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On Eigenfunctions of the Boundary Value Problems for Second Order Differential Equations with Involution

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Abstract: We give a definition of Green's function of the general boundary value problems for non-self-adjoint second order differential equation with involution. The sufficient conditions for the basis property of system of eigenfunctions are established in the terms of the boundary conditions. Uniform equiconvergence of spectral expansions related to the second-order differential equations with involution:$-y''(x) + \alpha y''(-x) + q(x)y(x) = \lambda y(x), -1 < x < 1$, with the boundary conditions $y'(-1) + b_1 y(-1) = 0,\ y'(1) + b_2 y(1) = 0$, is obtained. As a corollary, it is proved that the eigenfunctions of the perturbed boundary value problems form the basis in $L_2(-1,1)$ for any complex-valued coefficient $q(x) \in L_1(-1,1)$.

Keywords: differential equation; involution; boundary value problem; Green's function; eigenvalue; eigenfunction; basis

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1. Introduction

In this paper we consider in the Hilbert space $L_2(-1,1)$ a second-order differential operator L defined by differential expression

$$l_q y = -y''(x) + \alpha y''(-x) + q(x)y(x), -1 < x < 1, \tag{1}$$

with domain $D(L) \subset L_2(-1,1)$, where $q(x) \in L_1(-1,1)$ is a complex-valued function. The parameter α satisfies the condition $-1 < \alpha < 1$. Then, this operator is a semi-bonded operator. The differential expression (1) contains an involution transformation of the form $Sy = y(-x)$ for any function $y(x) \in L_2(-1,1)$. The graph of each f such $f(f(x)) = x$ is symmetric about the line $x = t$ in the (x,t) plane.

We denote by $AC[-1,1]$ the space of absolute continuous functions on $[-1,1]$ and denoted

$AC^1[-1,1] = \{y(x) \in C^1[-1,1] | y'(x) \in AC[-1,1]\}$. The functions $y(x) \in D(L)$ satisfy the conditions: $y(x)$ belongs to $AC^1[-1,1]$ and

$$y'(-1) + b_1 y(-1) = 0,\ y'(1) + b_2 y(1) = 0, \tag{2}$$

where b_i are complex constants.

Along with operator L, we also consider an operator L_0 defined by differential expression

$$l_0 y = -y''(x) + \alpha y''(-x) \tag{3}$$

with domain $D(L_0) = D(L) \subset L_2(-1,1)$, and an operator$\hat{L}_0$ defined by expression (3) and boundary conditions

$$y'(-1) = 0, \quad y'(1) = 0. \tag{4}$$

Uniform equiconvergence of spectral expansions related to the operators L_0 and L given by (3), (1), respectively, is studied.

Differential equations with involution form a special class of linear functional-differential equations, with their theory having been developed since the middle of the last century. Among a variety of studies in this direction, one can mention the books [1–3]. The existence of a solution of the partial differential equation with involution has been studied in [2] by the separation of variables method. As in the case of classical equations, applying the Fourier method to partial differential equations with involution leads to the related spectral problems for differential operators with involution. The study of spectral problems for differential operators with involution started relatively recently. In [4–7], the spectral problems for the first-order differential operators with an involution have been studied. In [8] (see also references therein), Ref. [9], the spectral problems for differential operators with involution in the lower terms have been considered. The spectral problems related to the second-order differential operators with involution have been studied in [10–14]. The Green's function of the boundary value problems for the first order equations (and a system of equations) with involution have been derived in [3,15–17]. In [12,13,18], the Green's functions of the second-order differential operators with involution have been investigated and theorems on basicity of eigenfunctions are proved. Theorems on basicity of eigenfunctions of the second order differential operators with involution [14] have been used to solving inverse problems in [19–21]. Solvability of problems for partial and ordinary differential equations with involution is discussed in [22–26].

The operator L defined by (1), (2) generalizes Sturm–Liouville operators, which have been studied fairly completely (see, for example, [27,28]). Spectral properties of the operator L with non self-adjoint boundary conditions in the form (2) have not been so well-studied yet, since this case is more complex for investigation. The first results about the basis property of eigenfunctions of boundary value problems for equation $-y''(-x)+q(x)y(x)=\lambda y(x), -1<x<1$, have been obtained in [12,18]. In [29], the basis property of eigenfunctions of operators (1) with periodic boundary conditions have been studied.

In this paper, the integral Cauchy method [27] (well-known in the spectral theory of ordinary differential operators) is modified for the case of differential operators with involution (1), (2) (and (3), (2)). The method is based on proving the equiconvergence of the known expansion with the eigenfunction expansion of the considered problem. We obtain our main results by developing the integral Cauchy method and by using the estimates for Green's functions.

The paper is organized as follows. In Section 2, we define the Green's function of the general boundary value problems. We give the formula for the Green's function of the operator $\hat{L}_0$ defined by (3), (4), and achieve the estimate for the Green's function. Section 3 is devoted to the estimate of the Green's function of the operator L_0 given by (3), (2). Finally, we discuss the basicity of eigenfunctions in Section 4.

2. Green's Function of the Operator $\hat{L}_0 - \lambda I$

Let us introduce the definition of the Green's function of the general boundary value problem $l_q y = \lambda y$ with boundary conditions

$$U_i(y) = a_{i1}y'(-1) + a_{i2}y(-1) + a_{i3}y'(1) + a_{i4}y(1) = 0, (i = 1,2), \tag{5}$$

where a_{ij} are complex constants, λ is a complex spectral parameter. Let the boundary value problem not have a non-trivial solution. However, there can exist a function $G_q(x,t,\lambda)$, such that:

(1) $G_q(x,t,\lambda)$ is continuous on the rectangle $-1 \le x,t \le 1$;
(2) The function $G_q(x,t,\lambda)$ has the continuous derivative $\left(G_q(x,t,\lambda)\right)'_x$ for $x \ne \mp t$ and satisfies the conditions:

$$\left(G_q(x,t,\lambda)\right)'_x\Big|_{t=-x-0} - \left(G_q(x,t,\lambda)\right)'_x\Big|_{t=-x+0} = \frac{\alpha}{\sqrt{1-\alpha^2}},$$

$$\left(G_q(x,t,\lambda)\right)'_x\Big|_{t=x-0} - \left(G_q(x,t,\lambda)\right)'_x\Big|_{t=x+0} = \frac{-1}{\sqrt{1-\alpha^2}};$$

(3) The function $G_q(x,t,\lambda)$ has the derivative $\left(G_q(x,t,\lambda)\right)''_{xx}$, satisfies $l_q y = \lambda y$ (except at $x \neq \mp t$) and (5).

The function $G_q(x,t,\lambda)$ is called the Green's function of the considered boundary value problem (of the operator $L - \lambda I$, defined by (1), (5), where I is the identity operator).

If the function $G_q(x,t,\lambda)$ is the Green's function of the operator $L - \lambda I$, then the function

$$y(x) = \int_{-1}^{1} G_q(x,t,\lambda) f(t)dt$$

gives the solution to the problem

$$-y''(x) + \alpha y''(-x) + q(x)y(x) = \lambda y(x) + f(x), \ -1 < x < 1,$$

with boundary conditions (5), for any function $f(x) \in C[-1,1]$ (this statement, existence and uniqueness of the Green's function can be proved by standard methods (see [28], chapter 1).

In order to study the basis property of system of eigenfunctions of the operator L (defined by (1), (2)), we construct the Green's function $G(x,t,\lambda)$ of the problem $l_0 y = \lambda y$, (4). Let us denote by $y_1(x) = \cos\alpha_0\rho x$ and $y_2(x) = \sin\alpha_1\rho x$, where $\sqrt{\lambda} = \rho$, $\alpha_0 = \sqrt{\frac{1}{1-\alpha}}$, $\alpha_1 = \sqrt{\frac{1}{1+\alpha}}$, the linearly independent solutions of the homogeneous equation $l_0 y = \lambda y(x)$.

Lemma 1. *If λ is not an eigenvalue of the operator $\hat{L}_0 - \lambda I$, then the function*

$$y(x) = -\frac{\alpha_0}{2\rho}\frac{\cos\alpha_0\rho}{\sin\alpha_0\rho}\cos(\alpha_0\rho x)\int_{-1}^{1}\cos(\alpha_0\rho t)f(t)dt$$

$$-\frac{\alpha_1}{2\rho}\frac{\sin\alpha_1\rho}{\cos\alpha_1\rho}\sin(\alpha_1\rho x)\int_{-1}^{1}\sin(\alpha_1\rho t)f(t)dt + g_0(x)$$

is the solution of non-homogeneous problem $l_0 y = \lambda y(x) + f(x)$, (4) for any continuous function $f(x)$, where

$$g_0(x) = \tfrac{1}{2\rho}\int_{-1}^{-x}[\alpha_0\cos(\alpha_0\rho x)\sin(\alpha_0\rho t) - \alpha_1\sin(\alpha_1\rho x)\cos(\alpha_1\rho t)]f(t)dt$$
$$+\tfrac{1}{2\rho}\int_{-x}^{x}[-\alpha_0\cos(\alpha_0\rho t)\sin(\alpha_0\rho x) + \alpha_1\sin(\alpha_1\rho t)\cos(\alpha_1\rho x)]f(t)dt$$
$$+\tfrac{1}{2\rho}\int_{x}^{1}[-\alpha_0\cos(\alpha_0\rho x)\sin(\alpha_0\rho t) + \alpha_1\sin(\alpha_1\rho x)\cos(\alpha_1\rho t)]f(t)dt.$$

This Lemma 1 can be proved by direct calculations. From Lemma 1, we find the following

Corollary 1. *The Green's function of the operator $\hat{L}_0 - \lambda I$ can be represented in the form*

$$\widehat{G}(x,t,\lambda) = -\frac{\alpha_0}{2\rho}\frac{\cos\alpha_0\rho}{\sin\alpha_0\rho}\cos(\alpha_0\rho x)\cos(\alpha_0\rho t) + \frac{\alpha_1}{2\rho}\frac{\sin\alpha_1\rho}{\cos\alpha_1\rho}\sin(\alpha_1\rho x)\sin(\alpha_1\rho t) + g(x) \quad (6)$$

where

$$g(x) = \frac{1}{2\rho}\begin{cases} \alpha_0 \cos(\alpha_0\rho x)\sin(\alpha_0\rho t) - \alpha_1 \sin(\alpha_1\rho x)\cos(\alpha_1\rho t), t \le -x, \\ -\alpha_0 \cos(\alpha_0\rho t)\sin(\alpha_0\rho x) + \alpha_1 \sin(\alpha_1\rho t)\cos(\alpha_1\rho x), -x \le t \le x, \\ -\alpha_0 \cos(\alpha_0\rho x)\sin(\alpha_0\rho t) + \alpha_1 \sin(\alpha_1\rho x)\cos(\alpha_1\rho t), x \le t. \end{cases}$$

The Green's function of the operator $\hat{L}_0 - \lambda I$ has the following properties:

(1) $\hat{G}(x,t,\lambda)$ is symmetric: $\hat{G}(x,t,\lambda) = \hat{G}(t,x,\lambda)$, for all $-1 \le x,t \le 1$;
(2) $\hat{G}(x,t,\lambda)$ is continuous on the rectangle$-1 \le x,t \le 1$;
(3) The function $\hat{G}(x,t,\lambda)$ has the continuous derivative $\hat{G}'_x(x,t,\lambda)$ for $x \ne \mp t$, and satisfies the conditions:

$$\hat{G}'_x(x,t,\lambda)|_{t=-x-0} - \hat{G}'_x(x,t,\lambda)|_{t=-x+0} = \frac{\alpha}{\sqrt{1-\alpha^2}},$$

$$\hat{G}'_x(x,t,\lambda)|_{t=x-0} - \hat{G}'_x(x,t,\lambda)|_{t=x+0} = \frac{-1}{\sqrt{1-\alpha^2}};$$

(4) The function $\hat{G}(x,t,\lambda)$ has the derivative $\hat{G}''_{xx}(x,t,\lambda)$, satisfies $l_0 y = \lambda y$ except at $x \ne \mp t$) and (4).

The operator $\hat{L}_0$ defined by (3), (4) has the eigenvalues

$$\lambda_{k1} = (1+\alpha)\left(k+\frac{1}{2}\right)^2\pi^2,\ k = 0,1,2,\ldots,;\ \lambda_{k2} = (1-\alpha)(k\pi)^2,\ k = 0,1,2,\ldots.$$

The system of eigenfunctions $\{y_{k1} = \sin\left(k+\frac{1}{2}\right)\pi x, y_{k2} = \cos k\pi x,\ k = 0,1,2,\ldots\}$, of the operator $\hat{L}_0$ is complete and orthogonal in $L_2(-1,1)$. Denote

$\rho_{k1} = \sqrt{(1+\alpha)}\left(k+\frac{1}{2}\right)\pi, k = 0,1,2,\ldots, \rho_{k2} = \sqrt{(1-\alpha)}k\pi, k = 0,1,2,\ldots.$.

Since $\rho_{k+1,1} - \rho_{k1} = \sqrt{(1+\alpha)}\pi$; $\rho_{k+1,2} - \rho_{k2} = \sqrt{(1-\alpha)}\pi$, we denote by $O_\xi(\rho_{kj}) = \left\{\rho : |\rho - \rho_{kj}| < \xi, k = 0,1,2,\ldots; j = 1,2,\right\}$ a circle of radius $\xi = \frac{1}{4}min((1-\alpha)\pi,(1+\alpha)\pi)$. Then, the circles $C_{kj}, k = 1,2,\ldots; j = 1,2,$ with equations $\rho = \frac{\xi}{2}, \rho = \rho_{kj} + \frac{\xi}{2}$ do not intersect the circles $O_\xi(\rho_{kj})$ for large k.

Further, we need an estimate of the Green's function for operator $\hat{L}_0 - \lambda I$.

Lemma 2. *Let $\rho \notin O_\xi(\rho_{kl})$ and $|\rho| > 1$. Then, the Green's function $\hat{G}(x,t,\lambda)$ of the operator $\hat{L}_0 - \lambda I$ satisfies the uniformly with respect to $-1 \le x,t \le 1$ the following estimate*

$$\left|\hat{G}(x,t,\lambda)\right| \le c_0(\alpha,\xi)|\rho|^{-1}r(x,t,\rho), \tag{7}$$

where

$$r(x,t,\rho) = \left(e^{-\alpha_2|\rho_0|(2-|x|-|t|)} + e^{-\alpha_2|\rho_0|||x|-|t||}\right), \alpha_2 = min\{\alpha_1,\alpha_0\}, \rho_0 = Im\rho.$$

Proof. Let us examine three cases: $t \ge x$, $-x \le t \le x$ and $t \le -x$ separately. In the first case, when $t \ge x$, the relation (7) can be rewritten in the form

$$\hat{G}(x,t,\lambda) = \frac{\alpha_0}{4i\rho}\left\{\frac{e^{-i\alpha_0\rho}}{e^{i\alpha_0\rho} - e^{-i\alpha_0\rho}}\left[e^{i\alpha_0\rho(x+t)} + e^{i\alpha_0\rho(t-x)}\right] + \right.$$

$$\left. + \frac{e^{i\alpha_0\rho}}{e^{i\alpha_0\rho} - e^{-i\alpha_0\rho}}\left[e^{i\alpha_0\rho(x-t)} + e^{i\alpha_0\rho(-x-t)}\right]\right\} +$$

$$+\frac{\alpha_1}{4i\rho}\left\{\frac{-e^{i\alpha_1\rho}}{e^{i\alpha_1\rho} + e^{-i\alpha_1\rho}}\left[e^{i\alpha_1\rho(x+t)} - e^{i\alpha_1\rho(t-x)}\right] + \frac{e^{i\alpha_1\rho}}{e^{i\alpha_1\rho} + e^{-i\alpha_1\rho}}\left[e^{i\alpha_1\rho(x-t)} - e^{i\alpha_1\rho(-x-t)}\right]\right\}.$$

For sufficiently large $|\rho|$, we find the following estimate

$$\begin{aligned}
&\left|\hat{G}(x,t,\lambda)\right| \le \tfrac{\alpha_0}{4|\rho|}\left\{\tfrac{e^{\alpha_0\rho_0}}{|e^{-\alpha_0\rho_0}-e^{\alpha_0\rho_0}|}\left[e^{-\alpha_0\rho_0(x+t)}+e^{-\alpha_0\rho_0(t-x)}\right]+\right.\\
&\left.+\tfrac{e^{-\alpha_0\rho_0}}{|e^{-\alpha_0\rho_0}-e^{\alpha_0\rho_0}|}\left[e^{-\alpha_0\rho_0(x-t)}+e^{-\alpha_0\rho_0(-x-t)}\right]\right\}+\\
&+\tfrac{\alpha_1}{4|\rho|}\left\{\tfrac{e^{\alpha_1\rho_0}}{e^{-\alpha_1\rho_0}+e^{\alpha_1\rho_0}}\left[e^{-\alpha_1\rho_0(x+t)}+e^{-\alpha_1\rho_0(t-x)}\right]+\right.\\
&\left.+\tfrac{e^{-\alpha_1\rho_0}}{e^{-\alpha_1\rho_0}+e^{\alpha_1\rho_0}}\left[e^{-\alpha_1\rho_0(x-t)}+e^{-\alpha_1\rho_0(-x-t)}\right]\right\}.
\end{aligned} \tag{8}$$

Let $\rho_0 > 0$ and γ be arbitrary positive number (depends only on α). For sufficiently large $\rho_0 > 0$, we find the relations

$$\tfrac{e^{\gamma\rho_0}}{|e^{-\gamma\rho_0}-e^{\gamma\rho_0}|} \sim 1, \quad \tfrac{e^{-\gamma\rho_0}}{|e^{-\gamma\rho_0}-e^{\gamma\rho_0}|} \sim e^{-2\gamma\rho_0}. \tag{9}$$

Applying inequalities (9) to (8), we find

$$\left|\hat{G}(x,t,\lambda)\right| \le \frac{\alpha_0}{4|\rho|}\left[e^{-\alpha_0\rho_0(2-x-t)}+e^{-\alpha_0\rho_0(t-x)}\right]+\frac{\alpha_1}{4|\rho|}\left[e^{-\alpha_1\rho_0(2-x-t)}+e^{-\alpha_1\rho_0(t-x)}\right].$$

Hence,

$$\left|\hat{G}(x,t,\lambda)\right| \le \frac{M_1}{|\rho|}\left(e^{-\alpha_2|\rho_0|(2-x-t)}+e^{-\alpha_2|\rho_0|(t-x)}\right), \quad \alpha_2 = min\{\alpha_0,\alpha_1\}.$$

In a similar manner, we can show that

$$\left|\hat{G}(x,t,\lambda)\right| \le \frac{M_1}{|\rho|}\left(e^{\alpha_2|\rho_0|(2-x-t)}+e^{\alpha_2|\rho_0|(t-x)}\right), \quad \alpha_2 = min\{\alpha_0,\alpha_1\}.$$

for $\rho_0 < 0$. Thus, for $t \ge x > 0$ the Green's function satisfies the estimate (7). The completion of the proof is a result of simple computations (see [29]). Lemma 2 is proved. □

3. Green's Function of the Operator $L_0 - \lambda I$

As we have done earlier (Lemma 2), we obtain the Green's function of the operator $L_0 - \lambda I$

$$\begin{aligned}
&G(x,t,\lambda) = \tfrac{1}{\Delta(\rho)}\left\{\left[-\alpha_1^2\alpha_0\rho\cos\alpha_1\rho\cos\alpha_0\rho+\tfrac{b_1-b_2}{2}\alpha_0\alpha_1\cos\alpha_1\rho\sin\alpha_0\rho+\right.\right.\\
&\left.+\tfrac{b_2-b_1}{2}\alpha_0^2\sin\alpha_1\rho\cos\alpha_0\rho+\tfrac{b_1b_2}{\rho}\alpha_0\sin\alpha_1\rho\sin\alpha_0\rho\right]\cos\alpha_0\rho t\cos\alpha_0\rho x+\\
&+\left[\alpha_1^2\alpha_0\rho\sin\alpha_0\rho\sin\alpha_1\rho-\tfrac{b_1b_2}{\rho}\alpha_1\cos\alpha_0\rho\cos\alpha_1\rho+\tfrac{b_1-b_2}{2}\alpha_0\alpha_1\cos\alpha_1\rho\sin\alpha_0\rho+\right.\\
&\left.+\tfrac{b_1-b_2}{2}\alpha_1^2\sin\alpha_1\rho\cos\alpha_0\rho\right]\sin\alpha_1\rho t\sin\alpha_1\rho x+\\
&\left.+\tfrac{b_1+b_2}{2}\alpha_1^2\sin\alpha_1\rho t\cos\alpha_0\rho x+\tfrac{b_1+b_2}{2}\alpha_0^2\cos\alpha_0\rho t\sin\alpha_1\rho x\right\}+g(x),
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
&\Delta(\rho) = 2\alpha_0\alpha_1\rho^2\sin\alpha_0\rho\cos\alpha_1\rho-\\
&-\tfrac{\rho\alpha_0(b_2-b_1)}{4}\left(e^{i\rho(\alpha_0+\alpha_1)}-e^{i\rho(\alpha_0-\alpha_1)}-e^{i\rho(\alpha_1-\alpha_0)}+e^{i\rho(-\alpha_0-\alpha_1)}\right)+\\
&+\tfrac{\rho\alpha_1(b_1-b_2)}{4}\left(e^{i\rho(\alpha_0+\alpha_1)}+e^{i\rho(\alpha_0-\alpha_1)}+e^{i\rho(\alpha_1-\alpha_0)}+e^{i\rho(-\alpha_0-\alpha_1)}\right)+\\
&+\tfrac{b_1b_2}{2i}\left(e^{i\rho(\alpha_0+\alpha_1)}-e^{i\rho(\alpha_0-\alpha_1)}+e^{i\rho(\alpha_1-\alpha_0)}-e^{i\rho(-\alpha_0-\alpha_1)}\right)
\end{aligned}$$

is the characteristic determinant of the operator L_0. If $b_1 = b_2 = 0$, we find the Green's function of $\hat{L}_0$. The zeros of function $\Delta(\rho)$ are the eigenvalues of L_0. For large $|\rho|$ the zeros of $\Delta(\rho)$ are close to

$$\rho_{k1} = \sqrt{(1+\alpha)}\left(k+\frac{1}{2}\right)\pi, k = 0,1,2,\ldots; \rho_{k2} = \sqrt{(1-\alpha)}k\pi, k = 0,1,2,\ldots.$$

Each zero of $\Delta(\rho)$ belongs to a certain set $O_\xi(\rho_{kj}) = \left\{\rho : |\rho - \rho_{kj}| < \xi, k = 0,1,2,\ldots; j = 1,2\right\}$. The function $\Delta(\rho)$ can be written in the form

$$\Delta(\rho) = 2\alpha_0\alpha_1\rho^2 \sin\alpha_0\rho \cos\alpha_1\rho + O\left(|\rho| e^{|\rho_0|(\alpha_0+\alpha_1)}\right)$$

for $\rho \notin O_\xi(\rho_{kj})$, and

$$\frac{1}{\Delta(\rho)} = \frac{1}{2\alpha_0\alpha_1\rho^2 \sin\alpha_0\rho\cos\alpha_1\rho} + O\left(\frac{e^{-|\rho_0|(\alpha_0+\alpha_1)}}{|\rho|^3}\right).$$

Thus, if $\rho \notin O_\xi(\rho_{kj})$ for large $|\rho|$, from (10) it follows that

$$G(x,t,\lambda) = \hat{G}(x,t,\lambda) + O\left(\frac{e^{-|\rho_0|(\alpha_0+\alpha_1)||x|-|t||}}{|\rho|^2}\right) + O\left(\frac{e^{-|\rho_0|(\alpha_0+\alpha_1)(2-|x|-|t|)}}{|\rho|^2}\right), \tag{11}$$

where $\hat{G}(x,t,\lambda)$ is the Green's function of the operator $\hat{L}_0 - \lambda I$. Thus, we have proved validity of following lemma.

Lemma 3. *Suppose all assumptions of Lemma 2 hold true. Then, for Green's function $G(x,t,\lambda)$ of $L_0 - \lambda I$, the inequality (7) holds true.*

4. Basis Property of Eigenfunctons

Denote by

$$\hat{s}_m(f) = -\frac{1}{2\pi i}\int\limits_{\hat{C}_{mj}}\left(\int\limits_{-1}^{1}\hat{G}(x,t,\lambda)f(t)dt\right)d\lambda,\ s_m(f) = -\frac{1}{2\pi i}\int\limits_{\hat{C}_{mj}}\left(\int\limits_{-1}^{1}G(x,t,\lambda)f(t)dt\right)d\lambda$$

the partial sums of eigenfunction expansions for the operators $\hat{L}_0$ and L_0, respectively, where $\hat{C}_{mj}$ is a circles with equations $\hat{C}_{m1} : |\lambda| = \left(\rho_{m1} + \frac{1}{2}\right)^2$, $\hat{C}_{m2} : |\lambda| = \left(\rho_{m2} + \frac{1}{2}\right)^2$ in the λ-plane, $\forall f(x) \in L_1(-1,1)$. These representations hold true in the case when all eigenvalues of the operator $\hat{L}_0$ and L_0 are simple. Note that all eigenvalues of the operator $\hat{L}_0$ are simple if $\sqrt{\frac{1+\alpha}{1-\alpha}} \neq p_1$, $\sqrt{\frac{1-\alpha}{1+\alpha}} \neq p_2$ for any integers p_1, p_2.

We say that the sequence $s_m(f)$ equiconverges with $\hat{s}_m(f)$ on the interval $-1 \leq x \leq 1$ if $s_m - \hat{s}_m \to 0$ uniformly on this interval as $m \to \infty$.

Theorem 1. *Let all eigenvalues of operators L_0, $\hat{L}_0$ are simple. Then, for any function $f(x) \in L_1(-1,1)$ the sequence $s_m(f)$ equiconverges with $\hat{s}_m(f)$ on the interval $-1 \leq x \leq 1$.*

Proof. To prove Theorem 1, we consider the difference

$$s_m - \hat{s}_m = -\frac{1}{2\pi i}\int\limits_{\hat{C}_{mj}}\left(\int\limits_{-1}^{1}\left[G(x,t,\lambda) - \hat{G}(x,t,\lambda)\right]f(t)dt\right)d\lambda =$$

$$= -\frac{1}{2\pi i}\int\limits_{C_{mj}}\left(\int\limits_{-1}^{1}\left[G(x,t,\lambda) - \hat{G}(x,t,\lambda)\right]f(t)dt\right)2\rho d\rho,$$

where C_{mj} are the circles with equations $\rho = \frac{\xi}{2}$, $\rho = \rho_{mj} + \frac{\xi}{2}$. By virtue of (11) there exists a constant M_1, such that

$$|s_m - \hat{s}_m| \leq \frac{M_1}{2\pi}\int\limits_{C_{mj}}\left(\int\limits_{-1}^{1} r(x,t)|f(t)|dt\right)\left|\frac{d\rho}{\rho}\right|.$$

The proof of $\int\limits_{C_{mj}}\left(\int\limits_{-1}^{1} r(x,t)|f(t)|dt\right)\left|\frac{d\rho}{\rho}\right| \to 0$ (uniformly in $x \in [-1,1]$ as $m \to \infty$) is analogous to that given for inequality (26) in [29]. The proof of the theorem is complete. □

From Theorem 1 derives the following result.

Corollary 2. *Suppose all assumptions of Theorem 1 hold true. Then, the system of eigenfunctions of the operator L_0 forms the basis in $L_2(-1,1)$.*

Here the following result holds.

Corollary 3. *Suppose all assumptions of Theorem 1 hold true and b_1, b_2 in (2) are real numbers. Then, the system of eigenfunctions of the operator L_0 is orthonormal basis in $L_2(-1,1)$.*

To prove this assertion it suffices to show that the operator L_0 is self-adjoint in $L_2(-1,1)$.

Now we consider the operator L. Let denote by $G_q(x,t,\lambda)$ the Green's function of the operator $L - \lambda I$, where I is the identity operator. Denote by

$$S_m(f) = -\frac{1}{2\pi i}\int_{-1}^{1}\left[\int_{C_{mj}} G_q(x,t,\lambda)2\rho d\rho\right] f(t)dt$$

the partial sums of eigenfunction expansions for the operator L, $\forall f(x) \in L_1(-1,1)$.

Theorem 2. *Let all eigenvalues of operators L and L_0 are simple. Then, for any function $\forall f(x) \in L_1(-1,1)$ the sequence $S_m(f)$ equiconverges with $s_m(f)$ on the interval $-1 \le x \le 1$.*

The proof is analogous to that given for Theorem 1 in [29]. From Theorem 1 follows the following:

Corollary 4. *Suppose all assumptions of Theorem 1 hold true. Then, the system of eigenfunctions of the operator L forms the basis in $L_2(-1,1)$.*

Now let us turn to the self-adjoint operator L.

Corollary 5. *Suppose all assumptions of Theorem 1 hold true. If the coefficient $q(x) \in L_1(-1,1)$ in (1) is the real-valued function and b_1, b_2 in (2) are real numbers, then the system of eigenfunctions of the operator L forms orthonormal basis in $L_2(-1,1)$.*

Example 1. *Consider the spectral problem*

$$-y''(x) + \alpha y''(-x) + cy(x) = \lambda y(x), -1 < x < 1,$$

with boundary conditions (2), where c is a constant, $\sqrt{\frac{1+\alpha}{1-\alpha}} \neq p_1$, $\sqrt{\frac{1-\alpha}{1+\alpha}} \neq p_2$ for any integers p_1, p_2. It is easy to see that the system of eigenfunctions of spectral problem is simultaneously the system of eigenfunctions for the operator L_0 defined by (3), (2). Using Corollary 2 (Corollary 4), we conclude that the system of eigenfunctions of spectral problem forms the basis in $L_2(-1,1)$.

5. Conclusions

Summarizing the investigation carried out, we note that the Green's function of the second order differential operators (3), (4) with involution has been constructed. The estimates of the Green's functions of operators (3), (4), and (3), (2) have been established. The equiconvergence theorems (Theorem 1, Theorem 2) for operators (3), (2), (1), and (2) have been proven. As a corollary, results on the basicity of eigenfunctions to the problems under consideration have been proven. These theorems might be useful in the theory of solvability of mixed problems for partial differential equations with involution. For example:

Problem 1. *Find a sufficiently smooth function* $u(x,t)$*, satisfying the conditions:*

$$u_t(x,t) = u_{xx}(x,t) - \alpha u_{xx}(-x,t) - q(x)u(x,t)\,;\ -1 < x < 1,\ t > 0;$$

$$u(0,x) = \varphi(x),\ u_x(-1,t) + b_1 u(-1,t) = 0,\ u_x(1,t) + b_2 u(1,t) = 0.$$

Problem 2. *Find a sufficiently smooth function* $u(x,t)$*, satisfying the conditions:*

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} - \alpha \frac{\partial^2 u(-x,t)}{\partial x^2} - q(x)u(x,t),\ \ -1 < x < 1,\ \ t > 0,$$

$$u(x,0) = \varphi(x),\ \ u_t(x,0) = \psi(x),$$

$$u_x(-1,t) + b_1 u(-1,t) = 0,\quad u_x(1,t) + b_2 u(1,t) = 0.$$

The described problems are the subject of further work and we are going to consider them in our next articles. In the future, we also plan to investigate the inverse spectral problems.

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