{}^{p}I^{\eta_i+\delta_2,\rho,\psi}H_i(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0))\\
&-{}^{p}I^{\delta_2,\rho,\psi}\mathfrak{f}(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0))^{p}I^{\delta_1,\rho,\psi}P(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0))\Bigg]\\
&+M_1\Bigg[\theta_2\sum_{i=1}^{n}{}^{p}I^{\eta_i+\delta_2,\rho,\psi}H_i(\xi_2,\mathfrak{u}(\xi_2),\mathfrak{s}(\xi_2))
\end{aligned}$$

$$
\begin{aligned}
&+\theta_2{}^pI^{\delta_2,\rho,\psi}\mathfrak{f}(\xi_2,\mathfrak{u}(\xi_2),\mathfrak{s}(\xi_2))^pI^{\delta_1,\rho,\psi}P(\xi_2,\mathfrak{u}(\xi_2),\mathfrak{s}(\xi_2))\\
&-\sum_{j=1}^{m}{}^pI^{\overline{\eta}_j+\delta_4,\rho,\psi}G_j(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0))\\
&-{}^pI^{\delta_4,\rho,\psi}\mathfrak{g}(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0))^pI^{\delta_3,\rho,\psi}Q(\mathfrak{b}_0,\mathfrak{u}(\mathfrak{b}_0),\mathfrak{s}(\mathfrak{b}_0))\Bigg]\Bigg\}.
\end{aligned}
\tag{18}
$$

Our main existence result is given in the next theorem.

Theorem 1. *Assume that:*

(H_1) *The functions* $\mathfrak{f},\mathfrak{g} : [\mathfrak{a}_0,\mathfrak{b}_0]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$, $P,Q : [\mathfrak{a}_0,\mathfrak{b}_0]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *and* $H_i,G_j : [\mathfrak{a}_0,\mathfrak{b}_0]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *for* $i=1,2,\ldots,n,\ j=1,2,\ldots,m$, *are continuous and there exist positive functions* ϕ, χ, π, ϖ, h_i, z_j, $i=1,2,\ldots,n$ $j=1,2,\ldots,m$, *with bounds* $\|\phi\|$, $\|\chi\|$, $\|\pi\|$, $\|\varpi\|$, *and* $\|h_i\|$, $i=1,2,\ldots,m$, $\|z_j\|$, $j=1,2,\ldots,m$, *respectively, such that*

$$
\begin{aligned}
&|\mathfrak{f}(\mathfrak{t},x_1,x_2)-\mathfrak{f}(\mathfrak{t},\overline{x}_1,\overline{x}_2)|\le\phi(\mathfrak{t})\big(|x_1-\overline{x}_1|+|x_2-\overline{x}_2|\big),\\
&|\mathfrak{g}(\mathfrak{t},x_1,x_2)-\mathfrak{g}(\mathfrak{t},\overline{x}_1,\overline{x}_2)|\le\chi(\mathfrak{t})\big(|x_1-\overline{x}_1|+|x_2-\overline{x}_2|\big),\\
&|P(\mathfrak{t},x_1,x_2)-P(\mathfrak{t},\overline{x}_1,\overline{x}_2|\le\pi(\mathfrak{t})(|x_1-\overline{x}_1|+|x_2-\overline{x}_2|),\\
&|Q(\mathfrak{t},x_1,x_2)-Q(\mathfrak{t},\overline{x}_1,\overline{x}_2|\le\varpi(\mathfrak{t})(|x_1-\overline{x}_1|+|x_2-\overline{x}_2|),\\
&|H_i(\mathfrak{t},x_1,x_2)-H_i(\mathfrak{t},\overline{x}_1,\overline{x}_2)|\le h_i(\mathfrak{t})\big(|x_1-\overline{x}_1|+|x_2-\overline{x}_2|\big),\\
&|G_j(\mathfrak{t},x_1,x_2)-G_j(\mathfrak{t},\overline{x}_1,\overline{x}_2)|\le z_j(\mathfrak{t})\big(|x_1-\overline{x}_1|+|x_2-\overline{x}_2|\big),
\end{aligned}
$$

for all $\mathfrak{t}\in[\mathfrak{a}_0,\mathfrak{b}_0]$ *and* $x_i,\overline{x}_i\in\mathbb{R}$, $i=1,2$.

(H_2) *There exist continuous functions* σ, τ, ℓ, m, λ_i, μ_j, $i=1,2,\ldots,n,\ j=1,2,\ldots,m$ *such that*

$$
\begin{aligned}
&|\mathfrak{f}(\mathfrak{t},x_1,x_2)|\le\sigma(\mathfrak{t}), &&|\mathfrak{g}(\mathfrak{t},x_1,x_2)|\le\tau(\mathfrak{t}),\\
&|H_i(\mathfrak{t},x_1,x_2)|\le\lambda_i(\mathfrak{t}), &&|G_j(\mathfrak{t},x_1,x_2)|\le\mu_j(\mathfrak{t}),\\
&|P(\mathfrak{t},x_1,x_2)|\le\ell(\mathfrak{t}), &&|Q(\mathfrak{t},x_1,x_2)|\le m(\mathfrak{t}),
\end{aligned}
$$

for all $\mathfrak{t}\in[\mathfrak{a}_0,\mathfrak{b}_0]$ *and* $x_1,x_2\in\mathbb{R}$.

(H_3) *Assume that*

$$
\begin{aligned}
K:\ =\ &\Big\{\Psi(\mathfrak{b}_0,\delta_2)\Big[1+(N_2+M_2|\theta_2|)\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_2)\Big]\\
&+(N_1+M_1|\theta_2|)\Psi(\mathfrak{b}_0,\delta_2)\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_4)\Big\}\Big[\|\sigma\|\|\pi\|+\|\ell\|\Psi(\mathfrak{b}_0,\delta_1)\|\phi\|\Big]\\
&+\Big\{(N_2|\theta_1|+M_2)\Psi(\mathfrak{b}_0,\delta_4)\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_2)\\
&+\Psi(\mathfrak{b}_0,\delta_4)\Big[1+(N_1|\theta_1|+M_1)\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_4)\Big]\Big\}\Big[\|\tau\|\|\varpi\|+\|m\|\Psi(\mathfrak{b}_0,\delta_3)\|\chi\|\Big]<1,
\end{aligned}
$$

where for convenience we have put

$$
\Psi(x,y)=\frac{(\psi(x)-\psi(\mathfrak{a}_0))^y}{\rho^y\Gamma(y+1)}
\tag{19}
$$

and $\|\epsilon\|=\sup_{t\in[\mathfrak{a}_0,\mathfrak{b}_0]}|\epsilon(\mathfrak{t})|$, $\epsilon\in\{\sigma,\tau,\pi,\varpi,\ell,m\}$.

Then the ψ*-Hilfer proportional coupled system (3) and (4) has at least one solution on* $[\mathfrak{a}_0,\mathfrak{b}_0]$.

Proof. Firstly, we consider a subset $\mathbb{S}$ of $\mathbb{Y}\times\mathbb{Y}$ defined by $\mathbb{S}=\{(\mathfrak{u},\mathfrak{s})\in\mathbb{Y}\times\mathbb{Y} : \|(\mathfrak{u},\mathfrak{s})\|\le r\}$, where r is given by

$$
r=R_1+R_2,
\tag{20}
$$

where

$$\begin{aligned} R_1 &= \Big[1+\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_2)(N_2+M_2|\theta_2|)\Big] \\ &\quad\times\Big[\sum_{i=1}^{n}\|\lambda_i\|\Psi(\mathfrak{b}_0,\eta_i+\delta_2)+\|\sigma\|\|\ell\|\Psi(\mathfrak{b}_0,\delta_1)\Psi(\mathfrak{b}_0,\delta_2)\Big] \\ &\quad+(N_2|\theta_1|+M_2)\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_2)\Big[\sum_{j=1}^{m}\|\mu_j\|\Psi(\mathfrak{b}_0,\overline{\eta}_j+\delta_4) \\ &\quad+\Psi(\mathfrak{b}_0,\delta_4)\|\tau\|\|L_2\|\Psi(\mathfrak{b}_0,\delta_3)\Big], \end{aligned}$$

and

$$\begin{aligned} R_2 &= \Big[1+\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_4)(N_1|\theta_1|+M_1)\Big] \\ &\quad\times\Big[\sum_{j=1}^{m}\|\mu_j\|\Psi(\mathfrak{b}_0,\overline{\eta}_j+\delta_4)+\|\tau\|\|m\|\Psi(\mathfrak{b}_0,\delta_4)\Psi(\mathfrak{b}_0,\delta_3)\Big] \\ &\quad+(N_1|+M_1\theta_2|)\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_4)\Big[\sum_{i=1}^{n}\|\lambda_i\|\Psi(\mathfrak{b}_0,\eta_i+\delta_2) \\ &\quad+\Psi(\mathfrak{b}_0,\delta_2)\|\sigma\|\|L_1\|\Psi(\mathfrak{b}_0,\delta_1)\Big], \end{aligned}$$

where $\sup_{t\in[\mathfrak{a}_0,\mathfrak{b}_0]}|\lambda_i(t)|=\|\lambda_i\|$, $i=1,2,\ldots,n$, $\sup_{t\in[\mathfrak{a}_0,\mathfrak{b}_0]}|\mu_j(t)|=\|\mu_j\|$, $j=1,2,\ldots,m$.

Let us define the operators

$$\mathcal{H}_i(\mathfrak{u},\mathfrak{s})(\mathfrak{t})=\sum_{i=1}^{n}{}^{p}I^{\eta_i+\delta_2,\rho,\psi}H_i(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})),\ \ \mathfrak{t}\in[\mathfrak{a}_0,\mathfrak{b}_0],$$

$$\mathcal{G}_j(\mathfrak{u},\mathfrak{s})(\mathfrak{t})=\sum_{j=1}^{m}{}^{p}I^{\overline{\eta}_j+\delta_4,\rho,\psi}G_j(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})),\ \ \mathfrak{t}\in[\mathfrak{a}_0,\mathfrak{b}_0],$$

$$\mathcal{Y}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{t})={}^{p}I^{\delta_1,\rho,\psi}P(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})),\ \ \mathfrak{t}\in[\mathfrak{a}_0,\mathfrak{b}_0],$$
$$\mathcal{Y}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{t})={}^{p}I^{\delta_3,\rho,\psi}Q(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})),\ \ \mathfrak{t}\in[\mathfrak{a}_0,\mathfrak{b}_0],$$

and

$$\mathcal{F}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{t})=\mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})),\ \ \mathfrak{t}\in[\mathfrak{a}_0,\mathfrak{b}_0],$$
$$\mathcal{F}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{t})=\mathfrak{g}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})),\ \ \mathfrak{t}\in[\mathfrak{a}_0,\mathfrak{b}_0].$$

Then we have

$$\begin{aligned} |\mathcal{H}_i(\mathfrak{u}_1,\mathfrak{s}_1))(\mathfrak{t})-\mathcal{H}_i(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| &\le \sum_{i=1}^{n}{}^{p}I^{\eta_i+\delta_2,\rho,\psi}|H_i(\mathfrak{t},\mathfrak{u}_1(\mathfrak{t}),\mathfrak{s}_1(\mathfrak{t}))-H_i(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))| \\ &\le \sum_{i=1}^{n}\|h_i\|\Psi(\mathfrak{b}_0,\eta_i+\delta_2)(\|\mathfrak{u}_1-\mathfrak{u}\|+\|\mathfrak{s}_1-\mathfrak{s}\|), \end{aligned}$$

and

$$\begin{aligned} |\mathcal{H}_i(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| &\le \sum_{i=1}^{n}{}^{p}I^{\eta_i+\delta_2,\rho,\psi}|H_i(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))| \\ &\le \sum_{i=1}^{n}\|\lambda_i\|\Psi(\mathfrak{b}_0,\eta_i+\delta_2). \end{aligned}$$

Also, we obtain

$$\begin{aligned}|\mathcal{G}_j(\mathfrak{u}_1,\mathfrak{s}_1))(\mathfrak{t})-\mathcal{G}_j(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| &\leq \sum_{j=1}^{m} {}^{p}I^{\overline{\eta}_j+\delta_4,\rho,\psi}|G_j(\mathfrak{t},\mathfrak{u}_1(\mathfrak{t}),\mathfrak{s}_1(\mathfrak{t}))-G_j(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))| \\ &\leq \sum_{j=1}^{m}\|z_j\|\Psi(\mathfrak{b}_0,\overline{\eta}_j+\delta_4)(\|\mathfrak{u}_1-\mathfrak{u}\|+\|\mathfrak{s}_1-\mathfrak{s}\|),\end{aligned}$$

and

$$\begin{aligned}|\mathcal{G}_j(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| &\leq \sum_{j=1}^{m} {}^{p}I^{\overline{\eta}_j+\delta_4,\rho,\psi}|G_j(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))| \\ &\leq \sum_{j=1}^{m}\|\mu_j\|\Psi(\mathfrak{b}_0,\overline{\eta}_j+\delta_4).\end{aligned}$$

Moreover, we have

$$\begin{aligned}|\mathcal{Y}_1(\mathfrak{u}_1,\mathfrak{s}_1))(\mathfrak{t})-\mathcal{Y}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| &\leq {}^{p}I^{\delta_1,\rho,\psi}|P(\mathfrak{t},\mathfrak{u}_1(\mathfrak{t}),\mathfrak{s}_1(\mathfrak{t}))-P(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))| \\ &\leq \|\pi\|\Psi(\mathfrak{b}_0,\delta_1)(\|\mathfrak{u}_1-\mathfrak{u}\|+\|\mathfrak{s}_1-\mathfrak{s}\|),\end{aligned}$$

$$\begin{aligned}|\mathcal{Y}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| &\leq {}^{p}I^{\delta_1,\rho,\psi}|P(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))| \\ &\leq \|\ell\|\Psi(\mathfrak{b}_0,\delta_1),\end{aligned}$$

and

$$\begin{aligned}|\mathcal{Y}_2(\mathfrak{u}_1,\mathfrak{s}_1))(\mathfrak{t})-\mathcal{Y}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| &\leq {}^{p}I^{\delta_3,\rho,\psi}|Q(\mathfrak{t},\mathfrak{u}_1(\mathfrak{t}),\mathfrak{s}_1(\mathfrak{t}))-Q(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))| \\ &\leq \|\varpi\|\Psi(\mathfrak{b}_0,\delta_3)(\|\mathfrak{u}_1-\mathfrak{u}\|+\|\mathfrak{s}_1-\mathfrak{s}\|),\end{aligned}$$

$$\begin{aligned}|\mathcal{Y}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| &\leq {}^{p}I^{\delta_1,\rho,\psi}|Q(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))| \\ &\leq \|m\|\Psi(\mathfrak{b}_0,\delta_1).\end{aligned}$$

Finally, we obtain

$$\begin{aligned}|\mathcal{F}_1(\mathfrak{u}_1,\mathfrak{s}_1))(\mathfrak{t})-\mathcal{F}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| &\leq |\mathfrak{f}(\mathfrak{t},\mathfrak{u}_1(\mathfrak{t}),\mathfrak{s}_1(\mathfrak{t}))-\mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))| \\ &\leq \|\phi\|(\|\mathfrak{u}_1-\mathfrak{u}\|+\|\mathfrak{s}_1-\mathfrak{s}\|),\end{aligned}$$

$$\begin{aligned}|\mathcal{F}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| &\leq |\mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))| \\ &\leq \|\sigma\|,\end{aligned}$$

and

$$\begin{aligned}|\mathcal{F}_2(\mathfrak{u}_1,\mathfrak{s}_1))(\mathfrak{t})-\mathcal{F}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| &\leq |\mathfrak{g}(\mathfrak{t},\mathfrak{u}_1(\mathfrak{t}),\mathfrak{s}_1(\mathfrak{t}))-\mathfrak{g}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))| \\ &\leq \|\chi\|(\|\mathfrak{u}_1-\mathfrak{u}\|+\|\mathfrak{s}_1-\mathfrak{s}\|),\end{aligned}$$

$$\begin{aligned}|\mathcal{F}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| &\leq |\mathfrak{g}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))| \\ &\leq \|\tau\|.\end{aligned}$$

Now we split the operator $\mathbb{U}$ as

$$\mathbb{U}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{t})=\mathbb{U}_{1,1}(\mathfrak{u},\mathfrak{s})(\mathfrak{t})+\mathbb{U}_{1,2}(\mathfrak{u},\mathfrak{s})(\mathfrak{t}),$$

$$\mathbb{U}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{t}) = \mathbb{U}_{2,1}(\mathfrak{u},\mathfrak{s})(\mathfrak{t}) + \mathbb{U}_{2,2}(\mathfrak{u},\mathfrak{s})(\mathfrak{t}),$$

with

$$\begin{aligned}
& \mathbb{U}_{1,1}(\mathfrak{u},\mathfrak{s})(\mathfrak{t}) \\
= \; & \mathcal{H}_i(\mathfrak{u},\mathfrak{s})(\mathfrak{t}) \\
& + \frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma_2-1}}{\Theta\rho^{\gamma_2-1}\Gamma(\gamma_2)}\Big\{N_2\Big[\theta_1\mathcal{G}_i(\mathfrak{u},\mathfrak{s})(\xi_1) - \mathcal{H}_i(\mathfrak{u},\mathfrak{s})(\mathfrak{b}_0)\Big] \\
& + M_2\Big[\theta_2\mathcal{H}_i(\mathfrak{u},\mathfrak{s})(\xi_2)) - G_j(\mathfrak{u},\mathfrak{s})(\mathfrak{b}_0))\Big]\Big\},
\end{aligned}$$

$$\begin{aligned}
& \mathbb{U}_{1,2}(\mathfrak{u},\mathfrak{s})(\mathfrak{t}) \\
= \; & {}^pI^{\delta_2,\rho,\psi}\mathcal{F}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{t})\mathcal{Y}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{t}) + \frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma_2-1}}{\Theta\rho^{\gamma_2-1}\Gamma(\gamma_2)} \\
& \times\Big\{N_2\Big[\theta_1{}^pI^{\delta_4,\rho,\psi}\mathcal{F}_2(\mathfrak{u},\mathfrak{s})(\xi_1)\mathcal{Y}_2(\mathfrak{u},\mathfrak{s})(\xi_1) - {}^pI^{\delta_2,\rho,\psi}\mathcal{F}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{b}_0)\mathcal{Y}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{b}_0)\Big] \\
& + M_2\Big[\theta_2{}^pI^{\delta_2,\rho,\psi}\mathcal{F}_1(\mathfrak{u},\mathfrak{s})(\xi_2)\mathcal{Y}_1(\mathfrak{u},\mathfrak{s})(\xi_2) - {}^pI^{\delta_4,\rho,\psi}\mathcal{F}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{b}_0)\mathcal{Y}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{b}_0)\Big]\Big\},
\end{aligned}$$

$$\begin{aligned}
& \mathbb{U}_{2,1}(\mathfrak{u},\mathfrak{s})(\mathfrak{t}) \\
= \; & \mathcal{G}_j(\mathfrak{u},\mathfrak{s})(\mathfrak{t}) \\
& + \frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma_4-1}}{\Theta\rho^{\gamma_4-1}\Gamma(\gamma_4)}\Big\{N_1\Big[\theta_1\mathcal{G}_j(\mathfrak{u},\mathfrak{s})(\xi_1) - \mathcal{H}_i(\mathfrak{u},\mathfrak{s})(\mathfrak{b}_0)\Big] \\
& + M_1\Big[\theta_2\mathcal{H}_i(\mathfrak{u},\mathfrak{s})(\xi_2) - \mathcal{G}_j(\mathfrak{u},\mathfrak{s})(\mathfrak{b}_0)\Big]\Big\},
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{U}_{2,2}(\mathfrak{u},\mathfrak{s})(\mathfrak{t}) \\
= \; & {}^pI^{\delta_4,\rho,\psi}\mathcal{F}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{t})\mathcal{Y}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{t}) + \frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma_4-1}}{\Theta\rho^{\gamma_4-1}\Gamma(\gamma_4)} \\
& \times\Big\{N_1\Big[\theta_1{}^pI^{\delta_4,\rho,\psi}\mathcal{F}_2(\mathfrak{u},\mathfrak{s})(\xi_1)\mathcal{Y}_2(\mathfrak{u},\mathfrak{s})(\xi_1) - {}^pI^{\delta_2,\rho,\psi}\mathcal{F}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{b}_0)\mathcal{Y}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{b}_0)\Big] \\
& + M_1\Big[\theta_2{}^pI^{\delta_2,\rho,\psi}\mathcal{F}_1(\mathfrak{u},\mathfrak{s})(\xi_2)\mathcal{Y}_1(\mathfrak{u},\mathfrak{s})(\xi_2) - {}^pI^{\delta_4,\rho,\psi}\mathcal{F}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{b}_0)\mathcal{Y}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{b}_0)\Big]\Big\}.
\end{aligned}$$

In the following steps, we will prove that the operators $\mathbb{U}_1, \mathbb{U}_2$ fulfill the assumptions of Lemma 3.

Step 1. In the first step we will prove that the operators $\mathbb{U}_{1,2}$ and $\mathbb{U}_{2,2}$ are contraction mappings. For all $(\mathfrak{u},\mathfrak{s}), (\mathfrak{u}_1,\mathfrak{s}_1) \in \mathbb{Y}\times\mathbb{Y}$ we have

$$\begin{aligned}
& |\mathbb{U}_{1,2}(\mathfrak{u}_1,\mathfrak{s}_1)(\mathfrak{t}) - \mathbb{U}_{1,2}(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| \\
\leq \; & \Psi(\mathfrak{b}_0,\delta_2)|\mathcal{F}_1(\mathfrak{u}_1,\mathfrak{s}_1)(\mathfrak{t})\mathcal{Y}_1(\mathfrak{u}_1,\mathfrak{s}_1)(\mathfrak{t}) - \mathcal{F}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{t})\mathcal{Y}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| \\
& + \frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_2)\Big\{N_2|\theta_1|\Psi(\mathfrak{b}_0,\delta_4)|\mathcal{F}_2(\mathfrak{u}_1,\mathfrak{s}_1)(\mathfrak{t})\mathcal{Y}_2(\mathfrak{u}_1,\mathfrak{s}_1)(\mathfrak{t}) - \mathcal{F}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{t})\mathcal{Y}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| \\
& + (N_2 + M_2|\theta_2|)\Psi(\mathfrak{b}_0,\delta_2)|\mathcal{F}_1(\mathfrak{u}_1,\mathfrak{s}_1)(\xi_2)\mathcal{Y}_1(\mathfrak{u}_1,\mathfrak{s}_1)(\xi_2) - \mathcal{F}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{t})\mathcal{Y}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| \\
& + M_2\Psi(\mathfrak{b}_0,\delta_4)|\mathcal{F}_2(\mathfrak{u}_1,\mathfrak{s}_1)(\mathfrak{t})\mathcal{Y}_2(\mathfrak{u}_1,\mathfrak{s}_1)(\mathfrak{t}) - \mathcal{F}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{t})\mathcal{Y}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{t})|\Big\}
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \Psi(\mathfrak{b}_0,\delta_2)\Big[1+(N_2+M_2|\theta_2|)\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_2)\Big] \\
& \times|\mathcal{F}_1(\mathfrak{u}_1,\mathfrak{s}_1)(\mathfrak{t})\mathcal{Y}_1(\mathfrak{u}_1,\mathfrak{s}_1)(\mathfrak{t})-\mathcal{F}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{t})\mathcal{Y}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| \\
& +(N_2|\theta_1|+M_2)\Psi(\mathfrak{b}_0,\delta_4)\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_2) \\
& \times|\mathcal{F}_2(\mathfrak{u}_1,\mathfrak{s}_1)(\mathfrak{t})\mathcal{Y}_2(\mathfrak{u}_1,\mathfrak{s}_1)(\mathfrak{t})-\mathcal{F}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{t})\mathcal{Y}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| \\
\leq\ & \Psi(\mathfrak{b}_0,\delta_2)\Big[1+(N_2+M_2|\theta_2|)\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_2)\Big]\Big[|\mathcal{F}_1(\mathfrak{u}_1,\mathfrak{s}_1)(\mathfrak{t})||\mathcal{Y}_1(\mathfrak{u}_1,\mathfrak{s}_1)(\mathfrak{t})-\mathcal{Y}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| \\
& +|\mathcal{Y}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{t})||\mathcal{F}_1(\mathfrak{u}_1,\mathfrak{s}_1)(\mathfrak{t})-\mathcal{F}_1(\mathfrak{u},\mathfrak{s})(\mathfrak{t})\Big] \\
& +(N_2|\theta_1|+M_2)\Psi(\mathfrak{b}_0,\delta_4)\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_2)\Big[|\mathcal{F}_2(\mathfrak{u}_1,\mathfrak{s}_1)(\mathfrak{t})||\mathcal{Y}_2(\mathfrak{u}_1,\mathfrak{s}_1)(\mathfrak{t})-\mathcal{Y}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| \\
& +|\mathcal{Y}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{t})||\mathcal{F}_2(\mathfrak{u}_1,\mathfrak{s}_1)(\mathfrak{t})-\mathcal{F}_2(\mathfrak{u},\mathfrak{s})(\mathfrak{t})|\Big] \\
\leq\ & \Psi(\mathfrak{b}_0,\delta_2)\Big[1+(N_2+M_2|\theta_2|)\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_2)\Big]\Big[\|\sigma\|\|\pi\|+\|\ell\|\Psi(\mathfrak{b}_0,\delta_1)\|\phi\|\Big] \\
& \times\Big[\|\mathfrak{u}_1-\mathfrak{u}\|+\|\mathfrak{s}_1-\mathfrak{s}\|\Big] \\
& +(N_2|\theta_1|+M_2)\Psi(\mathfrak{b}_0,\delta_4)\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_2)\Big[\|\tau\|\|\varpi\|+\|m\|\Psi(\mathfrak{b}_0,\delta_3)\|\chi\|\Big] \\
& \times\Big[\|\mathfrak{u}_1-\mathfrak{u}\|+\|\mathfrak{s}_1-\mathfrak{s}\|\Big] \\
=\ & \Big\{\Psi(\mathfrak{b}_0,\delta_2)\Big[1+(N_2+M_2|\theta_2|)\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_2)\Big]\Big[\|\sigma\|\|\pi\|+\|\ell\|\Psi(\mathfrak{b}_0,\delta_1)\|\phi\|\Big] \\
& +(N_2|\theta_1|+M_2)\Psi(\mathfrak{b}_0,\delta_4)\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_2)\Big[\|\tau\|\|\varpi\|+\|m\|\Psi(\mathfrak{b}_0,\delta_3)\|\chi\|\Big]\Big\} \\
& \times\Big[\|\mathfrak{u}_1-\mathfrak{u}\|+\|\mathfrak{s}_1-\mathfrak{s}\|\Big].
\end{aligned}
$$

Similarly we can find

$$
\begin{aligned}
& |\mathbb{U}_{2,2}(\mathfrak{u}_1,\mathfrak{s}_1)(\mathfrak{t})-\mathbb{U}_{2,2}(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| \\
\leq\ & \Big\{\Psi(\mathfrak{b}_0,\delta_4)\Big[1+(N_1|\theta_1|+M_1)\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_4)\Big]\Big[\|\tau\|\|\varpi\|+\|m\|\Psi(\mathfrak{b}_0,\delta_3)\|\chi\|\Big] \\
& +(N_1+M_1|\theta_2|)\Psi(\mathfrak{b}_0,\delta_2)\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_4)\Big[\|\sigma\|\|\pi\|+\|\ell\|\Psi(\mathfrak{b}_0,\delta_2)\|\phi\|\Big]\Big\} \\
& \times\Big[\|\mathfrak{u}_1-\mathfrak{u}\|+\|\mathfrak{s}_1-\mathfrak{s}\|\Big].
\end{aligned}
$$

Consequently, we obtain

$$\|(\mathbb{U}_{1,2},\mathbb{U}_{2,2})(\mathfrak{u}_1,\mathfrak{s}_1)-(\mathbb{U}_{1,2},\mathbb{U}_{2,2})(\mathfrak{u},\mathfrak{s})\|\leq K(\|\mathfrak{u}_1-\mathfrak{u}\|+\|\mathfrak{s}_1-\mathfrak{s}\|),$$

which means that $(\mathbb{U}_{1,2},\mathbb{U}_{2,2})$ is a contraction.

Step 2. In the second step we will prove that the operator $(\mathbb{U}_{1,1},\mathbb{U}_{2,1})$ is completely continuous on $\mathbb{S}$. For continuity of $\mathbb{U}_{1,1}$, take any sequence of points $(\mathfrak{u}_n,\mathfrak{s}_n)$ in $\mathbb{S}$ converging to a point $(\mathfrak{u},\mathfrak{s})\in\mathbb{S}$. Then, by Lebesgue dominated convergence theorem, we have

$$
\begin{aligned}
\lim_{n\to\infty}\mathbb{U}_{1,1}(\mathfrak{u}_n,\mathfrak{s}_n)(\mathfrak{t}) \quad = \quad & \lim_{n\to\infty}\mathcal{H}_i(\mathfrak{u}_n,\mathfrak{s}_n)(\mathfrak{t})+\frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma_2-1}}{\rho^{\gamma_2-1}\Gamma(\gamma_2)} \\
& \times\Big\{N_2\Big[\theta_1\lim_{n\to\infty}\mathcal{G}_i(\mathfrak{u}_n,\mathfrak{s}_n)(\xi_1)-\lim_{n\to\infty}\mathcal{H}_i(\mathfrak{u}_n,\mathfrak{s}_n)(\mathfrak{b}_0)\Big] \\
& +M_2\Big[\theta_2\lim_{n\to\infty}\mathcal{H}_i(\mathfrak{u}_n,\mathfrak{s}_n)(\xi_2))-\lim_{n\to\infty}G_j(\mathfrak{u}_n,\mathfrak{s}_n)(\mathfrak{b}_0))\Big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \mathcal{H}_i(\mathfrak{u},\mathfrak{s})(\mathfrak{t}) + \frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma_2-1}}{\rho^{\gamma_2-1}\Gamma(\gamma_2)} \\
&\quad \times\Bigg\{ N_2\Big[\theta_1\mathcal{G}_i(\mathfrak{u},\mathfrak{s})(\xi_1) - \mathcal{H}_i(\mathfrak{u},\mathfrak{s})(\mathfrak{b}_0)\Big] \\
&\quad +M_2\Big[\theta_2\mathcal{H}_i(\mathfrak{u},\mathfrak{s})(\xi_2)) - G_j(tp,q)(\mathfrak{b}_0))\Big]\Bigg\} \\
&= \mathbb{U}_{1,1}(\mathfrak{u},\mathfrak{s})(\mathfrak{t}),
\end{aligned}
$$

for all $\mathfrak{t} \in [\mathfrak{a}_0,\mathfrak{b}_0]$. Similarly, we prove $\lim_{n\to\infty}\mathbb{U}_{2,1}(\mathfrak{u}_n,\mathfrak{s}_n)(\mathfrak{t}) = \mathbb{U}_{2,1}(\mathfrak{u},\mathfrak{s})(\mathfrak{t})$ for all $\mathfrak{t} \in [\mathfrak{a}_0,\mathfrak{b}_0]$. Thus $(\mathbb{U}_{1,1}(\mathfrak{u}_n,\mathfrak{s}_n),\mathbb{U}_{2,1}(\mathfrak{u}_n,\mathfrak{s}_n))$ converges to $(\mathbb{U}_{1,1}(\mathfrak{u},\mathfrak{s}),\mathbb{U}_{2,1}(\mathfrak{u},\mathfrak{s}))$ on $[\mathfrak{a}_0,\mathfrak{b}_0]$, which shows that $(\mathbb{U}_{1,2},\mathbb{U}_{2,2})$ is continuous.

Next, we show that the operator $(\mathbb{U}_{1,1},\mathbb{U}_{2,1})$ is uniformly bounded on $\mathbb{S}$. For any $(\mathfrak{u},\mathfrak{s}) \in S$ we have

$$
\begin{aligned}
|\mathbb{U}_{1,1}(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| &\leq |\mathcal{H}_i(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| + \frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma_2-1}}{|\Theta|\rho^{\gamma_2-1}\Gamma(\gamma_2)} \\
&\quad \times\Bigg\{ N_2\Big[|\theta_1||\mathcal{G}_i(\mathfrak{u},\mathfrak{s})(\xi_1)| + |\mathcal{H}_i(\mathfrak{u},\mathfrak{s})(\mathfrak{b}_0)|\Big] \\
&\quad +M_2\Big[|\theta_2||\mathcal{H}_i(\mathfrak{u},\mathfrak{s})(\xi_2))| + |G_j(\mathfrak{u},\mathfrak{s})(\mathfrak{b}_0))|\Big]\Bigg\} \\
&\leq \sum_{i=1}^{n}\|\lambda_i\|\Psi(\mathfrak{b}_0,\eta_i+\delta_2) + \frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_2) \\
&\quad \times\Bigg\{ N_2|\theta_1|\sum_{j=1}^{m}\|\mu_j\|\Psi(\mathfrak{b}_0,\eta_i+\delta_4) + N_2\sum_{i=1}^{n}\|\lambda_i\|\Psi(\mathfrak{b}_0,\eta_i+\delta_2) \\
&\quad +M_2|\theta_2|\sum_{i=1}^{n}\|\lambda_i\|\Psi(\mathfrak{b}_0,\eta_i+\delta_2) + M_2\sum_{j=1}^{m}\|\mu_j\|\Psi(\mathfrak{b}_0,\overline{\eta}_j+\delta_4)\Bigg\} := \Lambda_1.
\end{aligned}
$$

Similarly we can prove that

$$
\begin{aligned}
|\mathbb{U}_{2,1}(\mathfrak{u},\mathfrak{s})(\mathfrak{t})| &\leq \sum_{j=1}^{m}\|\mu_j\|\Psi(\mathfrak{b}_0,\overline{\eta}_i+\delta_4) + \frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_2)\Bigg\{ N_1|\theta_1|\sum_{j=1}^{m}\|\mu_j\|\Psi(\mathfrak{b}_0,\overline{\eta}_i+\delta_4) \\
&\quad +N_1\sum_{i=1}^{n}\|\lambda_i\|\Psi(\mathfrak{b}_0,\eta_i+\delta_2) + M_1|\theta_2|\sum_{i=1}^{n}\|\lambda_i\|\Psi(\mathfrak{b}_0,\eta_i+\delta_2) \\
&\quad +M_1\sum_{j=1}^{m}\|\mu_j\|\Psi(\mathfrak{b}_0,\overline{\eta}_i+\delta_4)\Bigg\} := \Lambda_2.
\end{aligned}
$$

Therefore $\|\mathbb{U}_{1,1}\| + \|\mathbb{U}_{2,1}\| \leq \Lambda_1 + \Lambda_2, (\mathfrak{u},\mathfrak{s}) \in \mathbb{S}$, which shows that the operator $(\mathbb{U}_{1,1},\mathbb{U}_{2,1})$ is uniformly bounded on $\mathbb{S}$. Finally we show that the operator $(\mathbb{U}_{1,1},\mathbb{U}_{2,1})$ is equicontinuous. Let $\tau_1 < \tau_2$ and $(\mathfrak{u},\mathfrak{s}) \in \mathbb{S}$. Then, we have

$$
\begin{aligned}
&|\mathbb{U}_{1,1}(\mathfrak{u},\mathfrak{s})(\tau_2) - \mathbb{U}_{1,1}(\mathfrak{u},\mathfrak{s})(\tau_1)| \\
\leq\ &\Bigg|\sum_{i=1}^{n}\frac{1}{\rho^{\eta_i+\delta_2}\Gamma(\eta_i+\delta_2)}\int_{\mathfrak{a}_0}^{\tau_1}\psi'(s)\Big[(\psi(\tau_2)-\psi(s))^{\eta_i+\delta_2-1} - (\psi(\tau_1)-\psi(s))^{\eta_i+\delta_2-1}\Big] \\
&\times|H_i(s,\mathfrak{u}(s),\mathfrak{s}(s))|ds \\
&+\sum_{i=1}^{n}\frac{1}{\rho^{\eta_i+\delta_2}\Gamma(\eta_i+\delta_2)}\int_{\tau_1}^{\tau_2}\psi'(s)(\psi(\tau_2)-\psi(s))^{\eta_i+\delta_2-1}|H_i(s,\mathfrak{u}(s),\mathfrak{s}(s))|ds\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\left|(\psi(\tau_2)-\psi(\mathfrak{a}_0))^{\gamma_2-1}-(\psi(\tau_1)-\psi(\mathfrak{a}_0))^{\gamma_2-1}\right|}{|\Theta|\rho^{\gamma_2-1}\Gamma(\gamma_2)}\mathbb{W}\\
\leq\ &\sum_{i=1}^{n}\frac{1}{\rho^{\eta_i+\delta_2}\Gamma(\eta_i+\delta_2+1)}\|\lambda_i\|\Big[\left|(\psi(\tau_2)-\psi(\mathfrak{a}_0))^{\eta_i+\delta_2}-(\psi(\tau_1)-\psi(\mathfrak{a}_0))^{\eta_i+\delta_2}\right|\\
&+2(\psi(\tau_2)-\psi(\tau_1))^{\eta_i+\delta_2}\Big]+\frac{\left|(\psi(\tau_2)-\psi(\mathfrak{a}_0))^{\gamma_2-1}-(\psi(\tau_1)-\psi(\mathfrak{a}_0))^{\gamma_2-1}\right|}{|\Theta|\rho^{\gamma_2-1}\Gamma(\gamma_2)}\mathbb{W},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{W}\ =\ &N_2|\theta_1|\sum_{j=1}^{m}\|\mu_j\Psi(\mathfrak{b}_0,\overline{\eta}_j+\delta_4)+N_2\sum_{i=1}^{n}\|\lambda_i\|\Psi(\mathfrak{b}_0,\eta_i+\delta_2)\\
&+M_2|\theta_2|\sum_{i=1}^{n}\|\lambda_i\|\Psi(\mathfrak{b}_0,\eta_i+\delta_2)+M_2\sum_{j=1}^{m}\|\mu_j\|\Psi(\mathfrak{b}_0,\overline{\eta}_j+\delta_4).
\end{aligned}
$$

As $\tau_2-\tau_1\to 0$, the right-hand side of the above inequality tends to zero, independently of $(\mathfrak{u},\mathfrak{s})$. Similarly we have $|\mathbb{U}_{2,1}(\mathfrak{u},\mathfrak{s})(\tau_2)-\mathbb{U}_{2,1}(\mathfrak{u},\mathfrak{s})(\tau_1)|\to 0$ as $\tau_2-\tau_1\to 0$. Thus $(\mathbb{U}_{1,1},\mathbb{U}_{2,1})$ is equicontinuous. Therefore, it follows by Aezelá-Ascoli theorem that $(\mathbb{U}_{1,1},\mathbb{U}_{2,1})$ is a completely continuous operator on S.

Step 3. In the third step we will prove that condition (iii) of Lemma 3 is fulfilled. Let $(\mathfrak{u},\mathfrak{s})\in\mathbb{Y}\times\mathbb{Y}$ be such that, for all $(\mathfrak{u}_1,\mathfrak{u}_2)\in S$

$$
(\mathfrak{u},\mathfrak{s})=(\mathbb{U}_{1,1}(\mathfrak{u},\mathfrak{s}),\mathbb{U}_{2,1}(\mathfrak{u},\mathfrak{s}))+(\mathbb{U}_{1,2}(\mathfrak{u}_1,\mathfrak{s}_1),\mathbb{U}_{2,2}(\mathfrak{u}_1,\mathfrak{s}_1)).
$$

Then, we have

$$
\begin{aligned}
|\mathfrak{u}(\mathfrak{t})|\ \leq\ &|\mathbb{U}_{1,1}(\mathfrak{u},\mathfrak{s})(\mathfrak{t})|+|\mathbb{U}_{1,2}(\mathfrak{u}_1,\mathfrak{s}_1)(\mathfrak{t})|\\
\leq\ &|\mathcal{H}_i(\mathfrak{u},\mathfrak{s})|(\mathfrak{t})+\frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma_2-1}}{|\Theta|\rho^{\gamma_2-1}\Gamma(\gamma_2)}\\
&\times\bigg\{N_2\Big[|\theta_1||\mathcal{G}_i(\mathfrak{u},\mathfrak{s})|(\xi_1)+|\mathcal{H}_i(\mathfrak{u},\mathfrak{s})|(\mathfrak{b}_0)\Big]\\
&+M_2\Big[|\theta_2||\mathcal{H}_i(\mathfrak{u},\mathfrak{s})|(\xi_2))+|G_j(\mathfrak{u},\mathfrak{s})|(\mathfrak{b}_0))\Big]\bigg\}\\
&+{}^{p}I^{\delta_2,\rho,\psi}|\mathcal{F}_1(\mathfrak{u}_1,\mathfrak{s}_1)|(\mathfrak{t})|\mathcal{Y}_1(\mathfrak{u}_1,\mathfrak{s}_1)|(\mathfrak{t})+\frac{e^{\frac{\rho-1}{\rho}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))}(\psi(\mathfrak{t})-\psi(\mathfrak{a}_0))^{\gamma_2-1}}{|\Theta|\rho^{\gamma_2-1}\Gamma(\gamma_2)}\\
&\times\bigg\{N_2\Big[|\theta_1|{}^{p}I^{\delta_4,\rho,\psi}|\mathcal{F}_2(\mathfrak{u}_1,\mathfrak{s}_1)|(\xi_1)|\mathcal{Y}_2(\mathfrak{u}_1,\mathfrak{s}_1)|(\xi_1)\\
&+{}^{p}I^{\delta_2,\rho,\psi}|\mathcal{F}_1(\mathfrak{u}_1,\mathfrak{s}_1)|(\mathfrak{b}_0)|\mathcal{Y}_1(\mathfrak{u}_1,\mathfrak{s}_1)|(\mathfrak{b}_0)\Big]\\
&+M_2\Big[|\theta_2|{}^{p}I^{\delta_2,\rho,\psi}|\mathcal{F}_1(\mathfrak{u}_1,\mathfrak{s}_1)|(\xi_2)|\mathcal{Y}_1(\mathfrak{u}_1,\mathfrak{s}_1)|(\xi_2)\\
&+{}^{p}I^{\delta_4,\rho,\psi}|\mathcal{F}_2(\mathfrak{u}_1,\mathfrak{s}_1)|(\mathfrak{b}_0)|\mathcal{Y}_2(\mathfrak{u}_1,\mathfrak{s}_1)|(\mathfrak{b}_0)\Big]\bigg\}\\
\leq\ &\sum_{i=1}^{n}\|\lambda_i\|\Psi(\mathfrak{b}_0,\eta_i+\delta_2)+\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_2)\Big\{N_2|\theta_1|\sum_{j=1}^{m}\|\mu_j\Psi(\mathfrak{b}_0,\overline{\eta}_j+\delta_4)\\
&+N_2\sum_{i=1}^{n}\|\lambda_i\|\Psi(\mathfrak{b}_0,\eta_i+\delta_2)+M_2|\theta_2|\sum_{i=1}^{n}\|\lambda_i\|\Psi(\mathfrak{b}_0,\eta_i+\delta_2)
\end{aligned}
$$

$$
\begin{aligned}
& +M_2\sum_{j=1}^{m}\|\mu_j\|\Psi(\mathfrak{b}_0,\overline{\eta}_j+\delta_4)\Big\}+\Psi(\mathfrak{b}_0,\delta_2)\|\sigma\|\|\ell\|\Psi(\mathfrak{b}_0,\delta_1)\\
& +\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_2)\Big\{N_2|\theta_1|\Psi(\mathfrak{b}_0,\delta_4)\|\tau\|\|L_2\|\Psi(\mathfrak{b}_0,\delta_3)+N_2\Psi(\mathfrak{b}_0,\delta_2)\|\sigma\|\|\ell\|\Psi(\mathfrak{b}_0,\delta_1)\\
& +M_2|\theta_2|\Psi(\mathfrak{b}_0,\delta_2)\|\sigma\|\|\ell\|\Psi(\mathfrak{b}_0,\delta_1)+M_2\Psi(\mathfrak{b}_0,\delta_4)\|\tau\|\|L_2\|\Psi(\mathfrak{b}_0,\delta_3)\Big\}\\
= & \Big[1+\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_2)(N_2+M_2|\theta_2|)\Big]\Big[\sum_{i=1}^{n}\|\lambda_i\|\Psi(\mathfrak{b}_0,\eta_i+\delta_2)\\
& +\|\sigma\|\|\ell\|\Psi(\mathfrak{b}_0,\delta_1)\Psi(\mathfrak{b}_0,\delta_2)\Big]+(N_2|\theta_1|+M_2)\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_2)\Big[\sum_{j=1}^{m}\|\mu_j\|\Psi(\mathfrak{b}_0,\overline{\eta}_j+\delta_4)\\
& +\Psi(\mathfrak{b}_0,\delta_4)\|\tau\|\|L_2\|\Psi(\mathfrak{b}_0,\delta_3)\Big]=R_1.
\end{aligned}
$$

By a similar way we found

$$
\begin{aligned}
|\mathfrak{s}(\mathfrak{t})| \le & |\mathbb{U}_{2,1}(\mathfrak{u},\mathfrak{s})(\mathfrak{t})|+|\mathbb{U}_{2,2}(\mathfrak{u}_1,\mathfrak{s}_1)(\mathfrak{t})|\\
\le & \Big[1+\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_4)(N_1|\theta_1|+M_1)\Big]\Big[\sum_{j=1}^{m}\|\mu_j\|\Psi(\mathfrak{b}_0,\overline{\eta}_j+\delta_4)\\
& +\|\tau\|\|m\|\Psi(\mathfrak{b}_0,\delta_4)\Psi(\mathfrak{b}_0,\delta_3)\Big]+(N_1|+M_1\theta_2|)\frac{1}{|\Theta|}\Psi(\mathfrak{b}_0,\gamma_4)\Big[\sum_{i=1}^{n}\|\lambda_i\|\Psi(\mathfrak{b}_0,\eta_i+\delta_2)\\
& +\Psi(\mathfrak{b}_0,\delta_2)\|\sigma\|\|L_1\|\Psi(\mathfrak{b}_0,\delta_1)\Big]=R_2.
\end{aligned}
$$

Adding the previous inequalities, we obtain

$$\|\mathfrak{u}\|+\|\mathfrak{s}\|\le R_1+R_2=r.$$

As $\|(\mathfrak{u},\mathfrak{s})\|=\|\mathfrak{u}\|+\|\mathfrak{s}\|$, we have that $\|(\mathfrak{u},\mathfrak{s})\|\le r$ and so condition (iii) of Lemma 3 holds.

By Lemma 3, the ψ-Hilfer proportional coupled system (3) and (4) has at least one solution on $[\mathfrak{a}_0,\mathfrak{b}_0]$. The proof is finished. □

4. An Example

This section demonstrates the application to a given nonlocal coupled system of sequential ψ-Hilfer-type proportional fractional differential equations of the form:

$$
\begin{cases}
{}^{H}D^{\frac{3}{7},\frac{8}{9},\frac{3}{4},t+\sqrt{t}}\left[\dfrac{{}^{H}D^{\frac{11}{7},\frac{7}{9},\frac{3}{4},t+\sqrt{t}}\mathfrak{u}(\mathfrak{t})-\sum_{i=1}^{3}{}^{p}I^{\frac{2i+1}{4},\frac{3}{4},t+\sqrt{t}}H_i(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}{\mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}\right]\\
\qquad=P(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})),\qquad \mathfrak{t}\in\left[\frac{1}{8},\frac{13}{8}\right],\\
{}^{H}D^{\frac{5}{7},\frac{5}{9},\frac{3}{4},t+\sqrt{t}}\left[\dfrac{{}^{H}D^{\frac{9}{7},\frac{4}{9},\frac{3}{4},t+\sqrt{t}}\mathfrak{s}(\mathfrak{t})-\sum_{j=1}^{2}{}^{p}I^{\frac{2(j+1)}{5},\frac{3}{4},t+\sqrt{t}}G_j(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}{\mathfrak{g}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t}))}\right]\\
\qquad=Q(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{s}(\mathfrak{t})),\qquad \mathfrak{t}\in\left[\frac{1}{8},\frac{13}{8}\right],\\
\mathfrak{u}\left(\frac{1}{8}\right)={}^{H}D^{\frac{11}{7},\frac{7}{9},\frac{3}{4},t+\sqrt{t}}\mathfrak{u}\left(\dfrac{1}{8}\right)=0,\ \ \mathfrak{u}\left(\dfrac{13}{8}\right)=\dfrac{3}{11}\mathfrak{s}\left(\dfrac{5}{8}\right),\\
\mathfrak{s}\left(\frac{1}{8}\right)={}^{H}D^{\frac{9}{7},\frac{4}{9},\frac{3}{4},t+\sqrt{t}}\mathfrak{s}\left(\dfrac{1}{8}\right)=0,\ \ \mathfrak{s}\left(\dfrac{13}{8}\right)=\dfrac{4}{11}\mathfrak{u}\left(\dfrac{9}{8}\right),
\end{cases}
\tag{21}
$$

where

$$\mathfrak{f}(\mathfrak{t},\mathfrak{u},\mathfrak{s}) = \frac{1}{(8\mathfrak{t}+3)^2}\left(\frac{|\mathfrak{u}|}{2+|\mathfrak{u}|}\right)+\frac{1}{(8\mathfrak{t}+5)^2}\left(\frac{|\mathfrak{s}|}{1+|\mathfrak{s}|}\right)+\frac{1}{16},$$

$$\begin{aligned}
\mathfrak{g}(\mathfrak{t},\mathfrak{u},\mathfrak{s}) &= \frac{1}{16(\mathfrak{t}+1)}\left(\frac{|\mathfrak{u}|}{3+|\mathfrak{u}|}\right)+\frac{1}{16\mathfrak{t}+15}\left(\frac{|\mathfrak{s}|}{2+|\mathfrak{s}|}\right)+\frac{1}{17}, \\
H_i(\mathfrak{t},\mathfrak{u},\mathfrak{s}) &= \frac{i}{8\mathfrak{t}+i}\tan^{-1}|\mathfrak{u}|+\frac{i}{8\mathfrak{t}+2i}\sin|\mathfrak{s}|+\frac{i}{2}, \\
G_j(\mathfrak{t},\mathfrak{u},\mathfrak{s}) &= \frac{1}{8\mathfrak{t}+3j}\frac{|\mathfrak{u}|}{(j+|\mathfrak{u}|)}+\frac{1}{8\mathfrak{t}+4j}\frac{|\mathfrak{s}|}{(j+|\mathfrak{s}|)}+\frac{j}{3}, \\
P(\mathfrak{t},\mathfrak{u},\mathfrak{s}) &= \frac{1}{8(4\mathfrak{t}+3)}\left(\frac{|\mathfrak{u}|}{1+|\mathfrak{u}|}\right)+\frac{1}{4(8\mathfrak{t}+2\pi)}\tan^{-1}|\mathfrak{s}|+\frac{1}{18}, \\
Q(\mathfrak{t},\mathfrak{u},\mathfrak{s}) &= \frac{1}{2(8\mathfrak{t}+15)}\sin|\mathfrak{u}|+\frac{1}{(8\mathfrak{t}+14)}\left(\frac{|\mathfrak{s}|}{2+|\mathfrak{s}|}\right)+\frac{1}{25}.
\end{aligned}$$

In the given system (21), $\delta_1 = 3/7$, $\delta_2 = 11/7$, $\delta_3 = 5/7$, $\delta_4 = 9/7$, $\vartheta_1 = 8/9$, $\vartheta_2 = 7/9$, $\vartheta_3 = 5/9$, $\vartheta_4 = 4/9$, $\rho = 3/4$, $\psi(t) = t + \sqrt{t}$, $\mathfrak{a}_0 = 1/8$, $\mathfrak{b}_0 = 13/8$, $\theta_1 = 3/11$, $\theta_2 = 4/11$, $\xi_1 = 5/8$, $\xi_2 = 9/8$, $\eta_i = (2i+1)/4$, $\overline{\eta}_j = 2(j+1)/5$, $i = 1,2,3$, $j+1,2$. These settings lead to compute constants as $\gamma_1 = 59/63$, $\gamma_2 = 120/63$, $\gamma_3 = 55/63$, $\gamma_4 = 101/63$, $M_1 \approx 1.337156409$, $M_2 \approx 0.2553420381$, $N_1 \approx 0.4496860114$, $N_2 \approx 1.012080564$, $\Theta \approx 1.238486270$. In addition, some terms in assumption (H_3) can be computed as $\Psi(\mathfrak{b}_0,\delta_1) \approx 1.864913369$, $\Psi(\mathfrak{b}_0,\delta_2) \approx 4.506335230$, $\Psi(\mathfrak{b}_0,\delta_3) \approx 2.534134245$, $\Psi(\mathfrak{b}_0,\delta_4) \approx 3.900538887$, $\Psi(\mathfrak{b}_0,\gamma_2) \approx 5.079441906$, $\Psi(\mathfrak{b}_0,\gamma_4) \approx 4.567817507$. For the functions $\mathfrak{f}$ and $\mathfrak{g}$ we have

$$|\mathfrak{f}(\mathfrak{t},\mathfrak{u},\mathfrak{s}) - \mathfrak{f}(\mathfrak{t},\hat{\mathfrak{u}},\hat{\mathfrak{s}})| \leq \frac{1}{2(8\mathfrak{t}+3)^2}(\|\mathfrak{u}-\hat{\mathfrak{u}}\| + \|\mathfrak{s}-\hat{\mathfrak{s}}\|),$$

and

$$|\mathfrak{g}(\mathfrak{t},\mathfrak{u},\mathfrak{s}) - \mathfrak{g}(\mathfrak{t},\hat{\mathfrak{u}},\hat{\mathfrak{s}})| \leq \frac{1}{2(16\mathfrak{t}+15)}(\|\mathfrak{u}-\hat{\mathfrak{u}}\| + \|\mathfrak{s}-\hat{\mathfrak{s}}\|),$$

and thus $\phi(t) = 1/(2(8\mathfrak{t}+3)^2)$ and $\chi(t) = 1/(2(16\mathfrak{t}+15))$. Then we receive $\|\phi\| = 1/32$ and $\|\chi\| = 1/34$. The bound of these two functions can be shown that

$$|\mathfrak{f}(\mathfrak{t},\mathfrak{u},\mathfrak{s})| \leq \frac{1}{(8\mathfrak{t}+3)^2} + \frac{1}{16} \quad \text{and} \quad |\mathfrak{f}(\mathfrak{t},\mathfrak{u},\mathfrak{s})| \leq \frac{1}{(16\mathfrak{t}+15)} + \frac{1}{17}. \tag{22}$$

Then we obtain $\|\sigma\| = 1/8$ and $\|\tau\| = 2/17$ by choosing $\sigma(\mathfrak{t}) = 1/(8\mathfrak{t}+3)^2 + (1/16)$ and $\tau(\mathfrak{t}) = 1/(16\mathfrak{t}+15) + (1/17)$.

For the two nonlinear functions H_i and G_j we obtain

$$|H_i(\mathfrak{t},\mathfrak{u},\mathfrak{s})| \leq \frac{1}{8\mathfrak{t}+i}\left(\frac{\pi}{2}+1\right) + \frac{i}{2} := \lambda_i(\mathfrak{t}) \text{ and } |G_j(\mathfrak{t},\mathfrak{u},\mathfrak{s})| \leq \frac{2}{8\mathfrak{t}+3j} + \frac{j}{3} := \mu_j(\mathfrak{t}).$$

Both of them satisfy the Lipschitz condition as

$$|H_i(\mathfrak{t},\mathfrak{u},\mathfrak{s}) - H_i(\mathfrak{t},\hat{\mathfrak{u}},\hat{\mathfrak{s}})| \leq \frac{2i}{8\mathfrak{t}+i}(\|\mathfrak{u}-\hat{\mathfrak{u}}\| + \|\mathfrak{s}-\hat{\mathfrak{s}}\|),$$

and

$$|G_j(\mathfrak{t},\mathfrak{u},\mathfrak{s}) - G_j(\mathfrak{t},\hat{\mathfrak{u}},\hat{\mathfrak{s}})| \leq \frac{2}{j(8\mathfrak{t}+3j)}(\|\mathfrak{u}-\hat{\mathfrak{u}}\| + \|\mathfrak{s}-\hat{\mathfrak{s}}\|),$$

by setting $h_i(\mathfrak{t}) = 2i/(8\mathfrak{t}+i)$ and $z_j(\mathfrak{t}) = 2/(j(8\mathfrak{t}+3j))$. Finally for the functions appeared in right hand-sides of nonlinear functions in (21), we see that

$$|P(\mathfrak{t},\mathfrak{u},\mathfrak{s}) - P(\mathfrak{t},\hat{\mathfrak{u}},\hat{\mathfrak{s}})| \leq \frac{1}{8(4\mathfrak{t}+3)}(\|\mathfrak{u}-\hat{\mathfrak{u}}\| + \|\mathfrak{s}-\hat{\mathfrak{s}}\|),$$

and

$$|Q(\mathfrak{t},\mathfrak{u},\mathfrak{s})-Q(\mathfrak{t},\hat{\mathfrak{u}},\hat{\mathfrak{s}})| \leq \frac{1}{2(8\mathfrak{t}+14)}(\|\mathfrak{u}-\hat{\mathfrak{u}}\|+\|\mathfrak{s}-\hat{\mathfrak{s}}\|),$$

with bounds

$$|P(\mathfrak{t},\mathfrak{u},\mathfrak{s})| \leq \frac{1}{8(4\mathfrak{t}+3)}+\frac{\pi}{8(8\mathfrak{t}+2\pi)}+\frac{1}{18}:=\ell(\mathfrak{t})$$

and

$$|Q(\mathfrak{t},\mathfrak{u},\mathfrak{s})| \leq \frac{1}{2(8\mathfrak{t}+15)}+\frac{1}{8\mathfrak{t}+14}+\frac{1}{25}:=m(\mathfrak{t}).$$

Therefore, we set $\pi(t)=1/(8(4\mathfrak{t}+3))$, $\omega(\mathfrak{t})=1/(2(8\mathfrak{t}+14))$ and then we have $\|\pi\|=1/28$, $\|\omega\|=1/30$, $\|\ell\|=(1/28)+(\pi)/(8(1+2\pi))+(1/18)$ and $\|m\|=(1/32)+(1/15)+(1/25)$. These information leads to compute the constant K in assumption (H_3) by

$$K \approx 0.9976072624 < 1.$$

Hence the nonlocal sequential Hilfer-type coupled system of nonlinear proportional fractional differential quations (21) satisfies all conditions in Theorem 1, and thus it has at least one solution $(\mathfrak{u},\mathfrak{s})$ on $[1/8, 13/8]$.

5. Conclusions

In this research, we have presented the existence result for a new class of coupled systems of sequential ψ-Hilfer proportional fractional differential equations with nonlocal boundary conditions. The main existence result was proved via a Burton's generalization of Krasnosel'skiĭ's fixed point theorem. The main result was illustrated by a numerical constructed example. Our results are new and enrich the existing literature on coupled systems of ψ-Hilfer proportional fractional differential equations. For special values of the parameters involved in the system at hand, we cover many new problems. Thus, by taking $\psi(\mathfrak{t})=\mathfrak{t}$, our problem reduces to a coupled system of Hilfer generalized proportional fractional differential equations with boundary conditions, while if $\rho=1$, reduces to a coupled system of ψ-Hilfer fractional differential equations. In future work, we can implement these techniques on different boundary value problems equipped with complicated integral multi-point boundary conditions.

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