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Parameter Choice Strategy That Computes Regularization Parameter before Computing the Regularized Solution

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Abstract: The modeling of many problems of practical interest leads to nonlinear ill-posed equations (for example, the parameter identification problem (see the Numerical section)). In this article, we introduce a new source condition (SC) and a new parameter choice strategy (PCS) for the Tikhonov regularization (TR) method for nonlinear ill-posed problems. The new PCS is introduced using a new SC to compute the regularization parameter (RP) before computing the regularized solution. The theoretical results are verified using a numerical example.

Keywords: source condition; parameter choice strategy; Tikhonov regularization method; ill-posed problems

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1. Introduction

Many problems of practical interest lead to nonlinear ill-posed equations. For example, consider the inverse problem of identifying the distributed growth law $x(t), t \in (0,1)$ in the initial value problem

$$\frac{dy}{dt} = x(t)y(t), \;\; y(0) = c, \;\; t \in (0,1) \tag{1}$$

from the noisy data $y^\delta(t) \in L^2(0,1)$.

If it is the exact case, we can use the variable separable method and obtain that $x(t) = \frac{d}{dt} \ln y$. Assume there is a fidelity term $\delta \sin(\frac{t}{\delta^2})$ added to $\ln y$ so that

$$\ln y^\delta = \ln y + \delta \sin\left(\frac{t}{\delta^2}\right). \tag{2}$$

Taking the derivative with respect to t for finding new x^δ, we obtain

$$x^\delta(t) = \frac{d}{dt} \ln y + \frac{1}{\delta} \cos\left(\frac{t}{\delta^2}\right). \tag{3}$$

Note that the magnitude of noise is small (if δ is small) in (2), but it is large in (3). This is typical of an ill-posed problem (the violation of Hadamard's criterion [1]). One can reformulate the above problems as an ill-posed operator equation $\mathcal{L}(x) = y$ with

$$[\mathcal{L}(x)](t) = ce^{\int_0^t x(\theta)d\theta}, \;\; x \in L^2(0,1), \;\; t \in (0,1). \tag{4}$$

The problem is to find x for a given y, when y is not exactly known. The modeling of problems in acoustics, electrodynamics, gravimetry, phase retrieval, etc., that leads to the solving of ill-posed equations can be found in [2].

Another real-life application occurs in the parameter identification problem when mathematical models used in biology, physics, economics, etc., are often defined by a Partial Differential Equation (PDE) (see Example 1) [3,4]. It is known that in general the solution of such a PDE need not be an elementary function. So, based on the experimental data, one need to obtain the parameters of the mathematical model. This type of problem is known as the parameter identification problem [5].

In this paper, we consider the abstract nonlinear ill-posed equation

$$\mathcal{L}(f) = g, \tag{5}$$

where $\mathcal{L} : D(\mathcal{L}) \subset U \longrightarrow V$ is a nonlinear operator and U, V are Hilbert spaces. Throughout the paper, it is assumed that $\mathcal{L}$ is weakly/sequentially closed, the continuous operator $D(\mathcal{L})$ is a subset of U, $\mathcal{L}$ has the Fréchet derivative at all $f \in D(\mathcal{L})$ and is denoted by $\mathcal{L}'(f)$, and $\mathcal{L}'(f)^*$ is the adjoint of the linear operator $\mathcal{L}'(f)$. We are interested in an f_0-minimum norm solution $\hat{f}$ (f_0-MNS) (see [5,6]) of (5) (here, f_0 is an apriori estimate in the interior of $D(\mathcal{L})$, see [5,7,8]). Recall that a solution $\hat{f}$ of (5) is called an $f_0 - MNS$ of (5) if

$$\|\hat{f} - f_0\| = \min\{\|f - f_0\| : \mathcal{L}(f) = g,\ f \in D(\mathcal{L})\}.$$

We assume that $\hat{f}$ does not depend continuously on the data g, and the available data are g^δ with

$$\|g - g^\delta\| \leq \delta. \tag{6}$$

In such a situation, regularization methods are employed to obtain approximation for $\hat{f}$. TR is the well-known regularization method [5,6,8–13]. In this method, the minimizer f_α^δ of the Tikhonov functional

$$J_\alpha(f) = \|\mathcal{L}(f) - g^\delta\|^2 + \alpha\|f - f_0\|^2, \quad f \in D(\mathcal{L}) \tag{7}$$

for some $\alpha > 0$ is taken as an approximation. It is known [5,9] that f_α^δ satisfies the equation

$$\mathcal{L}'(f_\alpha^\delta)^*(\mathcal{L}(f_\alpha^\delta) - g^\delta) + \alpha(f_\alpha^\delta - f_0) = 0. \tag{8}$$

The convergence and rate of convergence of $\|f_\alpha^\delta - \hat{f}\|$ are obtained [5,9,14] under the so-called source conditions (SCs) on $f_0 - \hat{f}$. Recall that apriori assumptions about the unknown solution $\hat{f}$ are called source conditions [15]. The most commonly used SCs for the TR method are [5,9];

$$f_0 - \hat{f} = (\Gamma^*\Gamma)^\nu w \text{ for some } w \in N(\Gamma^*\Gamma)^\perp,\ \|w\| \leq \rho, \tag{9}$$

where $\Gamma = \mathcal{L}'(\hat{f})$, Γ^* is the adjoint of the linear operator Γ, and [16,17]

$$f_0 - \hat{f} = (\mathcal{L}'(f_0)^*\mathcal{L}'(f_0))^\nu w \text{ for some } w \in N(\mathcal{L}'(f_0)^*\mathcal{L}'(f_0))^\perp,\ \|w\| \leq \rho \tag{10}$$

for $\rho > 0$, $0 < \nu \leq 1$.

Other types of SCs are also studied in the literature, for example, the generalized source condition [16–20] and variational source condition [21–25].

In this paper, we introduce a new SC, i.e, we assume that

$$f_0 - \hat{f} = (L^*L)^\nu w \text{ for some } w \in N(L^*L)^\perp,\ \|w\| \leq \rho, \tag{11}$$

where $L = \int_0^1 \mathcal{L}'(\hat{f} + t(f_0 - \hat{f}))dt, \rho > 0,\ 0 < \nu \leq 1$. It is known that [5,8,14,16], under the SCs (9) and (10) the best possible rate of convergence of $\|f_\alpha^\delta - \hat{f}\|$ is $O(\delta^{\frac{2\nu}{2\nu+1}})$. We shall prove that the SC (11) also gives the convergence rate $O(\delta^{\frac{2\nu}{2\nu+1}})$ (hereafter, we call ν the

Hölder-type parameter). We formulate the new SC to introduce a new PCS (this stategy is apriori in the sense that the RP α is chosen depending on δ and g^δ before computing the regularized solution f_α^δ) to choose α. The new PCS gives the order

$$\|f_\alpha^\delta - \hat{f}\| = \begin{cases} O(\delta^{\frac{2\nu}{2\nu+1}}), 0 < \nu \leq \frac{1}{2} \\ O(\delta^{\frac{1}{2}}), \nu > \frac{1}{2} \end{cases}.$$

Note that most of the apriori PCS depends on the unknown ν in the SC. The advantages of our proposed PCS are (i) it is independent of the parameter ν, (ii) it provides the order $O(\delta^{\frac{2\nu}{2\nu+1}})$ for $0 < \nu \leq \frac{1}{2}$, and (iii) it is apriori in the sense that it is computed before computing the regularized solution f_α^δ.

In earlier studies such as [10,11,20,26–28], the regularization parameter $\alpha = \alpha(n, \delta)$, depending on the iteration step, is computed in each iteration, and the stopping index is determined using some stopping criteria [11,20,26–28]. This apprach is computationally very expensive, but our approach requires the computation of $\alpha = \alpha(\delta)$ only once (here, α is independent of the iteration step); hence, one can also fix the stopping index for a given tolerence level in the beginning of the computation (see the comparison table in Example 1).

The above-mentioned advantages are obtained without actually using the operator L for computing α and f_α^δ (or the iteratively regularized solution).

Another class of regularization methods is the so-called iterative regularization methods [26–36] (and the reference therein). Since our aim in this paper is to introduce a new PCS that allows us to compute the RP α (depending on g^δ and δ) before computing the regularized solution f_α^δ, we leave the details of the above-mentioned (except (11)) source conditions and iterative regularization methods to motivated readers.

The rest of the paper is arranged as follows. An error analysis under the new SC is given in Section 2. A new PCS is given in Section 3, the numerical results are given in Section 4, and the paper ends with a conclusion in Section 5, followed by the Appendix.

2. Error Analysis

The proof of our results is based on the following assumptions (cf. [5,9]).

(i) $\exists$ constant $k_0 > 0$ and a continuous function $\varphi : D(\mathcal{L}) \times D(\mathcal{L}) \times U \longrightarrow U$ such that for $(f, z, v) \in D(\mathcal{L}) \times D(\mathcal{L}) \times U$, there is a $\varphi(f, z, v) \in U$ such that

$$(\mathcal{L}'(f) - \mathcal{L}'(z))v = \mathcal{L}'(z)\varphi(f, z, v), \tag{12}$$

where
$$\|\varphi(f, z, v)\| \leq k_0\|f - z\|\|v\|.$$

(ii) $\exists$ constant $k_1 > 0$ and a continuous function $\varphi_1 : D(\mathcal{L}) \times D(\mathcal{L}) \times V \longrightarrow V$ such that for $(f, z, g) \in D(\mathcal{L}) \times D(\mathcal{L}) \times V$, there is a $\varphi_1(f, z, g) \in V$ such that

$$(\mathcal{L}'(f)^* - \mathcal{L}'(z)^*)g = \mathcal{L}'(z)^*\varphi_1(f, z, g), \tag{13}$$

where
$$\|\varphi_1(f, z, g)\| \leq k_1\|f - z\|\|g\|.$$

(iii) $\exists$ constant $k_2 > 0$ and a continuous function $\varphi_2 : D(\mathcal{L}) \times D(\mathcal{L}) \times U \longrightarrow U$ such that for $(f, z, v) \in D(\mathcal{L}) \times D(\mathcal{L}) \times U$, there is a $\varphi_2(f, z, v) \in U$ such that

$$(\mathcal{L}'(f)^* - \mathcal{L}'(z)^*)\mathcal{L}'(z)v = \mathcal{L}'(z)^*\mathcal{L}'(z)\varphi_2(f, z, v), \tag{14}$$

where
$$\|\varphi_2(f, z, v)\| \leq k_2\|f - z\|\|v\|.$$

(iv) $\exists$ constant $k_3 > 0$ and a continuous function $\varphi_3 : D(\mathcal{L}) \times D(\mathcal{L}) \times V \longrightarrow V$ such that for $(f,z,g) \in D(\mathcal{L}) \times D(\mathcal{L}) \times V$, there is a $\varphi_3(f,z,g) \in V$ such that

$$(\mathcal{L}'(z) - \mathcal{L}'(f))\mathcal{L}'(z)^* g = \mathcal{L}'(f)\mathcal{L}'(z)^* \varphi_3(f,z,g), \tag{15}$$

where

$$\|\varphi_3(f,z,g)\| \leq k_3 \|f - z\| \|g\|.$$

Remark 1. *(a) Note that, by (ii) above, we have*

$$\mathcal{L}'(f')^* h = \mathcal{L}'(f)^* R(f', f, h) \tag{16}$$

where $\|R(f', f, h)\| \leq C_R \|h\|$ for some constant $C_R > 0$ provided $\|f - f'\|$ is bounded.

(b) Using the above assumptions, one can prove the following identities (proof of which is given in Appendix A). Let $\Pi = \int_0^1 \mathcal{L}'(u + t(v - u))dt$. Then,

$$\begin{aligned} &\|(\mathcal{L}'(f)^*\mathcal{L}'(f) + \alpha I)^{-1}\mathcal{L}'(f)^*(\Pi - \mathcal{L}'(f))\xi\| \\ \leq\ &\begin{cases} k_0\Big(\|u - f\| + \frac{\|v-u\|}{2}\Big)\|\xi\|, & v \neq f \\ k_0 \frac{\|v-u\|}{2}\|\xi\|, & v = f \end{cases} \end{aligned} \tag{17}$$

$$\begin{aligned} &\|(\mathcal{L}'(f)^*\mathcal{L}'(f) + \alpha I)^{-1}(\Pi^* - \mathcal{L}'(f)^*)\mathcal{L}'(f)\xi\| \\ \leq\ &\begin{cases} k_2\Big(\|u - f\| + \frac{\|v-u\|}{2}\Big)\|\xi\|, & v \neq f \\ k_2 \frac{\|v-u\|}{2}\|\xi\|, & v = f, \end{cases} \end{aligned} \tag{18}$$

$$\begin{aligned} &\|(\mathcal{L}'(f)^*\mathcal{L}'(f) + \alpha I)^{-1}(\Pi^* - \mathcal{L}'(f)^*)(\Pi - \mathcal{L}'(f))\xi\| \\ \leq\ &\begin{cases} k_2 k_0\Big(\|u - f\| + \frac{\|v-u\|}{2}\Big)^2\|\xi\|, & v \neq f \\ 3k_2 k_0 \frac{\|v-u\|^2}{4}\|\xi\|, & v = f \end{cases} \end{aligned} \tag{19}$$

and

$$\begin{aligned} &\|\alpha[(\mathcal{L}'(f)\mathcal{L}'(f)^* + \alpha I)^{-1} - (LL^* + \alpha I)^{-1}]\zeta\| \\ \leq\ &\begin{cases} (k_3 C_R + k_1)(\|f - \hat{f}\| + \frac{\|f_0 - \hat{f}\|}{2})\|\alpha(LL^* + \alpha I)^{-1}\zeta\|, & f \neq f_0 \\ (k_3 C_R + k_1)\frac{\|f_0 - \hat{f}\|}{2}\|\alpha(LL^* + \alpha I)^{-1}\zeta\|, & f = f_0. \end{cases} \end{aligned} \tag{20}$$

(c) We will be using the following estimates:

$$\|(L^*L + \alpha I)^{-1}(L^*L)^\nu\| \leq \alpha^{\nu - 1},\ 0 < \nu \leq 1, \tag{21}$$

$$\|(\mathcal{L}'(f)^*\mathcal{L}'(f) + \alpha I)^{-1}\mathcal{L}'(f)^*\mathcal{L}'(f)\| \leq 1,\ f \in D(\mathcal{L}) \tag{22}$$

and

$$\|(\mathcal{L}'(f)^*\mathcal{L}'(f) + \alpha I)^{-1}\mathcal{L}'(f)^*\| \leq \frac{1}{\sqrt{\alpha}},\ f \in D(\mathcal{L}). \tag{23}$$

Let $r := \frac{\delta}{\sqrt{\alpha}} + 2r_0$, where $r_0 = \|f_0 - \hat{f}\|$. Then, since f_α^δ is the minimizer of (7), we have

$$\begin{aligned} \|\mathcal{L}(f_\alpha^\delta) - g^\delta\|^2 + \alpha\|f_\alpha^\delta - f_0\|^2 &\leq \|\mathcal{L}(\hat{f}) - g^\delta\|^2 + \alpha\|\hat{f} - f_0\|^2 \\ &= \delta^2 + \alpha\|\hat{f} - f_0\|^2, \end{aligned}$$

hence,

$$\|f_\alpha^\delta - f_0\| \leq \frac{\delta}{\sqrt{\alpha}} + \|\hat{f} - f_0\| = \frac{\delta}{\sqrt{\alpha}} + r_0.$$

Similarly, we have

$$\|f_\alpha - f_0\| \leq \|f_0 - \hat{f}\| = r_0.$$

First, we shall prove that $f_0 - \hat{f} \in R(L^*L)$ implies $f_0 - \hat{f} \in R(\Gamma^*\Gamma)$ and $f_0 - \hat{f} \in R((L^*L)^\nu)$ implies $f_0 - \hat{f} \in R((\Gamma^*\Gamma)^{\nu_1})$ for $0 < \nu_1 < \nu$.

Proposition 1. *Suppose (i) and (iii) hold. Then, the following hold:*

(P_1) $f_0 - \hat{f} = L^*Lz, \|z\| \leq \rho \Rightarrow f_0 - \hat{f} = \Gamma^*\Gamma\xi_z, \|\xi_z\| \leq \rho_0$ *for some* $\rho_0 > 0$.

(P_2) $f_0 - \hat{f} = (L^*L)^\nu z, \|z\| \leq \rho \Rightarrow f_0 - \hat{f} = (\Gamma^*\Gamma)^{\nu_1}\xi_z, \|\xi_z\| \leq \rho_1$ *for some* $\rho_1 > 0, 0 < \nu_1 < \nu < 1$.

Proof. The proof is given in Appendix B. □

Remark 2. *Similarly, one can prove*

(P_1') $f_0 - \hat{f} = L^*Lz, \|z\| \leq \rho \Rightarrow f_0 - \hat{f} = \mathcal{L}'(f_0)^*\mathcal{L}'(f_0)\xi_z, \|\xi_z\| \leq \rho_0$ *for some* $\rho_0 > 0$ *and*

(P_2') $f_0 - \hat{f} = (L^*L)^\nu z, \|z\| \leq \rho \Rightarrow f_0 - \hat{f} = (\mathcal{L}'(f_0)^*\mathcal{L}'(f_0)^{\nu_1}\xi_z, \|\xi_z\| \leq \rho_1$ *for some* $\rho_1 > 0, 0 < \nu_1 < \nu < 1$.

Remark 3. *Proposition 1 shows that SC (11) is not a severe restriction, but it almost follows from SC (9) or SC (10). But the advantage of using SC (11), as mentioned in the introduction, is that one can compute the regularization parameter α (depending on g^δ and δ) before computing the regularized solution f_α^δ (see Section 3).*

Lemma 1. *If we suppose $k_0 r < 2$, then assumptions (i) and (iii) hold. Let f_α^δ be as in (8) and f_α be the solution of (8) with g in place of g^δ. Then,*

$$\|f_\alpha^\delta - f_\alpha\| \leq \frac{2}{2 - k_0 r}\left[\frac{\delta}{\sqrt{\alpha}} + k_2 r(k_0(r + \frac{r_0}{2}) + 1)\|f_\alpha - \hat{f}\|\right].$$

Proof. The proof is given in Appendix C. □

Lemma 2. *Suppose $k_0 r_0 < 2$, (11) and the assumptions (i)–(iii) hold. Then,*

$$\|f_\alpha - \hat{f}\| \leq \frac{2\|w\|}{2 - k_0 r_0}[(k_3 C_R + k_1)\frac{3r_0}{2} + 1]\alpha^\nu.$$

Proof. The proof is given in Appendix D. □

Next, we prove the main result of this Section using Lemma 1 and Lemma 2.

Theorem 1. *Let the assumptions in Lemmas 1 and 2 hold. Then,*

$$\|f_\alpha^\delta - \hat{f}\| \leq q(\frac{\delta}{\sqrt{\alpha}} + \alpha^\nu),$$

where $q = \frac{2}{2-k_0 r}\max\left\{1, k_2 r(k_0(r + \frac{r_0}{2}) + 1)\frac{2\|w\|}{2-k_0 r_0}[(k_3 C_R + k_1)\frac{3r_0}{2} + 1]\right\}$. *In particular, for* $\alpha = \delta^{\frac{2}{2\nu+1}}$, *we have*

$$\|f_\alpha^\delta - \hat{f}\| = O(\delta^{\frac{2\nu}{2\nu+1}}).$$

Proof. Since,
$$\|f_\alpha^\delta - \hat{f}\| \leq \|f_\alpha^\delta - f_\alpha\| + \|f_\alpha - \hat{f}\|,$$

the result follows from Lemma 1 and Lemma 2. □

Remark 4. *Note that the apriori parameter choice* $\alpha = \delta^{\frac{2}{2\nu+1}}$ *gives the order* $O(\delta^{\frac{2\nu}{2\nu+1}})$*, for* $0 \leq \nu \leq 1$*. But,* ν *is unknown, so such a choice is impossible when it comes to practical cases. So, we consider a new PCS that does not require knowledge of the unknown parameter* ν *and provide the order* $O(\delta^{\frac{2\nu}{2\nu+1}})$*, for* $0 \leq \nu \leq \frac{1}{2}$ *and* $O(\delta^{\frac{1}{2}})$*, for* $\frac{1}{2} < \nu \leq 1$*.*

3. New Parameter Choice Strategy

Let
$$d(\alpha, g^\delta) = \|\alpha(L_0L_0^* + \alpha I)^{-1}(\mathcal{L}(f_0) - g^\delta)\|, \tag{24}$$
where $L_0 = \mathcal{L}'(f_0)$.

Theorem 2. *The function* $\alpha \to d(\alpha, g^\delta)$ *for* $\alpha > 0$*, defined in (24), is monotonically increasing, continuous, and*

$$\lim_{\alpha\to 0} d(\alpha, g^\delta) = \|P(\mathcal{L}(f_0) - g^\delta)\|, \quad \lim_{\alpha\to\infty} d(\alpha, g^\delta) = \|\mathcal{L}(f_0) - g^\delta\|,$$

where P *is the orthogonal projection onto the null space* $N(L_0^*)$ *of* L_0^**.*

Proof. See Lemma 1 in [18]. □

Further, we assume

$$\|P(\mathcal{L}(f_0) - g^\delta)\| \leq c\delta \leq \|\mathcal{L}(f_0) - g^\delta\|, \tag{25}$$

for some $c > 1$.

The application of the intermediate value theorem gives the following theorem.

Theorem 3. *If* g^δ *satisfies (6) and (25), then* $\exists$ *is a unique* α *such that*

$$d(\alpha, g^\delta) = c\,\delta. \tag{26}$$

We will be using the following moment inequality:

$$\|B^u x\| \leq \|B^v x\|^{\frac{u}{v}}\, \|x\|^{1-\frac{u}{v}}, \quad 0 \leq u \leq v, \tag{27}$$

where B is positive selfadjoint operator (see [37]).

Lemma 3. *Let* $\alpha = \alpha(\delta)$ *be the unique solution of (26) and* $(k_3C_R + k_1)\frac{3r_0}{2} < 1$*. Suppose that (11) holds and* g^δ *satisfies (6) and (25). Then, under the assumptions (i), (ii), (iii), and (iv):*

$$\|f_\alpha - \hat{f}\| = O(\delta^{\frac{2\nu}{2\nu+1}}),\ 0 \leq \nu \leq 1.$$

Proof. The proof is given in Appendix E. □

Lemma 4. *Let* g^δ *satisfy (6) and (25) and let* $\alpha = \alpha(\delta)$ *satisfy (26). Further, suppose (11) holds and assumptions (i)–(iv) hold. Then,*

$$\frac{\delta}{\sqrt{\alpha}} = \begin{cases} O(\delta^{\frac{2\nu}{2\nu+1}}), \nu \leq \frac{1}{2} \\ O(\delta^{\frac{1}{2}}), \nu > \frac{1}{2} \end{cases}.$$

Proof. The proof is given in Appendix F. □

Theorem 4. *Suppose that the assumptions in Lemmas 1–4 hold. Then,*

$$\|f_\alpha^\delta - \hat{f}\| = \begin{cases} O(\delta^{\frac{2\nu}{2\nu+1}}), \nu \leq \frac{1}{2} \\ O(\delta^{\frac{1}{2}}), \nu > \frac{1}{2} \end{cases}.$$

Proof. Since,

$$\|f_\alpha^\delta - \hat{f}\| \leq \|f_\alpha^\delta - f_\alpha\| + \|f_\alpha - \hat{f}\|,$$

the proof follows from Lemmas 1–4. □

Remark 5. *Note that $\alpha = \alpha(\delta)$ satisfies (26) and is independentof ν and gives the order $O(\delta^{\frac{2\nu}{2\nu+1}})$ for $0 \leq \nu \leq \frac{1}{2}$ and $O(\delta^{\frac{1}{2}})$ for $\frac{1}{2} < \nu \leq 1$. Also, observe that the PCS does not depend on the operator L and that the regularization parameter α is computed before computing f_α^δ.*

4. Numerical Example

Next, we provide an example satisfying the assumptions (i)–(iv).

Example 1. *Here, the problem is to find q satisfying the two-point boundary value problem*

$$\begin{aligned} -u'' + qu &= f, \ \ t \in (0,1) \\ u(0) &= g_0, \ \ u(1) = g_1, \end{aligned} \tag{28}$$

where g_0, g_1 and $f \in L^2[0,1]$ are given. This problem can be written as an operator equation of the form $\mathcal{L}(q) = u(q)$, where $\mathcal{L} : D(\mathcal{L}) \subset L^2[0,1] \longrightarrow L^2[0,1]$ is a nonlinear operator and $u(q)$ satisfies (28). Here,

$$D(\mathcal{L}) := \{q \in L^2[0,1] : \|q - q_0\| \leq \epsilon \text{ for some } q_0 \in U \text{ and small enough } \epsilon > 0\},$$

where

$$U = \{q \in L^2[0,1] : q \geq 0 \ \text{ a.e.}\}.$$

Then,

$$\mathcal{L}'(q)h = -T_q^{-1}(\mathcal{L}(q)), \ \mathcal{L}'(q)^* w = -\mathcal{L}(q) T_q^{-1}(w),$$

for $q \in D(\mathcal{L}), h, w \in L^2[0,1]$, where $T_q : H^2(0,1) \cap H_0^1(0,1) \longrightarrow L^2[0,1]$ satisfies

$$T_q u = - \triangle u + qu, \ u \in H^2 \cap H_0^1.$$

Assumptions (i) and (ii) are verified in [5]. The verification of assumptions (iii) and (iv) is given in Appendix G.

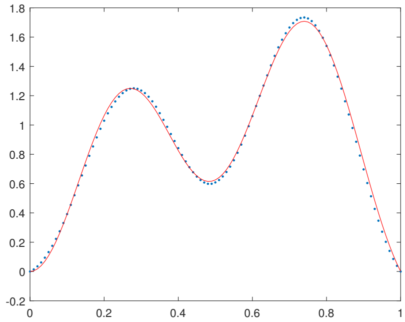
We estimate the parameter α using PCS (26). To compute f_α^δ in (8), we use the Gauss–Newton method, which defines the iterate $\{f_{k,\alpha}^\delta\}$ for $k = 1, 2, \dots$ by

$$f_{k+1,\alpha}^\delta = f_{k,\alpha}^\delta - (\mathcal{L}'(f_{k,\alpha}^\delta)^* \mathcal{L}'(f_{k,\alpha}^\delta) + \alpha I)^{-1} [\mathcal{L}'(f_{k,\alpha}^\delta)^* (\mathcal{L}(f_{k,\alpha}^\delta) - g^\delta) + \alpha (f_{k,\alpha}^\delta - f_0)]. \tag{29}$$

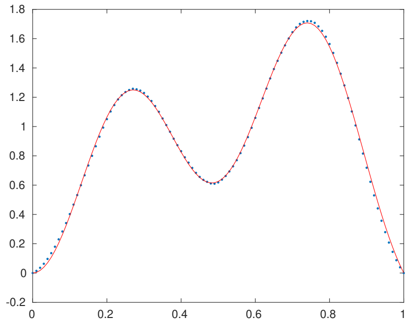
Since we are estimating q, we will use the notation $q_{k,\alpha}^\delta$ for $f_{k,\alpha}^\delta$, $\hat{q}$ for $\hat{f}$, and u^δ for g^δ in the example.

We take $f = 100e^{-10(t-0.5)^2}$ and $g_0 = 1$, $g_1 = 2$ as in [28]. Then, $\hat{q} = 5t^2(1-t) + \sin(2\pi t)$. For our computation, we use random noise data u^δ so that $\|u - u^\delta\| \leq \delta$. Further, we have taken the initial approximation as $q_0 = 0$. We have used a finite difference method for solving the differential equations involved in the computation by dividing $[0,1]$ into 100 subintervals of equal length, and the resulting tridiagonal system has been solved by the Thomas algorithm [38].

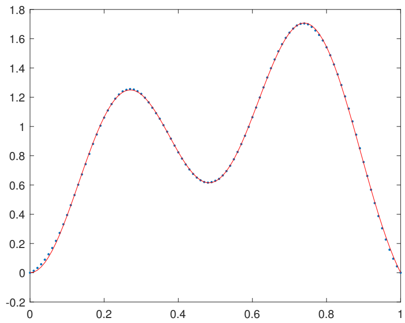
We have taken $c = 4$ in (26) to compute α. Table 1 gives the values of δ, the parameter α computed using (26), and the error $\|q^{\delta}_{k,\alpha} - \hat{q}\|$ and time taken to compute α for different values of δ. The corresponding figures are provided in Figure 1.



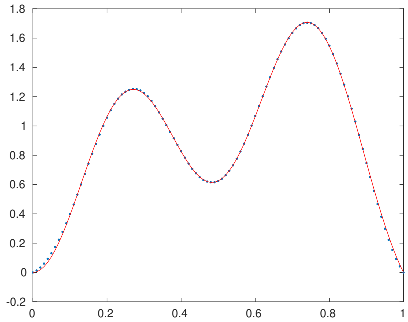
(**a**): method (29), $\delta = 0.01$



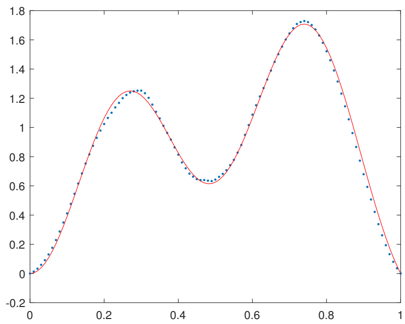
(**b**): method (30) $\delta = 0.01$



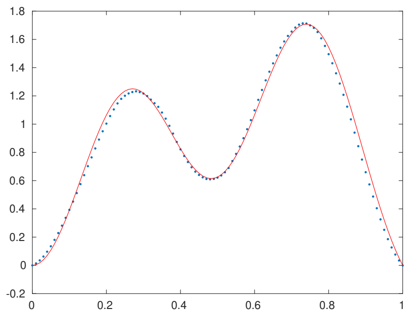
(**c**): method (29), $\delta = 0.001$



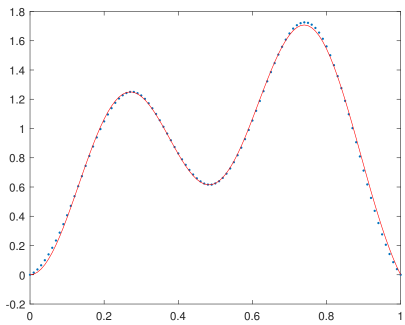
(**d**): method (30), $\delta = 0.001$



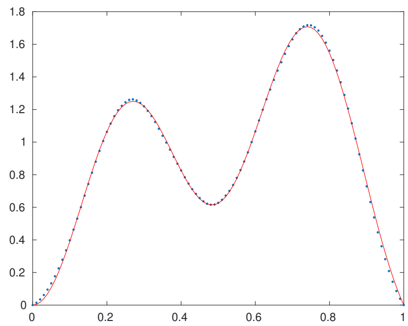
(**e**): method (29), $\delta = 0.05$



(**f**): method (30), $\delta = 0.05$



(**g**): method (29), $\delta = 0.005$



(**h**): method (30), $\delta = 0.005$

Figure 1. Exact $(- - --)$ and computed solutions $(* * *)$ for various parameters given against each subfigure.

Table 1. Computed α and computed error.

Method	δ	α	$\|q^\delta_{k,\alpha} - \hat{q}\|$	Elapsed Time in Seconds
(29)	0.01	3.8147×10^{-6}	0.0255	0.1664
	0.001	3.7253×10^{-9}	0.0081	0.3884
	0.05	3.0518×10^{-5}	0.0298	0.1277
	0.005	9.5367×10^{-7}	0.0178	0.1865
(30)	0.01	1.9073×10^{-6}	0.0138	0.2190
	0.001	3.7253×10^{-9}	0.0086	0.5289
	0.05	6.1035×10^{-5}	0.0404	0.3383
	0.005	4.7684×10^{-7}	0.0143	0.3988

We compare our method with that of the most widely used iterative method [26] for (5), which is the regularized Gauss Newton method, in which the iterations $x^\delta_{\alpha,k}$ are defined for $k = 0, 1, 2, \ldots$ by

$$f^\delta_{k+1,\alpha_{k+1}} = f^\delta_{k,\alpha_k} - (\mathcal{L}'(f^\delta_{k,\alpha_k})^* \mathcal{L}'(f^\delta_{k,\alpha_k}) + \alpha_k I)^{-1}[\mathcal{L}'(f^\delta_{k,\alpha_k})^*(\mathcal{L}(f^\delta_{k,\alpha_k}) - y^\delta) + \alpha(f^\delta_{k,\alpha_k} - f_0)], \tag{30}$$

where $f^\delta_{0,\alpha} := x_0$. Here, (α_k) is a given sequence of numbers such that

$$\alpha_k > 0, \quad 1 < \frac{\alpha_k}{\alpha_{k+1}} \leq r \quad \text{and} \quad \lim_{k\to\infty} \alpha_k = 0$$

for some constant $r > 1$.

Stopping index: Choose k_δ as the first positive integer that satisfies

$$\frac{1}{2}(\|F(x^\delta_{k_\delta}) - y^\delta\| + \|F(x^\delta_{k_\delta - 1}) - y^\delta\| \leq \tau\delta, \tag{31}$$

where $\tau > 1$ is a sufficiently large constant not depending on δ. We have taken $\lambda = 1.05$ and $\alpha_k = 1/k$ in our computations.

We use a 4-core 64 bit Windows machine with 11th Gen Intel(R) Core(TM) i5-1135G7 CPU @ 2.40GHz for all our computations (using MATLAB).

Clearly, the table shows that our approach requires less computational time than that of method (30).

5. Conclusions

We introduced a new SC and a new PCS for the TR of nonlinear ill-posed problems. Our PCS does not require knowledge of ν, and it gives the error estimate

$$\|f^\delta_\alpha - \hat{f}\| = \begin{cases} O(\delta^{\frac{2\nu}{2\nu+1}}), 0 < \nu \leq \frac{1}{2} \\ O(\delta^{\frac{1}{2}}), \nu > \frac{1}{2} \end{cases}.$$

The advantage of our method is that one can compute the RP α before computing the regularized solution f^δ_α. We also applied the method to the parameter identification problem modeled as in Example 1 and obtained favourable numerical results.

Author Contributions: Conceptualization, S.G., J.P., A.K., I.K.A. and S.R.; methodology, S.G., J.P., A.K., I.K.A. and S.R.; software, S.G., J.P., A.K., I.K.A. and S.R.; validation, S.G., J.P., A.K., I.K.A. and S.R.; formal analysis, S.G., J.P., A.K., I.K.A. and S.R.; investigation, S.G., J.P., A.K., I.K.A. and S.R.; resources, S.G., J.P., A.K., I.K.A. and S.R.; data curation, S.G., J.P., A.K., I.K.A. and S.R.; writing—original draft preparation, S.G., J.P., A.K., I.K.A. and S.R.; writing—review and editing, S.G., J.P., A.K., I.K.A. and S.R.; visualization, S.G., J.P., A.K., I.K.A. and S.R.; supervision, S.G., J.P., A.K., I.K.A. and S.R.; project administration, S.G., J.P., A.K., I.K.A. and S.R.; funding acquisition, S.G., J.P., A.K., I.K.A. and S.R. All authors have read and agreed to the published version of the manuscript.

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Appendix A. Proof of the Identities (17)–(20)

Using assumption (i), we have

$$\begin{aligned}
& \|(\mathcal{L}'(f)^*\mathcal{L}'(f)+\alpha I)^{-1}\mathcal{L}'(f)^*(\Pi-\mathcal{L}'(f))\xi\| \\
= & \|(\mathcal{L}'(f)^*\mathcal{L}'(f)+\alpha I)^{-1}\mathcal{L}'(f)^*\mathcal{L}'(f)\int_0^1 \varphi(u+t(v-u),f,\xi)dt\| \\
\leq & \begin{cases} k_0\left(\|u-f\|+\frac{\|v-u\|}{2}\right)\|\xi\|, & v\neq f \\ k_0\frac{\|v-u\|}{2}\|\xi\|, & v=f \end{cases}
\end{aligned}$$

and using (iii), we have

$$\begin{aligned}
& \|(\mathcal{L}'(f)^*\mathcal{L}'(f)+\alpha I)^{-1}(\Pi^*-\mathcal{L}'(f)^*)\mathcal{L}'(f)\xi\| \\
= & \|(\mathcal{L}'(f)^*\mathcal{L}'(f)+\alpha I)^{-1}\mathcal{L}'(f)^*\mathcal{L}'(f) \\
& \times\int_0^1 \varphi_2(u+t(v-u),f,\xi)dt\| \\
\leq & \begin{cases} k_2\left(\|u-f\|+\frac{\|v-u\|}{2}\right)\|\xi\|, & v\neq f \\ k_2\frac{\|v-u\|}{2}\|\xi\|, & v=f \end{cases}
\end{aligned}$$

and by (i) and (iii);

$$\begin{aligned}
& \|(\mathcal{L}'(f)^*\mathcal{L}'(f)+\alpha I)^{-1}(\Pi^*-\mathcal{L}'(f)^*)(\Pi-\mathcal{L}'(f))\xi\| \\
= & \|(\mathcal{L}'(f)^*\mathcal{L}'(f)+\alpha I)^{-1}(\Pi^*-\mathcal{L}'(f)^*)\mathcal{L}'(f) \\
& \times\int_0^1 \varphi(u+t(v-u),f,\xi)dt\| \\
= & \|(\mathcal{L}'(f)^*\mathcal{L}'(f)+\alpha I)^{-1}\mathcal{L}'(f)^*\mathcal{L}'(f) \\
& \times\int_0^1 \varphi_2(u+\tau(v-u),f,\int_0^1 \varphi(u+t(v-u),f,\xi)dt)d\tau\| \\
\leq & k_2(\|u-f\|+\frac{\|v-u\|}{2})\|\int_0^1 \varphi(u+t(v-u),f,\xi)dt\| \\
\leq & \begin{cases} k_2k_0\left(\|u-f\|+\frac{\|v-u\|}{2}\right)^2\|\xi\|, & v\neq f \\ k_2k_0\frac{\|v-u\|^2}{4}\|\xi\|, & v=f. \end{cases}
\end{aligned}$$

Further, using (ii), (iv), and Remark 1 (a) and (c), we obtain

$$\begin{aligned}
& \|\alpha[(\mathcal{L}'(f)\mathcal{L}'(f)^*+\alpha I)^{-1}-(LL^*+\alpha I)^{-1}]\zeta\| \\
= & \|(\mathcal{L}'(f)\mathcal{L}'(f)^*+\alpha I)^{-1}[LL^*-\mathcal{L}'(f)\mathcal{L}'(f)^*]\alpha(LL^*+\alpha I)^{-1}\zeta\| \\
= & \|(\mathcal{L}'(f)\mathcal{L}'(f)^*+\alpha I)^{-1}[(L-\mathcal{L}'(f))L^*+\mathcal{L}'(f)(L^*-\mathcal{L}'(f)^*)] \\
& \times\alpha(LL^*+\alpha I)^{-1}\zeta\| \\
= & \left\|(\mathcal{L}'(f)\mathcal{L}'(f)^*+\alpha I)^{-1}\left[(L-\mathcal{L}'(f))\mathcal{L}'(f)^*\right.\right. \\
& \times\int_0^1 R(\hat{f}+\tau(f_0-\hat{f}),f,\alpha(LL^*+\alpha I)^{-1}\zeta)d\tau \\
& \left.\left.+\mathcal{L}'(f)(L^*-\mathcal{L}'(f)^*)\alpha(LL^*+\alpha I)^{-1}\zeta\right]\right\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad \|(\mathcal{L}'(f)\mathcal{L}'(f)^* + \alpha I)^{-1}\mathcal{L}'(f)\mathcal{L}'(f)^* \int_0^1 R(\hat{f} + \tau(f_0 - \hat{f}), f, \\
&\qquad \int_0^1 \varphi_3(\hat{f} + t(f_0 - \hat{f}), f, \alpha(LL^* + \alpha I)^{-1}\zeta))d\tau dt\| \\
&\qquad + \|(\mathcal{L}'(f)\mathcal{L}'(f)^* + \alpha I)^{-1}\mathcal{L}'(f)\mathcal{L}'(f)^* \\
&\qquad \times \int_0^1 \varphi_1(\hat{f} + t(f_0 - \hat{f}), f, \alpha(LL^* + \alpha I)^{-1}\zeta)dt\| \\
&\leq \quad \begin{cases} (k_3 C_R + k_1)(\|f - \hat{f}\| + \frac{\|f_0 - \hat{f}\|}{2})\|\alpha(LL^* + \alpha I)^{-1}\zeta\|, & f \neq f_0 \\ (k_3 C_R + k_1)\frac{\|f_0 - \hat{f}\|}{2}\|\alpha(LL^* + \alpha I)^{-1}\zeta\|, & f = f_0. \end{cases}
\end{aligned}
$$

Appendix B. Proof Proposition 1

Suppose $f_0 - \hat{f} = L^*Lz$, $\|z\| \leq \rho$. Then, by (i) and (iii) we have

$$
\begin{aligned}
f_0 - \hat{f} &= L^*Lz \\
&= [L^*L - \Gamma^*\Gamma]z + \Gamma^*\Gamma z \\
&= [(L^* - \Gamma^*)L + \Gamma^*(L - \Gamma)]z + \Gamma^*\Gamma z, \\
&= [(L^* - \Gamma^*)(L - \Gamma + \Gamma) + \Gamma^*(L - \Gamma)]z + \Gamma^*\Gamma z, \\
&= [(L^* - \Gamma^*)(L - \Gamma) + (L^* - \Gamma^*)\Gamma + \Gamma^*(L - \Gamma)]z + \Gamma^*\Gamma z, \\
&= (L^* - \Gamma^*)\Gamma \int_0^1 \varphi(\hat{f} + \tau(f_0 - \hat{f}), \hat{f}, z)d\tau + \int_0^1 \Gamma^*\Gamma\varphi_2(\hat{f} + t(f_0 - \hat{f}), \hat{f}, z)dt \\
&\quad + \int_0^1 \Gamma^*\Gamma\varphi(\hat{f} + t(f_0 - \hat{f}), \hat{f}, z)dt + \Gamma^*\Gamma z, \\
&= \int_0^1 \Gamma^*\Gamma\varphi_2(\hat{f} + t(f_0 - \hat{f}), \hat{f}, \int_0^1 \varphi(\hat{f} + \tau(f_0 - \hat{f}), \hat{f}, z)d\tau)dt \\
&\quad + \int_0^1 \Gamma^*\Gamma\varphi_2(\hat{f} + t(f_0 - \hat{f}), \hat{f}, z)dt + \int_0^1 \Gamma^*\Gamma\varphi(\hat{f} + t(f_0 - \hat{f}), \hat{f}, z)dt + \Gamma^*\Gamma z, \\
&= \Gamma^*\Gamma\Psi(\hat{f}, z, \hat{f}),
\end{aligned}
$$

where $\Psi(\hat{f}, z, \hat{f}) = \int_0^1 \varphi_2\left(\hat{f} + t(f_0 - \hat{f}), \hat{f}, \int_0^1 \varphi(\hat{f} + \tau(f_0 - \hat{f}), \hat{f}, z)d\tau\right)dt + \int_0^1 \varphi_2(\hat{f} + t(f_0 - \hat{f}), \hat{f}, z)dt + \int_0^1 \varphi(\hat{f} + t(f_0 - \hat{f}), \hat{f}, z)dt + z$. Further, we have

$$
\begin{aligned}
\|\Psi(\hat{f}, z, \hat{f})\| &\leq \| \int_0^1 \varphi_2\left(\hat{f} + t(f_0 - \hat{f}), \hat{f}, \int_0^1 \varphi(\hat{f} + \tau(f_0 - \hat{f}), \hat{f}, z)d\tau\right)dt\| \\
&\quad + \| \int_0^1 \varphi_2(\hat{f} + t(f_0 - \hat{f}), \hat{f}, z)dt\| \\
&\quad + \| \int_0^1 \varphi(\hat{f} + t(f_0 - \hat{f}), \hat{f}, z)dt\| + \|z\| \\
&\leq \left[\left(\frac{k_2 k_0}{2}\|f_0 - \hat{f}\| + (k_2 + k_0)\right)\frac{\|f_0 - \hat{f}\|}{2} + 1\right]\|z\| \\
&\leq \left[\left(\frac{k_2 k_0 r_0}{2} + (k_2 + k_0)\right)\frac{r_0}{2} + 1\right]\rho =: \rho_0.
\end{aligned}
$$

This proves (P_1). To prove (P_2), we use the formula ([37], p. 287) for the fractional power of positive self-adjoint operators $\mathcal{B}$ given by

$$\begin{aligned}\mathcal{B}^{\varrho}x &= \frac{\sin \pi\varrho}{\pi}\int_0^{\infty}\tau^{\varrho}\left[(\mathcal{B}+\tau I)^{-1}x - \frac{\Theta(\tau)}{\tau}x + \ldots + (-1)^n\frac{\Theta(\tau)}{\tau^n}\mathcal{B}^{n-1}x\right]d\tau \\ &\quad + \frac{\sin \pi\varrho}{\pi}\left[\frac{x}{\varrho} - \frac{\mathcal{B}x}{\varrho-1} + \ldots + (-1)^{n-1}\frac{\mathcal{B}^{n-1}x}{\varrho-n+1}\right],\ x \in \mathcal{U},\end{aligned}$$

where

$$\Theta(\varsigma) = \begin{cases} 0 & \text{if} \quad 0 \le \varsigma \le 1 \\ 1 & \text{if} \quad 1 < \varsigma < \infty \end{cases}$$

and ϱ is a complex number such that $0 < Re\varrho < n$.

Suppose that $f_0 - \hat{f} = (L^*L)^{\nu}z, 0 \le \nu < 1$. Then, by using the above formula, we have

$$\begin{aligned} f_0 - \hat{f} &= [(L^*L)^{\nu} - (\Gamma^*\Gamma)^{\nu}]z + (\Gamma^*\Gamma)^{\nu}z \\ &= -\frac{\sin \pi(\nu)}{\pi}\int_0^{\infty}\tau^{\nu}(\Gamma^*\Gamma+\tau I)^{-1} \times (L^*L - \Gamma^*\Gamma)(L^*L+\tau I)^{-1}z d\tau + (\Gamma^*\Gamma)^{\nu}z, \\ &= -\frac{\sin \pi(\nu)}{\pi}\int_0^{\infty}\tau^{\nu}(\Gamma^*\Gamma+\tau I)^{-1}\Big[(L^*-\Gamma^*)(L-\Gamma) + (L^*-\Gamma^*)\Gamma \\ &\quad +\Gamma^*(L-\Gamma)\Big](L^*L+\tau I)^{-1}z d\tau + (\Gamma^*\Gamma)^{\nu}z. \end{aligned}$$

So, by using (i) and (iii) we have

$$\begin{aligned} f_0 - \hat{f} &= -\frac{\sin \pi(\nu)}{\pi}\int_0^{\infty}\tau^{\nu}(\Gamma^*\Gamma+\tau I)^{-1} \\ &\quad \times\Big[(L^*-\Gamma^*)\Gamma\int_0^1 \varphi(\hat{f}+s(f_0-\hat{f}),\hat{f},(L^*L+\tau I)^{-1}z)ds \\ &\quad + \int_0^1 \Gamma^*\Gamma\varphi_2(\hat{f}+t(f_0-\hat{f}),\hat{f},(L^*L+\tau I)^{-1}z)dt \\ &\quad + \int_0^1 \Gamma^*\Gamma\varphi(\hat{f}+t(f_0-\hat{f}),\hat{f},(L^*L+\tau I)^{-1}z)dt\Big]d\tau + (\Gamma^*\Gamma)^{\nu}z \\ &\quad \text{(again, using (iii) we have)} \\ &= -\frac{\sin \pi(\nu)}{\pi}\int_0^{\infty}\tau^{\nu}(\Gamma^*\Gamma+\tau I)^{-1} \\ &\quad \times\Big[\int_0^1 \Gamma^*\Gamma\varphi_2\Big(\hat{f}+t(f_0-\hat{f}),\hat{f}, \\ &\quad \int_0^1 \varphi(\hat{f}+s(f_0-\hat{f}),\hat{f},(L^*L+\tau I)^{-1}z)ds\Big)dt \\ &\quad + \int_0^1 \Gamma^*\Gamma\varphi_2(\hat{f}+t(f_0-\hat{f}),\hat{f},(L^*L+\tau I)^{-1}z)dt \\ &\quad + \int_0^1 \Gamma^*\Gamma\varphi(\hat{f}+t(f_0-\hat{f}),\hat{f},(L^*L+\tau I)^{-1}z)dt\Big]d\tau + (\Gamma^*\Gamma)^{\nu}z \end{aligned}$$

so, for $0 < \nu_1 < \nu$ we have

$$\begin{aligned} f_0 - \hat{f} &= (\Gamma^*\Gamma)^{\nu_1}\Bigg[-\frac{\sin \pi(\nu)}{\pi} \int_0^\infty \tau^\nu (\Gamma^*\Gamma)^{1-\nu_1} (\Gamma^*\Gamma + \tau I)^{-1} \\ &\quad \times \Bigg\{ \int_0^1 \varphi_2\Bigg(\hat{f} + t(f_0 - \hat{f}), \hat{f}, \int_0^1 \varphi(\hat{f} + s(f_0 - \hat{f}), \hat{f}, (L^*L + \tau I)^{-1} z) ds \Bigg) dt d\tau \\ &\quad + \int_0^1 \varphi_2(\hat{f} + t(f_0 - \hat{f}), \hat{f}, (L^*L + \tau I)^{-1} z) dt \\ &\quad + \int_0^1 \varphi(\hat{f} + t(f_0 - \hat{f}), \hat{f}, (L^*L + \tau I)^{-1} z) dt \Bigg\} \Bigg] d\tau + (\Gamma^*\Gamma)^\nu z, \\ &= (\Gamma^*\Gamma)^{\nu_1} \xi_z, \end{aligned}$$

where

$$\begin{aligned} \xi_z &= -\frac{\sin \pi(\nu)}{\pi} \int_0^\infty \tau^\nu (\Gamma^*\Gamma)^{1-\nu_1} (\Gamma^*\Gamma + \tau I)^{-1} \\ &\quad \times \Bigg\{ \int_0^1 \varphi_2\Bigg(\hat{f} + t(f_0 - \hat{f}), \hat{f}, \int_0^1 \varphi(\hat{f} + s(f_0 - \hat{f}), \hat{f}, (L^*L + \tau I)^{-1} z) ds \Bigg) dt \\ &\quad + \int_0^1 \varphi_2(\hat{f} + t(f_0 - \hat{f}), \hat{f}, (L^*L + \tau I)^{-1} z) dt d\tau \\ &\quad + \int_0^1 \varphi(\hat{f} + t(f_0 - \hat{f}), \hat{f}, (L^*L + \tau I)^{-1} z) dt d\tau \Bigg\} + (\Gamma^*\Gamma)^{\nu - \nu_1} z. \end{aligned}$$

Further, by (i) and (iii) we have

$$\begin{aligned} \|\xi_z\| &\leq \frac{1}{\pi} \Bigg(\int_0^\infty \tau^\nu \|(\Gamma^*\Gamma)^{1-\nu_1} (\Gamma^*\Gamma + \tau I)^{-1}\| \\ &\quad \times \Bigg[\frac{k_2 k_0}{4} \|f_0 - \hat{f}\|^2 + \frac{k_2 + k_0}{2} \|f_0 - \hat{f}\| \Bigg] \|(L^*L + \tau I)^{-1}\| \|z\| d\tau \\ &\quad + \|(\Gamma^*\Gamma)^{1-\nu_1}\| \|z\|. \end{aligned}$$

By spliting the limit of intergration and rearranging the terms, we obtain

$$\begin{aligned} \|\xi_z\| &= \frac{1}{\pi} \Bigg[\frac{k_2 k_0}{4} \|f_0 - \hat{f}\|^2 + \frac{k_2 + k_0}{2} \|f_0 - \hat{f}\| \Bigg] \\ &\quad \times \Bigg[\int_0^1 \tau^\nu \|(\Gamma^*\Gamma)^{1-\nu_1} (\Gamma^*\Gamma + \tau I)^{-1+\nu_1}\| \\ &\quad \times \|(\Gamma^*\Gamma + \tau I)^{-\nu_1}\| \|(L^*L + \tau I)^{-1}\| \|z\| d\tau \\ &\quad + \int_1^\infty \tau^\nu \|(\Gamma^*\Gamma)^{1-\nu_1} (\Gamma^*\Gamma + \tau I)^{-1}\| \|(L^*L + \tau I)^{-1}\| \|z\| d\tau \Bigg] \\ &\quad + \|(\Gamma^*\Gamma)^{1-\nu_1}\| \|z\|. \end{aligned}$$

Now, using the relations $\|(\Gamma^*\Gamma)^{1-\nu_1}(\Gamma^*\Gamma + \tau I)^{-1+\nu_1}\| \leq 1$, $\|(\Gamma^*\Gamma + \tau I)^{-1}\| \leq \tau^{-1}$, $\|(\Gamma^*\Gamma + \tau I)^{-\nu_1}\| \leq \tau^{-\nu_1}$, and $\|(L^*L + \tau I)^{-1}\| \leq \tau^{-1}$ we have

$$\begin{aligned}
\|\xi_z\| &\le \frac{1}{\pi}\left[\frac{k_2k_0}{4}\|f_0-\hat{f}\|^2+\frac{k_2+k_0}{2}\|f_0-\hat{f}\|\right] \\
&\quad \times\left[\int_0^1 \tau^{\nu-\nu_1-1}d\tau+\int_1^\infty \tau^{\nu-2}d\tau\|(\Gamma^*\Gamma)^{1-\nu_1}\|\right]\|z\| \\
&\quad +\|(\Gamma^*\Gamma)^{1-\nu_1}\|\|z\| \\
&= \frac{1}{\pi}\left[\frac{k_2k_0}{4}r_0^2+\frac{k_2+k_0}{2}r_0\right]\left(\frac{1}{\nu-\nu_1}+\frac{\|(\Gamma^*\Gamma)^{1-\nu_1}\|}{1-\nu}\right)\rho \\
&\quad +\|(\Gamma^*\Gamma)^{1-\nu_1}\|\rho =: \rho_1.
\end{aligned}$$

This proves (P_2).

Appendix C. Proof of Lemma 1

Observe that,

$$\mathcal{L}'(f_\alpha^\delta)^*[\mathcal{L}(f_\alpha^\delta)-g^\delta]+\alpha(f_\alpha^\delta-f_0)=0$$

and

$$\mathcal{L}'(f_\alpha)^*[\mathcal{L}(f_\alpha)-g]+\alpha(f_\alpha-f_0)=0.$$

So, we have

$$\mathcal{L}'(f_\alpha^\delta)^*\mathcal{L}(f_\alpha^\delta)-\mathcal{L}'(f_\alpha)^*\mathcal{L}(f_\alpha)+\alpha(f_\alpha^\delta-f_\alpha)=\mathcal{L}'(f_\alpha^\delta)^*g^\delta-\mathcal{L}'(f_\alpha)^*g$$

or

$$\begin{aligned}
& \mathcal{L}'(f_\alpha^\delta)^*[\mathcal{L}(f_\alpha^\delta)-\mathcal{L}(f_\alpha)]+[\mathcal{L}'(f_\alpha^\delta)^*-\mathcal{L}'(f_\alpha)^*]\mathcal{L}(f_\alpha)+\alpha(f_\alpha^\delta-f_\alpha) \\
= \;& \mathcal{L}'(f_\alpha^\delta)^*(g^\delta-g)+[\mathcal{L}'(f_\alpha^\delta)^*-\mathcal{L}'(f_\alpha)^*]g. \qquad \text{(A1)}
\end{aligned}$$

Let

$$M_\alpha^\delta=\int_0^1 \mathcal{L}'(f_\alpha+t(f_\alpha^\delta-f_\alpha))dt.$$

Then, by (A1) we have,

$$\begin{aligned}
f_\alpha^\delta-f_\alpha &= (\mathcal{L}'(f_\alpha^\delta)^*\mathcal{L}'(f_\alpha^\delta)+\alpha I)^{-1}[\mathcal{L}'(f_\alpha^\delta)^*(\mathcal{L}'(f_\alpha^\delta)-M_\alpha^\delta)((f_\alpha^\delta-f_\alpha) \\
&\quad +\mathcal{L}'(f_\alpha^\delta)^*(g^\delta-g)+(\mathcal{L}'(f_\alpha^\delta)^*-\mathcal{L}'(f_\alpha)^*)(g-\mathcal{L}(f_\alpha))] \\
&= (\mathcal{L}'(f_\alpha^\delta)^*\mathcal{L}'(f_\alpha^\delta)+\alpha I)^{-1}\Bigg[\mathcal{L}'(f_\alpha^\delta)^*(\mathcal{L}'(f_\alpha^\delta)-M_\alpha^\delta)((f_\alpha^\delta-f_\alpha) \\
&\quad +\mathcal{L}'(f_\alpha^\delta)^*(g^\delta-g)+(\mathcal{L}'(f_\alpha^\delta)^*-\mathcal{L}'(f_\alpha)^*) \\
&\quad \times\int_0^1[\mathcal{L}'(f_\alpha+t(\hat{f}-f_\alpha))-\mathcal{L}'(f_\alpha^\delta)+\mathcal{L}'(f_\alpha^\delta)]dt(\hat{f}-f_\alpha)\Bigg] \\
&= (\mathcal{L}'(f_\alpha^\delta)^*\mathcal{L}'(f_\alpha^\delta)+\alpha I)^{-1}\Bigg[\mathcal{L}'(f_\alpha^\delta)^*(\mathcal{L}'(f_\alpha^\delta)-M_\alpha^\delta)((f_\alpha^\delta-f_\alpha) \\
&\quad +\mathcal{L}'(f_\alpha^\delta)^*(g^\delta-g)+(\mathcal{L}'(f_\alpha^\delta)^*-\mathcal{L}'(f_\alpha)^*)\mathcal{L}'(f_\alpha^\delta) \\
&\quad \times\left(\int_0^1\varphi(f_\alpha+t(\hat{f}-f_\alpha),f_\alpha^\delta,\hat{f}-f_\alpha)dt+\hat{f}-f_\alpha\right)\Bigg].
\end{aligned}$$

By (17) (with $\Pi = M_\alpha^\delta$, i.e, $u = f_\alpha, f = v = f_\alpha^\delta$, $\xi = f_\alpha^\delta - f_\alpha$), (iii), (22), and (23), we have

$$\begin{aligned}\|f_\alpha^\delta - f_\alpha\| &\leq \frac{k_0 r}{2}\|f_\alpha^\delta - f_\alpha\| + \frac{\delta}{\sqrt{\alpha}} \\ &+k_2 r\Big[k_0(r+\frac{r_0}{2})+1\Big]\|f_\alpha - \hat{f}\| \ \ (\because \|f_\alpha^\delta - f_\alpha\| \leq r).\end{aligned}$$

Therefore,

$$(1-\frac{k_0 r}{2})\|f_\alpha^\delta - f_\alpha\| \leq \frac{\delta}{\sqrt{\alpha}} + k_2 r\Big[k_0(r+\frac{r_0}{2})+1\Big]\|f_\alpha - \hat{f}\|. \tag{A2}$$

Appendix D. Proof of Lemma 2

Since $\mathcal{L}'(f_\alpha)^*(\mathcal{L}(f_\alpha) - g) + \alpha(f_\alpha - f_0) = 0$ and $\mathcal{L}(\hat{f}) = g$, we have

$$[\mathcal{L}'(f_\alpha)^*\int_0^1 \mathcal{L}'(\hat{f} + t(f_\alpha - \hat{f}))dt + \alpha I](f_\alpha - \hat{f}) \ = \ \alpha(f_0 - \hat{f}),$$

and

$$\begin{aligned}&[\mathcal{L}'(f_\alpha)^*\mathcal{L}'(f_\alpha) + \alpha I](f_\alpha - \hat{f}) \\ = \ &\mathcal{L}'(f_\alpha)^*\int_0^1 [\mathcal{L}'(f_\alpha) - \mathcal{L}'(\hat{f} + t(f_\alpha - \hat{f}))]dt(f_\alpha - \hat{f}) + \alpha(f_0 - \hat{f}).\end{aligned}$$

So,

$$\begin{aligned}f_\alpha - \hat{f} \ &= \ (\mathcal{L}'(f_\alpha)^*\mathcal{L}'(f_\alpha) + \alpha I)^{-1} \\ &\times\Big[\mathcal{L}'(f_\alpha)^*\int_0^1 [\mathcal{L}'(f_\alpha) - \mathcal{L}'(\hat{f} + t(f_\alpha - \hat{f}))]dt(f_\alpha - \hat{f}) + \alpha(f_0 - \hat{f})\Big]\end{aligned}$$

and hence by (17) (with $\Pi = \int_0^1 \mathcal{L}'(\hat{f} + t(f_\alpha - \hat{f})))$ we have

$$\begin{aligned}\|f_\alpha - \hat{f}\| \ &\leq \ \frac{k_0}{2}\|f_\alpha - \hat{f}\|^2 + \|\alpha(\mathcal{L}'(f_\alpha)^*\mathcal{L}'(f_\alpha) + \alpha I)^{-1}(f_0 - \hat{f})\| \\ &\leq \ \frac{k_0 r_0}{2}\|f_\alpha - \hat{f}\| + \|\alpha(\mathcal{L}'(f_\alpha)^*\mathcal{L}'(f_\alpha) + \alpha I)^{-1}(f_0 - \hat{f})\|.\end{aligned} \tag{A3}$$

Since $\|f_\alpha - \hat{f}\| \leq r_0$, by (A3) we have

$$(1-\frac{k_0 r_0}{2})\|f_\alpha - \hat{f}\| \leq \|\alpha(\mathcal{L}'(f_\alpha)^*\mathcal{L}'(f_\alpha) + \alpha I)^{-1}(f_0 - \hat{f})\|. \tag{A4}$$

Next, we shall prove that $\|\alpha(\mathcal{L}'(f_\alpha)^*\mathcal{L}'(f_\alpha) + \alpha I)^{-1}(f_0 - \hat{f})\| = O(\alpha^\nu)$ under the assumption (11).

Note that

$$\begin{aligned}&\|\alpha(\mathcal{L}'(f_\alpha)^*\mathcal{L}'(f_\alpha) + \alpha I)^{-1}(f_0 - \hat{f})\| \\ \leq \ &\|\alpha[(\mathcal{L}'(f_\alpha)^*\mathcal{L}'(f_\alpha) + \alpha I)^{-1} - (L^*L + \alpha I)^{-1}](f_0 - \hat{f})\| \\ &+\|\alpha(L^*L + \alpha I)^{-1}(f_0 - \hat{f})\| \\ \leq \ &[(k_3 C_R + k_1)(\|f_\alpha - \hat{f}\| + \frac{\|f_0 - \hat{f}\|}{2}) + 1]\|\alpha(L^*L + \alpha I)^{-1}(f_0 - \hat{f})\| \\ &(\text{by } (20)) \\ \leq \ &[(k_3 C_R + k_1)\frac{3r_0}{2} + 1]\|\alpha(L^*L + \alpha I)^{-1}(L^*L)^\nu w\| \\ \leq \ &[(k_3 C_R + k_1)\frac{3r_0}{2} + 1]\alpha^\nu\|w\|.\end{aligned} \tag{A5}$$

Appendix E. Proof of Lemma 3

Note that, by (A4) and (A5), we have

$$\|f_\alpha - \hat{f}\| \le \frac{2r_0}{2-k_0r_0}((k_3C_R+k_1))\frac{3r_0}{2}+1)\|\alpha(L^*L+\alpha I)^{-1}(L^*L)^\nu w\|. \tag{A6}$$

Let $B=(L^*L)^{\frac{1}{2}}$, $x=\alpha(L^*L+\alpha I)^{-1}w$. Then, by (27), we have

$$\begin{aligned}
\|B^{2\nu}x\| &= \|\alpha(L^*L+\alpha I)^{-1}(f_0-\hat{f})\| \\
&\le \|B^{1+2\nu}x\|^{\frac{2\nu}{2\nu+1}}\|x\|^{\frac{1}{2\nu+1}} \\
&\le \|\alpha(L^*L)^{\frac{1}{2}}(L^*L+\alpha I)^{-1}(f_0-\hat{f})\|^{\frac{2\nu}{2\nu+1}}\|w\|^{\frac{1}{2\nu+1}} \\
&\le \|\alpha(LL^*+\alpha I)^{-1}A(f_0-\hat{f})\|^{\frac{2\nu}{2\nu+1}}\|w\|^{\frac{1}{2\nu+1}} \\
&= (\|\alpha(LL^*+\alpha I)^{-1}(\mathcal{L}(f_0)-g^\delta+g^\delta-g)\|)^{\frac{2\nu}{2\nu+1}}\|w\|^{\frac{1}{2\nu+1}} \\
&\le (\|\alpha(LL^*+\alpha I)^{-1}(\mathcal{L}(f_0)-g^\delta)\|+\delta)^{\frac{2\nu}{2\nu+1}}\|w\|^{\frac{1}{2\nu+1}}.
\end{aligned} \tag{A7}$$

Here, we have used the relations $(L^*L)^{\frac{1}{2}}=\mathbb{U}L$, where $\mathbb{U}$ is the unitary operator and $L(f_0-\hat{f})=\mathcal{L}(f_0)-g$. Observe that,

$$\begin{aligned}
& \|\alpha(LL^*+\alpha I)^{-1}(\mathcal{L}(f_0)-g^\delta)\| \\
\le\ & \|\alpha[(LL^*+\alpha I)^{-1}-(L_0L_0^*+\alpha I)^{-1}](\mathcal{L}(f_0)-g^\delta)\| \\
& +\|\alpha(L_0L_0^*+\alpha I)^{-1}(\mathcal{L}(f_0)-g^\delta)\| \\
\le\ & (k_3C_R+k_1)\frac{r_0}{2}\|\alpha(LL^*+\alpha I)^{-1}(\mathcal{L}(f_0)-g^\delta)\| \\
& +\|\alpha(L_0L_0^*+\alpha I)^{-1}(\mathcal{L}(f_0)-g^\delta)\| \text{ (by (20))} \\
\le\ & \frac{2}{(2-(k_3C_R+k_1)r_0)}\|\alpha(L_0L_0^*+\alpha I)^{-1}(\mathcal{L}(f_0)-g^\delta)\| \\
=\ & \frac{2}{(2-(k_3C_R+k_1)r_0)}d(\alpha,g^\delta) \\
=\ & \frac{2}{(2-(k_3C_R+k_1)r_0)}c\delta.
\end{aligned} \tag{A8}$$

The result now follows from (A6), (A7), and (A8).

Appendix F. Proof of Lemma 4

Note that,

$$\begin{aligned}
c\delta &= d(\alpha,g^\delta) \\
&\le \|\alpha(L_0L_0^*+\alpha I)^{-1}(g-g^\delta)\|+\|\alpha(L_0L_0^*+\alpha I)^{-1}(\mathcal{L}(f_0)-g)\| \\
&\le \delta+\|\alpha(L_0L_0^*+\alpha I)^{-1}(\mathcal{L}(f_0)-g)\|.
\end{aligned}$$

So,

$$\begin{aligned}
(c-1)\delta &\le \|\alpha[(L_0L_0^*+\alpha I)^{-1}-(LL^*+\alpha I)^{-1}](\mathcal{L}(f_0)-g)\| \\
&\quad +\|\alpha(LL^*+\alpha I)^{-1}(\mathcal{L}(f_0)-g)\| \\
&\le \left[(k_3C_R+k_1)\frac{r_0}{2}+1\right]\|\alpha(L^*L+\alpha I)^{-1}(L^*L)^{\frac{1}{2}+\nu}w\| \\
&\quad \text{(by (20))} \\
&\le \left[(k_3C_R+k_1)\frac{r_0}{2}+1\right]\begin{cases}\alpha^{\frac{1}{2}+\nu}\|w\|, & \nu\le\frac{1}{2} \\ \alpha\|L^*L\|^{\nu-\frac{1}{2}}\|w\|, & \nu>\frac{1}{2}\end{cases}.
\end{aligned} \tag{A9}$$

Therefore, since

$$\frac{\delta}{\sqrt{\alpha}} = \begin{cases} \left(\frac{\delta}{\alpha^{\frac{1}{2}+\nu}}\right)^{\frac{1}{2\nu+1}} \delta^{\frac{2\nu}{2\nu+1}}, \nu \leq \frac{1}{2} \\ \left(\frac{\delta}{\alpha}\right)^{\frac{1}{2}} \delta^{\frac{1}{2}}, \nu > \frac{1}{2} \end{cases} \tag{A10}$$

by (A9) and (A10), we have

$$\frac{\delta}{\sqrt{\alpha}} = \begin{cases} O(\delta^{\frac{2\nu}{2\nu+1}}), \nu \leq \frac{1}{2} \\ O(\delta^{\frac{1}{2}}), \nu > \frac{1}{2} \end{cases}.$$

Appendix G. Verification of Assumptions (iii) and (iv)

As in [5], we use the following assumptions:

(A1) Let $q_0 \in D(\mathcal{L})$, and assume that $\exists\, \kappa > 0$ with $|\mathcal{L}(q_0)(t)| \geq \kappa \;\forall\, t \in (0,1)$.
Then, $\exists\, U(q_0)$ of q_0 in $L^2[0,1]$ such that

(A2) $|\mathcal{L}(q)(t)| \geq \frac{\kappa}{2}$ for all $q \in U(q_0) \cap D(\mathcal{L})$ and $t \in (0,1)$.

Note that,

$$(T_z - T_f)\mathcal{L}'(z)\mathcal{L}'(z)^* w + T_f(\mathcal{L}'(z) - \mathcal{L}'(f))\mathcal{L}'(z)^* w = (\mathcal{L}(f) - \mathcal{L}(z))\mathcal{L}'(z)^* w,$$

so, we have for $x, z \in U(q_0) \cap D(\mathcal{L})$:

$$\begin{aligned} & (\mathcal{L}'(z) - \mathcal{L}'(f))\mathcal{L}'(z)^* w \\ = \;& T_f^{-1}\Big[(T_f - T_z)\mathcal{L}'(z)\mathcal{L}'(z)^* w + (\mathcal{L}(f) - \mathcal{L}(z))\mathcal{L}'(z)^* w\Big] \\ = \;& -T_f^{-1}\left[\frac{-1}{\mathcal{L}(f)}((T_f - T_z)\mathcal{L}'(z)\mathcal{L}'(z)^* w + (\mathcal{L}(f) - \mathcal{L}(z))\mathcal{L}'(z)^* w)\right]\mathcal{L}(f) \\ = \;& -T_f^{-1}\Bigg[-\mathcal{L}(z)T_z^{-1}T_z\frac{-1}{\mathcal{L}(f)\mathcal{L}(z)}((T_f - T_z)\mathcal{L}'(z)\mathcal{L}'(z)^* w \\ & +(\mathcal{L}(f) - \mathcal{L}(z))\mathcal{L}'(z)^* w)\Bigg]\mathcal{L}(f) \\ = \;& \mathcal{L}'(f)\mathcal{L}'(z)^* \varphi_2(z, f, w), \end{aligned}$$

where $\varphi_2(z, f, w) = \frac{-1}{\mathcal{L}(f)\mathcal{L}(z)} T_z((T_f - T_z)\mathcal{L}'(z)\mathcal{L}'(z)^* w + (\mathcal{L}(f) - \mathcal{L}(z))\mathcal{L}'(z)^* w)$. Then, as in Lemma 2.4 in [5], one can prove that $\|\varphi_2(z, f, w)\| \leq k_2\|z - f\|\|w\|$. Further, observe that

$$\begin{aligned} & [\mathcal{L}'(f)^* - \mathcal{L}'(z)^*]\mathcal{L}'(z)v \\ = \;& -\mathcal{L}(z)T_z^{-1}\Bigg[T_z\frac{1}{\mathcal{L}(z)}\Bigg(\mathcal{L}(f)(T_f^{-1} - T_z^{-1})(\mathcal{L}'(z)v) \\ & +(\mathcal{L}(f) - \mathcal{L}(z))T_z^{-1}(\mathcal{L}'(z)v)\Bigg)\Bigg] \\ = \;& -\mathcal{L}(z)T_z^{-1}\Bigg[-T_z\frac{1}{\mathcal{L}(z)^2}\Bigg(\mathcal{L}(f)(T_f^{-1} - T_z^{-1})(\mathcal{L}'(z)v) \\ & \times(\mathcal{L}(f) - \mathcal{L}(z))T_z^{-1}(\mathcal{L}'(z)v)\Bigg)\Bigg]\mathcal{L}(z) \\ = \;& \mathcal{L}'(z)^*\mathcal{L}'(z)\varphi_3(z, f, v), \end{aligned}$$

where $\varphi_3(z,f,v) = \frac{1}{\mathcal{L}(z)^2}\Big(\mathcal{L}(f)(T_f^{-1} - T_z^{-1})(\mathcal{L}'(z)v) + (\mathcal{L}(f) - \mathcal{L}(z))T_z^{-1}(\mathcal{L}'(z)v)\Big)$. Again, as in Lemma 2.4 in [5], one can prove that $\|\varphi_3(z,f,v)\| \le k_3\|z-f\|\|v\|$.

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