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Functional Solutions of Stochastic Differential Equations

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Abstract: We present an integration condition ensuring that a stochastic differential equation $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t$, where μ and σ are sufficiently regular, has a solution of the form $X_t = Z(t, B_t)$. By generalizing the integration condition we obtain a class of stochastic differential equations that again have a functional solution, now of the form $X_t = Z(t, Y_t)$, with Y_t an Ito process. These integration conditions, which seem to be new, provide an a priori test for the existence of functional solutions. Then path-independence holds for the trajectories of the process. By Green's Theorem, it holds also when integrating along any piece-wise differentiable path in the plane. To determine Z at any point (t, x), we may start at the initial condition and follow a path that is first horizontal and then vertical. Then the value of Z can be determined by successively solving two ordinary differential equations. Due to a Lipschitz condition, this value is unique. The differential equations relate to an earlier path-dependent approach by H. Doss, which enables the expression of a stochastic integral in terms of a differential process.

Keywords: stochastic differential equations; Ito's Lemma; systems of partial differential equations; path-independence

MSC: 35A05; 35F20; 60H10

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1. Introduction

Consider a stochastic differential equation

$$\begin{cases} dX_t &= \mu(t, X_t)dt + \sigma(t, X_t)dB_t \qquad 0 \leq t < T \\ X_0 &= x_0 \end{cases}, \tag{1}$$

where B_t is Brownian Motion, μ has continuous first-order partial derivatives, and $\sigma \neq 0$ has continuous partial derivatives of the first order in time and the second order in space. The random variables $X_t : \Omega \to \mathbb{R}$ should be defined on a sufficiently rich probability space Ω, where the initial condition $X_0 = x_0$ could be itself a random variable, or simply a constant. We give a condition, involving partial derivatives of μ and σ (Condition (2) below) for the existence of a functional solution of the form $X_t = Z(t, B_t)$, or more generally of the form $X_t = Z(t, Y_t)$, where Y_t is an Ito process. Then such a solution may be found by solving ordinary differential equations. The main results are stated in Theorems 3 and 4 of Section 3.

In case $X_t = Z(t, B_t)$, where Z has continuous partial derivatives up to the second order, the necessity of the condition follows from the equality of mixed second-order derivatives of Z. Indeed, Ito's Lemma generates a system of two first-order partial differential equations for Z in terms of μ and σ, and then it follows from the equality of mixed derivatives that they are related by

$$\Gamma \equiv \Gamma(t, X) \equiv \sigma \frac{\partial \mu}{\partial X} - \left(\frac{\partial \sigma}{\partial t} + \mu \frac{\partial \sigma}{\partial X} + \frac{\sigma^2}{2} \frac{\partial^2 \sigma}{\partial X^2} \right) = 0. \tag{2}$$

We show that (2) is also an *integration condition*, i.e., it is a sufficient condition for the existence of a solution Z of (1), which is a function only of time t and of the value taken by a trajectory of Brownian motion at time t.

If (2) does not hold, the stochastic differential Equation (1) still may have a functional solution, now of the form $Z(t, Y_t)$, where Y_t is an Ito process. It is given by a stochastic integral

$$Y_t = y_0 + \int_0^t F(s)dt + \int_0^t G(s)dB_s, \tag{3}$$

with $y_0 \in \mathbb{R}$, where F and G are of class C^1 in time and G is supposed to be non-zero. In fact, this is the case if $\Gamma(t, X)/\sigma(t, X)$ only depends on t. Then we may take the *integrating factor* G to be equal to

$$G(t) = g_0 \exp\left(-\int_0^t \frac{\Gamma}{\sigma}(s)ds\right),$$

with $g_0 \neq 0$. The integrating factor does not depend on the drift F appearing in (3); hence, (1) has a family of functional solutions. If we choose $F = 0$, the solution is a function of a martingale. However, Section 5.2 contains an example for which it is convenient to choose $F \neq 0$.

Our approach is a sort of converse to Ito's Lemma. Given an Ito process Y_t, Ito's Lemma derives a stochastic differential equation for a process of the form $Z(t, Y_t)$, such that its trend μ and conditional standard deviation σ are defined by two equalities in terms of partial derivatives of the function Z. Here, we consider μ and σ as given, and the two equalities are now seen as a system of partial differential equations. In case an integration condition holds, we look for a functional solution Z of time and an appropriate Ito process. In a sense, this means that within the class of stochastic differential equations, the applicability of Ito's Lemma is not completely generic, as the existence of such a function Z is subject to the validity of an integration condition.

Still, the method sketched above enables a general approach to solve a rather comprehensive class of stochastic differential equations, starting with a test in terms of partial derivatives of its coefficients for the existence of such a functional solution. To determine the functional solution $Z(t, Y_t)$ at a value $Y_t = y$, one may integrate along any path $\{(s, \lambda(s)) | 0 \leq s \leq t\}$, going from $(0, y_0)$ to (t, y) such that $\lambda : [0, t] \to \mathbb{R}$ is (piece-wise) continuously differentiable; this property of path independence follows from Green's Theorem. In particular, we may integrate first along a horizontal path, and then a vertical path. The theorems of Section 3 show that by this method, the value of Z can be determined by successively solving two ordinary differential equations in a rather simple form.

Once one disposes of a functional expression $Z(t, Y_t)$ of a process, it may be easier to determine important properties of the random variables of the process.

To start with, the process has the property of path-independence. This means that if ω_1 and $\omega_2 \in \Omega$ are such that at some time t it holds that $Y_t(\omega_1) = Y_t(\omega_2) \equiv y$, the value taken by $Z(t, y)$ solely depends on the value of y, and not on the history of the trajectories at time t; i.e., it again holds that $Z_t(\omega_1) = Z_t(\omega_2)$.

Secondly, this property of path-independence may simplify the determination of important properties of the random variables of the process. Indeed, its probability law will be a function of the Gaussian distribution. So integrals of Gaussian type may be used to determine expectations [1], moments [2], or conditional variance and volatility [3].

Thirdly, dynamical properties of the trajectories of a process $X_t = Z(t, Y_t)$ could be more easily studied, like local or asymptotic stability of the solutions. This may follow from the study of partial derivatives [4], or even by studying the function $Z(t, Y_t)$ directly. The Geometrical Brownian Motion $S(t, B_t) = \exp((\mu - \sigma^2)t + \sigma B_t)$, with $\sigma > 0$, is a striking example. Note that, by the Law of the Iterated Logarithm [1], the growth of nearly every trajectory of Brownian motion is at most of order $\sqrt{2t \log \log t}$. Hence, even in the case that $\mu = 0$, almost surely $S(t, B_t)$ is of exponential decay, i.e., a martingale with mean 1 can have a concentration of trajectories exponentially close to 0.

Fourthly, a functional representation has numerical relevance. Indeed, a numerical simulation of the Brownian motion involves a discretization in time and a numerical representation of the Gaussian distribution, such as the Monte-Carlo method [5]. This may be adapted in a straightforward way to a process that is a deterministic function of Brownian motion, and possibly of time. So, *mutatis mutandis*, a numerical simulation could again use the properties of the common Gaussian distribution.

Of course most integrals along a path, or the ordinary differential equations figuring in the Main Theorems, do not have solutions in closed form, and have to be approximated by numerical discretizations. Still, there exist many approaches, like Euler-methods, trapezium rules or Runge–Kutta methods [6]. In these deterministic cases, such methods are susceptible to be more effective than corresponding numerical approaches to stochastic differential equations, in particular the Euler–Maruyama method, as errors originating from the chosen discretization method may intervene with the effects of randomizing [7]. The above observations on the numerical relevance of functional solutions generally hold true for processes that are functions of time and an Ito process, though additional complications may arise from numerical approximations of such a process.

The integration Condition (2) has been derived earlier in first approximation for stochastic difference equations with infinitesimal steps, with the use of Taylor-expansions [8]. Under this condition it was shown that a solution is nearly equivalent, in the sense of [9], to a stochastic process that is a function of time and the discrete Wiener Walk. In the context of stochastic differential equations, to our knowledge, the integration Condition (2) corresponding to functional solutions in terms of Brownian Motion, and the integrating factor leading to functional solutions in terms of Ito processes are new.

The search for functional solutions may be compared to two other methods of solution of certain classes of stochastic differential equations.

In [10], H. Doss shows that, under some conditions of regularity for the coefficients, a solution of an autonomous stochastic differential equation can be given in the form of a function $h(D_t, M_t)$ of a solution of a stochastic process $D_t(\omega)$, where ω ranges over some probability space Ω, with trajectories that are differentiable in the ordinary sense, and a continuous semi-martingale M_t; in particular he considers the case of Brownian Motion, i.e., solutions of the form $h(D_t, B_t)$. The results are extended to non-autonomous equations. By leveraging the differentiability, he improves some properties of approximation of processes derived earlier in, among other works, [11]. The stochastic variables $D_t(\omega)$ satisfy ordinary differential equations, and the function h satisfies a partial differential equation, which is similar to the differential equations along the horizontal and vertical paths used to determine the functional solutions mentioned above. Similar to in our approach, they are derived from Ito's Lemma. In fact, our approach may be recognized as a special case, where the process D_t no longer depends on the random variable ω, i.e., has become a deterministic function, with the property that $D_t(\omega) = t$ holds uniformly. Consequently, the integration Condition (2) may be derived as well.

Our method of searching for a functional solution is essentially relevant for some classes of non-linear stochastic differential equations. There exists a general method for the solution of linear stochastic differential equations; see, e.g., [2]. As with linear ordinary differential equations, it is based on finding first a solution for the associated homogeneous equation, which is then used to find a solution for the inhomogeneous equation. The principal tool is Ito's product Theorem. In Section 5.2, we will see that, generally speaking, the solution is not functional.

Introductions to stochastic differential equations are found, for example, in [1,2,12]; in particular, ref. [2] discusses many special classes of stochastic differential equations that are explicitly solvable. For introductions to systems of first-order partial differential equations, we refer to [13,14]. Integration along paths in higher dimension is treated, for instance, in [15].

Section 2 recalls Ito's Lemma, some essential properties of systems of partial differential equations and Green's Theorem on path-independence. Section 3 presents the main results.

Section 4 is devoted to the proofs of the main theorems. Section 5 contains some special cases and examples. In Section 5.1, we consider the autonomous case, and in Section 5.2, a class of linear stochastic differential equations. We treat the Geometric Brownian motion as a special case of autonomous equations and the Ornstein–Uhlenbeck process as a special case of linear equations. Section 5.3 presents two non-linear stochastic differential equations that have a functional solution. Section 6 shows briefly how a stochastic process may be a function of a particular path-dependent differentiable process. In Section 7, we resume the principal results, and suggest some possible applications.

2. Background on Ito's Lemma and Systems of Partial Differential Equations

To prove the main theorems, Ito's Lemma is needed, as is a well-known condition [13,14] on the existence of a common solution to a system of two partial differential equations. The latter is stated in Proposition 1. We recall also Ito's Lemma, where we single out the case of processes depending on time and Brownian motion.

A system of partial differentiable equations is *compatible* if they have a common solution. The following proposition states a condition for compatibility for a system of two first-order partial differential equations; the condition is also called an *integration condition*.

Proposition 1. *Let $t_0, x_0, z_0 \in \mathbb{R}$. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be of class C^{111} and be uniformly Lipschitz in the third variable. Consider the system of two first-order partial differential equations*

$$\begin{cases} \frac{\partial Z}{\partial t} = f(t,x,Z) \\ \frac{\partial Z}{\partial x} = g(t,x,Z) \\ Z(t_0,x_0) = z_0 \end{cases}. \tag{4}$$

1. *The system (4) has a unique solution $Z : \mathbb{R}^2 \to \mathbb{R}$ of class C^{22} if and only if*

$$\frac{\partial f}{\partial x} + g\frac{\partial f}{\partial Z} - \frac{\partial g}{\partial t} - f\frac{\partial g}{\partial Z} = 0 \tag{5}$$

   holds for all $t, x \in \mathbb{R}$.
2. *Assume (5) holds for all $t, x \in \mathbb{R}$. Let $t_1, x_1, b \in \mathbb{R}$ with $b \geq 0$, and $\varphi : [0,b] \to \mathbb{R}^2$ be a piecewise continuously differentiable simple curve such that $\varphi(0) = (t_0, x_0), \varphi(b) = (t_1, x_1)$. Then*

$$Z(t_1,x_1) = Z(t_0,x_0) + \int_\varphi \left(\frac{\partial Z}{\partial t}dt + \frac{\partial Z}{\partial x}dx\right).$$

   In particular, for $(t_0, x_0) = (0, 0)$ and $(t_1, x_1) \in \mathbb{R}^2$ with $t_1 > 0$ one has

$$Z(t_1,x_1) = Z^{(v)}(x_1),$$

   which is obtained by solving successively the ordinary differential equations

$$\frac{dZ^{(h)}}{dt} = f\left(t, 0, Z^{(h)}\right) \qquad Z^{(h)}(0) = z_0, \tag{6}$$

   along the horizontal path from $(0,0)$ to $(t_1, 0)$, and

$$\frac{dZ^{(v)}}{dx} = f\left(t_1, x, Z^{(v)}\right) \qquad Z^{(v)}(0) = Z^{(h)}(t_1), \tag{7}$$

   along the vertical path from $(t_1, 0)$ to (t_1, x_1).

The necessity of the integration Condition (5) of Part 1 follows easily from the equality of second-order mixed partial derivatives

$$\frac{\partial^2 Z(t,x)}{\partial x \partial t} = \frac{\partial^2 Z(t,x)}{\partial t \partial x}. \tag{8}$$

Together with the regularity conditions imposed on f and g, the integration condition is also sufficient for the existence and uniqueness of a solution for the system (4), noting that the equations figuring in (4) may be seen as parameterized ordinary differential equations.

Part 2 states that the value of the solution at a particular point (t_1, x_1) may be calculated along any sufficiently regular curve leading from the initial condition (t_0, x_0) to this point; this is a consequence of (8) and Green's Theorem [15]. Indeed, with the usual notations for an integral along a piece-wise continuously differentiable simple closed path φ, and the double integral over a domain $\Delta \subset \mathbb{R}^2$ delimited by the path, we have

$$\oint_{\varphi}\left(\frac{\partial Z}{\partial t}dt + \frac{\partial Z}{\partial x}dx\right) = \iint_{\Delta}\left(\frac{\partial^2 Z(t,x)}{\partial x \partial t} - \frac{\partial^2 Z(t,x)}{\partial t \partial x}\right)dtdx = \iint_{\Delta} 0dtdx = 0.$$

In the special case given by (6) and (7), the integrals correspond to ordinary differential equations. In fact, the equations figuring in (4) may be seen as parameterized ordinary differential equations, first along a horizontal path, and then along a vertical path.

We now recall Ito's Lemma, first for processes depending on Brownian motion.

Theorem 1 (Ito's Lemma, processes of the form $Z(t, B_t)$)**.** *Let $Z : [0, T] \times \mathbb{R} \to \mathbb{R}$ be of class C^{12}. The stochastic process $Z(t, B_t)$ satisfies the stochastic differential equation*

$$\begin{cases} dZ_t &= \left(\frac{\partial Z}{\partial t} + \frac{1}{2}\frac{\partial^2 Z}{\partial x^2}\right)dt + \frac{\partial Z}{\partial x}dB_t \qquad 0 \le t \le T \\ Z_0 &= Z(0, x_0) \end{cases}. \tag{9}$$

Theorem 2 (Ito's Lemma, processes of the form $Z(t, Y_t)$)**.** *Let Y_t be an Ito process of the form (3), with initial condition $y_0 \in \mathbb{R}$. Let $Z : [0, T] \times \mathbb{R} \to \mathbb{R}$ be of class C^{12}. The stochastic process $Z(t, Y_t)$ satisfies the stochastic differential equation*

$$\begin{cases} dZ_t &= \left(\frac{\partial Z}{\partial t} + F\frac{\partial Z}{\partial x} + \frac{1}{2}G^2\frac{\partial^2 Z}{\partial x^2}\right)dt + G\frac{\partial Z}{\partial x}dB_t \qquad 0 \le t \le T \\ Z_0 &= Z(0, y_0) \end{cases}.$$

3. Main Theorems: Solutions by Functionals of Brownian Motion and of Ito Processes

Let $T > 0$. We will always work within an appropriate probability space $(\Omega, \mathcal{F}, P)$, where Ω is a sufficiently rich set, P is a probability and $\mathcal{F} = (\mathcal{F}_t)_{t \in [0,T]}$ is the natural filtration associated to a Standard Brownian Motion B_t on $[0, T]$. For $\omega \in \Omega$ and $x_0 \in \mathbb{R}$, we use the notation of stochastic differential Equation (1) for the stochastic integral

$$X(T, \omega) = x_0 + \int_0^T \mu(t, X_t(\omega))dt + \int_0^T \sigma(t, X_t(\omega))dB_t(\omega).$$

We recall that a stochastic process X_t is an *Ito process* with respect to B_t if it is of the form

$$X(T, \omega) = x_0(\omega) + \int_0^T F(t, \omega)dt + \int_0^T G(t, \omega)dB_t(\omega), \tag{10}$$

where x_0 is $\mathcal{F}_0$-measurable, F and G are at any time t adapted to $\mathcal{F}_t$, and $\int_0^T |F(t, \omega)|dt$ and $\int_0^T |G(t, \omega)|^2 dt$ exist almost surely.

For $i, j \in \mathbb{N}$, a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is said to be *of class C^{ij}* if all partial derivatives exist and are continuous up to order i in the first variable, and j in the second variable. For $i, j, k \in \mathbb{N}$, functions $\varphi : \mathbb{R}^3 \to \mathbb{R}$ of class C^{ijk} are defined by analogy, also some indices may take value at infinity.

Theorem 3 (Functional solution in terms of Brownian Motion)**.** *Let $T > 0$ and $x_0 \in \mathbb{R}$. Let $\mu : [0, T] \times \mathbb{R} \to \mathbb{R}$ be of class C^{11}, and $\sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ be of class C^{12}, both uniformly*

Lipschitz in the second variable, and with $\sigma \neq 0$, $\frac{\partial \sigma}{\partial X}$ bounded. Consider the stochastic differential Equation (1).

1. *If the integration condition (2) holds, there exists a unique function $Z : [0,T] \times \mathbb{R} \to \mathbb{R}$ of class C^{23} such that $Z(t, B_t)$ satisfies (1). Moreover, for all $(\bar{t}, \bar{x}) \in [0,T] \times \mathbb{R}$ the value $Z(\bar{t}, \bar{x})$ may be determined by solving successively the ordinary differential equations*

$$\begin{cases} \frac{dZ^{(h)}}{dt} &= \mu\left(t, Z^{(h)}\right) - \frac{1}{2}\sigma\left(t, Z^{(h)}\right)\frac{\partial\sigma\left(t, Z^{(h)}\right)}{\partial Z} \\ Z^{(h)}(0) &= x_0 \end{cases} \tag{11}$$

and

$$\begin{cases} \frac{dZ^{(v)}}{dx} &= \sigma\left(\bar{t}Z^{(v)}(x)\right) \\ Z^{(v)}(0) &= Z^{(h)}(\bar{t}) \end{cases}, \tag{12}$$

with $Z(\bar{t}, \bar{x}) = \widetilde{Z}^{(v)}(\bar{x})$.

2. *If (1) has a solution of the form $X_t = Z(t, B_t)$, where $Z : [0,T] \times \mathbb{R} \to \mathbb{R}$ is of class C^{23}, the integration condition (2) holds for all (t, X) such that $0 < t < T, X \in \mathrm{Im}(Z(t,.))$.*

The integration Condition (2) may be written in the form

$$\Gamma = \sigma\frac{\partial\mu}{\partial X} - D\sigma = 0, \tag{13}$$

with

$$D = \left(\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial X} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial X^2}\right)$$

the operator of Dynkin. As we will see, it is a consequence of the equality

$$\frac{\partial^2 Z(t,x)}{\partial t \partial x} = \frac{\partial^2 Z(x,t)}{\partial x \partial t},$$

where $\frac{\partial Z}{\partial t}$ and $\frac{\partial Z}{\partial x}$ are derived from Ito's formula; in fact they are given by (11) and (12), when written in the form of partial differential equations.

If Γ/σ depends only on t and is always non-zero, the stochastic differential equation still has a functional solution, now of the form $Z(t, Y_t)$, where Y_t is an Ito process given by (10). In fact, G has the role of an "integrating factor", and can be given in closed form, as stated in Theorem 4.

Theorem 4 (Functional solution in terms of Ito processes)**.** *Let $T > 0$ and $x_0 \in \mathbb{R}$. Let $\mu : [0,T] \times \mathbb{R} \to \mathbb{R}$ be of class C^{11}, and $\sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$ be of class C^{12}, both uniformly Lipschitz in the second variable, and with $\sigma \neq 0$, $\frac{\partial \sigma}{\partial X}$ bounded. Let $\Gamma(t, X)$ be given by (2).*

1. *Assume Γ/σ depends only on t on $[0,T]$. Let Y_t be an Ito process given by (3), with*

$$G(t) = g_0 \exp\left(-\int_0^t \frac{\Gamma}{\sigma}(s)ds\right) \qquad g_0 \neq 0, \tag{14}$$

where $y_0 \in \mathbb{R}$ and $F : [0,T] \to \mathbb{R}$ is any function of class C^1. Then there exists a unique function $Z : [0,T] \times \mathbb{R} \to \mathbb{R}$ of class C^{23} such that $Z(t, Y_t)$ satisfies (1). Moreover, for all $(\bar{t}, \bar{x}) \in [0,T] \times \mathbb{R}$, the value $Z(\bar{t}, \bar{x})$ may be determined by solving successively the ordinary differential equations

$$\begin{cases} \frac{dZ^{(h)}}{dt} &= \mu\left(t, Z^{(h)}\right) - \sigma\left(t, Z^{(h)}\right)\left(\frac{1}{2}\frac{\partial\sigma\left(t, Z^{(h)}\right)}{\partial Z^{(h)}} + \frac{F(t)}{G(t)}\right) \\ Z^{(h)}(0) &= x_0 \end{cases} \tag{15}$$

and

$$\left\{ \begin{array}{lcl} \frac{dZ^{(v)}}{dx} & = & \frac{\sigma\left(\bar{t}Z^{(v)}(x)\right)}{G(\bar{t})} \\ Z^{(v)}(0) & = & Z^{(h)}(\bar{t}) \end{array} \right. , \tag{16}$$

with $Z(\bar{t},\bar{x}) = Z^{(v)}(\bar{x})$.

2. *Conversely, if* (1) *has a solution of the form* $X_t = Z(t, Y_t)$, *where* $Z : [0,T] \times \mathbb{R} \to \mathbb{R}$ *is of class* C^{23} *and* Y_t *is an Ito process of the form* (3) *such that* $F, G : [0,T] \to \mathbb{R}$ *are of class* C^1, *it holds that* Γ/σ *depends only on* t *for all* (t,X) *such that* $0 < t < T$, $X \in \mathrm{Im}(Z(t,.))$, *where* G *is of the form* (14).

The choice of F is free, and, of course, one may choose $F \equiv 0$. However, it may be convenient to choose $F \neq 0$, as will be seen in Section 5.2.

4. Proofs of the Main Theorems

Ito's Lemma shows that a solution Z of the stochastic differential Equation (1) should satisfy a system of partial differential equations. In particular, if a functional solution is of the form $Z(t, B_t)$, noting that μ, σ, F, and G all are supposed to be at least continuously differentiable, by Theorem 1, the function Z should satisfy

$$\left\{ \begin{array}{lcl} \frac{\partial Z}{\partial t} + \frac{1}{2}\frac{\partial^2 Z}{\partial x^2} & = & \mu(t, Z(t,x)) \\ \frac{\partial Z}{\partial x} & = & \sigma(t, Z(t,x)) \\ Z(0,0) & = & x_0 \end{array} \right. , \tag{17}$$

and in the general case of solutions of the form $Z(t, Y_t)$, with Y_t given by (3), according to Theorem 2, the function Z should satisfy the system of partial differential equations

$$\left\{ \begin{array}{lcl} \frac{\partial Z}{\partial t} + F\frac{\partial Z}{\partial x} + \frac{1}{2}G^2\frac{\partial^2 Z}{\partial x^2} & = & \mu(t, Z(t,x)) \\ G\frac{\partial Z}{\partial x} & = & \sigma(t, Z(t,x)) \\ Z(0,0) & = & x_0 \end{array} \right. . \tag{18}$$

With respect to system (17), the system (18) presents some additional complications. This motivates a separate proof of Theorem 3.

Proof of Theorem 3. Let $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\left\{ \begin{array}{lcl} f(t,x,Z) & = & \mu(t,Z) - \frac{1}{2}\sigma(t,Z)\frac{\partial \sigma(t,Z)}{\partial Z} \\ g(t,x,Z) & = & \sigma(t,Z) \end{array} \right. . \tag{19}$$

Then f is of class $C^{1\infty 1}$ and g is of class $C^{1\infty 2}$ and both are uniformly Lipschitz in the third variable.

1. Assume (2) holds. Then

$$\begin{array}{lcl} \frac{\partial f}{\partial x} + g\frac{\partial f}{\partial Z} - \frac{\partial g}{\partial t} - f\frac{\partial g}{\partial Z} & = & \sigma\frac{\partial\left(\mu - \frac{1}{2}\sigma\frac{\partial \sigma}{\partial Z}\right)}{\partial Z} - \frac{\partial \sigma}{\partial t} - \left(\mu - \frac{1}{2}\sigma\frac{\partial \sigma}{\partial Z}\right)\frac{\partial \sigma}{\partial Z} \\ & = & \sigma\frac{\partial \mu}{\partial Z} - \frac{\partial \sigma}{\partial t} - \mu\frac{\partial \sigma}{\partial Z} - \frac{\sigma^2}{2}\frac{\partial^2 \sigma}{\partial Z^2} \\ & = & \Gamma = 0. \end{array} \tag{20}$$

Hence, according to Proposition 1(1), the system of two partial differential equations

$$\left\{ \begin{array}{lcl} \frac{\partial Z}{\partial t} & = & f(t,x,Z) \\ \frac{\partial Z}{\partial x} & = & g(t,x,Z) \\ Z(0,0) & = & x_0 \end{array} \right. . \tag{21}$$

has a solution $Z : [0,T] \times \mathbb{R} \to \mathbb{R}$, which is at least of class C^{22}. Note that

$$\frac{\partial^2 Z}{\partial x^2} = \frac{\partial \sigma(t, Z(t,x))}{\partial x} = \frac{\partial \sigma(t, Z(t,x))}{\partial Z}\frac{\partial Z}{\partial x} = \sigma(t, Z(t,x))\frac{\partial \sigma(t, Z(t,x))}{\partial Z}. \tag{22}$$

It follows that the function Z is at least of class C^{23}. By substituting (21) and (22) into (19), we see that Z satisfies the system (17). Then Ito's Lemma in the form of Theorem 1 implies that the process $Z(t, B_t)$ solves the stochastic differential Equation (1). By Proposition 1(2), at any point $(\bar{t}, \bar{x})$ with $0 \le \bar{t} \le T, \bar{x} \in \mathbb{R}$, the value $Z(\bar{t}, \bar{x})$ may be determined by solving successively (11) and (12).
For uniqueness, let $\zeta : [0,T] \times \mathbb{R} \to \mathbb{R}$ be of class C^{23} such that $\zeta(t, B_t)$ is a stochastic process satisfying (1). Then ζ satisfies the system (17). This system is equivalent to the system

$$\begin{cases} \frac{\partial Z}{\partial t} &= \mu(t, Z(t,x)) - \frac{1}{2}\frac{\partial^2 Z}{\partial x^2} \\ \frac{\partial Z}{\partial x} &= \sigma(t, Z(t,x)) \\ Z(0,0) &= x_0 \end{cases},$$

and we derive from (22) that it is also equivalent to the system (21). The latter has a unique solution by Proposition 1(1). Hence, $\zeta = Z$.

2. Assume (1) has a solution of the form $X_t = Z(t, B_t)$, such that $Z : [0,T] \times \mathbb{R} \to \mathbb{R}$ is of class C^{23}. Then (8) holds for all (t,x) with $0 < t < T, x \in \mathbb{R}$. Similarly to the uniqueness part of the proof of Theorem 3(1), we derive that (21) holds, with f and g defined by (19). Differentiating as in (20), we derive that (2) holds for all (t, X) such that $0 < t < T, X \in \mathrm{Im}Z$.

□

Proof of Theorem 4. Let the functions $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\begin{cases} f(t,x,Z) &= \mu(t,Z) - \frac{1}{2}\sigma(t,Z)\frac{\partial \sigma(t,Z)}{\partial Z} - \frac{F(t)}{G(t)}\sigma(t,Z) \\ g(t,x,Z) &= \frac{\sigma(t,Z)}{G(t)} \end{cases}. \tag{23}$$

Due to the conditions on F and G, the function f is of class $C^{1\infty 1}$, the function g is of class $C^{1\infty 2}$ and both functions are uniformly Lipschitz in the third variable.

1. It follows from (14) that

$$G' = -\frac{\Gamma G}{\sigma}. \tag{24}$$

Then the conditions for the integration of the system

$$\begin{cases} \frac{\partial Z}{\partial t} &= f(t,x,Z) \\ \frac{\partial Z}{\partial x} &= g(t,x,Z) \\ Z(0,0) &= x_0 \end{cases} \tag{25}$$

are satisfied, as we derive from (2) and (24) that

$$\begin{aligned} \frac{\partial f}{\partial x} + g\frac{\partial f}{\partial Z} - \frac{\partial g}{\partial t} - f\frac{\partial g}{\partial Z} &= \frac{\sigma}{G}\frac{\partial\left(\mu - \frac{1}{2}\sigma\frac{\partial \sigma}{\partial Z} - \frac{F}{G}\sigma\right)}{\partial Z} - \frac{\partial(\sigma/G)}{\partial t} - \left(\mu - \frac{1}{2}\sigma\frac{\partial \sigma}{\partial Z} - \frac{F}{G}\sigma\right)\frac{\partial(\sigma/G)}{\partial Z} \\ &= \frac{\sigma}{G}\frac{\partial \mu}{\partial Z} - \frac{1}{G}\frac{\partial \sigma}{\partial t} - \frac{\mu}{G}\frac{\partial \sigma}{\partial Z} - \frac{\sigma^2}{2G}\frac{\partial^2 \sigma}{\partial Z^2} + \frac{G'}{G^2}\sigma \\ &= \frac{\Gamma - \Gamma}{G} = 0. \end{aligned} \tag{26}$$

Hence, (25) has a solution Z of class C^{22}. It follows from (13) that

$$\frac{\partial Z}{\partial x} = \frac{\sigma(t, Z(t,x))}{G(t)}. \tag{27}$$

Hence,

$$\frac{\partial^2 Z}{\partial x^2} = \frac{\sigma(t, Z(t,x))}{G^2(t)} \frac{\partial \sigma(t, Z(t,x))}{\partial Z}, \tag{28}$$

which implies that Z is in fact of class C^{23}. According to Proposition 1(2), for $(\bar{t}, \bar{x})$ with $0 \le \bar{t} \le T, \bar{x} \in \mathbb{R}$, we may determine $Z(\bar{t}, \bar{x})$ by solving successively (15) and (16). It follows from (23) and (28) that Z satisfies (18), and then Theorem 2 ensures that the process $Z_t \equiv Z(t, Y_t)$ satisfies the stochastic differential Equation (1).
To prove the uniqueness part, let $\zeta : [0,T] \times \mathbb{R} \to \mathbb{R}$ be of class C^{23} such that $\zeta(t, Y_t)$ is a stochastic process satisfying (1). Then ζ satisfies the system (18). This system is equivalent to the system

$$\begin{cases} \frac{\partial Z}{\partial t} & = & \mu(t, Z(t,x)) - F\frac{\partial Z}{\partial x} - \frac{1}{2}G^2\frac{\partial^2 Z}{\partial x^2} \\ \frac{\partial Z}{\partial x} & = & \sigma(t, Z(t,x)) \\ Z(0,0) & = & x_0 \end{cases}, \tag{29}$$

By (27) and (28), the system (29) reduces to the system (25), which has a unique solution by Proposition 1(1). From this, we derive that $\zeta = Z$.

2. Assume (1) has a solution of the form $X_t = Z(t, Y_t)$, such that $Z : [0,T] \times \mathbb{R} \to \mathbb{R}$ is of class C^{23}, and Y_t is given by (3), with F and G of class C^1 in time. Then (8) holds for all (t,x) with $0 < t < T, x \in \mathbb{R}$. As in the proof of the uniqueness part of Theorem 4 (1), we see that also (25) holds, with f and g defined by (23). Then we differentiate as in (26). We derive that for all (t, X) with $0 < t < T, X \in \operatorname{Im} Z$, it holds that

$$\frac{\Gamma + \frac{G'}{G}\sigma}{G} = 0;$$

hence, also

$$\frac{\Gamma}{\sigma} = -\frac{G'}{G}.$$

We conclude that Γ/σ depends only on t and G is of the form (14).

□

5. Special Cases and Examples

The Sections 5.1 and 5.3 concern stochastic differentiable equations with functional solutions in terms of Brownian Motion. Section 5.1 considers the autonomous case and Section 5.3 two specific non-linear equations. Section 5.2 considers a class of linear equations. They may have solutions that are a function of a stochastic integral with respect to Brownian motion, where it is convenient to take the trend non-zero. A specific example of this class is the Ornstein–Uhlenbeck process.

5.1. Autonomous Case

We will only study solutions of the form $Z(t, B_t)$. Then the integration condition is given by (2), and we will see that it may be solved explicitly for μ. A special case is given by Geometric Brownian Motion, which satisfies a linear autonomous equation.

Consider the stochastic differential equation

$$\begin{cases} dX_t & = & \mu(X_t)dt + \sigma(X_t)dB_t & 0 \le t < T \\ X_0 & = & x_0 \end{cases}, \tag{30}$$

with $x_0 \in \mathbb{R}$, μ of class C^1 and $\sigma \neq 0$ of class C^2. The functions μ and σ have only ordinary derivatives with respect to X, which we indicate by primes.

The integration condition (2) takes the form

$$\Gamma = \sigma\mu' - \mu\sigma' - \frac{1}{2}\sigma^2\sigma'' = 0. \tag{31}$$

Observe that only linear equations can have a solution that is a martingale, as $\mu = 0$ amounts to $\sigma'' = 0$.

One may reduce (31) to an ordinary linear differential equation for μ, i.e.,

$$\mu' = \frac{\sigma'}{\sigma}\mu + \frac{1}{2}\sigma\sigma''. \tag{32}$$

Solving (32) for μ, we see that in order for a functional solution $Z(t, B_t)$ of class C^{12} to exist, the trend μ should satisfy

$$\mu(X_t) = \left(\frac{\mu(x_0)}{\sigma(x_0)} + \frac{1}{2}\left(\sigma'(X_t) - \sigma'(x_0)\right)\right)\sigma(X_t). \tag{33}$$

For $0 \leq \bar{t} \leq T$ and $B_{\bar{t}} \equiv \bar{x}$, the ordinary differential equations used to determine $Z(\bar{t}, \bar{x})$ also become autonomous. Equation (11) takes the form

$$\begin{cases} \frac{dZ^{(h)}}{dt} &= \mu\left(Z^{(h)}\right) - \frac{1}{2}\sigma\left(Z^{(h)}\right)\frac{d\sigma\left(Z^{(h)}\right)}{dZ} \\ Z^{(h)}(0) &= x_0 \end{cases} \tag{34}$$

and is of separable variables. Equation (12) becomes

$$\begin{cases} \frac{dZ^{(v)}}{dx} &= \sigma\left(Z^{(v)}(x)\right) \\ Z^{(v)}(0) &= Z^{(h)}(\bar{t}) \end{cases}, \tag{35}$$

and then $Z(\bar{t}, \bar{x}) = Z^{(v)}(\bar{x})$.

Example 1. *As a simple example, we consider the linear stochastic differential equation with constant coefficients*

$$\begin{cases} dS_t &= \overline{\mu}S_t + \overline{\sigma}S_t dB_t \\ S_0 &= s_0 \end{cases}, \tag{36}$$

with $\overline{\mu}, \overline{\sigma}, s_0 \in \mathbb{R}, \overline{\sigma} \neq 0$. It is well-known that the solution is a Geometric Brownian Motion. In fact, it can be expressed as

$$S(t, B_t) = \exp\left[\left(\overline{\mu} - \frac{\overline{\sigma}^2}{2}\right)t + \overline{\sigma}B_t\right]. \tag{37}$$

Indeed, if we substitute $S(t, B_t)$ into (9), it follows directly from Ito's Lemma that $S(t, B_t)$ satisfies (36).

We give here a more a priori derivation of (37). We show first that $\mu(S) \equiv \overline{\mu}S$ and $\sigma(S) \equiv \overline{\sigma}S$ satisfy the integration condition (33) and then determine the solution by solving the differential Equations (34) and (35).

The integration condition (33) amounts to the identity

$$\overline{\mu}S = \frac{\overline{\mu}}{\overline{\sigma}}\overline{\sigma}S.$$

With abuse of language, the first-order differential Equation (34) becomes

$$\begin{cases} \frac{dS}{dt} & = \left(\overline{\mu} - \frac{\overline{\sigma}^2}{2}\right)S \\ S(0) & = s_0 \end{cases},$$

and (35) becomes

$$\begin{cases} \frac{dS}{dx} & = \sigma S \\ S(0) & = s_0 \exp\left[\left(\overline{\mu} - \frac{\overline{\sigma}^2}{2}\right)t\right] \end{cases}. \tag{38}$$

Then (37) is obtained by solving (38), where we put $x = B_t$.

A nonlinear autonomous equation with a functional solution is presented in Example 4 of Section 5.3.

5.2. Linear Stochastic Differential Equations

We consider linear stochastic differential equations with variable coefficients of the form

$$\begin{cases} dX_t & = (\alpha(t) + \beta(t)X_t)dt + (\gamma(t) + \delta(t)X_t)dB_t \qquad 0 \le t \le T \\ X_0 & = x_0 \end{cases}, \tag{39}$$

where $\alpha, \beta, \gamma, \delta : [0, T] \to \mathbb{R}$ are of class C^1, and $x_0 \in \mathbb{R}$. Like ordinary linear stochastic differential equations they can be solved in general. We present conditions on the coefficients ensuring that the solution is a function of time and an Ito process, in particular this is the case for homogeneous equations. However, in most cases the solution is not a function of an Ito process. We will illustrate this with the help of linear stochastic differential equations with constant coefficients.

Theorem 5 gives the solution formula for (39), which is a special case of [2] Theorem 8.4.2.

Theorem 5. *Consider the linear stochastic differential Equation (39).*

1. *The solution of the associated homogeneous equation*

$$\begin{cases} dY_t & = \beta(t)Y_t dt + \delta(t)Y_t dB_t \qquad 0 \le t \le T \\ Y_0 & = 1 \end{cases} \tag{40}$$

is given by

$$Y_t = \exp\left(\int_0^t \left(\beta(s) - \frac{\delta^2(s)}{2}\right)ds + \int_0^t \delta(s)dB_s\right). \tag{41}$$

2. *The solution of (39) is given by*

$$X_t = Y_t\left(x_0 + \int_0^t Y_s^{-1}(\alpha(s) - \gamma(s)\delta(s))ds + \int_0^t Y_s^{-1}\gamma(s)dB_s\right). \tag{42}$$

The solution of the homogeneous Equation (40) of Theorem 5(1) is an Ito process. Observe that if β, δ are constant, Equation (40) reduces to (36). The proof of (41) is similar to the proof of (37). The proof of Theorem 5(1) uses Ito's product formula [1].

We study now whether (42) represents a functional solution. For simplicity, we consider the case where $\alpha, \beta, \gamma, \delta$ are constants.

Proposition 2. *Consider the linear stochastic differential equation with constant coefficients*

$$\begin{cases} dX_t & = (\alpha + \beta X_t)dt + (\gamma + \delta X_t)dB_t \qquad 0 \le t \le T \\ X_0 & = x_0 \end{cases}, \tag{43}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, and γ, δ are not both zero.

1. *If $\alpha\delta - \beta\gamma = 0$, the solution of (43) is a function of time and Brownian Motion. We have the following subcases:*
   (a) $\delta = \beta = 0$. *Then* $X_t = x_0 + \alpha + \gamma B_t$.
   (b) $\gamma = \alpha = 0$. *Then* $X_t = \exp\left[\left(\beta - \frac{\delta^2}{2}\right)t + \delta B_t\right]$, *i.e.,* X_t *is a Geometrical Brownian Motion.*
   (c) $\alpha, \beta, \gamma, \delta \neq 0$. *Again the solution is a Geometrical Brownian Motion.*
2. *If $\beta, \gamma \neq 0, \delta = 0$, the solution may be written as a function $Z(t, Y_t)$ of time and the Ito process*

$$Y_t = \int_0^t \exp\beta(t-s)dB_s. \tag{44}$$

   In fact,

$$X_t = x_0 e^{\beta t} + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \gamma\int_0^t e^{\beta(t-s)}dB_s. \tag{45}$$

   In particular, if $\alpha \neq 0$, the solution is an Ornstein-Uhlenbeck process.
3. *If $\delta \neq 0$ and $\alpha\delta - \beta\gamma \neq 0$, the Equation (43) does not have a functional solution in terms of time and an Ito process.*

Proof. The value of Γ as defined by (2) is given by

$$\Gamma = (\gamma + \delta X)\beta - (\alpha + \beta X)\delta = \beta\gamma - \alpha\delta. \tag{46}$$

1. Assume that $\alpha\delta - \beta\gamma = 0$. Then it follows from (46) that $\Gamma = 0$. According to Theorem 3, the Equation (43) has a solution that is a function of time and Brownian Motion. If $\alpha\delta = \beta\gamma = 0$, only the cases (1a) and (1b) are relevant, else (43) is not a stochastic differential equation. In both cases the solution formulas follow in a straightforward way from Theorem 5.
   If $\alpha\delta \neq 0$, also $\beta\gamma \neq 0$. Then we have indeed $\alpha, \beta, \gamma, \delta \neq 0$. Then there exists $c \neq 0$ such that $\alpha = c\beta, \gamma = c\delta$. By the change of variable $Y_t = (c+1)X_t$, one obtains a homogeneous linear stochastic differential equation with constant coefficients. It follows that the solution is a Geometrical Brownian Motion.
2. Formula (45) easily follows from Theorem 5. It shows that the solution is a function of time and the Ito process (44). In Example 2 below, we establish the relation with the common formulation of the Ornstein–Uhlenbeck process.
3. It holds that

$$\frac{\Gamma}{\sigma} = \frac{\beta\gamma - \alpha\delta}{\gamma + \delta X}.$$

   Hence, Γ/σ does not only depend on time. According to Theorem 4(2), the Equation (43) cannot have a solution that is a function of time and an Ito process.

□

Example 2. *The well-known Ornstein–Uhlenbeck process corresponds to Part 2 of Proposition 2. Indeed, by putting $\theta = -\beta, \overline{\mu} = \alpha/\theta, \overline{\sigma} = \gamma$ and $R_t = X_t$, the stochastic differential Equation (43) takes the more common form*

$$\begin{cases} dR_t &= \theta(\overline{\mu} - R_t)dt + \overline{\sigma}dB_t \qquad 0 \leq t < T \\ R_0 &= r_0 \end{cases},$$

where $\theta, \overline{\mu}, \overline{\sigma} \neq 0, r_0 \in \mathbb{R}$. Its solution is given by the well-known formula

$$R_t = r_0 e^{-\theta t} + \overline{\mu}(1 - e^{-\theta t}) + \int_0^t e^{\theta(s-t)}dB_s. \tag{47}$$

Below, we derive (47) by applying Theorem 4(1).

The integrating factor takes the form

$$G(t) = ce^{\theta t},$$

for some $c \in \mathbb{R}$*; hence,* $R_t = Z(t, Y_t)$*, where* Y_t *is of the form*

$$Y_t = y_0 + \int_0^t F(s)ds + c\int_0^t e^{\theta s}dB_s,$$

with $y_0 \in \mathbb{R}$ *and* F *of class* C^1 *on* $[0, T]$*. To simplify, we assume that* $y_0 = 0$ *and* $c = 1$*.*

The differential Equations (15) and (16) become

$$\begin{cases} \frac{dZ^{(h)}}{ds} &= \theta\overline{\mu} - \overline{\sigma}F(s)e^{-\theta s} - \theta Z^{(h)} \\ Z^{(h)}(0) &= r_0 \end{cases}$$

and

$$\begin{cases} \frac{dZ^{(v)}}{dx} &= \overline{\sigma}e^{-\theta t} \\ Z^{(v)}(0) &= Z^{(h)}(t) \end{cases}.$$

To simplify, we take F *to be non-zero; in fact,*

$$F(s) = \frac{\theta\overline{\mu}}{\overline{\sigma}}e^{\theta s}.$$

Then the differential equation for $Z^{(h)}$ *becomes homogeneous. The solution* Z *is given by*

$$Z(t, Y_t) = r_0e^{-\theta t} + \overline{\sigma}e^{-\theta t}Y_t,$$

where $Y_t = \frac{\theta\overline{\mu}}{\overline{\sigma}}\int_0^t e^{\theta s}ds + \int_0^t e^{\theta s}dB_s = \frac{\overline{\mu}}{\overline{\sigma}}(e^{\theta t} - 1) + \int_0^t e^{\theta s}dB_s$*. This reduces to (47).*

Example 3. *The stochastic process*

$$X_t = \exp(B_t)\left(\int_0^t \exp(-B_s)ds + \int_0^t \exp(-B_s)dB_s\right).$$

does not have the property of path-independence with respect to the trajectories of an Ito process. Indeed, it follows from (42) that it is the solution of the linear stochastic differential equation with constant coefficients (43), as $\alpha = 2, \beta = 1/2, \gamma = 1, \delta = 1$ *and* $x_0 = 0$*. Then* $\alpha\delta - \beta\gamma = 3/2 \neq 0$*. By Proposition 2(3), the process* X_t *cannot be a function of time and an Ito process.*

5.3. On Nonlinear Stochastic Differential Equations

The most obvious non-linear equations are polynomial equations. However, they are not appropriate in our context, because they do not satisfy the Lipschitz condition. This particularity remains if the powers are fractional. The coefficients of Example 4 below are rational functions with quadratic decay and the coefficients of Example 5 have quadratic exponential decay.

Example 4. *Consider the autonomous stochastic differential equation*

$$\begin{cases} dX_t &= -\frac{X_t}{(1+X_t^2)^3}dt + \frac{1}{1+X_t^2}dB_t & 0 \le t < T \\ X_0 &= 0 \end{cases}.$$

We put

$$\begin{cases} \mu(X_t) &= -\frac{X_t}{(1+X_t^2)^3} \\ \sigma(X_t) &= \frac{1}{1+X_t^2} \end{cases}.$$

In a straightforward way, one verifies that μ and σ satisfy (33). Hence, according to Theorem 3, the stochastic differential Equation (1) has a functional solution of the form $Z(t, B_t)$.

We will see that Z depends only on B_t. The function Z may be determined by solving successively the ordinary differential Equations (34) and (35). The solution of (34) is simply $Z^{(h)} = 0$. Let $0 \leq \bar{t} \leq T$ and $\bar{x} = B_{\bar{t}}$. The Equation (35) becomes

$$\begin{cases} \frac{dZ^{(v)}}{dx} & = & \frac{1}{1+(Z^{(v)})^2} \\ Z^{(v)}(0) & = & 0 \end{cases}.$$

One finds $Z + Z^3/3 = B_{\bar{t}}$. Then Cardano's formula [16] yields

$$Z(B_{\bar{t}}) = \sqrt[3]{\frac{3B_{\bar{t}} + \sqrt{9B_{\bar{t}}^2 + 4}}{2}} + \sqrt[3]{\frac{3B_{\bar{t}} - \sqrt{9B_{\bar{t}}^2 + 4}}{2}}.$$

Example 5. *Consider the stochastic differential Equation (1) with*

$$\begin{cases} \mu(t, X_t) & = & \frac{\exp\left(-\frac{X^2}{2}\right)}{t+1} \int_0^X \exp \frac{\xi^2}{2} d\xi - \frac{X(t+1)^2}{2} \exp(-X^2) \\ \sigma(t, X_t) & = & (t+1) \exp\left(-\frac{X^2}{2}\right) \end{cases}.$$

All the terms are infinitely differentiable. Using the fact that they have a factor with exponential decay, one verifies that the functions σ and $\frac{\partial \sigma}{\partial X}$ are uniformly bounded and $\mu(t, X)$ is uniformly Lipschitz in X. One verifies also that μ and σ satisfy the integration condition (2). Hence, according to Theorem 3, the stochastic differential Equation (1) has a solution $Z(\bar{t}, B_{\bar{t}})$, which may be found by solving the ordinary differential equations with separable variables

$$\begin{cases} \frac{dZ^{(h)}}{dt} & = & \frac{1}{t+1} \exp\left(-\frac{\left(Z^{(h)}\right)^2}{2}\right) \int_0^{Z^{(h)}} \exp \frac{\xi^2}{2} d\xi \\ Z^{(h)}(0) & = & x_0 \end{cases}$$

and

$$\begin{cases} \frac{dZ^{(v)}}{dx} & = & (\bar{t} + 1) \exp\left(-\frac{\left(Z^{(v)}\right)^2}{2}\right) \\ Z^{(v)}(0) & = & Z^{(h)}(\bar{t}) \end{cases}.$$

6. On Path-Dependent Functional Solutions

In general, it seems not to be very relevant to study a general condition enabling the expression of the solution of the stochastic differential Equation (1) as a function of time and a continuous semi-martingale, i.e., a process given by another stochastic differential equation. Indeed, the equality of mixed derivatives (8) would lead to a system of coupled partial differential equations involving two processes.

However, in [10], H. Doss expresses the solution of a stochastic differential equation successfully as a function

$$X_t = h(D_t, M_t) \tag{48}$$

of a stochastic process with differential trajectories D_t and a continuous semi-martingale M_t. In general, these processes are path-dependent. It is shown that the function h satisfies a partial differential equation, and the random variables $D_t(\omega)$ satisfy an ordinary differential equation, where $\omega \in \Omega$ for some probability space Ω. Though the approach of [10] is valid for time-dependent stochastic differential equations, it is treated in detail for autonomous equations, i.e., equations of the form (30). Below, we sketch briefly this approach. For

simplicity we take M_t to be equal to Brownian motion B_t, corresponding to [10] (Theorem 3), i.e., (48) becomes
$$X_t = h(D_t, B_t).$$

As in [10], we denote the derivative a function f by f', and also, with some abuse of language, for every $\omega \in \Omega$, the derivative of the random variable $D_t(\omega)$ with respect to time will be written $D_t'(\omega)$. Like in the Main Theorems, we assume that μ and σ, figuring in (30), are of class C^2, and satisfy the Lipschitz condition.

The function $h : \mathbb{R}^2 \to \mathbb{R}$ is defined by
$$\begin{cases} \frac{\partial h(u,x)}{\partial x} &= \sigma(h(u,x)) \\ h(u,0) &= u \end{cases}. \tag{49}$$

We recognize the differential equation for $Z^{(v)}$ as given in (12), with a different initial condition. It is shown in [10] (Theorem 3.i, Equation (V)) that the random variables $D_t'(\omega)$ satisfy almost surely
$$D_t'(\omega) = \exp\left(-\int_0^{B_t(\omega)} \sigma'(h(D_s(\omega), s))ds\right)\left(\mu(h(D_t, (B_t(\omega))) - \frac{1}{2}\sigma'\sigma(h(D_t, (B_t(\omega)))\right). \tag{50}$$

In the special case of path-independence corresponding to
$$D_t(\omega) = t \qquad \forall \omega \in \Omega, \tag{51}$$

we will derive from the differential Equation (50) a partial differential equation for the solution h, i.e.,
$$\frac{\partial h(t,x)}{\partial t} = \mu(h(t,x)) - \frac{1}{2}\sigma'\sigma(h(t,x)). \tag{52}$$

We will see that also the integration condition (31) is satisfied.

As indicated in [10] (Lemme 2), the function h is of class C^{22}. Then the equality of mixed derivatives
$$\frac{\partial^2 h(u,x)}{\partial u \partial x} = \frac{\partial^2 h(u,x)}{\partial x \partial u}.$$

holds. Then also
$$\frac{\partial \frac{\partial h(u,x)}{\partial u}}{\partial x} = \frac{\partial \sigma(h(u,x))}{\partial u} = \sigma'(h(u,x))\frac{\partial h(u,x)}{\partial u},$$

hence,
$$\frac{\partial h(u,x)}{\partial u} = \exp\left(\int_0^x \sigma'(h(u,v))dv\right). \tag{53}$$

One derives from Ito's Lemma, (49), (53) and (30) that
$$\begin{aligned} dh(D_t, B_t) &= \tfrac{\partial h}{\partial D} D_t' dt + \tfrac{\partial h}{\partial B} dB_t + \tfrac{1}{2}\tfrac{\partial^2 h}{\partial B^2} dt \\ &= \left(\exp\left(\int_0^x \sigma'(h(D_s, s))ds\right) D_t' + \tfrac{1}{2}\sigma'\sigma\right)dt + \sigma dB_t \\ &= \mu dt + \sigma dB_t. \end{aligned}$$

It follows that
$$D_t' \exp\left(\int_0^x \sigma'(h(D_s, s))ds\right) + \frac{1}{2}\sigma'\sigma = \mu.$$

This implies (50).

Finally, we assume that D is given by (51), and derive (52) and the integration condition (31). Observe that $D_t' = 1$, so (52) follows from (53) and (50). Hence, using (49)
$$\frac{\partial^2 h(t,x)}{\partial x \partial t} = \sigma\mu'(h(t,x)) - \frac{1}{2}\sigma''\sigma^2(h(t,x)) - \frac{1}{2}\sigma(\sigma')^2(h(t,x)).$$

Also, again using (49) and (52)

$$\frac{\partial^2 h(t,x)}{\partial t \partial x} = \sigma'(h(t,x))\left(\mu(h(t,x)) - \frac{1}{2}\sigma'\sigma(h(t,x))\right).$$

Hence, (31) holds, for

$$\sigma\mu'(h(t,x)) - \sigma'(h(t,x))\mu(h(t,x) - \frac{1}{2}\sigma''\sigma^2(h(t,x)) = \frac{\partial^2 h(t,x)}{\partial x \partial t} - \frac{\partial^2 h(t,x)}{\partial t \partial x} = 0.$$

7. Conclusions

A necessary and sufficient criterion was given for the existence and uniqueness of a solution of a stochastic differential equation, which is a function of time and an Ito process. This criterion determines the martingale part of the Ito process, and seems to be new. The deterministic part of the Ito process may be chosen by convenience. The solution being functional, path-independence holds for the trajectories of the process, and because it is given in terms of a twice continuously differentiable function, path-independence also holds when determining its value along a piece-wise differentiable curve starting at an initial condition. Choosing a horizontal path, followed by a vertical path, this value can be determined by solving successively two ordinary differential equations. The special case where the Ito process reduces to Brownian motion leads to some simplifications.

A functional solution, in particular in the case of Brownian motion with its Gaussian probability distribution, can be used as a tool to study the properties of the random variables of the process, like expectation and variance. A well-known application in this sense concerns the Black–Scholes model [17], in economics. This model states that the price of an option on an asset is determined by a self-financing stochastic process, which has the form of a stochastic differential equation. Its solution is a Geometric Brownian Motion, hence is a function of time and Brownian Motion. In particular the Black–Scholes formula for the price of a European option is given by an expectation, in terms of an integral of this function. This motivates research for applications of functional solutions in other areas outside mathematics. In particular functional solutions facilitate the study of the behavior of trajectories, which is relevant for, say, population dynamics. The presence of formulas for densities and distribution functions may also be interesting from a numerical point-of-view. For instance, usually it is easier to estimate the conditional variance of a process than its trend. As a primary guess, one could choose a trend satisfying an integration condition, and test whether the corresponding distribution function leads to a reasonable fit.

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