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Ill-Posedness of a Three-Component Novikov System in Besov Spaces

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Abstract: In this paper, we consider the Cauchy problem for a three-component Novikov system on the line. We give a construction of the initial data $(\rho_0, u_0, v_0) \in B^{\sigma-1}_{p,\infty}(\mathbb{R}) \times B^{\sigma}_{p,\infty}(\mathbb{R}) \times B^{\sigma}_{p,\infty}(\mathbb{R})$ with $\sigma > \max\left\{3 + \frac{1}{p}, \frac{7}{2}\right\}, 1 \le p \le \infty$, such that the corresponding solution to the three-component Novikov system starting from (ρ_0, u_0, v_0) is discontinuous at $t = 0$ in the metric of $B^{\sigma-1}_{p,\infty}(\mathbb{R}) \times B^{\sigma}_{p,\infty}(\mathbb{R}) \times B^{\sigma}_{p,\infty}(\mathbb{R})$, which implies the ill-posedness for this system in $B^{\sigma-1}_{p,\infty}(\mathbb{R}) \times B^{\sigma}_{p,\infty}(\mathbb{R}) \times B^{\sigma}_{p,\infty}(\mathbb{R})$.

Keywords: ill-posedness; three-component Novikov system; Besov spaces

MSC: 35Q53; 37K10

1. Introduction

In this paper, we investigate the Cauchy problem of a three-component Novikov system which takes the form

$$\begin{cases} \rho_t + (\rho u v)_x = 0, \ t > 0, \ x \in \mathbb{R}, \\ m_t + 3 m u_x v + m_x u v + \rho^2 u = 0, \ t > 0, \ x \in \mathbb{R}, \\ n_t + 3 n u v_x + n_x u v - \rho^2 v = 0, \ t > 0, \ x \in \mathbb{R}, \\ m = u - u_{xx}, \ n = v - v_{xx}, \\ \rho(0, x) = \rho_0(x), \ u(0, x) = u_0(x), \ v(0, x) = v_0(x), \ x \in \mathbb{R}. \end{cases} \tag{1}$$

System (1) was proposed by Li [1]; it can be derived from the following spectral problem:

$$\phi_x = U\phi, \ \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \ M = \begin{pmatrix} 0 & 1 & 0 \\ 1 + \lambda\rho^2 & 0 & m \\ \lambda n & 0 & 0 \end{pmatrix},$$

where λ is a constant spectral parameter. The authors showed that system (1) is equivalent to the zero-curvature equation

$$U_t - V_x + [U, V] = 0,$$

where the spacial and temporal 3×3 matrices U and V are

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 1 + \lambda\rho^2 & 0 & m \\ \lambda n & 0 & 0 \end{pmatrix}, \ V = \begin{pmatrix} \frac{1}{3\lambda} + u v_x & -uv & \frac{u}{\lambda} \\ u_x v_x - \lambda\rho^2 uv & \frac{1}{3\lambda} - u_x v & \frac{u_x}{\lambda} - muv \\ -\lambda nuv - v_x & v & u_x v - u v_x - \frac{2}{3\lambda} \end{pmatrix}.$$

The authors also constructed a bi-Hamiltonian structure and infinitely many conserved quantities. Li-Hu [2] established the local well-posedness of the system in Besov spaces $\left(B^s_{p,r}(\mathbb{R})\right)^3$ $\left(s > \max\left\{\frac{1}{p}, \frac{1}{2}\right\}, 1 \le p, r \le \infty\right)$, and derived two blow-up criteria for the system.

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For $\rho = 0$, system (1) degenerates into the following Geng-Xue (GX) system:

$$\begin{cases} m_t + 3muu_x + m_x uv = 0, \\ n_t + 3nuv_x + n_x uv = 0, \\ m = u - u_{xx},\ n = v - v_{xx}. \end{cases} \tag{2}$$

Mi-Mu-Tao [3] and Tang-Liu [4] established the local well-posedness of a GX system (2) in the supercritical Besov spaces $B^s_{p,r}(\mathbb{R}) \times B^s_{p,r}(\mathbb{R})$ $(s > \max\{2 + \frac{1}{p}, \frac{5}{2}\},\ 1 \le p, r \le \infty)$ and the critical Besov space $B^{\frac{5}{2}}_{2,1}(\mathbb{R}) \times B^{\frac{5}{2}}_{2,1}(\mathbb{R})$. Based on the local well-posedness results of a GX system (2) in Besov spaces, Wang-Chong-Wu [5] further established nonuniform continuous dependence on initial data for a GX system (2) in Besov spaces: $B^s_{p,r}(\mathbb{T}) \times B^s_{p,r}(\mathbb{T})$ $(s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}, 1 \le p, r \le \infty)$, $B^{\frac{5}{2}}_{2,1}(\mathbb{T}) \times B^{\frac{5}{2}}_{2,1}(\mathbb{T})$ and $B^s_{p,r}(\mathbb{R}) \times B^s_{p,r}(\mathbb{R})$ $(s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}, 1 \le p, r \le \infty)$.

For $v = 1$, the GX system (2) degenerates into the following Degasperis–Procesi (DP) equation:

$$m_t + 3muu_x + m_x u = 0,\ m = u - u_{xx}. \tag{3}$$

The DP Equation (3) can be regarded as a nonlinear shallow water wave dynamics model, and its asymptotic accuracy is the same as that of the Camassa–Holm (CH) shallow water system [6]. By constructing the Lax pair [7], the formal integrability of DP equation was proved, which has a bi-Hamiltonian structure and infinitely many conserved quantities. The Cauchy problem of DP Equation (3) is locally well-posed in certain Sobolev spaces and Besov spaces in [8–10]. Li-Yu-Zhu studied the local existence and uniqueness of the solution of the DP Equation (3) on the critical Besov space $B^1_{\infty,1}(\mathbb{R})$ in [11], and further proved the nonuniform continuous dependence of the data-to-solution map. Zhou [12] gave the local well-posedness of the CH-DP system in the nonhomogeneous Besov spaces $B^s_{p,r}(\mathbb{R}) \times B^s_{p,r}(\mathbb{R})$ $(s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}, 1 \le p, r \le \infty)$ and the critical Besov space $B^{\frac{5}{2}}_{2,1}(\mathbb{R}) \times B^{\frac{5}{2}}_{2,1}(\mathbb{R})$ by using the transport equation theory and the classical Friedrichs regularization method. Zhu-Li-Li [13] proved the ill-posedness of the two-component DP system in the Besov space $B^1_{\infty,1}(\mathbb{R}) \times B^0_{\infty,1}(\mathbb{R})$.

When $u = v$, the GX system (2) degenerates into the following Novikov equation:

$$m_t + 3muu_x + m_x u^2 = 0,\ m = u - u_{xx}. \tag{4}$$

Novikov Equation (4) is a new integrabe equation with cubic nonlinearities derived by Novikov [14]. It has similar properties to the CH equation, such as a Lax pair in matrix form and a bi-Hamiltonian structure, it is integrable in the with infinitely many conserved quantities, and admits peakon solutions given by the formula $u(x,t) = \pm\sqrt{C}e^{|x-ct|}$ [15]. The local well-posedness, global existence, and asymptotic behavior of the Novikov Equation (4) in Sobolev spaces and Besov spaces were established in [16–20]. Ni-Zhou established the local well-posedness in the critical Besov space $B^{\frac{3}{2}}_{2,1}(\mathbb{R})$ in [17]. Yan-Li-Zhang generalized well-posed spaces to a larger class of Besov spaces $B^s_{p,r}(\mathbb{R})$ $(s > \max\left\{1 + \frac{1}{p}, \frac{3}{2}\right\}, 1 \le p \le \infty,$ $1 \le r \le \infty)$, and the local well-posedness was proved to be invalid in Besov space $B^{\frac{3}{2}}_{2,\infty}(\mathbb{R})$ by using the peakon traveling wave solution in [20]. Li-Yu-Zhu [21] established the ill-posedness of the Novikov equation (4) in Besov spaces $B^s_{2,\infty}(\mathbb{R})$ $(s > \frac{7}{2})$. Subsequently, Wu-Li extended the above results to the following two-component Novikov system in [22].

$$\begin{cases} \rho_t = \rho_x u^2 + \rho u u_x,\ t > 0,\ x \in \mathbb{R}, \\ m_t = 3u_x u m + u^2 m_x - \rho(u\rho)_x,\ t > 0,\ x \in \mathbb{R}, \\ m = u - u_{xx}, \\ \rho(0,x) = \rho_0,\ u(0,x) = u_0,\ x \in \mathbb{R}. \end{cases} \tag{5}$$

Wu-Li proved that the two-component Novikov system (5) is ill-posed in Besov spaces $B^{s-1}_{p,\infty}(\mathbb{R}) \times B^{s}_{p,\infty}(\mathbb{R})$ $(s > \max\left\{1+\frac{1}{p}, \frac{3}{2}\right\}, 1 \le p, r \le \infty)$ by constructing the initial data (ρ_0, u_0) in [22]. The result of the nonuniform continuous dependence of the two-component Novikov system (5) was established by Wu-Cao in [23].

The inverse scattering approach is a powerful tool to study the CH equation and analyze its dynamics [24]. The inverse scattering transform (IST) problem of the CH equation is a complicated issue [25]. An algorithm was proposed by Constantin and Lenells in [26] and slightly modificated in [27] to solve the inverse scattering problem for the CH equation; while other approaches were subsequently introduced [28], this method turns out to be quite effective in giving a closed form to the CH solitary waves. The IST method for the DP Equation (3) was presented in [29]; the basic aspects of the IST such as the construction of fundamental analytic solutions, the formulation of a Riemann–Hilbert (RH) problem, and the implementation of the dressing method were introduced. The authors in [30] developed the IST method for the Novikov Equation (4) in the case of nonzero constant background; the approach was based on the analysis of an associated RH problem, which in this case was a 3×3 matrix problem. A new IST method corresponding to a RH problem was also formulated for the two-component generalization of the CH equation [31].

From the above discussion, it can be seen that the nonuniform dependence and well-posedness results of the GX system (2), DP Equation (3), Novikov equation (4), and the two-component Novikov system (5), have been established in Sobolev spaces and Besov spaces. However, the ill-posedness for the three-component Novikov system (1) has yet to be systematically addressed. Using the ideas in [22,32,33], based on the existing results, we investigate ill-posedness of the solution for the system (1) in Besov spaces.

To facilitate the proof of the main result, we rewrite (1) into the following equivalent nonlocal form:

$$\begin{cases} \rho_t = uv\rho_x + \rho v u_x + \rho u v_x,\ t > 0,\ x \in \mathbb{R}, \\ u_t = uvu_x + P_1(u,v) + R_1(u,\rho),\ t > 0,\ x \in \mathbb{R}, \\ v_t = uvv_x + P_2(u,v) + R_2(v,\rho),\ t > 0,\ x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x),\ u(0,x) = u_0(x),\ v(0,x) = v_0(x),\ x \in \mathbb{R}, \end{cases} \tag{6}$$

where

$$\begin{aligned}
&P_1(u,v) = P_{11}(u,v) + P_{12}(u,v) + P_{13}(u,v) + P_{14}(u,v),\\
&P_2(u,v) = P_{21}(u,v) + P_{22}(u,v) + P_{23}(u,v) + P_{24}(u,v),\\
&P_{11}(u,v) = \mathcal{P} * (3uu_x v),\ P_{12}(u,v) = \mathcal{P} * (u_x^2 v_x),\ P_{13}(u,v) = \mathcal{P} * (uu_{xx}v_x),\ P_{14}(u,v) = \partial_x \mathcal{P} * (uu_x v_x),\\
&P_{21}(u,v) = \mathcal{P} * (3uvv_x),\ P_{22}(u,v) = \mathcal{P} * (u_x v_x^2),\ P_{23}(u,v) = \mathcal{P} * (u_x v v_{xx}),\ P_{24}(u,v) = \partial_x \mathcal{P} * (u_x v v_x),\\
&R_1(u,\rho) = \mathcal{P} * (u\rho^2),\ R_2(v,\rho) = -\mathcal{P} * (\rho^2 v).
\end{aligned}$$

where we denote by $*$ the convolution, for all $f \in L^2$,

$$\mathcal{P} * f = (1-\partial_x^2)^{-1} f,\ \ \mathcal{P}(x) = \frac{1}{2}e^{-|x|}.$$

We can now state our main result as follows.

Theorem 1. *Let $\sigma > \max\left\{3+\frac{1}{p}, \frac{7}{2}\right\}, 1 \le p \le \infty$. Then the three-component Novikov system (1) is ill-posed in the Besov spaces $B^{\sigma-1}_{p,\infty}(\mathbb{R}) \times B^{\sigma}_{p,\infty}(\mathbb{R}) \times B^{\sigma}_{p,\infty}(\mathbb{R})$. More precisely, there exists $(\rho_0, u_0, v_0) \in B^{\sigma-1}_{p,\infty}(\mathbb{R}) \times B^{\sigma}_{p,\infty}(\mathbb{R}) \times B^{\sigma}_{p,\infty}(\mathbb{R})$ and a positive constant $\varepsilon_0 > 0$ such that the data-to-solution map $(\rho_0, u_0, v_0) \longmapsto (\rho, u, v)$ of the Cauchy problem (6) satisfies*

$$\limsup_{t \to 0^+} \left(\|\rho - \rho_0\|_{B^{\sigma-1}_{p,\infty}(\mathbb{R})} + \|u - u_0\|_{B^{\sigma}_{p,\infty}(\mathbb{R})} + \|v - v_0\|_{B^{\sigma}_{p,\infty}(\mathbb{R})} \right) \ge \varepsilon_0.$$

Remark 1. *Theorem 1 demonstrates the ill-posedness of the three-component Novikov system in* $B^{\sigma-1}_{p,\infty}(\mathbb{R})\times B^{\sigma}_{p,\infty}(\mathbb{R})\times B^{\sigma}_{p,\infty}(\mathbb{R})$. *More precisely, there exists initial data* $(\rho_0,u_0,v_0)\in B^{\sigma-1}_{p,\infty}(\mathbb{R})\times B^{\sigma}_{p,\infty}(\mathbb{R})\times B^{\sigma}_{p,\infty}(\mathbb{R})$ *such that corresponding solution to the Cauchy problem (6) that starts from* (ρ_0,u_0,v_0) *does not converge back to* (ρ_0,u_0,v_0) *in the metric of* $B^{\sigma-1}_{p,\infty}(\mathbb{R})\times B^{\sigma}_{p,\infty}(\mathbb{R})\times B^{\sigma}_{p,\infty}(\mathbb{R})$ *as time goes to zero.*

This paper is structured as follows. In Section 2, we list some notations and known results and recall some lemmas which will be used in the sequel. In Section 3, by establishing some technical lemmas and propositions, we present the proof of Theorem 1.

2. Preliminaries

In this section, we list some basic concepts and useful lemmas, which will be frequently used in proving our main results.

Denote $\mathcal{F}$ and $\mathcal{F}^{-1}$ by the Fourier transform and the Fourier inverse transform, respectively, as follows:

$$\mathcal{F}u(\xi)=\hat{u}(\xi)=\int_{\mathbb{R}^d}e^{-ix\xi}u(x)\mathrm{d}x,$$
$$u(x)=\mathcal{F}^{-1}\hat{u}(x)=\frac{1}{2\pi}\int_{\mathbb{R}^d}e^{ix\xi}\hat{u}(\xi)\mathrm{d}\xi.$$

For any $u\in\mathcal{S}'(\mathbb{R}^d)$ and all $j\in\mathbb{Z}$, define $\Delta_j u=0$ for $j\le -2$; $\Delta_{-1}u=\mathcal{F}^{-1}(\chi\mathcal{F}u)$; $\Delta_j u=\mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\mathcal{F}u)$ for $j\ge 0$; and $S_j u=\sum_{j'<j}\Delta_{j'}u$.

Let $s\in\mathbb{R}$, $1\le p,\ r\le\infty$. We define the nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^d)$ as

$$B^s_{p,r}=B^s_{p,r}(\mathbb{R}^d)=\left\{u\in S'(\mathbb{R}^d):\|u\|_{B^s_{p,r}}=\left\|(2^{js}\|\Delta_j u\|_{L^p})_j\right\|_{l^r(\mathbb{Z})}<\infty\right\}.$$

Lemma 1 (See [34]). *The following estimates hold:*

(1) *For any $s>0$ and any $p,r\in[1,\infty]$, the space $L^\infty\cap B^s_{p,r}$ is an algebra and a constant $C=C(s)$ exists such that*

$$\|uv\|_{B^s_{p,r}}\le C(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}}+\|u\|_{B^s_{p,r}}\|v\|_{L^\infty}).$$

(2) *If $1\le p,r\le\infty$, $s_1\le s_2$, $s_2>\frac{1}{p}$ ($s_2\ge\frac{1}{p}$ if $r=1$) and $s_1+s_2>\max\{0,\frac{2}{p}-1\}$, there exists $C=C(s_1,s_2,p,r)$, such that*

$$\|uv\|_{B^{s_1}_{p,r}}\le C\|u\|_{B^{s_1}_{p,r}}\|v\|_{B^{s_2}_{p,r}}.$$

Lemma 2 (See [34,35]). *Let $1\le p,r\le\infty$, $\theta>-\min(\frac{1}{p},\frac{1}{p'})$, Then there exists a constant C such that for all solutions $f\in L^\infty(0,T;B^\theta_{p,r})$ of the following problem:*

$$\begin{cases}\partial_t+\partial_x f=g,\\ f(0,x)=f_0(x),\end{cases}$$

with initial data $f_0\in B^\theta_{p,r}$ and $g\in L^1(0,T;B^\theta_{p,r})$, we have, for a.e. $t\in[0,T]$,

$$\|f(t)\|_{B^\theta_{p,r}}\le\|f_0\|_{B^\theta_{p,r}}+\int_0^t\|g(t')\|_{B^\theta_{p,r}}\mathrm{d}t'+\int_0^t V'(t')\|f(t')\|_{B^\theta_{p,r}}\mathrm{d}t'$$

or

$$\|f(t)\|_{B^\theta_{p,r}}\le e^{CV(t)}\left(\|f_0\|_{B^\theta_{p,r}}+\int_0^t e^{-CV(t')}\|g(t')\|_{B^\theta_{p,r}}\mathrm{d}t'\right),$$

with

$$V'(t) = \begin{cases} \|\partial_x v(t)\|_{B^{\frac{1}{p}}_{p,\infty} \cap L^\infty}, & \text{if} \quad \theta < 1 + \frac{1}{p}, \\ \|\partial_x v(t)\|_{B^\theta_{p,r}}, & \text{if} \quad \theta = 1 + \frac{1}{p},\ r > 1, \\ \|\partial_x v(t)\|_{B^{\theta-1}_{p,r}}, & \text{if} \quad \theta > 1 + \frac{1}{p} \text{ or } \theta = 1 + \frac{1}{p},\ r = 1. \end{cases}$$

If $\theta > 0$, then there exists a positive constant $C = C(p, r, \theta)$ such that the following statement holds:

$$\|f(t)\|_{B^\theta_{p,r}} \le \|f_0\|_{B^\theta_{p,r}} + \int_0^t \|g(\tau)\|_{B^\theta_{p,r}} \mathrm{d}\tau + C\int_0^t \Big(\|f(\tau)\|_{B^\theta_{p,r}} \|\partial_x v(\tau)\|_{L^\infty} + \|\partial_x v(\tau)\|_{B^{\theta-1}_{p,r}} \|\partial_x f(\tau)\|_{L^\infty} \Big) \mathrm{d}\tau.$$

In particular, if $f = av + b$, $a, b \in \mathbb{R}$, then for all $\theta > 0$, $V'(t) = \|\partial_x v(t)\|_{L^\infty}$.

Lemma 3 (See [22])**.** *Let $s > 0$, $1 \le p \le \infty$, then we have*

$$\Big\| 2^{js} \|[\Delta_j, u]\partial_x v\|_{L^p} \Big\|_{L^\infty} \le C(\|\partial_x u\|_{L^\infty} \|v\|_{B^s_{p,r}} + \|\partial_x v\|_{L^\infty} \|u\|_{B^s_{p,r}}),$$

with

$$[\Delta_j, u]\partial_x v = \Delta_j(u\partial_x v) - u\Delta_j(\partial_x v).$$

Next we present the local well-posedness result of system (1) which was given in [2].

Lemma 4 (See [2])**.** *Let $\sigma > \max\{2 + \frac{1}{p}, \frac{5}{2}\}, 1 \le p \le \infty, 1 \le r < \infty$, and $(\rho_0, u_0, v_0) \in B^{s-1}_{p,r}(\mathbb{R}) \times B^s_{p,r}(\mathbb{R}) \times B^s_{p,r}(\mathbb{R})$. Then, there exists some $T > 0$ such that system (1) has a unique solution (ρ, u, v) in $C([0, T]; B^{s-1}_{p,r}(\mathbb{R}) \times B^s_{p,r}(\mathbb{R}) \times B^s_{p,r}(\mathbb{R})) \bigcap C^1([0, T]; B^{s-2}_{p,r}(\mathbb{R}) \times B^{s-1}_{p,r}(\mathbb{R}) \times B^{s-1}_{p,r}(\mathbb{R}))$.*

Let $\hat{\phi} \in \mathcal{C}^\infty_0(\mathbb{R})$ be an even, real-valued and non-negative function and satisfy

$$\hat{\phi}(x) = \begin{cases} 1, & \text{if} \quad |x| \le \frac{1}{4}, \\ 0, & \text{if} \quad |x| \ge \frac{1}{2}. \end{cases}$$

Define the function $g_n(x)$ by

$$g_n(x) := \phi(x) \cos\Big(\frac{17}{12} 2^n x\Big), n \ge 2.$$

Easy computations give that

$$\operatorname{supp} \widehat{\phi(\cdot)\cos\Big(\frac{17}{12} 2^n \cdot\Big)} \subset \Big\{ \xi : -\frac{1}{2} + \lambda 2^n \le |\xi| \le \frac{1}{2} + \lambda 2^n \Big\},$$

and

$$\Delta_j(g_n) = \begin{cases} g_n, & \text{if} \quad j = n, \\ 0, & \text{if} \quad j \ne n. \end{cases} \tag{7}$$

3. Proof of Theorem 1

In this section, we prove Theorem 1. Similar to the discussion in [21], the following lemma can be illustrated.

Lemma 5. *Let $l \ge 4$, $n \in \mathbb{N}^+$, and define the function $h^l_{m,n}(x)$ by*

$$h^l_{m,n}(x) := \phi(x) \cos\Big(\frac{17}{12}(2^{ln} \pm 2^{lm})x\Big), \quad 0 \le m \le n - 1.$$

Then we have

$$\Delta_j(h^l_{m,n}) = \begin{cases} h^l_{m,n}, & \text{if} \quad j = ln, \\ 0, & \text{if} \quad j \ne ln. \end{cases}$$

The proof of the above lemma is similar to that of [21], which is omitted here.

For $1 \leq p, r \leq \infty$, define the initial data (ρ_0, u_0, v_0) as

$$u_0(x) = v_0(x) := \sum_{n=0}^{\infty} 2^{-ln\sigma} \phi(x) \cos\left(\frac{17}{12} 2^{ln} x\right), \tag{8}$$

$$\rho_0(x) := \sum_{n=0}^{\infty} 2^{-ln(\sigma-1)} \phi(x) \cos\left(\frac{17}{12} 2^{ln} x\right). \tag{9}$$

In order to prove Theorem 1, we first give some estimation results.

Lemma 6. *Let $\sigma > 0$. Then for the above constructed initial data (ρ_0, u_0, v_0), we have*

$$\|u_0 v_0 \partial_x \Delta_{ln} u_0\|_{L^p} \geq C 2^{-ln(\sigma-1)}, \tag{10}$$

$$\|u_0 v_0 \partial_x \Delta_{ln} v_0\|_{L^p} \geq C 2^{-ln(\sigma-1)}, \tag{11}$$

$$\|u_0 v_0 \partial_x \Delta_{ln} \rho_0\|_{L^p} \geq C 2^{-ln(\sigma-2)}, \tag{12}$$

for some large enough n.

Proof. We just show (10) here, since (11) and (12) can be performed in a similar way. Firstly, according to Lemma 5, we derive that

$$\Delta_{ln} u_0(x) = 2^{-ln\sigma} \phi(x) \cos\left(\frac{17}{12} 2^{ln} x\right),$$

therefore,

$$\partial_x \Delta_{ln} u_0(x) = 2^{-ln\sigma} \phi'(x) \cos\left(\frac{17}{12} 2^{ln} x\right) - \frac{17}{12} 2^{-ln(\sigma-1)} \phi(x) \sin\left(\frac{17}{12} 2^{ln} x\right).$$

Thus, we have

$$u_0 v_0 \partial_x \Delta_{ln} u_0(x) = 2^{-ln\sigma} u_0 v_0 \phi'(x) \cos\left(\frac{17}{12} 2^{ln} x\right) - \frac{17}{12} 2^{-ln(\sigma-1)} u_0 v_0 \phi(x) \sin\left(\frac{17}{12} 2^{ln} x\right).$$

Since $u_0(x)$ and $v_0(x)$ are real-valued continuous functions on $\mathbb{R}$, then there exists some $\delta > 0$, such that

$$|u_0(x) v_0(x)| = |u_0^2(x)| \geq \frac{1}{2} |u^2(0)| = \frac{1}{2} \Big(\sum_{n=0}^{\infty} 2^{-ln\sigma} \phi(0) \Big)^2 = \frac{2^{l\sigma-1} \phi^2(0)}{(2^{l\sigma} - 1)^2}. \tag{13}$$

Thus, we have from (13)

$$\begin{aligned} \|u_0 v_0 \partial_x \Delta_{ln} u_0\|_{L_p} \geq & C 2^{-ln(\sigma-1)} \left\| \phi(x) \sin\left(\frac{17}{12} 2^{ln} x\right) \right\|_{L^p(B_\delta(0))} \\ & - C_1 2^{-ln\sigma} \left\| \phi'(x) \cos\left(\frac{17}{12} 2^{ln} x\right) \right\|_{L^p(B_\delta(0))} \\ \geq & (C 2^{ln} - C_1) 2^{-ln\sigma}. \end{aligned}$$

We choose n large enough such that $C_1 < \frac{C}{2} 2^{ln}$ and then finish the proof of (10). $\square$

Proposition 1. *Let* $s=\sigma-2$, $(\rho_0,u_0,v_0)\in B^{\sigma-1}_{p,\infty}(\mathbb{R})\times B^{\sigma}_{p,\infty}(\mathbb{R})\times B^{\sigma}_{p,\infty}(\mathbb{R})$ *with* $\sigma>\max\left\{3+\frac{1}{p},\frac{7}{2}\right\}, 1\le p\le\infty$. *Assume that* $(\rho,u,v)\in L^\infty\left(0,T;B^{\sigma-1}_{p,\infty}(\mathbb{R})\times B^{\sigma}_{p,\infty}(\mathbb{R})\times B^{\sigma}_{p,\infty}(\mathbb{R})\right)$ *is the solution to the Cauchy problem (6); we have*

$$\begin{aligned}
\|\rho(t)-\rho_0\|_{B^{s-1}_{p,\infty}} \le & Ct(\|u_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}\|\rho_0\|_{B^{s}_{p,\infty}}+\|u_0\|_{B^{s-1}_{p,\infty}}\|\rho_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}} \\
& +\|\rho_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}), \qquad (14)\\
\|u(t)-u_0\|_{B^{s-1}_{p,\infty}} \le & Ct(\|u_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}+\|u_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}} \\
& +\|\rho_0\|^2_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s-1}_{p,\infty}}+\|u_0\|^2_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}}), \qquad (15)\\
\|v(t)-v_0\|_{B^{s-1}_{p,\infty}} \le & Ct(\|v_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}}\|u_0\|_{B^{s-1}_{p,\infty}}+\|v_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}} \\
& +\|\rho_0\|^2_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}+\|v_0\|^2_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}), \qquad (16)\\
\|\rho(t)-\rho_0\|_{B^{s}_{p,\infty}} \le & Ct(\|u_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}\|\rho_0\|_{B^{s+1}_{p,\infty}}+\|u_0\|_{B^{s-1}_{p,\infty}}\|\rho_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s+1}_{p,\infty}} \\
& +\|\rho_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s+1}_{p,\infty}}+\|u_0\|_{B^{s-1}_{p,\infty}}\|\rho_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}} \\
& +\|\rho_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}}+\|\rho_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}), \qquad (17)\\
\|u(t)-u_0\|_{B^{s}_{p,\infty}} \le & Ct(\|u_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}+\|u_0\|^2_{B^{s}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}} \\
& +\|u_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}}+\|\rho_0\|^2_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s-1}_{p,\infty}}), \qquad (18)\\
\|v(t)-v_0\|_{B^{s}_{p,\infty}} \le & Ct(\|v_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}}\|u_0\|_{B^{s-1}_{p,\infty}}+\|v_0\|^2_{B^{s}_{p,\infty}}\|u_0\|_{B^{s-1}_{p,\infty}} \\
& +\|v_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}+\|\rho_0\|^2_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}), \qquad (19)\\
\|u(t)-u_0\|_{B^{s+1}_{p,\infty}} \le & Ct(\|u_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s+2}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}+\|u_0\|_{B^{s+1}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}} \\
& +\|u_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s+1}_{p,\infty}}+\|u_0\|^2_{B^{s}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}} \\
& +\|u_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s+1}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}}+\|\rho_0\|^2_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s-1}_{p,\infty}}), \qquad (20)\\
\|v(t)-v_0\|_{B^{s+1}_{p,\infty}} \le & Ct(\|v_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s+2}_{p,\infty}}\|u_0\|_{B^{s-1}_{p,\infty}}+\|v_0\|_{B^{s+1}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}}\|u_0\|_{B^{s-1}_{p,\infty}} \\
& +\|v_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}}\|u_0\|_{B^{s+1}_{p,\infty}}+\|v_0\|^2_{B^{s}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}} \\
& +\|v_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s+1}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}+\|\rho_0\|^2_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}). \qquad (21)
\end{aligned}$$

Proof. For $\delta>0$, according to the local existence result [33], the Cauchy problem (6) has a unique solution:

$$(\rho,u,v)\in L^\infty(0,T;B^{\delta}_{p,\infty}(\mathbb{R})\times B^{\delta}_{p,\infty}(\mathbb{R})\times B^{\delta}_{p,\infty}(\mathbb{R})),$$

and satisfies the following estimate:

$$\sup_{0\le t\le T}(\|\rho(t)\|_{B^{\delta}_{p,\infty}}+\|u(t)\|_{B^{\delta}_{p,\infty}}+\|v(t)\|_{B^{\delta}_{p,\infty}})\le C(\|\rho_0\|_{B^{\delta}_{p,\infty}}+\|u_0\|_{B^{\delta}_{p,\infty}}+\|v_0\|_{B^{\delta}_{p,\infty}}). \qquad (22)$$

Using the differential mean value theorem, the Minkowski inequality, (1) and (2) of Lemma 1, we deduce that

$$\begin{aligned}
&\|\rho(t)-\rho_0\|_{B^{s-1}_{p,\infty}} \le \int_0^t \|\partial_\tau \rho\|_{B^{s-1}_{p,\infty}} d\tau \\
\le& \int_0^t (\|uv\partial_x\rho\|_{B^{s-1}_{p,\infty}} + \|u\rho\partial_x v\|_{B^{s-1}_{p,\infty}} + \|\rho v\partial_x u\|_{B^{s-1}_{p,\infty}}) d\tau \\
\le& \int_0^t (\|u\|_{L_\infty}\|v\partial_x\rho\|_{B^{s-1}_{p,\infty}} + \|u\|_{B^{s-1}_{p,\infty}}\|v\partial_x\rho\|_{L_\infty} + \|u\|_{L_\infty}\|\rho\partial_x v\|_{B^{s-1}_{p,\infty}} \\
&+ \|u\|_{B^{s-1}_{p,\infty}}\|\rho\partial_x v\|_{L_\infty} + \|\rho\|_{L_\infty}\|v\partial_x u\|_{B^{s-1}_{p,\infty}} + \|\rho\|_{B^{s-1}_{p,\infty}}\|\partial_x uv\|_{L_\infty}) d\tau \\
\le& \int_0^t (\|u\|_{B^{s-1}_{p,\infty}}\|v\|_{B^{s-1}_{p,\infty}}\|\rho\|_{B^{s}_{p,\infty}} + \|u\|_{B^{s-1}_{p,\infty}}\|v\|_{B^{s-1}_{p,\infty}}\|\rho\|_{B^{s}_{p,\infty}} + \|u\|_{B^{s-1}_{p,\infty}}\|\rho\|_{B^{s-1}_{p,\infty}}\|v\|_{B^{s}_{p,\infty}} \\
&+ \|u\|_{B^{s-1}_{p,\infty}}\|\rho\|_{B^{s-1}_{p,\infty}}\|v\|_{B^{s}_{p,\infty}} + \|\rho\|_{B^{s-1}_{p,\infty}}\|v\|_{B^{s-1}_{p,\infty}}\|u\|_{B^{s}_{p,\infty}} + \|\rho\|_{B^{s-1}_{p,\infty}}\|u\|_{B^{s}_{p,\infty}}\|v\|_{B^{s-1}_{p,\infty}}) d\tau \\
\le& C\int_0^t (\|u\|_{B^{s-1}_{p,\infty}}\|v\|_{B^{s-1}_{p,\infty}}\|\rho\|_{B^{s}_{p,\infty}} + \|u\|_{B^{s-1}_{p,\infty}}\|\rho\|_{B^{s-1}_{p,\infty}}\|v\|_{B^{s}_{p,\infty}} + \|\rho\|_{B^{s-1}_{p,\infty}}\|v\|_{B^{s-1}_{p,\infty}}\|u\|_{B^{s}_{p,\infty}}) d\tau \\
\le& Ct(\|u_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}\|\rho_0\|_{B^{s}_{p,\infty}} + \|u_0\|_{B^{s-1}_{p,\infty}}\|\rho_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}} + \|\rho_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}).
\end{aligned}$$

Thus, we complete the proof of (14). Next we will present the proof of (15).

$$\begin{aligned}
&\|u(t)-u_0\|_{B^{s-1}_{p,\infty}} \le \int_0^t \|\partial_\tau u\|_{B^{s-1}_{p,\infty}} d\tau \\
\le& \int_0^t (\|u\partial_x uv\|_{B^{s-1}_{p,\infty}} + \|u\partial_x uv\|_{B^{s-3}_{p,\infty}} + \|u_x^2 v_x\|_{B^{s-3}_{p,\infty}} + \|u\partial_x^2 u\partial_x v\|_{B^{s-3}_{p,\infty}} + \|\rho^2 u\|_{B^{s-3}_{p,\infty}} + \|u\partial_x u\partial_x v\|_{B^{s-2}_{p,\infty}}) d\tau \\
\le& C\int_0^t (\|u\|_{B^{s-1}_{p,\infty}}\|u\|_{B^{s}_{p,\infty}}\|v\|_{B^{s-1}_{p,\infty}} + \|u\|_{B^{s-1}_{p,\infty}}\|u\|_{B^{s}_{p,\infty}}\|v\|_{B^{s}_{p,\infty}} + \|\rho\|^2_{B^{s-1}_{p,\infty}}\|u\|_{B^{s-1}_{p,\infty}} + \|u\|^2_{B^{s-1}_{p,\infty}}\|v\|_{B^{s}_{p,\infty}}) d\tau \\
\le& Ct(\|u_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}} + \|u_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}} + \|\rho_0\|^2_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s-1}_{p,\infty}} + \|u_0\|^2_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}}).
\end{aligned}$$

Thus, (15) is proved and (16) can be proved in a similar manner. Next, we will give the proof of (17).

$$\begin{aligned}
&\|\rho(t)-\rho_0\|_{B^{s}_{p,\infty}} \le \int_0^t \|\partial_\tau \rho\|_{B^{s}_{p,\infty}} d\tau \\
\le& C\int_0^t (\|u\|_{B^{s-1}_{p,\infty}}\|v\|_{B^{s-1}_{p,\infty}}\|\rho\|_{B^{s+1}_{p,\infty}} + \|u\|_{B^{s-1}_{p,\infty}}\|\rho\|_{B^{s-1}_{p,\infty}}\|v\|_{B^{s+1}_{p,\infty}} + \|\rho\|_{B^{s-1}_{p,\infty}}\|v\|_{B^{s-1}_{p,\infty}}\|u\|_{B^{s+1}_{p,\infty}} \\
&+ \|u\|_{B^{s-1}_{p,\infty}}\|\rho\|_{B^{s}_{p,\infty}}\|v\|_{B^{s}_{p,\infty}} + \|\rho\|_{B^{s-1}_{p,\infty}}\|u\|_{B^{s}_{p,\infty}}\|v\|_{B^{s}_{p,\infty}} + \|\rho\|_{B^{s}_{p,\infty}}\|v\|_{B^{s-1}_{p,\infty}}\|u\|_{B^{s}_{p,\infty}}) d\tau, \\
\le& Ct(\|u_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}\|\rho_0\|_{B^{s+1}_{p,\infty}} + \|u_0\|_{B^{s-1}_{p,\infty}}\|\rho_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s+1}_{p,\infty}} + \|\rho_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s+1}_{p,\infty}} \\
&+ \|u_0\|_{B^{s-1}_{p,\infty}}\|\rho_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}} + \|\rho_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}} + \|\rho_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}).
\end{aligned}$$

Thereafter, we shall present the proof of (18).

$$\begin{aligned}
&\|u(t)-u_0\|_{B^{s}_{p,\infty}} \le \int_0^t \|\partial_\tau u\|_{B^{s}_{p,\infty}} d\tau \\
\le& \int_0^t (\|u\partial_x uv\|_{B^{s}_{p,\infty}} + \|u\partial_x uv\|_{B^{s-2}_{p,\infty}} + \|(\partial_x u)^2\partial_x v\|_{B^{s-2}_{p,\infty}} + \|u\partial_x^2 u\partial_x v\|_{B^{s-2}_{p,\infty}} + \|\rho^2 u\|_{B^{s-2}_{p,\infty}} + \|u\partial_x u\partial_x v\|_{B^{s-1}_{p,\infty}}) d\tau \\
\le& C\int_0^t (\|u\|_{B^{s-1}_{p,\infty}}\|u\|_{B^{s}_{p,\infty}}\|v\|_{B^{s-1}_{p,\infty}} + \|u\|^2_{B^{s}_{p,\infty}}\|v\|_{B^{s-1}_{p,\infty}} + \|u\|_{B^{s-1}_{p,\infty}}\|u\|_{B^{s}_{p,\infty}}\|v\|_{B^{s}_{p,\infty}} + \|\rho\|^2_{B^{s-1}_{p,\infty}}\|u\|_{B^{s-1}_{p,\infty}}) d\tau \\
\le& Ct(\|u_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}} + \|u_0\|^2_{B^{s}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}} + \|u_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}} + \|\rho_0\|^2_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s-1}_{p,\infty}}).
\end{aligned}$$

Similarly, we can obtain (19). Following this, we will furnish the evidence for (20).

$$
\begin{aligned}
&\|u(t)-u_0\|_{B^{s+1}_{p,\infty}} \le \int_0^t \|\partial_\tau u\|_{B^{s+1}_{p,\infty}} d\tau \\
\le &\int_0^t (\|u\partial_x uv\|_{B^{s+1}_{p,\infty}} + \|u\partial_x uv\|_{B^{s-1}_{p,\infty}} + \|(\partial_x u)^2\partial_x v\|_{B^{s-1}_{p,\infty}} + \|u\partial_x^2 u\partial_x v\|_{B^{s-1}_{p,\infty}} + \|\rho^2 u\|_{B^{s-1}_{p,\infty}} + \|u\partial_x u\partial_x v\|_{B^{s}_{p,\infty}}) d\tau \\
\le &C\int_0^t (\|u\|_{B^{s-1}_{p,\infty}}\|u\|_{B^{s+2}_{p,\infty}}\|v\|_{B^{s-1}_{p,\infty}} + \|u\|_{B^{s+1}_{p,\infty}}\|u\|_{B^{s}_{p,\infty}}\|v\|_{B^{s-1}_{p,\infty}} + \|u\|_{B^{s-1}_{p,\infty}}\|u\|_{B^{s}_{p,\infty}}\|v\|_{B^{s+1}_{p,\infty}} \\
&+ \|u\|^2_{B^{s}_{p,\infty}}\|v\|_{B^{s}_{p,\infty}} + \|u\|_{B^{s-1}_{p,\infty}}\|u\|_{B^{s+1}_{p,\infty}}\|v\|_{B^{s}_{p,\infty}} + \|\rho\|^2_{B^{s-1}_{p,\infty}}\|u\|_{B^{s-1}_{p,\infty}}) d\tau \\
\le &Ct(\|u_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s+2}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}} + \|u_0\|_{B^{s+1}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}} + \|u_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s+1}_{p,\infty}} \\
&+ \|u_0\|^2_{B^{s}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}} + \|u_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s+1}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}} + \|\rho_0\|^2_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s-1}_{p,\infty}}).
\end{aligned}
$$

Equation (21) can be verified similarly. Thus, we complete the proof of Proposition 1. □

Proposition 2. *Let* $s = \sigma - 2$, $(\rho_0, u_0, v_0) \in B^{\sigma-1}_{p,\infty}(\mathbb{R}) \times B^{\sigma}_{p,\infty}(\mathbb{R}) \times B^{\sigma}_{p,\infty}(\mathbb{R})$ *with* $\sigma > \max\left\{3+\frac{1}{p}, \frac{7}{2}\right\}, 1 \le p \le \infty$. *Assume that* $(\rho, u, v) \in L^\infty\left(0, T; B^{\sigma-1}_{p,\infty}(\mathbb{R}) \times B^{\sigma}_{p,\infty}(\mathbb{R}) \times B^{\sigma}_{p,\infty}(\mathbb{R})\right)$ *is the solution of the Cauchy problem (6); we have*

$$
\begin{aligned}
\|\rho(t)-\rho_0-t\boldsymbol{\rho}_0\|_{B^{s-1}_{p,r}} \le &Ct^2(\|u_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}} + \|u_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}} \\
&+ \|u_0\|_{B^{s-1}_{p,\infty}}\|\rho_0\|_{B^{s}_{p,\infty}} + \|u_0\|_{B^{s}_{p,\infty}}\|\rho_0\|_{B^{s-1}_{p,\infty}} \\
&+ \|v_0\|_{B^{s}_{p,\infty}}\|\rho_0\|_{B^{s-1}_{p,\infty}} + \|v_0\|_{B^{s-1}_{p,\infty}}\|\rho_0\|_{B^{s}_{p,\infty}}), \qquad (23)\\
\|u(t)-u_0-t\mathbf{u}_0\|_{B^{s}_{p,r}} \le &Ct^2(\|u_0\|^2_{B^{s}_{p,\infty}} + \|v_0\|^2_{B^{s}_{p,\infty}} + \|\rho_0\|^2_{B^{s}_{p,\infty}} \\
&+ \|u_0\|_{B^{s+1}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}} + \|u_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s+1}_{p,\infty}}), \qquad (24)\\
\|v(t)-v_0-t\mathbf{u}_0\|_{B^{s}_{p,r}} \le &Ct^2(\|u_0\|^2_{B^{s}_{p,\infty}} + \|v_0\|^2_{B^{s}_{p,\infty}} + \|\rho_0\|^2_{B^{s}_{p,\infty}} \\
&+ \|u_0\|_{B^{s+1}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}} + \|u_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s+1}_{p,\infty}}). \qquad (25)
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{\rho}_0 &= u_0 v_0 \partial_x \rho_0 + u_0 \rho_0 \partial_x v_0 + \rho_0 v_0 \partial_x u_0, \\
\mathbf{u}_0 &= u_0 v_0 \partial_x u_0 + P_1(u_0, v_0) + R_1(u_0, \rho_0), \\
\mathbf{v}_0 &= u_0 v_0 \partial_x v_0 + P_2(u_0, v_0) + R_2(v_0, \rho_0).
\end{aligned}
$$

Proof. For simplicity, denote

$$
\begin{cases}
\boldsymbol{\rho} = \rho(t) - \rho_0 - t\boldsymbol{\rho}_0, \\
\mathbf{u} = u(t) - u_0 - t\mathbf{u}_0, \\
\mathbf{v} = v(t) - v_0 - t\mathbf{v}_0.
\end{cases}
$$

Combining the differential mean value theorem and the Minkowski inequality, we arrive at

$$
\begin{aligned}
&\|\boldsymbol{\rho}_0\|_{B^{s-1}_{p,\infty}} \le \int_0^t \|\partial_\tau \rho - \boldsymbol{\rho}_0\|_{B^{s-1}_{p,\infty}} d\tau \\
\le &\int_0^t (\|uv\partial_x\rho - u_0v_0\partial_x\rho_0\|_{B^{s-1}_{p,\infty}} + \|u\rho\partial_x v - u_0v_0\partial_x\rho_0\|_{B^{s-1}_{p,\infty}} + \|\rho v\partial_x u - \rho_0 v_0\partial_x u_0\|_{B^{s-1}_{p,\infty}}) d\tau \\
\triangleq &\int_0^t (I_1(\tau) + I_2(\tau) + I_3(\tau)) d\tau. \qquad (26)
\end{aligned}
$$

Note that

$$\begin{aligned}uv\partial_x\rho - u_0v_0\partial_x\rho_0 &= uv\partial_x\rho - u_0v_0\partial_x\rho + u_0v_0\partial_x\rho - u_0v_0\partial_x\rho_0\\ &= (uv-u_0v_0)\partial_x\rho + u_0v_0\partial_x(\rho-\rho_0)\\ &= (uv-u_0v+u_0v-u_0v_0)\partial_x\rho + u_0v_0\partial_x(\rho-\rho_0)\\ &= (u-u_0)v\partial_x\rho + u_0(v-v_0)\partial_x\rho + u_0v_0\partial_x(\rho-\rho_0).\end{aligned}$$

We can deduce

$$\begin{aligned}I_1(\tau) \leq& \|(u-u_0)v\partial_x\rho\|_{B^{s-1}_{p,\infty}} + \|u_0(v-v_0)\partial_x\rho\|_{B^{s-1}_{p,\infty}} + \|u_0v_0\partial_x(\rho-\rho_0)\|_{B^{s-1}_{p,\infty}}\\ \triangleq& \int_0^t (I_{11}(\tau)+I_{12}(\tau)+I_{13}(\tau))d\tau, \qquad (27)\\ I_{11}(\tau) \leq& \|u-u_0\|_{L^\infty}\|v\partial_x\rho\|_{B^{s-1}_{p,\infty}} + \|u-u_0\|_{B^{s-1}_{p,\infty}}\|\partial_x\rho\|_{L^\infty}\\ \leq& \|u-u_0\|_{B^{s-1}_{p,\infty}}(\|v\|_{L^\infty}\|\partial_x\rho\|_{B^{s-1}_{p,\infty}} + \|v\|_{B^{s-1}_{p,\infty}}\|\partial_x\rho\|_{L^\infty})\\ &+ \|u-u_0\|_{B^{s-1}_{p,\infty}}(\|v\|_{L^\infty}\|\partial_x\rho\|_{B^{s-1}_{p,\infty}} + \|v\|_{B^{s-1}_{p,\infty}}\|\partial_x\rho\|_{L^\infty})\\ \leq& \|u-u_0\|_{B^{s-1}_{p,\infty}}\|v\|_{B^{s-1}_{p,\infty}}\|\rho\|_{B^{s}_{p,\infty}} \leq C\|\rho_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}\|u-u_0\|_{B^{s-1}_{p,\infty}}\\ \leq& C\|\rho_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}\tau. \qquad (28)\end{aligned}$$

Analogously,

$$I_{12}(\tau) \leq \|u_0\|_{B^{s-1}_{p,\infty}}\|\rho\|_{B^{s}_{p,\infty}}\|v-v_0\|_{B^{s-1}_{p,\infty}} \leq \|u_0\|_{B^{s-1}_{p,\infty}}\|\rho_0\|_{B^{s}_{p,\infty}}\tau, \qquad (29)$$

$$I_{13}(\tau) \leq \|u_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}\|\rho-\rho_0\|_{B^{s}_{p,\infty}} \leq \|u_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}\tau. \qquad (30)$$

For the second term $I_2(\tau)$, since

$$\begin{aligned}u\rho\partial_x v - u_0\rho_0\partial_x v_0 =& u\rho\partial_x v - u_0\rho_0\partial_x v + u_0\rho_0\partial_x v - u_0\rho_0\partial_x v_0\\ =& (u\rho-u_0\rho_0)\partial_x v + u_0\rho_0\partial_x(v-v_0)\\ =& (u-u_0)\rho\partial_x v + u_0(\rho-\rho_0)\partial_x v + u_0\rho_0\partial_x(v-v_0),\end{aligned}$$

there holds

$$\begin{aligned}I_2(\tau) \leq& \|\rho\|_{B^{s-1}_{p,\infty}}\|v\|_{B^{s}_{p,\infty}}\|u-u_0\|_{B^{s-1}_{p,\infty}} + \|u_0\|_{B^{s-1}_{p,\infty}}\|v\|_{B^{s}_{p,\infty}}\|\rho-\rho_0\|_{B^{s-1}_{p,\infty}}\\ &+ \|u_0\|_{B^{s-1}_{p,\infty}}\|\rho_0\|_{B^{s-1}_{p,\infty}}\|v-v_0\|_{B^{s}_{p,\infty}}\\ \leq& C(\|\rho_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}} + \|u_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s}_{p,\infty}} + \|u_0\|_{B^{s-1}_{p,\infty}}\|\rho_0\|_{B^{s-1}_{p,\infty}})\tau. \qquad (31)\end{aligned}$$

For the third term $I_3(\tau)$, since

$$\rho v\partial_x u - \rho_0v_0\partial_x u_0 = (\rho-\rho_0)v\partial_x u + \rho_0(v-v_0)\partial_x u + \rho_0v_0\partial_x(u-u_0).$$

We obtain

$$\begin{aligned}I_3(\tau) \leq& \|u\|_{B^{s}_{p,\infty}}\|v\|_{B^{s-1}_{p,\infty}}\|\rho-\rho_0\|_{B^{s-1}_{p,\infty}} + \|\rho_0\|_{B^{s-1}_{p,\infty}}\|u\|_{B^{s}_{p,\infty}}\|v-v_0\|_{B^{s-1}_{p,\infty}}\\ &+ \|\rho_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}\|u-u_0\|_{B^{s}_{p,\infty}}\\ \leq& C(\|u_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}} + \|\rho_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s}_{p,\infty}} + \|\rho_0\|_{B^{s}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}})\tau. \qquad (32)\end{aligned}$$

Substituting (28)–(30) into (26), we derive

$$\begin{aligned}\|\rho_0\|_{B^{s-1}_{p,\infty}} \leq & Ct^2(\|u_0\|_{B^s_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}} + \|u_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^s_{p,\infty}} + \|u_0\|_{B^{s-1}_{p,\infty}}\|\rho_0\|_{B^s_{p,\infty}} \\ & + \|u_0\|_{B^s_{p,\infty}}\|\rho_0\|_{B^{s-1}_{p,\infty}} + \|v_0\|_{B^{s-1}_{p,\infty}}\|\rho_0|_{B^s_{p,\infty}} + \|v_0\|_{B^s_{p,\infty}}\|\rho_0\|_{B^{s-1}_{p,\infty}}).\end{aligned}$$

Thus, we finish the proof of (23). Next, we intend to establish the proof of (24).

$$\begin{aligned}\|\mathbf{u}_0\|_{B^s_{p,\infty}} \leq & \int_0^t \|\partial_\tau u - \mathbf{u}_0\|_{B^s_{p,\infty}} d\tau \\ \leq & \int_0^t \|uu_x v - u_0\partial_x u_0 v_0\|_{B^s_{p,\infty}} d\tau + \int_0^t \|P_1(u,v) - P_1(u_0,v_0)\|_{B^s_{p,\infty}} d\tau \\ & + \int_0^t \|R_1(u,\rho) - R_1(u_0,\rho_0)\|_{B^s_{p,\infty}} d\tau \\ \triangleq & \int_0^t (J_1(\tau) + J_2(\tau) + J_3(\tau)) d\tau. \end{aligned} \tag{33}$$

For the first term $J_1(\tau)$, since

$$\begin{aligned}uv\partial_x u - u_0 v_0 \partial_x u_0 =& uv\partial_x u - u_0 v\partial_x u + u_0 v\partial_x u - u_0 v_0 \partial_x u_0 \\ =&(u-u_0)v\partial_x u - u_0(v\partial_x u - v\partial_x u_0 + v\partial_x u_0 - v_0\partial_x u_0) \\ =&(u-u_0)v\partial_x u - u_0 v\partial_x(u-u_0) - u_0(v-v_0)\partial_x u_0.\end{aligned}$$

We can perform the following estimate:

$$\begin{aligned}J_1(\tau) \leq& \|(u-u_0)v\partial_x u\|_{B^s_{p,\infty}} + \|u_0 v\partial_x(u-u_0)\|_{B^s_{p,\infty}} + \|u_0(v-v_0)\partial_x u_0\|_{B^s_{p,\infty}} \\ \triangleq & \int_0^t (J_{11}(\tau) + J_{12}(\tau) + J_{13}(\tau)) d\tau.\end{aligned} \tag{34}$$

By applying Lemma 1, one obtains

$$\begin{aligned}J_{11}(\tau) \leq& \|u-u_0\|_{L^\infty}\|v\partial_x u\|_{B^s_{p,\infty}} + \|uu_o\|_{B^s_{p,\infty}}\|v\partial_x u\|_{L^\infty} \\ \leq& \|u-u_0\|_{L^\infty}\|\partial_x u\|_{L^\infty}\|v\|_{B^s_{p,\infty}} + \|u-u_0\|_{L^\infty}\|\partial_x u\|_{B^s_{p,\infty}}\|v\|_{L^\infty} + \|u-u_0\|_{B^s_{p,\infty}}\|\partial_x u\|_{L^\infty}\|v\|_{L^\infty} \\ \leq& \|u\|_{B^s_{p,\infty}}\|v\|_{B^s_{p,\infty}}\|u-u_0\|_{B^{s-1}_{p,\infty}} + \|u\|_{B^{s+1}_{p,\infty}}\|v\|_{B^{s-1}_{p,\infty}}\|u-u_0\|_{B^{s-1}_{p,\infty}} + \|u\|_{B^s_{p,\infty}}\|v\|_{B^{s-1}_{p,\infty}}\|u-u_0\|_{B^s_{p,\infty}} \\ \leq& C(\|u_0\|_{B^s_{p,\infty}}\|v_0\|_{B^s_{p,\infty}} + \|u_0\|_{B^{s+1}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}})\tau,\end{aligned} \tag{35}$$

$$\begin{aligned}J_{12}(\tau) \leq& \|u_0\|_{L^\infty}\|v\partial_x(u-u_0)\|_{B^s_{p,\infty}} + \|u_0\|_{B^s_{p,\infty}}\|v\partial_x(u-u_0)\|_{L^\infty} \\ \leq& \|u_0\|_{B^{s-1}_{p,\infty}}\|\partial_x(u-u_0)\|_{L^\infty}\|v\|_{B^s_{p,\infty}} + \|u_0\|_{B^{s-1}_{p,\infty}}\|\partial_x(u-u_0)\|_{B^s_{p,\infty}}\|v\|_{L^\infty} + \|u_0\|_{B^s_{p,\infty}}\|u-u_0\|_{B^s_{p,\infty}}\|v\|_{L^\infty} \\ \leq& \|u_0\|_{B^{s-1}_{p,\infty}}\|v\|_{B^s_{p,\infty}}\|u-u_0\|_{B^s_{p,\infty}} + \|u_0\|_{B^{s-1}_{p,\infty}}\|v\|_{B^{s-1}_{p,\infty}}\|u-u_0\|_{B^{s+1}_{p,\infty}} \\ & + \|u_0\|_{B^{s-1}_{p,\infty}}\|u\|_{B^{s-1}_{p,\infty}}\|u-u_0\|_{B^s_{p,\infty}} \\ \leq& C\|u_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^s_{p,\infty}}\tau,\end{aligned} \tag{36}$$

$$\begin{aligned}J_{13}(\tau) \leq& \|u_0\|_{L^\infty}\|(v-v_0)\partial_x u_0\|_{B^s_{p,\infty}} + \|u_0\|_{B^s_{p,\infty}}\|(v-v_0)\partial_x u_0\|_{L^\infty} \\ \leq& \|u_0\|_{L^\infty}(\|\partial_x u_0\|_{L^\infty}\|v-v_0\|_{B^s_{p,\infty}} + \|\partial_x u_0\|_{B^s_{p,\infty}}\|v-v_0\|_{L^\infty}) + \|u_0\|_{B^s_{p,\infty}}\|u_0\|_{B^s_{p,\infty}}\|v-v_0\|_{B^{s-1}_{p,\infty}} \\ \leq& \|u_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^s_{p,\infty}}\|v-v_0\|_{B^s_{p,\infty}} + \|u_0\|_{B^{s+1}_{p,\infty}}\|u_0\|_{B^{s-1}_{p,\infty}}\|v-v_0\|_{B^{s-1}_{p,\infty}} + \|u_0\|^2_{B^s_{p,\infty}}\|v-v_0\|_{B^{s-1}_{p,\infty}} \\ \leq& C(\|u_0\|^2_{B^s_{p,\infty}} + \|u_0\|_{B^{s+1}_{p,\infty}}\|u_0\|_{B^{s-1}_{p,\infty}})\tau.\end{aligned} \tag{37}$$

For the term $P_{12}(u,v) - P_{12}(u_0,v_0)$, it can be resolved into

$$(\partial_x u)^2\partial_x v - (\partial_x u_0)^2\partial_x v_0 = (\partial_x u + \partial_x u_0)\partial_x(u-u_0)\partial_x v + (\partial_x u_0)^2\partial_x(v-v_0).$$

We can find that

$$\begin{aligned}
&\|(\partial_x u)^2\partial_x v-(\partial_x u_0)^2\partial_x v_0\|_{B^{s-2}_{p,\infty}}\\
\leq&\|\partial_x(u+u_0)\partial_x(u-u_0)\partial_x v\|_{B^{s-2}_{p,\infty}}+\|(\partial_x u_0)^2\partial_x(v-v_0)\|_{B^{s-2}_{p,\infty}}\\
\leq&\|\partial_x(u+v)\partial_x v\|_{B^{s-1}_{p,\infty}}\|\partial_x(u-u_0)\|_{B^{s-2}_{p,\infty}}+\|(\partial_x u_0)^2\|_{B^{s-1}_{p,\infty}}\|\partial_x(v-v_0)\|_{B^{s-2}_{p,\infty}}\\
\leq&(\|u\|_{B^s_{p,\infty}}+\|v\|_{B^s_{p,\infty}})\|v\|_{B^s_{p,\infty}}\|u-u_0\|_{B^{s-1}_{p,\infty}}+\|u_0\|^2_{B^s_{p,\infty}}\|v-v_0\|_{B^{s-1}_{p,\infty}}\\
\leq&C(\|u_0\|^2_{B^s_{p,\infty}}+\|v_0\|^2_{B^s_{p,\infty}})\tau.
\end{aligned}\tag{38}$$

For the term $P_{13}(u,v)-P_{13}(u_0,v_0)$, since

$$u\partial_x^2 u\partial_x v-u_0\partial_x^2 u_0\partial_x v_0=(u-u_0)\partial_x^2 u\partial_x v+u_0\partial_x v\partial_x^2(u-u_0)+u_0\partial_x^2 u_0\partial_x(v-v_0),$$

we arrive at

$$\begin{aligned}
&\|u\partial_x^2 u\partial_x v-u_0\partial_x^2 u_0\partial_x v_0\|_{B^{s-2}_{p,\infty}}\\
\leq&\|(u-u_0)\partial_x^2 u\partial_x v\|_{B^{s-2}_{p,\infty}}+\|u_0\partial_x v\partial_x^2(u-u_0)\|_{B^{s-2}_{p,\infty}}+\|u_0\partial_x^2 u_0\partial_x(v-v_0)\|_{B^{s-2}_{p,\infty}}\\
\leq&\|\partial_x^2 u\partial_x v\|_{B^{s-2}_{p,\infty}}\|u-u_0\|_{B^{s-1}_{p,\infty}}+\|u_0\partial_x v\|_{B^{s-1}_{p,\infty}}\|u-u_0\|_{B^s_{p,\infty}}+\|u_0\partial_x^2 u_0\|_{B^{s-1}_{p,\infty}}\|v-v_0\|_{B^{s-1}_{p,\infty}}\\
\leq&\|u\|_{B^s_{p,\infty}}\|v\|_{B^s_{p,\infty}}\|u-u_0\|_{B^{s-1}_{p,\infty}}+\|u_0\|_{B^{s-1}_{p,\infty}}\|v\|_{B^s_{p,\infty}}\|u-u_0\|_{B^s_{p,\infty}}+\|u_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^{s+1}_{p,\infty}}\|v-v_0\|_{B^{s+1}_{p,\infty}}\\
\leq&C(\|u_0\|_{B^s_{p,\infty}}\|v_0\|_{B^s_{p,\infty}}+\|u_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s+1}_{p,\infty}})\tau.
\end{aligned}\tag{39}$$

For the term $R_1(u,\rho)-R_1(u_0,\rho_0)$, it can be decomposed as

$$\rho^2 u-\rho_0^2 u_0=(\rho+\rho_0)u(\rho-\rho_0)+\rho_0^2(u-u_0),$$

which gives us that

$$\begin{aligned}
&\|\rho^2 u-\rho_0^2 u_0\|_{B^{s-2}_{p,\infty}}\\
\leq&\|(\rho+\rho_0)(\rho-\rho_0)u\|_{B^{s-2}_{p,\infty}}+\|\rho_0^2(u-u_0)\|_{B^{s-2}_{p,\infty}}\\
\leq&\|(\rho+\rho_0)u\|_{B^{s-2}_{p,\infty}}\|\rho-\rho_0\|_{B^{s-1}_{p,\infty}}+\|\rho_0\|_{B^{s-2}_{p,\infty}}\|\rho_0\|_{B^{s-1}_{p,\infty}}\|u-u_0\|_{B^{s-1}_{p,\infty}}\\
\leq&(\|\rho\|_{B^s_{p,\infty}}+\|\rho_0\|_{B^s_{p,\infty}})\|u\|_{B^{s-1}_{p,\infty}}\|\rho-\rho_0\|_{B^{s-1}_{p,\infty}}+\|\rho_0\|^2_{B^s_{p,\infty}}\|u-u_0\|_{B^{s-1}_{p,\infty}}\\
\leq&C(\|\rho_0\|_{B^s_{p,\infty}}\|u_0\|_{B^{s-1}_{p,\infty}}+\|\rho_0\|^2_{B^s_{p,\infty}})\tau.
\end{aligned}\tag{40}$$

Finally, for the term $P_{14}(u,v)-P_{14}(u_0,v_0)$, since

$$u\partial_x u\partial_x v-u_0\partial_x u_0\partial_x v_0=(u-u_0)\partial_x u\partial_x v+u_0\partial_x v\partial_x(u-u_0)+u_0\partial_x u_0\partial_x(v-v_0),$$

we can derive

$$\begin{aligned}
&\|u\partial_x u\partial_x v-u_0\partial_x u_0\partial_x v_0\|_{B^{s-1}_{p,\infty}}\\
\leq&\|(u-u_0)\partial_x u\partial_x v\|_{B^{s-1}_{p,\infty}}+\|u_0\partial_x v\partial_x(u-u_0)\|_{B^{s-1}_{p,\infty}}+\|u_0\partial_x u_0\partial_x(v-v_0)\|_{B^{s-1}_{p,\infty}}\\
\leq&\|u_0\|_{B^s_{p,\infty}}\|v_0\|_{B^s_{p,\infty}}\|u-u_0\|_{B^s_{p,\infty}}+\|u_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^s_{p,\infty}}\|v-v_0\|_{B^s_{p,\infty}}\\
\leq&C(\|u_0\|_{B^{s-1}_{p,\infty}}\|u_0\|_{B^s_{p,\infty}}+\|u_0\|_{B^s_{p,\infty}}\|v_0\|_{B^s_{p,\infty}})\tau.
\end{aligned}\tag{41}$$

Inserting (35)–(41) into (33), we obtain

$$\|\mathbf{u}\|_{B^s_{p,\infty}}\leq Ct^2\Big(\|u_0\|^2_{B^s_{p,\infty}}+\|v_0\|^2_{B^s_{p,\infty}}+\|\rho_0\|^2_{B^s_{p,\infty}}+\|u_0\|_{B^{s+1}_{p,\infty}}\|v_0\|_{B^{s-1}_{p,\infty}}+\|u_0\|_{B^{s-1}_{p,\infty}}\|v_0\|_{B^{s+1}_{p,\infty}}\Big).$$

Thus, the proof of (24) is complete. Similarly, we can prove (25). □

With Lemma 5, Proposition 1, and Proposition 2 at hand, we will proceed to verify Theorem 1.

Proof of Theorem 1. By the definition of the Besov norm, we have

$$\begin{aligned}\|\rho-\rho_0\|_{B^{\sigma-1}_{p,\infty}} \geq & 2^{ln(\sigma-1)}\|\Delta_{ln}(\rho-\rho_0)\|_{L^p} = 2^{ln(\sigma-1)}\|\Delta_{ln}(\boldsymbol{\rho}+t\boldsymbol{\rho}_0)\|_{L^p} \\ \geq & t2^{ln(\sigma-1)}\|\Delta_{ln}(u_0v_0\partial_x\rho_0+u_0\rho_0\partial_x v_0+\rho_0v_0\partial_x u_0)\|_{L^p} - 2^{ln(\sigma-1)}\|\Delta_{ln}\boldsymbol{\rho}\|_{L^p} \\ \geq & t2^{ln(\sigma-1)}\|\Delta_{ln}(u_0v_0\partial_x\rho_0)\|_{L^p} - t2^{ln(\sigma-1)}\|\Delta_{ln}(\rho_0u_0\partial_x v_0)\|_{L^p} \\ & - t2^{ln(\sigma-1)}\|\Delta_{ln}(\rho_0v_0\partial_x u_0)\|_{L^p} - 2^{ln(\sigma-3)}\cdot 2^{2ln}\|\Delta_{ln}\boldsymbol{\rho}\|_{L^p} \\ \geq & t2^{ln(\sigma-1)}\|\Delta_{ln}(u_0v_0\partial_x\rho_0)\|_{L^p} - t\|\rho_0u_0\partial_x v_0\|_{B^{\sigma-1}_{p,\infty}} \\ & - t\|\rho_0v_0\partial_x u_0\|_{B^{\sigma-1}_{p,\infty}} - 2^{2ln}\|\boldsymbol{\rho}\|_{B^{\sigma-3}_{p,\infty}}. \end{aligned} \tag{42}$$

Since

$$\begin{aligned}\Delta_{ln}(u_0v_0\partial_x\rho_0) =& \Delta_{ln}(u_0v_0\partial_x\rho_0) - u_0v_0\partial_x\Delta_{ln}\rho_0 + u_0v_0\partial_x\Delta_{ln}\rho_0 \\ =& [\Delta_{ln}, u_0v_0\partial_x]\rho_0 + u_0v_0\partial_x\Delta_{ln}\rho_0.\end{aligned}$$

With the aid of Lemma 3, we infer that

$$\begin{aligned}\|2^{ln(\sigma-1)}[\Delta_{ln}, u_0v_0\partial_x]\rho_0\|_{L^\infty} \leq & C\|\partial_x(u_0v_0)\|_{L^\infty}\|\rho_0\|_{B^{\sigma-1}_{p,\infty}} + C\|\partial_x\rho_0\|_{L^\infty}\|u_0v_0\|_{B^{\sigma-1}_{p,\infty}} \\ \leq & C\|u_0\|_{B^{\sigma}_{p,\infty}}\|v_0\|_{B^{\sigma}_{p,\infty}}\|\rho_0\|_{B^{\sigma-1}_{p,\infty}} \leq C, \end{aligned} \tag{43}$$

$$\|\rho_0u_0\partial_x v_0\|_{B^{\sigma-1}_{p,\infty}} \leq \|\rho_0\|_{B^{\sigma-1}_{p,\infty}}\|u_0\|_{B^{\sigma-1}_{p,\infty}}\|v_0\|_{B^{\sigma}_{p,\infty}} \leq C, \tag{44}$$

$$\|\rho_0v_0\partial_x u_0\|_{B^{\sigma-1}_{p,\infty}} \leq \|\rho_0\|_{B^{\sigma-1}_{p,\infty}}\|v_0\|_{B^{\sigma-1}_{p,\infty}}\|u_0\|_{B^{\sigma}_{p,\infty}} \leq C. \tag{45}$$

Inserting (43)–(45) into (42), we obtain

$$\|\rho-\rho_0\|_{B^{\sigma-1}_{p,\infty}} \geq t\cdot 2^{ln(\sigma-1)}\|u_0v_0\partial_x\Delta_{ln}\rho_0\|_{L^p} - C_1t - 2^{2ln}\|\boldsymbol{\rho}\|_{B^{\sigma-3}_{p,\infty}}.$$

Making full use of (12) of Lemma 6, and (23) of Proposition 2, we draw the conclusion that

$$\|\rho-\rho_0\|_{B^{\sigma-1}_{p,\infty}} \geq C_3t\cdot 2^{ln} - C_1t - C_22^{2ln}t^2.$$

For $l \geq 4$, choosing n large enough such that $C_3\cdot 2^{ln} > 2C_1$, we have

$$\|\rho-\rho_0\|_{B^{\sigma-1}_{p,\infty}} \geq \frac{C_3}{2}t\cdot 2^{ln} - C_22^{2ln}\cdot t^2.$$

As time t tends to zero, picking t sufficiently small such that $t\cdot 2^n \approx \delta_0 < \frac{C_3}{4C_2}$, one can easily check that

$$\|\rho-\rho_0\|_{B^{\sigma-1}_{p,\infty}} \geq \frac{C_3}{4}\delta_0.$$

Similarly, by the definition of the Besov norm, we have

$$
\begin{aligned}
\|u-u_0\|_{B^{\sigma}_{p,\infty}} \geq & 2^{ln\sigma}\|\Delta_{ln}(u-u_0)\|_{L^p} = 2^{ln\sigma}\|\Delta_{ln}(\mathbf{u}+t\mathbf{u}_0)\|_{L^p} \\
\geq & t\cdot 2^{ln\sigma}\|\Delta_{ln}\{u_0v_0\partial_x u_0 + \mathcal{P}*[3u_0v_0\partial_x u_0 + (\partial_x u_0)^2\partial_x v_0 + u_0\partial_x^2 u_0\partial_x v_0 + \rho_0^2 u_0] \\
& + \partial_x\mathcal{P}*(u_0\partial_x u_0\partial_x v_0)\}\|_{L^p} - 2^{ln\sigma}\|\Delta_{ln}\mathbf{u}\|_{L^p} \\
\geq & t2^{ln\sigma}\|\Delta_{ln}(u_0v_0\partial_x u_0)\|_{L^p} - t\cdot 2^{ln\sigma}\|\Delta_{ln}[\mathcal{P}*(3u_0v_0\partial_x u_0)]\|_{L^p} \\
& - t\cdot 2^{ln\sigma}\|\Delta_{ln}[\mathcal{P}*((\partial_x u_0)^2\partial_x v_0)]\|_{L^p} - t\cdot 2^{ln\sigma}\|\Delta_{ln}[\mathcal{P}*(u_0\partial_x^2 u_0\partial_x v_0)]\|_{L^p} \\
& - t\cdot 2^{ln\sigma}\|\Delta_{ln}[\mathcal{P}*(\rho_0^2 u_0)]\|_{L^p} - t\cdot 2^{ln\sigma}\|\Delta_{ln}[\partial_x\mathcal{P}*(u_0\partial_x u_0\partial_x v_0)]\|_{L^p} \\
& - 2^{2ln}\cdot 2^{ln(\sigma-2)}\|\Delta_{ln}\mathbf{u}\|_{L^p}.
\end{aligned} \tag{46}
$$

Since

$$
\begin{aligned}
2^{ln\sigma}\|\Delta_{ln}[\mathcal{P}*(3u_0v_0\partial_x u_0)]\|_{L^p} \leq & \|\mathcal{P}*(3u_0v_0\partial_x u_0)\|_{B^{\sigma}_{p,\infty}} \leq C\|u_0v_0\partial_x v_0)\|_{B^{\sigma-2}_{p,\infty}} \\
\leq & C\|u_0\|^2_{B^{\sigma-1}_{p,\infty}}\|v_0\|_{B^{\sigma}_{p,\infty}} \leq C.
\end{aligned} \tag{47}
$$

By the same token, we have

$$
\begin{aligned}
2^{ln\sigma}\|\Delta_{ln}[\mathcal{P}*((\partial_x u_0)^2\partial_x v_0]\|_{L^p} \leq & C\|(\partial_x u_0)^2\partial_x v_0)\|_{B^{\sigma-2}_{p,\infty}} \\
\leq & \|u_0\|^2_{B^{\sigma}_{p,\infty}}\|v_0\|_{B^{\sigma-1}_{p,\infty}} \leq C,
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
2^{ln\sigma}\|\Delta_{ln}[\mathcal{P}*(u_0\partial_x^2 u_0\partial_x v_0)]\|_{L^p} \leq & C\|u_0\partial_x^2 u_0\partial_x v_0\|_{B^{\sigma-2}_{p,\infty}} \\
\leq & C\|u_0\|_{B^{\sigma-1}_{p,\infty}}\|u_0\|_{B^{\sigma-1}_{p,\infty}}\|v_0\|_{B^{\sigma}_{p,\infty}} \leq C,
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
2^{ln\sigma}\|\Delta_{ln}[\mathcal{P}*(\rho_0^2 u_0)]\|_{L^p} \leq & C\|\rho_0^2 u_0\|_{B^{\sigma-2}_{p,\infty}} \\
\leq & C\|\rho_0\|^2_{B^{\sigma-1}_{p,\infty}}\|u_0\|_{B^{\sigma-1}_{p,\infty}} \leq C,
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
2^{ln\sigma}\|\Delta_{ln}[\partial_x\mathcal{P}*(u_0\partial_x u_0\partial_x v_0)]\|_{L^p} \leq & C\|u_0\partial_x u_0\partial_x v_0\|_{B^{\sigma-1}_{p,\infty}} \\
\leq & C\|u_0\|_{B^{\sigma-1}_{p,\infty}}\|u_0\|_{B^{\sigma}_{p,\infty}}\|v_0\|_{B^{\sigma}_{p,\infty}} \leq C.
\end{aligned} \tag{51}
$$

Integrating (47)–(51) into (46), and repeating the discussion on $\|\rho-\rho_0\|_{B^{\sigma-1}_{p,\infty}}$, there then exist $C_4, C_5 > 0$ such that

$$
\|u-u_0\|_{B^{\sigma}_{p,\infty}} \geq \frac{C_4}{4}\delta_0, \; \|v-v_0\|_{B^{\sigma}_{p,\infty}} \geq \frac{C_5}{4}\delta_0.
$$

Picking $\varepsilon_0 = \min\{\frac{C_3}{4}, \frac{C_4}{4}, \frac{C_5}{4}\}\delta_0$, this completes the proof of Theorem 1. $\square$

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