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Direct and Fixed-Point Stability–Instability of Additive Functional Equation in Banach and Quasi-Beta Normed Spaces

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Abstract: Over the last few decades, a certain interesting class of functional equations were developed while obtaining the generating functions of many system distributions. This class of equations has numerous applications in many modern disciplines such as wireless networks and communications. The Ulam stability theorem can be applied to numerous functional equations in investigating the stability when approximated in Banach spaces, Banach algebra, and so on. The main focus of this study is to analyse the relationship between functional equations, Hyers–Ulam–Rassias stability, Banach space, quasi-beta normed spaces, and fixed-point theory in depth. The significance of this work is the incorporation of the stability of the generalised additive functional equation in Banach space and quasi-beta normed spaces by employing concrete techniques like direct and fixed-point theory methods. They are powerful tools for narrowing down the mathematical models that describe a wide range of events. Some classes of functional equations, in particular, have lately emerged from a variety of applications, such as Fourier transforms and the Laplace transforms. This study uses linear transformation to explain our functional equations while providing suitable examples.

Keywords: additive functional equations; generalised Ulam–Hyers stability; Banach space; quasi-beta normed spaces; fixed point

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1. Introduction

A function is conventionally defined in mathematics, particularly in functional analysis, as a map from a vector space to the field underlying the vector space, which is commonly the real numbers. In other words, a function accepts a vector as an argument and returns a scalar. A functional equation $F = G$, which means an equation between functionals, can be understood as an "equation to solve", with solutions being functions themselves.

The development of functional equations coincided with the contemporary definition of the function. J. D'Alembert [1] published the first papers to be published on functional equations between 1747 and 1750. Because of their apparent simplicity and harmonic nature, functional equations have attracted the attention of many notable mathematicians, including N.H. Abel, J. Bolyai, A.L. Cauchy, L. Euler, M. Frechet, C.F. Gauss, J.L.W.V. Jensen, A.N. Kolmogorov, N.I. Lobacevskii J.V. Pexider, and S.D. Poisson.

In 1940, S.M. Ulam [2] was the first to work on the issue of the stability of functional equations which gave rise to the question of "When is it true that the solution of an equation differing slightly from a given one must of necessity be close to the solution of the given equation?" and further studies are based upon it. D. H. Hyers [3] came out with a positive response to the issue of Ulam stability for Banach spaces in 1941. T. Aoki [4] explored additive mappings further in 1950. Th.M. Rassias [5] was successful in extending Hyer's

Theorem's result by weakening the condition for the Cauchy difference. Taking into account the significant effect of Ulam, Hyers, and Rassias on the development of stability issues of functional equations, the stability phenomena demonstrated by Th.M. Rassias is known as Hyers-Ulam-Rassias stability cited in [6–10]. In the spirit of the Rassias' method, P. Gavruta [11] explored further by substituting the unbounded Cauchy difference with a generic control function in 1994.

The historical background of the stability of functional equations and literature survey has been explained in the cited references (see [12–20]). The detailed results of Ulam stability are explained in [21–25]. Different types of additive functional equations and their Ulam stability are addressed in [26–30]. Stability analysis is important in mathematics, with Ulam stability being particularly important for functional equations, differential equations, and integral equations.

Fixed point method is one of the prominent methods for investigating the Ulam stability analysis and recalls a fundamental result in fixed-point theory. For more recent research on fixed-point theory, see [31–34].

Recently, A. Batool et al. [35], proved the Hyers–Ulam stability of the cubic and quartic functional equation

$$\begin{aligned} f(2x+y)+f(2x-y) &= 3f(x+y)+f(-x-y)+3f(x-y)+f(y-x) \\ &\quad +18f(x)+6f(-x)-3f(y)-3f(-y) \end{aligned} \tag{1}$$

and additive and quartic functional equation

$$\begin{aligned} f(2x+y)+f(2x-y) &= 2f(x+y)+2f(-x-y)+2f(x-y)+2f(y-x) \\ &\quad +14f(x)+10f(-x)-3f(y)-3f(-y) \end{aligned} \tag{2}$$

using the fixed-point method in matrix Banach algebras.

In [36], K. Tamilvanan et al. introduced a new mixed type quadratic-additive functional equation

$$\begin{aligned} &\phi\left(\sum_{1\le a\le m} as_a\right)+\sum_{1\le a\le m}\phi\left(-as_a+\sum_{b=1,a\neq b}^{m} bs_b\right)=(m-3)\sum_{1\le a<b\le m}\phi(as_a+bs_b) \\ &-\left(m^2-5m+2\right)\sum_{1\le a\le m}a^2\left[\frac{\phi(s_a)+\phi(-s_a)}{2}\right]-\left(m^2-5m+4\right)\sum_{1\le a\le m}^{s}a\left[\frac{\phi(s_a)-\phi(-s_a)}{2}\right] \end{aligned} \tag{3}$$

where $m>4$ is a fixed integer and investigated Ulam stability by using the Hyers method in random normed spaces.

In [37], N. Uthirasamy et al. considered the following new dimension additive functional equation

$$\begin{aligned} &\sum_{1\le a<b<c\le s}\phi\left(-\nu_a-\nu_b-\nu_c+\sum_{d=1,d\neq a\neq b\neq c}^{s}\nu_d\right) \\ &\qquad -\left(\frac{s^3-9s^2+20s-12}{6}\right)\sum_{a=1}^{s}\left[\frac{\phi(\nu_a)-\phi(-\nu_a)}{2}\right]=0 \end{aligned} \tag{4}$$

where $s>4$ is a fixed integer, to examine the Ulam stability of this equation in intuitionistic fuzzy normed spaces and 2-Banach spaces with the help of direct and fixed-point approaches.

The purpose of this research is to suggest a novel form of functional equation as below

$$\begin{aligned} &\omega\left(\eta^{\ell+\wp}\gamma+\eta^{\hbar}\kappa\right)+\omega\left(\eta^{\ell+\wp}\kappa+\eta^{\hbar}\mu\right)+\eta^{\ell+\wp}\omega(\gamma-\kappa)+\eta^{\hbar}\omega(\kappa-\mu) \\ &\quad =2\left(\eta^{\ell+\wp}\omega(\gamma)+\eta^{\hbar}\omega(\kappa)\right) \end{aligned} \tag{5}$$

In this article, the solution of this equation, as well as its Ulam stability, are determined with $\eta^{\ell+\wp}, \eta^{\hbar} \neq 0$ in Banach spaces and quasi β normed spaces using direct and fixed-point methods. The counter-example for non-stable cases is also demonstrated.

$$\mathbb{J}\omega(\gamma,\kappa,\mu) = \omega\left(\eta^{\ell+\wp}\gamma + \eta^{\hbar}\kappa\right) + \omega\left(\eta^{\ell+\wp}\kappa + \eta^{\hbar}\mu\right) + \eta^{\ell+\wp}\omega(\gamma-\kappa) + \eta^{\hbar}\omega(\kappa-\mu) \\ - 2\left(\eta^{\ell+\wp}\omega(\gamma) + \eta^{\hbar}\omega(\kappa)\right).$$

2. Banach Space Stability Results

2.1. Donald H. Hyers' Theorem (1941) for (5)

Theorem 1. *If $\omega : \mathbb{R} \to \mathbb{R}$ real map satisfying $\|\mathbb{J}\omega(\gamma,\kappa,\mu)\| \leq \mathfrak{U}$ for some $\mathfrak{U} \geq 0$ and for all $\gamma,\kappa,\mu \in \mathbb{R}$, then there exists a unique additive function $\mathbb{A} : \mathbb{R} \to \mathbb{R}$ and $\chi = \left(\eta^{\ell+\wp} + \eta^{\hbar}\right)$ such that $\|\omega(\gamma) - \mathbb{A}(\gamma)\| \leq \frac{\mathfrak{U}}{2(\chi-1)}$ for all $\gamma \in \mathbb{R}$.*

Proof. Let the real function $\omega : \mathbb{R} \to \mathbb{R}$ satisfying

$$\left\|\omega\left(\eta^{\ell+\wp}\gamma + \eta^{\hbar}\kappa\right) + \omega\left(\eta^{\ell+\wp}\kappa + \eta^{\hbar}\mu\right) + \eta^{\ell+\wp}\omega(\gamma-\kappa) + \eta^{\hbar}\omega(\kappa-\mu) \right. \\ \left. -2\left(\eta^{\ell+\wp}\omega(\gamma) + \eta^{\hbar}\omega(\kappa)\right)\right\| \leq \mathfrak{U} \tag{6}$$

for some $\mathfrak{U} \geq 0$. Instead of (γ,κ,μ) by $(0,0,0)$ in above inequality and $\left\|\left(2 - \eta^{\ell+\wp} - \eta^{\hbar}\right)\omega(0)\right\|$ $= 0$ or $\omega(0) = 0$ in place of (γ,κ,μ) by (γ,γ,γ) in the above inequality then

$$\|\omega(\chi\gamma) - \chi\omega(\gamma)\| \leq \frac{\mathfrak{U}}{2} \tag{7}$$

for all $\gamma \in \mathbb{R}$, where $\chi = \left(\eta^{\ell+\wp} + \eta^{\hbar}\right)$. Consider γ by $\chi^{s-1}\gamma$

$$\left\|\omega(\chi^s\gamma) - \chi\omega(\chi^{s-1}\gamma)\right\| \leq \frac{\mathfrak{U}}{2}$$

for all $\gamma \in \mathbb{R}$ and $s = 1,2,3\cdots n$, where $n \in \mathbb{N}$. Taking summation and multiply both side by $\dfrac{1}{\chi^s}$ then

$$\sum_{s=1}^{n}\frac{1}{\chi^s}\left\|\omega(\chi^s\gamma) - \chi\omega(\chi^{s-1}\gamma)\right\| \leq \frac{1}{2}\sum_{s=1}^{n}\frac{\mathfrak{U}}{\chi^s}$$

using

$$|l+m| \leq |l| + |m|$$

we have

$$\left\|\frac{1}{\chi^n}\omega(\chi^n\gamma) - \omega(\gamma)\right\| \leq \frac{\mathfrak{U}}{2}\sum_{s=1}^{n}\frac{1}{\chi^s}. \tag{8}$$

Since

$$\sum_{s=1}^{n}\chi^{-s} \leq \sum_{s=1}^{+\infty}\chi^{-s}$$

the inequality (8) yields

$$\left\|\frac{1}{\chi^n}\omega(\chi^n\gamma) - \omega(\gamma)\right\| \leq \frac{\mathfrak{U}}{2}\sum_{s=1}^{+\infty}\frac{1}{\chi^s}$$

which is

$$\left\|\frac{1}{\chi^n}\omega(\chi^n\gamma) - \omega(\gamma)\right\| \leq \frac{\mathfrak{U}}{2(\chi-1)} \tag{9}$$

for all $\gamma \in \mathbb{R}$. By replacing n by $m-n$ in (9), we get

$$\left\| \frac{1}{\chi^{m-n}} \omega(\chi^{m-n}\gamma) - \omega(\gamma) \right\| \leq \frac{\mathfrak{U}}{2(\chi-1)} \tag{10}$$

which is

$$\left\| \frac{1}{\chi^{m}} \omega(\chi^{m-n}\gamma) - \frac{1}{\chi^{n}}\omega(\gamma) \right\| \leq \frac{1}{\chi^{n}} \left(\frac{\mathfrak{U}}{2(\chi-1)} \right) \tag{11}$$

for all $\gamma \in \mathbb{R}$. Considering γ by $\chi^n\gamma$ in (11), we get

$$\left\| \frac{1}{\chi^{m}} \omega(\chi^{m}\gamma) - \frac{1}{\chi^{n}}\omega(\chi^{n}\gamma) \right\| \leq \left(\frac{\mathfrak{U}}{2(\chi-1)} \right) \frac{1}{\chi^{n}} \tag{12}$$

However,

$$\lim_{n\to+\infty} \chi^{-n} = 0$$

From (12), we get

$$\lim_{n\to+\infty} \left\| \frac{1}{\chi^{m}} \omega(\chi^{m}\gamma) - \frac{1}{\chi^{n}}\omega(\chi^{n}\gamma) \right\| = 0$$

Therefore

$$\left\{ \frac{\omega(\chi^n\gamma)}{\chi^n} \right\}_{n=1}^{+\infty}$$

is a Cauchy sequence. The additive is defined as

$$\mathbb{A}(\gamma) = \lim_{n\to+\infty} \frac{\omega(\chi^n\gamma)}{\chi^n}$$

for all $\gamma \in \mathbb{R}$. The following section proves that $\mathbb{A} : \mathbb{R} \to \mathbb{R}$ is an additive function.

Consider

$$\begin{aligned}
&\Big\| \mathbb{A}\left(\eta^{\ell+\wp}\gamma + \eta^{\hbar}\kappa\right) + \mathbb{A}\left(\eta^{\ell+\wp}\kappa + \eta^{\hbar}\mu\right) + \eta^{\ell+\wp}\mathbb{A}(\gamma-\kappa) + \eta^{\hbar}\mathbb{A}(\kappa-\mu) \\
&\qquad -2\left(\eta^{\ell+\wp}\mathbb{A}(\gamma) + \eta^{\hbar}\mathbb{A}(\kappa)\right) \Big\| \\
=&\frac{1}{\chi^n} \Big\| \omega\left(\chi^n\eta^{\ell+\wp}\gamma + \chi^n\eta^{\hbar}\kappa\right) + \omega\left(\chi^n\eta^{\ell+\wp}\kappa + \chi^n\eta^{\hbar}\mu\right) + \eta^{\ell+\wp}\omega(\chi^n\gamma - \chi^n\kappa) \\
&\qquad + \eta^{\hbar}\omega(\chi^n\kappa - \chi^n\mu) - 2\left(\eta^{\ell+\wp}\omega(\chi^n\gamma) + \eta^{\hbar}\omega(\chi^n\kappa)\right) \Big\| \\
\leq& \lim_{n\to+\infty} \frac{\mathfrak{U}}{\chi^n} = 0
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{A}\left(\eta^{\ell+\wp}\gamma + \eta^{\hbar}\kappa\right) + \mathbb{A}\left(\eta^{\ell+\wp}\kappa + \eta^{\hbar}\mu\right) + \eta^{\ell+\wp}A(\gamma-\kappa) + \eta^{\hbar}\mathbb{A}(\kappa-\mu) \\
= 2\left(\eta^{\ell+\wp}\mathbb{A}(\gamma) + \eta^{\hbar}\mathbb{A}(\kappa)\right)
\end{aligned}$$

for all $\gamma \in \mathbb{R}$. Let

$$\begin{aligned}
||\mathbb{A}(\gamma) - \omega(\gamma)|| &= || \lim_{n\to+\infty} \frac{\omega(\chi^n\gamma)}{\chi^n} - \omega(\gamma)|| \\
&= \lim_{n\to+\infty} || \frac{\omega(\chi^n\gamma)}{\chi^n} - \omega(\gamma)|| \\
&\leq \lim_{n\to+\infty} \frac{\mathfrak{U}}{2(\chi-1)}
\end{aligned}$$

and

$$||\mathbb{A}(\gamma) - \omega(\gamma)|| \leq \frac{\mathfrak{U}}{2(\chi - 1)}$$

for all $\gamma \in \mathbb{R}$.

$\mathbb{A}$ is a unique function, which is proved as follows:

$$||\mathbb{B}(\gamma) - \omega(\gamma)|| \leq \frac{\mathfrak{U}}{2(\chi - 1)}$$

Hence

$$\begin{aligned} ||\mathbb{B}(\gamma) - \omega(\gamma)|| &\leq ||\mathbb{B}(\gamma) - \omega(\gamma)|| + ||\mathbb{A}(\gamma) - \omega(\gamma)|| \\ &\leq \frac{\mathfrak{U}}{2(\chi - 1)} + \frac{\mathfrak{U}}{2(\chi - 1)} \\ &= \frac{\mathfrak{U}}{(\chi - 1)}. \end{aligned}$$

However, $\mathbb{A}$ and $\mathbb{B}$ are additive, hence

$$\begin{aligned} ||\mathbb{A}(\gamma) - \mathbb{B}(\gamma)|| &= \frac{1}{n}||\mathbb{A}(n\gamma) - \mathbb{B}(n\gamma)|| \\ &\leq \frac{1}{n}\frac{\mathfrak{U}}{(\chi - 1)} \end{aligned} \tag{13}$$

Taking $n \to +\infty$, from (13)

$$\lim_{n \to +\infty} ||\mathbb{A}(\gamma) - \mathbb{B}(\gamma)|| \leq \lim_{n \to +\infty} \frac{1}{n}\frac{\mathfrak{U}}{(\chi - 1)}$$

Hence

$$||\mathbb{A}(\gamma) - \mathbb{B}(\gamma)|| \leq 0$$

Therefore $\mathbb{A}(\gamma) = \mathbb{B}(\gamma)$ for all $\gamma \in \mathbb{R}$. Henceforth $\mathbb{A}$ is unique. □

2.2. Tosio Aoki's (1950) Theorem for (5)

Theorem 2. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a real map satisfying $\|\mathbb{J}\omega(\gamma, \kappa, \mu)\| \leq \mathfrak{U}\{||\gamma||^p + ||\kappa||^p + ||\mu||^p\}$ for some $\mathfrak{U} \geq 0$, $p \in [0, 1)$ and $\forall \gamma, \kappa \in \mathbb{R}$, then there exists a unique additive function $\mathbb{A} : \mathbb{R} \to \mathbb{R}$ and $\chi = \left(\eta^{\ell+\wp} + \eta^{\hbar}\right)$ such that $\|\omega(\gamma) - \mathbb{A}(\gamma)\| \leq \frac{3\,\mathfrak{U}}{2|\chi - \chi^p|}||\gamma||^p$ for all $\gamma \in \mathbb{R}$.*

Proof. Let $\omega : \mathbb{R} \to \mathbb{R}$ be a real function satisfying $\|\mathbb{J}\omega(\gamma, \kappa, \mu)\| \leq \mathfrak{U}\{||\gamma||^p + ||\kappa||^p + ||\mu||^p\}$ for some $\mathfrak{U} \geq 0$, $p \in [0, 1)$. Instead of (γ, κ, μ) by $(0, 0, 0)$ in the above, then we have $\left\|\left(2 - \eta^{\ell+\wp} - \eta^{\hbar}\right)\omega(0)\right\| = 0$ or $\omega(0) = 0$ in place of (γ, κ, μ) by (γ, γ, γ) in the above inequality, hence

$$\|\omega(\chi\gamma) - \chi\omega(\gamma)\| \leq \frac{3}{2}\,\mathfrak{U}||\gamma||^p \tag{14}$$

for all $\gamma \in \mathbb{R}$, where $\chi = \left(\eta^{\ell+\wp} + \eta^{\hbar}\right)$. Replacing γ by $\chi^{s-1}\gamma$

$$\left\|\omega(\chi^s\gamma) - \chi\omega(\chi^{s-1}\gamma)\right\| \leq \frac{3}{2}\,\mathfrak{U}\,\chi^{(s-1)p}||\gamma||^p$$

for all $\gamma \in \mathbb{R}$. Taking summation and multiplying by $\dfrac{1}{\chi^s}$

$$\sum_{s=1}^{n} \frac{1}{\chi^s}\left\|\omega(\chi^s\gamma) - \chi\omega(\chi^{s-1}\gamma)\right\| \leq \frac{3}{2}\,\mathfrak{U}\,||\gamma||^p \sum_{s=1}^{n} \frac{\chi^{sp-p}}{\chi^s}$$

Using

$$|l+m| \leq |l| + |m|$$

$$\left\| \frac{1}{\chi^n}\omega(\chi^n\gamma) - \omega(\gamma) \right\| \leq \frac{3}{2}\,\mathfrak{U}\,||\gamma||^p \sum_{s=1}^{n} \chi^{s(p-1)}\chi^{-p}. \tag{15}$$

Since

$$\sum_{s=1}^{n} \chi^{s(p-1)} \leq \sum_{s=1}^{+\infty} \chi^{s(p-1)}$$

the inequality (15) yields

$$\left\| \frac{1}{\chi^n}\omega(\chi^n\gamma) - \omega(\gamma) \right\| \leq \frac{3}{2}\,\mathfrak{U}\,||\gamma||^p \chi^{-p} \sum_{s=1}^{+\infty} \chi^{s(p-1)}$$

which is

$$\left\| \frac{1}{\chi^n}\omega(\chi^n\gamma) - \omega(\gamma) \right\| \leq \frac{3\,\mathfrak{U}}{2(\chi - \chi^p)}||\gamma||^p \tag{16}$$

for all $\gamma \in \mathbb{R}$. By replacing n by $m-n$ in (16), hence

$$\left\| \frac{1}{\chi^{m-n}}\omega(\chi^{m-n}\gamma) - \omega(\gamma) \right\| \leq \frac{3\,\mathfrak{U}}{2(\chi - \chi^p)}||\gamma||^p \tag{17}$$

which is

$$\left\| \frac{1}{\chi^m}\omega(\chi^{m-n}\gamma) - \frac{1}{\chi^n}\omega(\gamma) \right\| \leq \frac{1}{\chi^n}\left(\frac{3\,\mathfrak{U}}{2(\chi - \chi^p)}\right)||\gamma||^p \tag{18}$$

for all $\gamma \in \mathbb{R}$. Replacing γ by $\chi^n\gamma$ in (18),

$$\left\| \frac{1}{\chi^m}\omega(\chi^m\gamma) - \frac{1}{\chi^n}\omega(\chi^n\gamma) \right\| \leq \left(\frac{3\,\mathfrak{U}}{2(\chi - \chi^p)}\right)\frac{\chi^{np}}{\chi^n}||\gamma||^p \tag{19}$$

Since $0 \leq p < 1$,

$$\lim_{n\to+\infty} \chi^{n(p-1)} = 0$$

and using (19), we obtain

$$\lim_{n\to+\infty} \left\| \frac{1}{\chi^m}\omega(\chi^m\gamma) - \frac{1}{\chi^n}\omega(\chi^n\gamma) \right\| = 0$$

Therefore

$$\left\{ \frac{\omega(\chi^n\gamma)}{\chi^n} \right\}_{n=1}^{+\infty}$$

is a Cauchy sequence. The additive function is defined as

$$\mathbb{A}(\gamma) = \lim_{n\to+\infty} \frac{\omega(\chi^n\gamma)}{\chi^n}$$

for all $\gamma \in \mathbb{R}$. $A : \mathbb{R} \to \mathbb{R}$ is an addictive function, proved as follows.

Consider

$$\begin{aligned}
&\Big\|\mathbb{A}\Big(\eta^{\ell+\wp}\gamma+\eta^{\hbar}\kappa\Big)+\mathbb{A}\Big(\eta^{\ell+\wp}\kappa+\eta^{\hbar}\mu\Big)+\eta^{\ell+\wp}\mathbb{A}(\gamma-\kappa)+\eta^{\hbar}\mathbb{A}(\kappa-\mu)\\
&\qquad -2\Big(\eta^{\ell+\wp}\mathbb{A}(\gamma)+\eta^{\hbar}\mathbb{A}(\kappa)\Big)\Big\|\\
=&\frac{1}{\chi^n}\Big\|\omega\Big(\chi^n\eta^{\ell+\wp}\gamma+\chi^n\eta^{\hbar}\kappa\Big)+\omega\Big(\chi^n\eta^{\ell+\wp}\kappa+\chi^n\eta^{\hbar}\mu\Big)+\eta^{\ell+\wp}\omega(\chi^n\gamma-\chi^n\kappa)\\
&+\eta^{\hbar}\omega(\chi^n\kappa-\chi^n\mu)-2\Big(\eta^{\ell+\wp}\omega(\chi^n\gamma)+\eta^{\hbar}\omega(\chi^n\kappa)\Big)\Big\|\\
\le&\lim_{n\to+\infty}\frac{\mathfrak{U}\{(||\gamma||^p+||\kappa||^p+||\mu||^p)\chi^n\}}{\chi^n}=0
\end{aligned}$$

Since $p\in[0,1)$.
Hence

$$\begin{aligned}
&A\Big(\eta^{\ell+\wp}\gamma+\eta^{\hbar}\kappa\Big)+A\Big(\eta^{\ell+\wp}\kappa+\eta^{\hbar}\mu\Big)+\eta^{\ell+\wp}A(\gamma-\kappa)+\eta^{\hbar}A(\kappa-\mu)\\
&\qquad=2\Big(\eta^{\ell+\wp}\mathbb{A}(\gamma)+\eta^{\hbar}A(\kappa)\Big)
\end{aligned}$$

for all $\gamma\in\mathbb{R}$. Consider

$$\begin{aligned}
||\mathbb{A}(\gamma)-\omega(\gamma)|| &= ||\lim_{n\to+\infty}\frac{\omega(\chi^n\gamma)}{\chi^n}-\omega(\gamma)||\\
&=\lim_{n\to+\infty}||\frac{\omega(\chi^n\gamma)}{\chi^n}-\omega(\gamma)||\\
&\le\lim_{n\to+\infty}\frac{3\,\mathfrak{U}}{2(\chi-\chi^p)}||\gamma||^p
\end{aligned}$$

Hence

$$||\mathbb{A}(\gamma)-\omega(\gamma)||\le\frac{3\,\mathfrak{U}}{2(\chi-\chi^p)}||\gamma||^p$$

for all $\gamma\in\mathbb{R}$.
$\mathbb{A}$ is a unique function, proved as follows

$$||\mathbb{B}(\gamma)-\omega(\gamma)||\le\frac{3\,\mathfrak{U}}{2(\chi-\chi^p)}||\gamma||^p$$

Hence

$$\begin{aligned}
||\mathbb{B}(\gamma)-\omega(\gamma)|| &\le ||\mathbb{B}(\gamma)-\omega(\gamma)||+||\mathbb{A}(\gamma)-\omega(\gamma)||\\
&\le\frac{3\,\mathfrak{U}}{2(\chi-\chi^p)}||\gamma||^p+\frac{3\,\mathfrak{U}}{2(\chi-\chi^s)}||\gamma||^p\\
&=\frac{3\,\mathfrak{U}}{(\chi-\chi^p)}||\gamma||^p
\end{aligned}$$

But $\mathbb{A}$ and $\mathbb{B}$ are additive, hence

$$\begin{aligned}
||\mathbb{A}(\gamma)-\mathbb{B}(\gamma)|| &= \frac{1}{n}||\mathbb{A}(n\gamma)-\mathbb{B}(n\gamma)||\\
&\le\frac{1}{n}\frac{3\,\mathfrak{U}}{(\chi-\chi^p)}||\gamma||^p
\end{aligned}\tag{20}$$

Taking $n\to+\infty$, using (20)

$$\lim_{n\to+\infty}||\mathbb{A}(\gamma)-\mathbb{B}(\gamma)||\le\lim_{n\to+\infty}\frac{1}{n}\frac{3\,\mathfrak{U}}{(\chi-\chi^p)}||\gamma||^p$$

Hence

$$||\mathbb{A}(\gamma) - \mathbb{B}(\gamma)|| \leq 0$$

Therefore $\mathbb{A}(\gamma) = \mathbb{B}(\gamma)$ for all $\gamma \in \mathbb{R}$. Hence $\mathbb{A}$ is unique. □

To prove example for not stable in $p = 1$ the Equation (5).

Example 1. *Consider the mapping $\phi : \mathbb{R} \to \mathbb{R}$ be defined by*

$$\phi(\gamma) = \begin{cases} \mu\gamma, & \text{if } |\gamma| < 1 \\ \mu, & \text{otherwise} \end{cases}$$

let $\omega : \mathbb{R} \to \mathbb{R}$ be defined by

$$\omega(\gamma) = \sum_{n=0}^{+\infty} \frac{\phi(\chi^n\gamma)}{\chi^n} \qquad \text{for all} \quad \gamma \in \mathbb{R}.$$

Then ω satisfies

$$\|\mathbb{J}\omega(\gamma, \kappa, \mu)\| \leq \frac{\chi^2\left[2 + 3\left(\eta^{\ell+\wp} + \eta^{\hbar}\right)\right]}{\chi - 1}\mu(|\gamma| + |\kappa| + |\mu|). \tag{21}$$

Then there is no an additive mapping $\mathbb{A} : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|\omega(\gamma) - \mathbb{A}(\gamma)| \leq \beta|\gamma| \qquad \text{for all} \quad \gamma \in \mathbb{R}. \tag{22}$$

Proof. Now $|\omega(\gamma)| \leq \sum_{n=0}^{+\infty} \frac{|\phi(\chi^n\gamma)|}{|\chi^n|} = \sum_{n=0}^{+\infty} \frac{\mu}{\chi^n} = \frac{1}{1-\frac{1}{\chi}}\mu = \frac{\chi\mu}{\chi-1}$.

Thus, ω is bounded.

If $\gamma = \kappa = \mu = 0$ at that point (29) is minor. In the event that $|\gamma| + |\kappa| + |\mu| \geq 1$, at that point the left hand side of (29) is under $\frac{\chi^2\left[2+3\left(\eta^{\ell+\wp}+\eta^{\hbar}\right)\right]}{\chi-1}\mu$. It is assumed that $0 < |\gamma| + |\gamma| + |\mu| < 1$. Then there exists a positive integer k such that

$$\frac{1}{\chi^k} \leq |\gamma| + |\kappa| + |\mu| < \frac{1}{\chi^{k-1}}, \tag{23}$$

so that $\chi^{k-1}|\gamma| < 1$, $\chi^{k-1}|\kappa| < 1, \chi^{k-1}|\mu| < 1$ and consequently

$$\chi^{k-1}\left(\eta^{\ell+\wp}\gamma + \eta^{\hbar}\kappa\right), \chi^{k-1}\left(\eta^{\ell+\wp}\kappa + \eta^{\hbar}\mu\right), \chi^{k-1}(\gamma - \kappa), \chi^{k-1}(\kappa - \mu), \chi^{k-1}(\gamma), \chi^{k-1}(\kappa) \in (-1, 1).$$

here $n = 0, 1, \ldots, k-1$,

$$\chi^n\left(\eta^{\ell+\wp}\gamma + \eta^{\hbar}\kappa\right), \chi^n\left(\eta^{\ell+\wp}\kappa + \eta^{\hbar}\mu\right), \chi^n(\gamma - \kappa), \chi^n(\kappa - \mu), \chi^n(\gamma), \chi^n(\kappa) \in (-1, 1).$$

and

$$\begin{aligned} &\phi\left(\eta^{\ell+\wp}\gamma + \eta^{\hbar}\kappa\right) + \phi\left(\eta^{\ell+\wp}\kappa + \eta^{\hbar}\mu\right) + \eta^{\ell+\wp}\phi(\gamma - \kappa) + \eta^{\hbar}\phi(\kappa - \mu) \\ &- 2\left(\eta^{\ell+\wp}\phi(\gamma) + \eta^{\hbar}\phi(\kappa)\right) = 0 \end{aligned}$$

for $n = 0, 1, \ldots, k-1$. From the definition of ω,

$$\begin{aligned}
&\Big|\omega\Big(\eta^{\ell+\wp}\gamma + \eta^{\hbar}\kappa\Big) + \omega\Big(\eta^{\ell+\wp}\kappa + \eta^{\hbar}\mu\Big) + \eta^{\ell+\wp}\omega(\gamma-\kappa) + \eta^{\hbar}\omega(\kappa-\mu) \\
&-2\Big(\eta^{\ell+\wp}\omega(\gamma) + \eta^{\hbar}\omega(\kappa)\Big)\Big| \\
&= \sum_{n=0}^{+\infty}\frac{1}{\chi^n}\Big|\phi\Big(\eta^{\ell+\wp}\gamma + \eta^{\hbar}\kappa\Big) + \phi\Big(\eta^{\ell+\wp}\kappa + \eta^{\hbar}\mu\Big) + \eta^{\ell+\wp}\phi(\gamma-\kappa) + \eta^{\hbar}\phi(\kappa-\mu) \\
&- 2\Big(\eta^{\ell+\wp}\phi(\gamma) + \eta^{\hbar}\phi(\kappa)\Big)\Big| \\
&= \sum_{n=k}^{+\infty}\frac{1}{\chi^n}\Big|\phi\Big(\eta^{\ell+\wp}\gamma + \eta^{\hbar}\kappa\Big) + \phi\Big(\eta^{\ell+\wp}\kappa + \eta^{\hbar}\mu\Big) + \eta^{\ell+\wp}\phi(\gamma-\kappa) + \eta^{\hbar}\phi(\kappa-\mu) \\
&- 2\Big(\eta^{\ell+\wp}\phi(\gamma) + \eta^{\hbar}\phi(\kappa)\Big)\Big| \\
&\leq \sum_{n=k}^{+\infty}\frac{1}{\chi^n}\Big[2 + 3\Big(\eta^{\ell+\wp} + \eta^{\hbar}\Big)\Big]\mu = \Big[2 + 3\Big(\eta^{\ell+\wp} + \eta^{\hbar}\Big)\Big]\,\mu \times \frac{1}{\chi^k} \times \frac{\chi}{\chi - 1} \\
&\leq \frac{\chi^2\Big[2 + 3\Big(\eta^{\ell+\wp} + \eta^{\hbar}\Big)\Big]}{\chi - 1}\mu(|\gamma| + |\kappa| + |\mu|).
\end{aligned}$$

Thus ω satisfies (29) with $0 < |\gamma| + |\kappa| + |\mu| < 1$.
The additive functional Equation (5) is not stable for $p = 1$ in the inequality

$$\|\mathbb{J}\omega(\gamma,\kappa,\mu)\| \leq \mathfrak{U}\{||\gamma||^p + ||\kappa||^p + ||\mu||^p\}$$

Suppose on the contrary that there exists an additive mapping $\mathbb{A} : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ satisfying (30). Since ω is bounded and continuous for all $\gamma \in \mathbb{R}$, $\mathbb{A}$ is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 2, $\mathbb{A}$ must have the form $\mathbb{A}(\gamma) = c\gamma$ for any γ in $\mathbb{R}$. Thus we obtain that

$$|\omega(\gamma)| \leq (\beta + |c|)|\gamma|. \tag{24}$$

now m with $m\mu > \beta + |c|$.

If $\gamma \in \left(0, \frac{1}{\chi^{m-1}}\right)$, at that point $\chi^n\gamma \in (0,1)$ for all $n = 0, 1, \ldots, m-1$. For this γ, we get

$$\omega(\gamma) = \sum_{n=0}^{+\infty}\frac{\phi(\chi^n\gamma)}{\chi^n} \geq \sum_{n=0}^{m-1}\frac{\phi(\chi^n\gamma)}{\chi^n} = m\mu\gamma > (\beta + |c|)\gamma$$

which negates (32).Therefore the additive functional Equation (5) is not stable for $p = 1$ in the sense of Ulam, Hyers, and Rassias. □

2.3. John M. Rassias' Theorem (1982) for (5)

Theorem 3. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a real map satisfying $\|\mathbb{J}\omega(\gamma,\kappa,\mu)\| \leq \mathfrak{U}\{||\gamma||^p||\kappa||^p||\mu||^p\}$for some $\mathfrak{U} \geq 0$, $p \in \left[0, \frac{1}{3}\right)$ and for all $\gamma, \kappa, \mu \in \mathbb{R}$, then there exists a unique additive function $A : \mathbb{R} \to \mathbb{R}$ and $\chi = \left(\eta^{\ell+\wp} + \eta^{\hbar}\right)$ such that $\|\omega(\gamma) - \mathbb{A}(\gamma)\| \leq \frac{\mathfrak{U}}{2|\chi - \chi^{3p}|}||\gamma||^{3p}$ for all $\gamma \in \mathbb{R}$.*

Proof. Let $\omega : \mathbb{R} \to \mathbb{R}$ be a real function satisfying

$$\|\mathbb{J}\omega(\gamma,\kappa,\mu)\| \leq \mathfrak{U}\{||\gamma||^p||\kappa||^p||\mu||^p\} \tag{25}$$

for some $\mathfrak{U} \geq 0,\ p \in \left[0, \frac{1}{3}\right)$. Instead of (γ, κ, μ) by $(0,0,0)$ in (25), then $\left\|\left(2 - \eta^{\ell+\wp} - \eta^{\hbar}\right)\omega(0)\right\|$ $= 0$ or $\omega(0) = 0$ in place of (γ, κ, μ) by (γ, γ, γ) in (25),

$$\|\omega(\chi\gamma) - \chi\omega(\gamma)\| \leq \frac{1}{2}\,\mathfrak{U}||\gamma||^{3p} \tag{26}$$

for all $\gamma \in \mathbb{R}$, where $\chi = \left(\eta^{\ell+\wp} + \eta^{\hbar}\right)$. Replacing γ by $\chi^{s-1}\gamma$

$$\left\|\omega(\chi^{s}\gamma) - \chi\omega(\chi^{s-1}\gamma)\right\| \leq \frac{3}{2}\,\mathfrak{U}\,\chi^{3(s-1)p}||\gamma||^{3p}$$

for all $\gamma \in \mathbb{R}$. Taking summation and multiplying by $\dfrac{1}{\chi^{s}}$ in the above inequality

$$\sum_{s=1}^{n}\frac{1}{\chi^{s}}\left\|\omega(\chi^{s}\gamma) - \chi\omega(\chi^{s-1}\gamma)\right\| \leq \frac{1}{2}\,\mathfrak{U}\,||\gamma||^{3p}\sum_{s=1}^{n}\frac{\chi^{3(sp-p)}}{\chi^{s}}$$

use

$$|l + m| \leq |l| + |m|$$

$$\left\|\frac{1}{\chi^{n}}\omega(\chi^{n}\gamma) - \omega(\gamma)\right\| \leq \frac{1}{2}\,\mathfrak{U}\,||\gamma||^{3p}\sum_{s=1}^{n}\chi^{s(3p-1)}\chi^{-p} \tag{27}$$

Since

$$\sum_{s=1}^{n}\chi^{s(3p-1)} \leq \sum_{s=1}^{+\infty}\chi^{s(3p-1)}$$

the inequality (27) yields

$$\left\|\frac{1}{\chi^{n}}\omega(\chi^{n}\gamma) - \omega(\gamma)\right\| \leq \frac{1}{2}\,\mathfrak{U}\,||\gamma||^{3p}\chi^{-p}\sum_{s=1}^{+\infty}\chi^{s(3p-1)}$$

which is

$$\left\|\frac{1}{\chi^{n}}\omega(\chi^{n}\gamma) - \omega(\gamma)\right\| \leq \frac{\mathfrak{U}}{2(\chi - \chi^{3p})}||\gamma||^{3p}. \tag{28}$$

Thus the proof is similar to that of Theorem 2. □

To prove example for not stable in $p = 1$ the Equation (5).

Example 2. *Consider the mapping $\phi : \mathbb{R} \to \mathbb{R}$ be defined by*

$$\phi(\gamma) = \begin{cases} \mu\gamma, & \text{if } |\gamma| < 1 \\ \mu, & \text{otherwise} \end{cases}$$

let $\omega : \mathbb{R} \to \mathbb{R}$ be defined by

$$\omega(\gamma) = \sum_{n=0}^{+\infty}\frac{\phi(\chi^{n}\gamma)}{\chi^{n}} \qquad \text{for all} \quad \gamma \in \mathbb{R}.$$

Then ω satisfies

$$\|\mathbb{J}\omega(\gamma,\kappa,\mu)\| \leq \frac{\chi^{2}\left[2 + 3\left(\eta^{\ell+\wp} + \eta^{\hbar}\right)\right]}{\chi - 1}\mu(|\gamma| + |\kappa| + |\mu|). \tag{29}$$

Then there is no an additive mapping $\mathbb{A} : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|\omega(\gamma) - \mathbb{A}(\gamma)| \leq \beta|\gamma| \qquad \text{for all} \quad \gamma \in \mathbb{R}. \tag{30}$$

Proof. Now

$$|\omega(\gamma)| \leq \sum_{n=0}^{+\infty} \frac{|\phi(\chi^n \gamma)|}{|\chi^n|} = \sum_{n=0}^{+\infty} \frac{\mu}{\chi^n} = \frac{1}{1-\frac{1}{\chi}}\mu = \frac{\chi\mu}{\chi-1}.$$

Thus, ω is bounded.

If $\gamma = \kappa = \mu = 0$ at that point (29) is minor. In the event that $|\gamma| + |\kappa| + |\mu| \geq 1$, at that point the left hand side of (29) is under $\frac{\chi^2\left[2+3\left(\eta^{\ell+\wp}+\eta^{\hbar}\right)\right]}{\chi-1}\mu$. It is assumed that $0 < |\gamma| + |\gamma| + |\mu| < 1$. Then there exists a positive integer k such that

$$\frac{1}{\chi^k} \leq |\gamma| + |\kappa| + |\mu| < \frac{1}{\chi^{k-1}}, \tag{31}$$

so that $\chi^{k-1}|\gamma| < 1$, $\chi^{k-1}|\kappa| < 1, \chi^{k-1}|\mu| < 1$ and consequently

$$\chi^{k-1}\left(\eta^{\ell+\wp}\gamma+\eta^{\hbar}\kappa\right), \chi^{k-1}\left(\eta^{\ell+\wp}\kappa+\eta^{\hbar}\mu\right), \chi^{k-1}(\gamma-\kappa), \chi^{k-1}(\kappa-\mu), \chi^{k-1}(\gamma), \chi^{k-1}(\kappa) \in (-1,1).$$

here $n = 0, 1, \ldots, k-1$,

$$\chi^{n}\left(\eta^{\ell+\wp}\gamma+\eta^{\hbar}\kappa\right), \chi^{n}\left(\eta^{\ell+\wp}\kappa+\eta^{\hbar}\mu\right), \chi^{n}(\gamma-\kappa), \chi^{n}(\kappa-\mu), \chi^{n}(\gamma), \chi^{n}(\kappa) \in (-1,1).$$

and

$$\begin{aligned}\phi\left(\eta^{\ell+\wp}\gamma+\eta^{\hbar}\kappa\right)+\phi\left(\eta^{\ell+\wp}\kappa+\eta^{\hbar}\mu\right)+\eta^{\ell+\wp}\phi(\gamma-\kappa)+\eta^{\hbar}\phi(\kappa-\mu)\\ -2\left(\eta^{\ell+\wp}\phi(\gamma)+\eta^{\hbar}\phi(\kappa)\right)=0\end{aligned}$$

for $n = 0, 1, \ldots, k-1$. From the definition of ω (31),

$$\begin{aligned}&\Big|\omega\left(\eta^{\ell+\wp}\gamma+\eta^{\hbar}\kappa\right)+\omega\left(\eta^{\ell+\wp}\kappa+\eta^{\hbar}\mu\right)+\eta^{\ell+\wp}\omega(\gamma-\kappa)+\eta^{\hbar}\omega(\kappa-\mu)\\ &-2\left(\eta^{\ell+\wp}\omega(\gamma)+\eta^{\hbar}\omega(\kappa)\right)\Big|\\ &=\sum_{n=0}^{+\infty}\frac{1}{\chi^n}\Big|\phi\left(\eta^{\ell+\wp}\gamma+\eta^{\hbar}\kappa\right)+\phi\left(\eta^{\ell+\wp}\kappa+\eta^{\hbar}\mu\right)+\eta^{\ell+\wp}\phi(\gamma-\kappa)+\eta^{\hbar}\phi(\kappa-\mu)\\ &-2\left(\eta^{\ell+\wp}\phi(\gamma)+\eta^{\hbar}\phi(\kappa)\right)\Big|\\ &=\sum_{n=k}^{+\infty}\frac{1}{\chi^n}\Big|\phi\left(\eta^{\ell+\wp}\gamma+\eta^{\hbar}\kappa\right)+\phi\left(\eta^{\ell+\wp}\kappa+\eta^{\hbar}\mu\right)+\eta^{\ell+\wp}\phi(\gamma-\kappa)+\eta^{\hbar}\phi(\kappa-\mu)\\ &-2\left(\eta^{\ell+\wp}\phi(\gamma)+\eta^{\hbar}\phi(\kappa)\right)\Big|\\ &\leq\sum_{n=k}^{+\infty}\frac{1}{\chi^n}\left[2+3\left(\eta^{\ell+\wp}+\eta^{\hbar}\right)\right]\mu=\left[2+3\left(\eta^{\ell+\wp}+\eta^{\hbar}\right)\right]\mu\times\frac{1}{\chi^k}\times\frac{\chi}{\chi-1}\\ &\leq\frac{\chi^2\left[2+3\left(\eta^{\ell+\wp}+\eta^{\hbar}\right)\right]}{\chi-1}\mu(|\gamma|+|\kappa|+|\mu|).\end{aligned}$$

Thus ω satisfies (29) with $0 < |\gamma| + |\kappa| + |\mu| < 1$.

The additive functional Equation (5) is not stable for $p = 1$ in the inequality

$$\|\mathbb{J}\omega(\gamma,\kappa,\mu)\| \leq \mathfrak{U}\{||\gamma||^p + ||\kappa||^p + ||\mu||^p\}$$

Suppose on the contrary that there exists an additive mapping $\mathbb{A} : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ satisfying (30). Since ω is bounded and continuous for all $\gamma \in \mathbb{R}$, $\mathbb{A}$ is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 2, $\mathbb{A}$ must have the form $\mathbb{A}(\gamma) = c\gamma$ for any γ in $\mathbb{R}$. Thus we obtain that

$$|\omega(\gamma)| \leq (\beta + |c|)|\gamma|. \tag{32}$$

now m with $m\mu > \beta + |c|$.

If $\gamma \in \left(0, \frac{1}{\chi^{m-1}}\right)$, at that point $\chi^n\gamma \in (0,1)$ for all $n = 0, 1, \dots, m-1$. For this γ, we get

$$\omega(\gamma) = \sum_{n=0}^{+\infty} \frac{\phi(\chi^n\gamma)}{\chi^n} \geq \sum_{n=0}^{m-1} \frac{\phi(\chi^n\gamma)}{\chi^n} = m\mu\gamma > (\beta + |c|)\gamma$$

which negates (32).Therefore the additive functional Equation (5) is not stable for $p = 1$ in the sense of Ulam, Hyers, and Rassias. □

2.4. K. Ravi, M. Arunkumar, and John M. Rassias' Theorem (2008) for (5)

Theorem 4. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a real map satisfying $\|\mathbb{J}\omega(\gamma,\kappa,\mu)\| \leq \mathfrak{U}\left\{||\gamma||^p||\kappa||^p||\mu||^p + \left\{||\gamma||^{3p} + ||\kappa||^{3p} + ||\mu||^{3p}\right\}\right\}$ for some $\mathfrak{U} \geq 0$, $p \in \left[0, \frac{1}{3}\right)$ and for all $\gamma,\kappa,\mu \in \mathbb{R}$, then there exists a unique additive function $A : \mathbb{R} \to \mathbb{R}$ and $\chi = \left(\eta^{\ell+\wp} + \eta^{\hbar}\right)$ such that $\|\omega(\gamma) - \mathbb{A}(\gamma)\| \leq \frac{2\,\mathfrak{U}}{|\chi - \chi^{3p}|}||\gamma||^{3p}$ for all $\gamma \in \mathbb{R}$.*

Proof. Let $\omega : \mathbb{R} \to \mathbb{R}$ be a real function satisfying $\|\mathbb{J}\omega(\gamma,\kappa,\mu)\| \leq \mathfrak{U}\left\{||\gamma||^p||\kappa||^p||\mu||^p + \left\{||\gamma||^{3p} + ||\kappa||^{3p} + ||\mu||^{3p}\right\}\right\}$ for some $\mathfrak{U} \geq 0$, $p \in \left[0, \frac{1}{3}\right)$. Instead of (γ,κ,μ) by$(0,0,0)$ in the above $\left\|\left(2 - \eta^{\ell+\wp} - \eta^{\hbar}\right)\omega(0)\right\| = 0$ or $\omega(0) = 0$. Replacing (γ,κ,μ) by (γ,γ,γ) in the above we get

$$\|\omega(\chi\gamma) - \chi\omega(\gamma)\| \leq \frac{4}{2}\,\mathfrak{U}||\gamma||^{3p} \tag{33}$$

for all $\gamma \in \mathbb{R}$, where $\chi = \left(\eta^{\ell+\wp} + \eta^{\hbar}\right)$. Replacing γ by $\chi^{s-1}\gamma$

$$\left\|\omega(\chi^s\gamma) - \chi\omega(\chi^{s-1}\gamma)\right\| \leq 2\,\mathfrak{U}\,\chi^{3(s-1)p}||\gamma||^{3p}$$

for all $\gamma \in \mathbb{R}$. Multiplying the two sides by $\dfrac{1}{\chi^s}$ and taking summation

$$\sum_{s=1}^{n} \frac{1}{\chi^s}\left\|\omega(\chi^s\gamma) - \chi\omega(\chi^{s-1}\gamma)\right\| \leq 2\,\mathfrak{U}\,||\gamma||^{3p} \sum_{s=1}^{n} \frac{\chi^{3(sp-p)}}{\chi^s}.$$

By applying

$$|l + m| \leq |l| + |m|$$

$$\left\|\frac{1}{\chi^n}\omega(\chi^n\gamma) - \omega(\gamma)\right\| \leq 2\,\mathfrak{U}\,||\gamma||^{3p} \sum_{s=1}^{n} \chi^{s(3p-1)}\chi^{-p}. \tag{34}$$

Since

$$\sum_{s=1}^{n} \chi^{s(3p-1)} \leq \sum_{s=1}^{+\infty} \chi^{s(3p-1)}$$

the inequality (34) yields

$$\left\|\frac{1}{\chi^n}\omega(\chi^n\gamma) - \omega(\gamma)\right\| \leq 2\,\mathfrak{U}\,||\gamma||^{3p}\chi^{-p} \sum_{s=1}^{+\infty} \chi^{s(3p-1)}$$

which is
$$\left\|\frac{1}{\chi^n}\omega(\chi^n\gamma)-\omega(\gamma)\right\| \leq \frac{2\,\mho}{(\chi-\chi^{3p})}||\gamma||^{3p} \tag{35}$$
□

Thus the proof is similar to that of Theorem 2.

Example 3. *Consider the map $\omega:\mathbb{R}\to\mathbb{R}$ is defined by*
$$\phi(\gamma)=\begin{cases}\mu\gamma, & \text{if } |\gamma|<\frac{1}{3}\\ \frac{\mu}{3}, & \text{otherwise}\end{cases}$$
where $\mu>0$ is a constant, and let the function $\omega:\mathbb{R}\to\mathbb{R}$ be defined as
$$\omega(\gamma)=\sum_{n=0}^{+\infty}\frac{\phi(\chi^n\gamma)}{\chi^n} \qquad \text{for all} \quad \gamma\in\mathbb{R}.$$
Then ω satisfies the functional inequality
$$\|\mathbb{J}\omega(\gamma,\kappa,\mu)\| \leq \frac{\chi\left[2+3\left(\eta^{\ell+\wp}+\eta^{\hbar}\right)\right]}{3(\chi-1)}\mu\left(|\gamma|^{\frac{1}{3}}+|\kappa|^{\frac{1}{3}}+|\mu|^{\frac{1}{3}}+|\gamma|+|\kappa|+|\mu|\right) \tag{36}$$
Then there is no an additive mapping $\mathbb{A}:\mathbb{R}\to\mathbb{R}$ and a constant $\beta>0$ such that
$$|\omega(\gamma)-\mathbb{A}(\gamma)| \leq \beta|\gamma| \qquad \text{for all} \quad \gamma\in\mathbb{R}. \tag{37}$$

Proof. Presently
$$|\omega(\gamma)| \leq \sum_{n=0}^{+\infty}\frac{|\phi(\chi^n\gamma)|}{|\chi^n|}=\sum_{n=0}^{+\infty}\frac{\mu}{3}\times\frac{\mu}{\chi^n}=\frac{\mu}{3}.\frac{\chi\mu}{\chi-1}.$$
we see that ω is limited. We prove ω fulfills (36).

If $\gamma=\kappa=\mu=0$ at that point (36) is insignificant and $|\gamma|^{\frac{1}{3}}+|\kappa|^{\frac{1}{3}}+|\mu|^{\frac{1}{3}}+|\gamma|+|\kappa|+|\mu|\geq\frac{1}{3}$, at that point the left hand side of (36) is under $\frac{\chi\left[2+3\left(\eta^{\ell+\wp}+\eta^{\hbar}\right)\right]}{3(\chi-1)}\mu$. Presently assume that $0<|\gamma|^{\frac{1}{3}}+|\kappa|^{\frac{1}{3}}+|\mu|^{\frac{1}{3}}+|\gamma|+|\kappa|+|\mu|<\frac{1}{3}$. For k is an integer
$$\frac{1}{\chi^k}\leq|\gamma|^{\frac{1}{3}}+|\kappa|^{\frac{1}{3}}+|\mu|^{\frac{1}{3}}+|\gamma|+|\kappa|+|\mu|<\frac{1}{\chi^{k-1}}, \tag{38}$$
so that $\chi^{k-1}|\gamma|<1$, $\chi^{k-1}|\kappa|<1, \chi^{k-1}|\mu|<1$ and consequently
$$\chi^{k-1}\left(\eta^{\ell+\wp}\gamma+\eta^{\hbar}\kappa\right),\chi^{k-1}\left(\eta^{\ell+\wp}\kappa+\eta^{\hbar}\mu\right),\chi^{k-1}(\gamma-\kappa),\chi^{k-1}(\kappa-\mu),\chi^{k-1}(\gamma),\chi^{k-1}(\kappa)\in\left(-\frac{1}{\chi},\frac{1}{\chi}\right).$$
Therefore for each $n=0,1,\ldots,k-1$,
$$\chi^n\left(\eta^{\ell+\wp}\gamma+\eta^{\hbar}\kappa\right),\chi^n\left(\eta^{\ell+\wp}\kappa+\eta^{\hbar}\mu\right),\chi^n(\gamma-\kappa),\chi^n(\kappa-\mu),\chi^n(\gamma),\chi^n(\kappa)\in\left(-\frac{1}{\chi},\frac{1}{\chi}\right).$$
and
$$\begin{aligned}&\phi\left(\eta^{\ell+\wp}\gamma+\eta^{\hbar}\kappa\right)+\phi\left(\eta^{\ell+\wp}\kappa+\eta^{\hbar}\mu\right)+\eta^{\ell+\wp}\phi(\gamma-\kappa)+\eta^{\hbar}\phi(\kappa-\mu)\\&-2\left(\eta^{\ell+\wp}\phi(\gamma)+\eta^{\hbar}\phi(\kappa)\right)=0\end{aligned}$$

for $n = 0, 1, \ldots, k-1$. From the definition of ω and (38),

$$\begin{aligned}
&\Big|\omega\Big(\eta^{\ell+\wp}\gamma+\eta^{\hbar}\kappa\Big)+\omega\Big(\eta^{\ell+\wp}\kappa+\eta^{\hbar}\mu\Big)+\eta^{\ell+\wp}\omega(\gamma-\kappa)+\eta^{\hbar}\omega(\kappa-\mu)\\
&-2\Big(\eta^{\ell+\wp}\omega(\gamma)+\eta^{\hbar}\omega(\kappa)\Big)\Big|\\
&=\sum_{n=0}^{+\infty}\frac{1}{\chi^n}\Big|\phi\Big(\eta^{\ell+\wp}\gamma+\eta^{\hbar}\kappa\Big)+\phi\Big(\eta^{\ell+\wp}\kappa+\eta^{\hbar}\mu\Big)+\eta^{\ell+\wp}\phi(\gamma-\kappa)+\eta^{\hbar}\phi(\kappa-\mu)\\
&\qquad\qquad-2\Big(\eta^{\ell+\wp}\phi(\gamma)+\eta^{\hbar}\phi(\kappa)\Big)\Big|\\
&=\sum_{n=k}^{+\infty}\frac{1}{\chi^n}\Big|\phi\Big(\eta^{\ell+\wp}\gamma+\eta^{\hbar}\kappa\Big)+\phi\Big(\eta^{\ell+\wp}\kappa+\eta^{\hbar}\mu\Big)+\eta^{\ell+\wp}\phi(\gamma-\kappa)+\eta^{\hbar}\phi(\kappa-\mu)\\
&\qquad\qquad-2\Big(\eta^{\ell+\wp}\phi(\gamma)+\eta^{\hbar}\phi(\kappa)\Big)\Big|\\
&\le\sum_{n=k}^{+\infty}\frac{1}{\chi^n}\Big[2+3\Big(\eta^{\ell+\wp}+\eta^{\hbar}\Big)\Big]\frac{\mu}{3}=\Big[2+3\Big(\eta^{\ell+\wp}+\eta^{\hbar}\Big)\Big]\frac{\mu}{3}\times\frac{1}{\chi^k}\times\frac{\chi}{\chi-1}\\
&\le\frac{\chi\Big[2+3\Big(\eta^{\ell+\wp}+\eta^{\hbar}\Big)\Big]}{3(\chi-1)}\mu\Big(|\gamma|^{\frac{1}{3}}+|\kappa|^{\frac{1}{3}}+|\mu|^{\frac{1}{3}}+|\gamma|+|\kappa|+|\mu|\Big).
\end{aligned}$$

and $0 < |\gamma|^{\frac{1}{3}}+|\kappa|^{\frac{1}{3}}+|\mu|^{\frac{1}{3}}+|\gamma|+|\kappa|+|\mu| < \frac{1}{3}$.

Assume on the opposite that there exists an added substance mapping $\mathbb{A}:\mathbb{R}\to\mathbb{R}$ and a steady $\beta > 0$ fulfilling (37). Since ω is limited and ceaseless for all $\gamma\in\mathbb{R}$, $\mathbb{A}$ is limited on any open interim containing the inception and consistent at the root. Considering Theorem 4, $\mathbb{A}$ must have the structure $\mathbb{A}(\gamma)=c\gamma$ for any γ in $\mathbb{R}$. Along these lines, we acquire that

$$|\omega(\gamma)|\le(\beta+|c|)|\gamma|. \tag{39}$$

For m with $m\mu>\beta+|c|$.

If $\gamma\in\Big(0,\frac{1}{\chi^{m-1}}\Big)$, at that point $\chi^n\gamma\in(0,1)$ for all $n=0,1,\ldots,m-1$. For this γ, we get

$$\omega(\gamma)=\sum_{n=0}^{+\infty}\frac{\phi(\chi^n\gamma)}{\chi^n}\ge\sum_{n=0}^{m-1}\frac{\phi(\chi^n\gamma)}{\chi^n}=m\mu\gamma>(\beta+|c|)\gamma$$

which repudiates (39). In this manner the added substance practical condition (5) the sense of Ulam $p=\frac{1}{3}$, accepted in the disparity (35). □

2.5. P. Găvrută' Theorem for (5)

Theorem 5. *Let the mapping $\omega:\mathbb{R}\to\mathbb{R}$ satisfy the inequality $\|\mathbb{J}\omega(\gamma,\kappa,\mu)\|\le\mathfrak{M}(\gamma,\kappa,\mu)$ with the condition $\lim\limits_{n\to+\infty}\frac{\mathfrak{M}(\chi^n\gamma,\chi^n\kappa,\chi^n\mu)}{\chi^n}=0$ for all $\gamma,\kappa,\mu\in\mathbb{R}$, then there exists a unique additive function $\mathbb{A}:\mathbb{R}\to\mathbb{R}$ and $\chi=\Big(\eta^{\ell+\wp}+\eta^{\hbar}\Big)$ such that $||\mathbb{A}(\gamma)-\omega(\gamma)||\le\frac{1}{2}\sum\limits_{s=1}^{n}\frac{\mathfrak{M}(\chi^{s-1}\gamma,\chi^{s-1}\gamma,\chi^{s-1}\gamma)}{\chi^s}$ for all $\gamma\in\mathbb{R}$.*

Proof. Let $\omega:\mathbb{R}\to\mathbb{R}$ be a real function satisfying

$$\begin{aligned}
&\Big\|\omega\Big(\eta^{\ell+\wp}\gamma+\eta^{\hbar}\kappa\Big)+\omega\Big(\eta^{\ell+\wp}\kappa+\eta^{\hbar}\mu\Big)+\eta^{\ell+\wp}\omega(\gamma-\kappa)+\eta^{\hbar}\omega(\kappa-\mu)\\
&\qquad-2\Big(\eta^{\ell+\wp}\omega(\gamma)+\eta^{\hbar}\omega(\kappa)\Big)\Big\|\le\mathfrak{M}(\gamma,\kappa,\mu)
\end{aligned} \tag{40}$$

for all $\gamma, \kappa, \mu \in \mathbb{R}$ and for some $\mathfrak{M} \geq 0$. Instead of (γ, κ, μ) by $(0,0,0)$ in (40), then we have $\left\| \left(2 - \eta^{\ell+\wp} - \eta^{\hbar}\right)\omega(0) \right\| = 0$ or $\omega(0) = 0$ in place of (γ, κ, μ) by (γ, γ, γ) in (40),

$$\|\omega(\chi\gamma) - \chi\omega(\gamma)\| \leq \frac{1}{2}\mathfrak{M}(\gamma, \gamma, \gamma) \tag{41}$$

for all $\gamma \in \mathbb{R}$, where $\chi = \left(\eta^{\ell+\wp} + \eta^{\hbar}\right)$. Replacing γ by $\chi^{s-1}\gamma$

$$\left\|\omega(\chi^s\gamma) - \chi\omega(\chi^{s-1}\gamma)\right\| \leq \frac{1}{2}\mathfrak{M}\left(\chi^{s-1}\gamma, \chi^{s-1}\gamma, \chi^{s-1}\gamma\right)$$

for all $\gamma \in \mathbb{R}$. Multiplying $\frac{1}{\chi^s}$ on both sides and taking summation

$$\sum_{s=1}^{n} \frac{1}{\chi^s}\left\|\omega(\chi^s\gamma) - \chi\omega(\chi^{s-1}\gamma)\right\| \leq \frac{1}{2}\sum_{s=1}^{n} \frac{\mathfrak{M}(\chi^{s-1}\gamma, \chi^{s-1}\gamma, \chi^{s-1}\gamma)}{\chi^s}$$

$$|l + m| \leq |l| + |m|$$

$$\left\|\frac{1}{\chi^n}\omega(\chi^n\gamma) - \omega(\gamma)\right\| \leq \frac{1}{2}\sum_{s=1}^{n} \frac{\mathfrak{M}(\chi^{s-1}\gamma, \chi^{s-1}\gamma, \chi^{s-1}\gamma)}{\chi^s} \tag{42}$$

Since

$$\sum_{s=1}^{n} \frac{\mathfrak{M}(\chi^{s-1}\gamma, \chi^{s-1}\gamma, \chi^{s-1}\gamma)}{\chi^s} \leq \sum_{s=1}^{+\infty} \frac{\mathfrak{M}(\chi^{s-1}\gamma, \chi^{s-1}\gamma, \chi^{s-1}\gamma)}{\chi^s}$$

the inequality (42) yields

$$\left\|\frac{1}{\chi^n}\omega(\chi^n\gamma) - \omega(\gamma)\right\| \leq \frac{1}{2}\sum_{s=1}^{+\infty} \frac{\mathfrak{M}(\chi^{s-1}\gamma, \chi^{s-1}\gamma, \chi^{s-1}\gamma)}{\chi^s} \tag{43}$$

for all $\gamma \in \mathbb{R}$. Replacing n by $m - n$ in (43),

$$\left\|\frac{1}{\chi^{m-n}}\omega\left(\chi^{m-n}\gamma\right) - \omega(\gamma)\right\| \leq \frac{1}{2}\sum_{s=1}^{+\infty} \frac{\mathfrak{M}(\chi^{s-1}\gamma, \chi^{s-1}\gamma, \chi^{s-1}\gamma)}{\chi^s} \tag{44}$$

which is

$$\left\|\frac{1}{\chi^m}\omega\left(\chi^{m-n}\gamma\right) - \frac{1}{\chi^n}\omega(\gamma)\right\| \leq \frac{1}{2\chi^n}\sum_{s=1}^{+\infty} \frac{\mathfrak{M}(\chi^{s-1}\gamma, \chi^{s-1}\gamma, \chi^{s-1}\gamma)}{\chi^s} \tag{45}$$

for all $\gamma \in \mathbb{R}$. Replacing γ by $\chi^n\gamma$ in (18),

$$\left\|\frac{1}{\chi^m}\omega(\chi^m\gamma) - \frac{1}{\chi^n}\omega(\chi^n\gamma)\right\| \leq \frac{1}{2\chi^n}\sum_{s=1}^{+\infty} \frac{\mathfrak{M}(\chi^{s+n-1}\gamma, \chi^{s+n-1}\gamma, \chi^{s+n-1}\gamma)}{\chi^s}. \tag{46}$$

Since

$$\lim_{n\to+\infty} \frac{1}{2\chi^n} = 0$$

and hence from (46),

$$\lim_{n\to+\infty}\left\|\frac{1}{\chi^m}\omega(\chi^m\gamma) - \frac{1}{\chi^n}\omega(\chi^n\gamma)\right\| = 0$$

Therefore

$$\left\{\frac{\omega(\chi^n\gamma)}{\chi^n}\right\}_{n=1}^{+\infty}$$

is a Cauchy sequence. Then the sequence has a limit in $\mathbb{R}$. If

$$\mathbb{A}(\gamma) = \lim_{n\to+\infty} \frac{\omega(\chi^n\gamma)}{\chi^n}$$

for all $\gamma \in \mathbb{R}$ then $A : \mathbb{R} \to \mathbb{R}$ is additive.

Consider

$$\begin{aligned}
&\Big\|\mathbb{A}\Big(\eta^{\ell+\wp}\gamma + \eta^{\hbar}\kappa\Big) + \mathbb{A}\Big(\eta^{\ell+\wp}\kappa + \eta^{\hbar}\mu\Big) + \eta^{\ell+\wp}\mathbb{A}(\gamma-\kappa) + \eta^{\hbar}\mathbb{A}(\kappa-\mu) \\
&\qquad -2\Big(\eta^{\ell+\wp}\mathbb{A}(\gamma) + \eta^{\hbar}\mathbb{A}(\kappa)\Big)\Big\| \\
&= \frac{1}{\chi^n}\Big\|\omega\Big(\chi^n\eta^{\ell+\wp}\gamma + \chi^n\eta^{\hbar}\kappa\Big) + \omega\Big(\chi^n\eta^{\ell+\wp}\kappa + \chi^n\eta^{\hbar}\mu\Big) + \eta^{\ell+\wp}\omega(\chi^n\gamma - \chi^n\kappa) \\
&\qquad + \eta^{\hbar}\omega(\chi^n\kappa - \chi^n\mu) - 2\Big(\eta^{\ell+\wp}\omega(\chi^n\gamma) + \eta^{\hbar}\omega(\chi^n\kappa)\Big)\Big\| \\
&\le \lim_{n\to+\infty} \frac{1}{\chi^n}\mathfrak{M}(\chi^n\gamma, \chi^n\kappa, \chi^n\mu) = 0
\end{aligned}$$

Hence

$$\begin{aligned}
&A\Big(\eta^{\ell+\wp}\gamma + \eta^{\hbar}\kappa\Big) + A\Big(\eta^{\ell+\wp}\kappa + \eta^{\hbar}\mu\Big) + \eta^{\ell+\wp}A(\gamma-\kappa) + \eta^{\hbar}A(\kappa-\mu) \\
&\qquad = 2\Big(\eta^{\ell+\wp}\mathbb{A}(\gamma) + \eta^{\hbar}A(\kappa)\Big)
\end{aligned}$$

for all $\gamma \in \mathbb{R}$.

$$\begin{aligned}
||\mathbb{A}(\gamma) - \omega(\gamma)|| &= ||\lim_{n\to+\infty} \frac{\omega(\chi^n\gamma)}{\chi^n} - \omega(\gamma)|| \\
&= \lim_{n\to+\infty} ||\frac{\omega(\chi^n\gamma)}{\chi^n} - \omega(\gamma)|| \\
&\le \lim_{n\to+\infty} \frac{1}{2}\sum_{s=1}^{n} \frac{\mathfrak{M}(\chi^{s-1}\gamma, \chi^{s-1}\gamma, \chi^{s-1}\gamma)}{\chi^s}
\end{aligned}$$

Hence,

$$||\mathbb{A}(\gamma) - \omega(\gamma)|| \le \frac{1}{2}\sum_{s=1}^{n} \frac{\mathfrak{M}(\chi^{s-1}\gamma, \chi^{s-1}\gamma, \chi^{s-1}\gamma)}{\chi^s}$$

for all $\gamma \in \mathbb{R}$.

$$||\mathbb{B}(\gamma) - \omega(\gamma)|| \le \frac{1}{2}\sum_{s=1}^{n} \frac{\mathfrak{M}(\chi^{s-1}\gamma, \chi^{s-1}\gamma, \chi^{s-1}\gamma)}{\chi^s}$$

Hence

$$\begin{aligned}
||\mathbb{B}(\gamma) - \mathbb{A}(\gamma)|| &\le ||\mathbb{B}(\gamma) - \omega(\gamma)|| + ||\mathbb{A}(\gamma) - \omega(\gamma)|| \\
&\le \frac{1}{2}\sum_{s=1}^{n} \frac{\mathfrak{M}(\chi^{s-1}\gamma, \chi^{s-1}\gamma, \chi^{s-1}\gamma)}{\chi^s} + \frac{1}{2}\sum_{s=1}^{n} \frac{\mathfrak{M}(\chi^{s-1}\gamma, \chi^{s-1}\gamma, \chi^{s-1}\gamma)}{\chi^s} \\
&= \sum_{s=1}^{n} \frac{\mathfrak{M}(\chi^{s-1}\gamma, \chi^{s-1}\gamma, \chi^{s-1}\gamma)}{\chi^s}
\end{aligned}$$

$$\begin{aligned}
||\mathbb{A}(\gamma) - \mathbb{B}(\gamma)|| &= \frac{1}{\chi^n}||\mathbb{A}(\chi^n\gamma) - \mathbb{B}(\chi^n\gamma)|| \\
&\le \frac{1}{\chi^n}\sum_{s=1}^{n} \frac{\mathfrak{M}(\chi^{s+n-1}\gamma, \chi^{s+n-1}\gamma, \chi^{s+n-1}\gamma)}{\chi^s}
\end{aligned} \tag{47}$$

Taking $n \to +\infty$, using (47)

$$\lim_{n\to+\infty} ||\mathbb{A}(\gamma) - \mathbb{B}(\gamma)|| \leq \lim_{n\to+\infty} \frac{1}{\chi^n} \sum_{s=1}^{n} \frac{\mathfrak{M}(\chi^{s+n-1}\gamma, \chi^{s+n-1}\gamma, \chi^{s+n-1}\gamma)}{\chi^s}$$

Hence

$$||\mathbb{A}(\gamma) - \mathbb{B}(\gamma)|| \leq 0$$

Therefore $\mathbb{A}(\gamma) = \mathbb{B}(\gamma)$ for all $\gamma \in \mathbb{R}$. Hence $\mathbb{A}$ is unique. □

Corollary 1. *Consider the inequality with various general control functions such as*

$$\|\mathbb{J}\omega(\gamma,\kappa,\mu)\| \leq \begin{cases} \mathfrak{U}, & \\ \mathfrak{U}\{||\gamma||^p + ||\kappa||^p + ||\mu||^p\}, & p \neq 1; \\ \mathfrak{U}||\gamma||^p||\kappa||^p||\mu||^p, & 3p \neq 1; \\ \mathfrak{U}\{||\gamma||^p||\kappa||^p||\mu||^p + \{||\gamma||^{3p} + ||\kappa||^{3p} + ||\mu||^{3p}\}\}, & 3p \neq 1; \end{cases} \tag{48}$$

which gives

$$\|\omega(\gamma) - \mathbb{A}(\gamma)\| \leq \begin{cases} \dfrac{\mathfrak{U}}{2|\chi - 1|}, \\ \dfrac{3\,\mathfrak{U}||\gamma||^p}{2|\chi - \chi^p|}, \\ \dfrac{\mathfrak{U}||\gamma||^{3p}}{2|\chi - \chi^{3p}|}, \\ \dfrac{2\,\mathfrak{U}||\gamma||^{3p}}{|\chi - \chi^{3p}|} \end{cases} \tag{49}$$

2.6. V. Radus' Method for (5) (or) Fixed-Point Method

Theorem 6. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a mapping with the condition*

$$\lim_{\ell\to+\infty} \frac{\mathfrak{M}(\theta_i^\ell \gamma, \theta_i^\ell \kappa,\ \theta_i^\ell \mu)}{\theta_i^\ell} = 0, \tag{50}$$

where

$$\theta_i = \begin{cases} \chi, & i = 0, \\ \frac{1}{\chi}, & i = 1 \end{cases}$$

satisfies

$$\|\mathbb{J}\omega(\gamma,\kappa,\mu)\| \leq \mathfrak{M}(\gamma,\kappa,\mu) \tag{51}$$

If the function there exists $L = L(i) < 1$ such that

$$\gamma \to \pi(\gamma) = \frac{1}{2}\mathfrak{M}\left(\frac{\gamma}{\chi}, \frac{\gamma}{\chi}, \frac{\gamma}{\chi}\right),$$

and

$$\pi(\gamma) = L\,\theta_i\,\gamma\left(\frac{\gamma}{\theta_i}\right) \tag{52}$$

Then there exists a function $\mathbb{A} : \mathbb{R} \to \mathbb{R}$ that fulfills (5) and

$$\|\omega(\gamma) - \mathbb{A}(\gamma)\| \leq \frac{L^{1-i}}{1-L}\pi(\gamma) \tag{53}$$

3. Stability Results in Quasi-Beta Normed Spaces

3.1. Stability Results: Direct Method

Theorem 7. *Let the mapping $\omega : \mathbb{R} \to \mathbb{R}$ satisfy the inequality*

$$\|\mathbb{J}\omega(\gamma,\kappa,\mu)\|_Y \leq \mathfrak{M}(\gamma,\kappa,\mu) \tag{54}$$

with the condition

$$\lim_{n\to+\infty} \frac{\mathfrak{M}(\chi^{nj}\gamma,\chi^{nj}\kappa,\chi^{nj}\mu)}{\chi^{nj}} = 0$$

with $j \in \{-1,1\}$ and for all $\gamma,\kappa,\mu \in \mathbb{R}$, then there exists a function $\mathbb{A} : \mathbb{R} \to \mathbb{R}$ and $\chi = \left(\eta^{\ell+\wp} + \eta^{\hbar}\right)$ such that

$$\|\omega(\gamma) - \mathbb{A}(\gamma)\|_Y^s \leq \frac{M^{(n-1)s}}{(2\chi)^{\beta s}} \sum_{k=\frac{1-j}{2}}^{+\infty} \frac{\mathfrak{M}(\chi^{kj}\gamma,\chi^{kj}\gamma,\chi^{kj}\gamma)^s}{\chi^{kjs}} \tag{55}$$

for all $\gamma \in \mathbb{R}$.

Proof. Consider (γ,κ,μ) by (γ,γ,γ) in (54)

$$\left\|2h\left[\left(\eta^{\ell+\wp} + \eta^{\hbar}\right)x\right] - 2\left(\eta^{\ell+\wp} + \eta^{\hbar}\right)\omega(\gamma)\right\|_Y \leq \mathfrak{M}(\gamma,\gamma,\gamma) \tag{56}$$

and

$$\|2\chi\omega(\gamma) - 2\omega(\chi\gamma)\|_Y \leq \mathfrak{M}(\gamma,\gamma,\gamma) \tag{57}$$

Dividing both sides by 2χ in (57)

$$\left\|\omega(\gamma) - \frac{\omega(\chi\gamma)}{\chi}\right\|_Y \leq \frac{\mathfrak{M}(\gamma,\gamma,\gamma)}{(2\chi)^{\beta}} \tag{58}$$

let γ by $\chi\gamma$ and dividing by χ in (58),

$$\left\|\frac{\omega(\chi\gamma)}{\chi} - \frac{\omega(\chi^2\gamma)}{\chi^2}\right\|_Y \leq \frac{\mathfrak{M}(\chi\gamma,\chi\gamma,\chi\gamma)}{(2\chi)^{\beta}\chi} \tag{59}$$

by these inequalities (58) and (59)

$$\begin{aligned}\left\|\omega(\gamma) - \frac{\omega(\chi^2\gamma)}{\chi^2}\right\|_Y &\leq \left\|\omega(\gamma) - \frac{\omega(\chi\gamma)}{\chi}\right\|_Y + \left\|\frac{\omega(\chi\gamma)}{\chi} - \frac{\omega(\chi^2\gamma)}{\chi^2}\right\|_Y \\ &\leq \frac{M}{(2\chi)^{\beta}}\left[\mathfrak{M}(\gamma,\gamma,\gamma) + \frac{\mathfrak{M}(\chi\gamma,\chi\gamma,\chi\gamma)}{\chi}\right]\end{aligned} \tag{60}$$

For n,

$$\begin{aligned}\left\|\omega(\gamma) - \frac{\omega(\chi^n\gamma)}{\chi^n}\right\|_Y &\leq \frac{M^{(n-1)}}{(2\chi)\beta} \sum_{k=0}^{n-1} \frac{\mathfrak{M}(\chi^k\gamma,\chi^k\gamma,\chi^k\gamma)}{\chi^k} \\ &\leq \frac{M^{(n-1)}}{(2\chi)\beta} \sum_{k=0}^{+\infty} \frac{\mathfrak{M}(\chi^k\gamma,\chi^k\gamma,\chi^k\gamma)}{\chi^k}\end{aligned} \tag{61}$$

$$\left\{\frac{\omega(\chi^n\gamma)}{\chi^n}\right\},$$

Take γ by $\chi^m\gamma$ and divide by χ^m in (61),

$$\begin{aligned}\left\|\frac{\omega(\chi^m\gamma)}{\chi^m}-\frac{\omega(\chi^{n+m}\gamma)}{\chi^{(n+m)}}\right\|_\Upsilon &= \frac{M^{(n-1)}}{(2\chi)\beta}\left\|\omega(\chi^m\gamma)-\frac{\omega(\chi^n\cdot\chi^m\gamma)}{\chi^n}\right\|_\Upsilon \\ &\leq \frac{M^{(n-1)}}{(2\chi)\beta}\sum_{k=0}^{n-1}\frac{\mathfrak{M}(\chi^{k+m}\gamma,\chi^{k+m}\gamma,\chi^{k+m}\gamma)}{\chi^{k+m}} \\ &\leq \frac{M^{(n-1)}}{(2\chi)\beta}\sum_{k=0}^{+\infty}\frac{\mathfrak{M}(\chi^{k+m}\gamma,\chi^{k+m}\gamma,\chi^{k+m}\gamma)}{\chi^{k+m}} \\ &\to 0 \quad as \quad m\to+\infty\end{aligned}$$

Therefore

$$\left\{\frac{\omega(\chi^n\gamma)}{\chi^n}\right\}$$

is a Cauchy sequence. Then the sequence has a limit in $\mathbb{R}$. Defining

$$\mathbb{A}(\gamma)=\lim_{n\to+\infty}\frac{\omega(\chi^n\gamma)}{\chi^n}$$

for all $\gamma\in\mathbb{R}$. As $n\to+\infty$ in (61) then (55) holds for all $\gamma\in\mathbb{R}$.

Now A fulfills (5), take (γ,κ,μ) by $(\chi^n\gamma,\chi^n\kappa,\chi^n\mu)$ and dividing by χ^n in (54),

$$\frac{1}{\chi^n}\|\mathbb{J}\omega(\chi^n\gamma,\chi^n\kappa,\chi^n\mu)\|\leq\frac{1}{\chi^n}\mathfrak{M}(\chi^n\gamma,\chi^n\kappa,\chi^n\mu)$$

for all $\gamma,\kappa,\mu\in\mathbb{R}$. As $n\to+\infty$

$$\begin{aligned}A\Big(\eta^{\ell+\wp}\gamma+\eta^{\hbar}\kappa\Big)+A\Big(\eta^{\ell+\wp}\kappa+\eta^{\hbar}\mu\Big)+\eta^{\ell+\wp}A(\gamma-\kappa)+\eta^{\hbar}A(\kappa-\mu)\\ =2\Big(\eta^{\ell+\wp}\mathbb{A}(\gamma)+\eta^{\hbar}A(\kappa)\Big)\end{aligned}$$

hence A fulfills (5). To demonstrate that $\mathbb{A}$ is unique

$$\begin{aligned}\|\mathbb{A}(\gamma)-\mathbb{B}(\gamma)\|_\Upsilon &= \frac{1}{\chi^n}\|A(\chi^n\gamma)-\mathbb{B}(\chi^n\gamma)\|_\Upsilon \\ &\leq \frac{M}{\chi^n}\{\|A(\chi^n\gamma)-\omega(\chi^n\gamma)\|_\Upsilon+\|\omega(\chi^n\gamma)-\mathbb{B}(\chi^n\gamma)\|_\Upsilon\} \\ &\leq \frac{M^{(n-1)}}{(\chi^n)^\beta}\sum_{k=0}^{+\infty}\frac{\mathfrak{M}(\chi^{k+n}\gamma,\chi^{k+n}\gamma,\chi^{k+n}\gamma)}{\chi^{(k+n)}} \\ &\to 0 \quad as \quad n\to+\infty\end{aligned}$$

for all $\gamma\in\mathbb{R}$. Hence A is unique. □

Corollary 2. *Considering the inequality with various control functions*

$$\|\mathbb{J}\omega(\gamma,\kappa,\mu)\|_\Upsilon\leq\begin{cases}\mathfrak{U}, & \\ \mathfrak{U}\{||\gamma||^p+||\kappa||^p+||\mu||^p\}, & p\neq1; \\ \mathfrak{U}||\gamma||^p||\kappa||^p||\mu||^p, & 3p\neq1; \\ \mathfrak{U}\{||\gamma||^p||\kappa||^p||\mu||^p+\{||\gamma||^{3p}+||\kappa||^{3p}+||\mu||^{3p}\}\}, & 3p\neq1;\end{cases} \tag{62}$$

and

$$\|\omega(\gamma)-\mathbb{A}(\gamma)\|_Y^s \leq \begin{cases} \left(\dfrac{\mathfrak{U}\, M^{(n-1)}}{(2\chi)^{\beta}}\dfrac{\chi}{|\chi-1|}\right)^s, \\ \left(\dfrac{3\mathfrak{U}\, M^{(n-1)}}{(2\chi)^{\beta}}\dfrac{\chi||\gamma||^p}{|\chi-\chi^{\beta p}|}\right)^s, \\ \left(\dfrac{\mathfrak{U}\, M^{(n-1)}}{(2\chi)^{\beta}}\dfrac{\chi||\gamma||^{3p}}{|\chi-\chi^{3\beta p}|}\right)^s, \\ \left(\dfrac{4\mathfrak{U}\, M^{(n-1)}}{(2\chi)^{\beta}}\dfrac{\chi||\gamma||^{3p}}{|\chi-\chi^{3\beta p}|}\right)^s \end{cases} \tag{63}$$

3.2. Stability Results: Fixed-Point Method

Theorem 8. *Let the map $\omega:\mathbb{R}\to\mathbb{R}$ with the condition*

$$\lim_{\ell\to+\infty}\frac{\mathfrak{M}(\theta_i^{\ell}\gamma,\theta_i^{\ell}\kappa,\ \theta_i^{\ell}\mu)}{\theta_i^{\ell}}=0, \tag{64}$$

where

$$\theta_i=\begin{cases}\chi, & i=0,\\ \frac{1}{\chi}, & i=1\end{cases}$$

satisfy

$$\|\mathbb{J}\omega(\gamma,\kappa,\mu)\|_Y \leq \mathfrak{M}(\gamma,\kappa,\mu) \tag{65}$$

If the function $L=L(i)<1$ exists such that

$$\gamma\to\pi(\gamma)=\frac{1}{2}\mathfrak{M}\left(\frac{\gamma}{\chi},\frac{\gamma}{\chi},\frac{\gamma}{\chi}\right),$$

one has the property

$$\pi(\gamma)=L\,\theta_i\,\gamma\left(\frac{\gamma}{\theta_i}\right) \tag{66}$$

for all $\gamma\in\mathbb{R}$. Then there exists additive map $\mathbb{A}:\mathbb{R}\to\mathbb{R}$ fulfilling (5) and

$$\|\omega(\gamma)-\mathbb{A}(\gamma)\|_Y^s \leq \left(\frac{L^{1-i}}{1-L}\pi(\gamma)\right)^s \tag{67}$$

Proof. Assuming $\mathcal{B}=\{u/u:\mathbb{R}\to\mathbb{R},\ u(0)=0\}$ and introducing the generalised metric on $\mathcal{B}$,

$$d(u,v)=\inf\{K\in(0,+\infty):\| u(\gamma)-v(\gamma)\|\leq K\pi(\gamma),\gamma\in\mathbb{R}\}.$$

Define $T:\mathcal{B}\to\mathcal{B}$ by

$$Tu(\gamma)=\frac{1}{\theta_i}v(\theta_i\gamma),\forall\quad\gamma\in\mathbb{R}.$$

Now $u, v \in \mathcal{B}$,

$$
\begin{aligned}
d(u,v) \leq K \Rightarrow & \parallel u(\gamma) - v(\gamma) \parallel_Y \leq K\pi(\gamma), \gamma \in \mathbb{R}. \\
& \Rightarrow \left\| \frac{1}{\theta_i} u(\theta_i \gamma) - \frac{1}{\theta_i} v(\theta_i \gamma) \right\|_Y \leq \frac{1}{\theta_i} K\gamma(\theta_i \gamma), \gamma \in \mathbb{R}, \\
& \Rightarrow \left\| \frac{1}{\theta_i} u(\theta_i \gamma) - \frac{1}{\theta_i} v(\theta_i \gamma) \right\|_Y \leq LK\pi(\gamma), \gamma \in \mathbb{R}, \\
& \Rightarrow \parallel Tu(\gamma) - Tv(\gamma) \parallel_Y \leq LK\pi(\gamma), \gamma \in \mathbb{R}, \\
& \Rightarrow d(Tu, Tv) \leq LK.
\end{aligned}
$$

This implies

$$d(Tu, Tv) \leq Ld(u,v),$$

$\Longrightarrow$ T is a strictly contractive mapping on $\mathcal{B}$ with Lipschitz constant L.

From (58),

$$\left\| \omega(\gamma) - \frac{\omega(\chi\gamma)}{\chi} \right\|_Y \leq \frac{\mathfrak{M}(\gamma,\gamma,\gamma)}{2\chi} \tag{68}$$

where

$$\beta(\gamma) = \frac{\mathfrak{M}(\gamma,\gamma,\gamma)}{2\chi}$$

For $i = 0$,

$$\left\| \frac{1}{\chi}\omega(\chi\gamma) - \omega(\gamma) \right\|_Y \leq \frac{1}{\chi}\pi(\gamma)$$

for all $\gamma \in \mathbb{R}$.

$$\text{i.e.,} \qquad d(T\omega, \omega) \leq \frac{1}{\chi} = L = L^{1-0} = L^{1-i} < +\infty.$$

Take $\gamma = \dfrac{\gamma}{\chi}$ in (68),

$$\left\| \omega(\gamma) - \chi h\left(\frac{\gamma}{\chi}\right) \right\|_Y \leq \frac{1}{2}\mathfrak{M}\left(\frac{\gamma}{\chi}, \frac{\gamma}{\chi}, \frac{\gamma}{\chi}\right).$$

For $i = 1$,

$$\left\| \omega(\gamma) - \chi\omega\left(\frac{\gamma}{\chi}\right) \right\|_Y \leq \pi(\gamma)$$

for all $\gamma \in \mathbb{R}$.

$$\text{i.e.,} \qquad d(\omega, T\omega) \leq 1 = L^0 = L^{1-1} = L^{1-i} < +\infty.$$

In the above two cases,

$$d(\omega, T\omega) \leq L^{1-i}.$$

By the fixed-point condition, $\mathbb{A}$ of T in $\mathcal{B}$ such that

$$\mathbb{A}(\gamma) = \lim_{\ell \to +\infty} \frac{\omega(\theta_i^\ell \gamma)}{\theta_i^\ell}, \qquad \forall\ \gamma \in \mathbb{R}. \tag{69}$$

Claim that $\mathbb{A} : \mathbb{R} \to \mathbb{R}$ additive. Supplanting (γ, κ, μ) by $\left(\theta_i^\ell \gamma, \theta_i^\ell \kappa, \theta_i^\ell \mu\right)$ in (65) and dividing by θ_i^ℓ, it follows from (64) and (69), that $\mathbb{A}$ fulfills (5) for all $\gamma, \kappa, \mu \in \mathbb{R}$.

By the fixed-point condition, $\mathbb{A}$ is the unique fixed point of $\mathbb{A}$ in the set $Y = \{h \in \mathcal{B} : d(T\omega, \mathbb{A}) < +\infty\}$, using the fixed-point alternative result $\mathbb{A}$ is the unique function such that

$$\|\omega(\gamma) - \mathbb{A}(\gamma)\|_Y \leq K\pi(\gamma)$$

Finally by the fixed-point condition,

$$d(\omega, \mathbb{A}) \leq \frac{1}{1-L} d(\omega, T\omega)$$

implying

$$d(\omega, \mathbb{A}) \leq \frac{L^{1-i}}{1-L}.$$

Thus it is presumed that

$$\| \ \omega(\gamma) - \mathbb{A}(\gamma) \ \|_Y^s \leq \left(\frac{L^{1-i}}{1-L} \pi(\gamma) \right)^s.$$

□

Corollary 3. *Considering the inequality with various control functions*

$$\|\mathbb{J}\omega(\gamma, \kappa, \mu)\|_Y \leq \begin{cases} \mathfrak{U}, & \\ \mathfrak{U}\{||\gamma||^p + ||\kappa||^p + ||\mu||^p\}, & p \neq 1; \\ \mathfrak{U}||\gamma||^p||\kappa||^p||\mu||^p, & 3p \neq 1; \\ \mathfrak{U}\{||\gamma||^p||\kappa||^p||\mu||^p + \{||\gamma||^{3p} + ||\kappa||^{3p} + ||\mu||^{3p}\}\}, & 3p \neq 1; \end{cases} \tag{70}$$

and

$$\|\omega(\gamma) - \mathbb{A}(\gamma)\|_Y^s \leq \begin{cases} \left(\dfrac{\mathfrak{U}}{2\chi|1-\chi|} \right)^s, \\ \left(\dfrac{3\,\mathfrak{U}||\gamma||^p}{2|\chi - \chi^{\beta p}|} \right)^s, \\ \left(\dfrac{\mathfrak{U}||\gamma||^{3p}}{2|\chi - \chi^{3\beta p}|} \right)^s, \\ \left(\dfrac{2\,\mathfrak{U}||\gamma||^{3p}}{|\chi - \chi^{3\beta p}|} \right)^s \end{cases} \tag{71}$$

Proof. Let

$$\mathfrak{M}(\gamma, \kappa, \mu) = \begin{cases} \mathfrak{U}, \\ \mathfrak{U}\{||\gamma||^p + ||\kappa||^p + ||\mu||^p\}, \\ \mathfrak{U} \ ||\gamma||^p \ ||\kappa||^p \ ||\mu||^p, \\ \mathfrak{U}\Big\{||\gamma||^p||\kappa||^p||\mu||^p + (||\gamma||^{3p} + ||\kappa||^{3p} + ||\mu||^{3p})\Big\} \end{cases}$$

Now

$$\frac{\mathfrak{M}(\theta_i^\ell \gamma, \theta_i^\ell y, \theta_i^\ell z)}{\theta_i^\ell} = \begin{cases} \dfrac{\mathfrak{U}}{\theta_i^\ell}, \\ \dfrac{\mathfrak{U}}{\theta_i^\ell}\Big\{||\theta_i^\ell \gamma||^p + ||\theta_i^\ell \kappa||^p + ||\theta_i^\ell \mu||^p\Big\}, \\ \dfrac{\mathfrak{U}}{\theta_i^\ell}||\theta_i^\ell \gamma||^p \ ||\theta_i^\ell \kappa||^p \ ||\theta_i^\ell \mu||^p, \\ \dfrac{\mathfrak{U}}{\theta_i^\ell}\Big\{||\theta_i^\ell \gamma||^p \ ||\theta_i^\ell \kappa||^p \ ||\theta_i^\ell \mu||^p \Big\{||\theta_i^\ell \gamma||^{3s} + ||\theta_i^\ell \kappa||^{3p} + ||\theta_i^\ell \mu||^{3p}\Big\}\Big\} \end{cases}$$

$$= \begin{cases} \to 0 \text{ as } \ \ell \to +\infty, \\ \to 0 \text{ as } \ \ell \to +\infty, \\ \to 0 \text{ as } \ \ell \to +\infty, \\ \to 0 \text{ as } \ \ell \to +\infty. \end{cases}$$

For (64)

$$\pi(\gamma) = \frac{1}{2}\left[\mathfrak{M}\left(\frac{\gamma}{\chi}, \frac{\gamma}{\chi}, \frac{\gamma}{\chi}\right)\right].$$

Hence

$$\pi(\gamma) = \frac{1}{2}\left[\mathfrak{M}\left(\frac{\gamma}{\chi}, \frac{\gamma}{\chi}, \frac{\gamma}{\chi}\right)\right] = \begin{cases} \frac{\mathfrak{U}}{2}, \\ \frac{3\mathfrak{U}}{2\chi^{\beta s}}||\gamma||^{p}, \\ \frac{\mathfrak{U}}{2\chi^{3\beta s}}||\gamma||^{3p}, \\ \frac{2\mathfrak{U}}{\chi^{3\beta s}}||\gamma||^{3p}. \end{cases}$$

Additionally,

$$\frac{1}{\theta_i}\gamma(\theta_i\gamma) = \begin{cases} \frac{\mathfrak{U}}{\theta_i \cdot 2}, \\ \frac{3\mathfrak{U}}{\theta_i \cdot 2\chi^{p}}||\theta_i\gamma||^{p}, \\ \frac{\mathfrak{U}}{\theta_i \cdot 2\chi^{3p}}||\theta_i\gamma||^{3p}, \\ \frac{\mathfrak{U}}{\theta_i \cdot 2\chi^{3p}}||\theta_i\gamma||^{3p}. \end{cases}$$

$$= \begin{cases} \theta_i^{-1}\frac{\mathfrak{U}}{2}, \\ \theta_i^{\beta p-1}\frac{3\mathfrak{U}}{2\chi^{p}}||\gamma||^{p}, \\ \theta_i^{3\beta p-1}\frac{\mathfrak{U}}{2\chi^{3p}}||\gamma||^{3p}, \\ \theta_i^{3\beta p-1}\frac{4\mathfrak{U}}{2\chi^{3p}}||\gamma||^{3p}. \end{cases}$$

$$= \begin{cases} \theta_i^{-1}\pi(\gamma), \\ \theta_i^{\beta p-1}\pi(\gamma), \\ \theta_i^{3\beta p-1}\pi(\gamma), \\ \theta_i^{3\beta p-1}\pi(\gamma). \end{cases}$$

From (67) the following results are obtained
Criteria:1 $L = \chi^{-1}$ if $i = 0$

$$\begin{aligned} \|\omega(\gamma) - \mathbb{A}(\gamma)\|_Y^s \leq \frac{L^{1-i}}{1-L}\pi(\gamma) &= \left(\frac{(\chi^{-1})^{1-0}}{1-(\chi)^{-1}} \cdot \frac{\mathfrak{U}}{2}\right)^s \\ &= \left(\frac{\mathfrak{U}}{2(\chi-1)}\right)^s. \end{aligned}$$

Criteria:2 $L = \chi$ if $i = 1$

$$\begin{aligned} \|\omega(\gamma) - \mathbb{A}(\gamma)\|_Y^s \leq \frac{L^{1-i}}{1-L}\pi(\gamma) &= \left(\frac{(\chi)^{1-1}}{1-\chi} \cdot \frac{\mathfrak{U}}{2}\right)^s \\ &= \left(\frac{\mathfrak{U}}{2(1-\chi)}\right)^s. \end{aligned}$$

Criteria:1 $L = \chi^{\beta p-1}$ if $i = 0$

$$\begin{aligned}\|\omega(\gamma) - \mathbb{A}(\gamma)\|_Y^s \leq \frac{L^{1-i}}{1-L}\pi(\gamma) &= \left(\frac{\left(\chi^{\beta p-1}\right)^{1-0}}{1-\chi^{\beta p-1}}\frac{3\mathfrak{U}}{2\chi^{\beta p}}||\gamma||^p\right)^s \\ &= \left(\frac{\chi^{\beta p}}{\chi-\chi^{\beta p}}\frac{3\mathfrak{U}}{2\chi^{\beta p}}||\gamma||^p\right)^s \\ &= \left(\frac{3\mathfrak{U}||\gamma||^p}{2(\chi-\chi^{\beta p})}\right)^s.\end{aligned}$$

Criteria:2 $L = \frac{1}{\chi^{\beta p-1}}$ if $i = 1$

$$\begin{aligned}\|\omega(\gamma) - \mathbb{A}(\gamma)\|_Y^s \leq \frac{L^{1-i}}{1-L}\pi(\gamma) &= \left(\frac{\left(\frac{1}{\chi^{\beta p-1}}\right)^{1-1}}{1-\frac{1}{2\chi^{\beta p-1}}}\frac{3\mathfrak{U}}{2\chi^{p}}||\gamma||^p\right)^s \\ &= \left(\frac{\chi^{\beta p}}{\chi^{\beta p}-\chi}\frac{3\mathfrak{U}}{2\chi^{\beta p}}||\gamma||^p\right)^s \\ &= \left(\frac{3\mathfrak{U}||\gamma||^p}{2\chi(\chi^{\beta p}-\chi)}\right)^s.\end{aligned}$$

Criteria:1 $L = \chi^{3\beta p-1}$ if $i = 0$

$$\begin{aligned}\|\omega(\gamma) - \mathbb{A}(\gamma)\|_Y^s \leq \frac{L^{1-i}}{1-L}\pi(\gamma) &= \left(\frac{\left(\chi^{3\beta p-1}\right)^{1-0}}{1-\chi^{3\beta p-1}}\frac{\mathfrak{U}}{2\chi^{3\beta p}}||\gamma||^{3p}\right)^s \\ &= \left(\frac{\chi^{3\beta p}}{\chi-\chi^{3\beta p}}\frac{\mathfrak{U}}{2\chi^{3\beta p}}||\gamma||^{3p}\right)^s \\ &= \left(\frac{\mathfrak{U}||\gamma||^{3p}}{2(\chi-\chi^{3\beta p})}\right)^s.\end{aligned}$$

Criteria:2 $L = \frac{1}{\chi^{3\beta p-1}}$ if $i = 1$

$$\begin{aligned}\|\omega(\gamma) - \mathbb{A}(\gamma)\|_Y^s \leq \frac{L^{1-i}}{1-L}\pi(\gamma) &= \left(\frac{\left(\frac{1}{\chi^{3\beta p-1}}\right)^{1-1}}{1-\frac{1}{\chi^{3\beta p-1}}}\frac{\mathfrak{U}}{2\chi^{3\beta p}}||\gamma||^{3p}\right)^s \\ &= \left(\frac{\chi^{3\beta p}}{\chi^{3\beta p}-\chi}\frac{\mathfrak{U}}{2\chi^{3\beta p}}||\gamma||^{3p}\right)^s \\ &= \left(\frac{\mathfrak{U}||\gamma||^{3p}}{2(\chi^{3\beta p}-\chi)}\right)^s.\end{aligned}$$

Criteria:1 $L = \chi^{3\beta p-1}$ if $i = 0$

$$\|\omega(\gamma) - \mathbb{A}(\gamma)\|_Y^s \leq \frac{L^{1-i}}{1-L}\pi(\gamma) = \left(\frac{\left(\chi^{3\beta p-1}\right)^{1-0}}{1-\chi^{3\beta p-1}}\frac{4\mathfrak{U}}{2\chi^{3\beta p}}||\gamma||^{3p}\right)^s$$
$$= \left(\frac{\chi^{3\beta p}}{\chi - \chi^{3\beta p}}\frac{2\mathfrak{U}}{\chi^{3\beta p}}||\gamma||^{3p}\right)^s$$
$$= \left(\frac{2\mathfrak{U}||\gamma||^{3p}}{(\chi - \chi^{3\beta p})}\right)^s.$$

Criteria:2 $L = \frac{1}{\chi^{3\beta p-1}}$ if $i = 1$

$$\|\omega(\gamma) - \mathbb{A}(\gamma)\|_Y^s \leq \frac{L^{1-i}}{1-L}\pi(\gamma) = \left(\frac{\left(\frac{1}{\chi^{3\beta p-1}}\right)^{1-1}}{1-\frac{1}{\chi^{3\beta p-1}}}\frac{4\mathfrak{U}}{2\chi^{3\beta p}}||\gamma||^{3p}\right)^s$$
$$= \left(\frac{\chi^{3\beta p}}{\chi^{3\beta p} - \chi}\frac{2\mathfrak{U}}{\chi^{3\beta p}}||\gamma||^{3p}\right)^s$$
$$= \left(\frac{2\mathfrak{U}||\gamma||^{3p}}{(\chi^{3\beta p} - \chi)}\right)^s.$$

□

3.3. Remark

(*i*) The proof of Theorem 5 and 6 replaced by $s = M = \beta = 1$ in Theorem 7 and 8.
(*ii*) Replacing by $s = M = \beta = 1$ in Corollary 2, the Corollary 1 is obtained and satisfies Theorems 1–4.

3.4. Applications

Functional equations play an important role in linear algebra specifically in linear transformation. The relationship between functional equation and linear transformation is demonstrated.

Linear transformation:
Let $\mathbb{A}$ and $\mathbb{B}$ be real vector spaces (their dimensions are different) and let $\mathbb{T}$ be the function with domain $\mathbb{A}$ and range in $\mathbb{B}$ $\mathbb{T} : \mathbb{A} \longrightarrow \mathbb{B}$. $\mathbb{T}$ is said to be a linear transformation.

(a) $\forall \gamma, \kappa \in \mathbb{A},\ \mathbb{T}(\gamma + \kappa) = \mathbb{T}(\gamma) + \mathbb{T}(\kappa)$ ($\mathbb{T}$ is additive)
(b) $\forall \gamma \in \mathbb{A}\ , H \in \mathbb{R}\ \ \mathbb{T}(H\gamma) = H\mathbb{T}(\gamma)$ ($\mathbb{T}$ is homogeneous)

Example 4. $\mathbb{A} = \mathbb{B} = E^1$. *Define* $\mathbb{T}(\gamma) = m\gamma$, *where* m *is the fixed real number.*

$$\omega\left(\eta^{\ell+\wp}\gamma + \eta^{\hbar}\kappa\right) + \omega\left(\eta^{\ell+\wp}\kappa + \eta^{\hbar}\mu\right) + \eta^{\ell+\wp}\omega(\gamma-\kappa) + \eta^{\hbar}\omega(\kappa-\mu)$$
$$= 2\left(\eta^{\ell+\wp}\omega(\gamma) + \eta^{\hbar}\omega(\kappa)\right)$$

Solution: Let us take the given equation as

$$\mathbb{T}\left(\eta^{\ell+\wp}\gamma + \eta^{\hbar}\kappa\right) + \mathbb{T}\left(\eta^{\ell+\wp}\kappa + \eta^{\hbar}\mu\right) + \eta^{\ell+\wp}\mathbb{T}(\gamma-\kappa) + \eta^{\hbar}\mathbb{T}(\kappa-\mu)$$
$$= 2\left(\eta^{\ell+\wp}\mathbb{T}(\gamma) + \eta^{\hbar}\mathbb{T}(\kappa)\right)$$

$$
\begin{aligned}
LHS: \quad & \mathbb{T}\left(\eta^{\ell+\wp}\gamma+\eta^{\hbar}\kappa\right)+\mathbb{T}\left(\eta^{\ell+\wp}\kappa+\eta^{\hbar}\mu\right)+\eta^{\ell+\wp}\mathbb{T}(\gamma-\kappa)+\eta^{\hbar}\mathbb{T}(\kappa-\mu)\\
&=\mathbb{T}\left(\eta^{\ell+\wp}\gamma\right)+\mathbb{T}\left(\eta^{\hbar}\kappa\right)+\mathbb{T}\left(\eta^{\ell+\wp}\kappa\right)+\mathbb{T}\left(\eta^{\hbar}\mu\right)+\eta^{\ell+\wp}\mathbb{T}(\gamma-\kappa)+\eta^{\hbar}\mathbb{T}(\kappa-\mu)\\
&=\eta^{\ell+\wp}\mathbb{T}(\gamma)+\eta^{\hbar}\mathbb{T}(\kappa)+\eta^{\ell+\wp}\mathbb{T}(\kappa)+\eta^{\hbar}\mathbb{T}(\mu)+\eta^{\ell+\wp}\mathbb{T}(\gamma-\kappa)+\eta^{\hbar}\mathbb{T}(\kappa-\mu)\\
&=\eta^{\ell+\wp}m\gamma+\eta^{\hbar}m\kappa+\eta^{\ell+\wp}m\kappa+\eta^{\hbar}m\mu+\eta^{\ell+\wp}m(\gamma-\kappa)+\eta^{\hbar}m(\kappa-\mu)\\
&=\eta^{\ell+\wp}m\gamma+\eta^{\hbar}m\kappa+\eta^{\ell+\wp}m\kappa+\eta^{\hbar}m\mu+\eta^{\ell+\wp}m\gamma-\eta^{\ell+\wp}m\kappa+\eta^{\hbar}m\kappa-\eta^{\hbar}m\mu\\
&=2\eta^{\ell+\wp}m\gamma+2\eta^{\hbar}m\kappa\\
RHS: \quad & 2\left(\eta^{\ell+\wp}\mathbb{T}(\gamma)+\eta^{\hbar}\mathbb{T}(\kappa)\right)\\
&=2\eta^{\ell+\wp}\mathbb{T}(\gamma)+2\eta^{\hbar}\mathbb{T}(\kappa)\\
&=2\eta^{\ell+\wp}m\gamma+2\eta^{\hbar}m\kappa.
\end{aligned}
$$

Hence ω is a linear transformation.

Example 5. $\mathbb{A}=\mathbb{B}=E^1$. *For* $\gamma\in\mathbb{A}$, $\mathbb{T}(\gamma)=m\gamma+b$, *where* m *and* b *are the fixed real numbers and* $b\neq 0$.

$$
\begin{aligned}
&\omega\left(\eta^{\ell+\wp}\gamma+\eta^{\hbar}\kappa\right)+\omega\left(\eta^{\ell+\wp}\kappa+\eta^{\hbar}\mu\right)+\eta^{\ell+\wp}\omega(\gamma-\kappa)+\eta^{\hbar}\omega(\kappa-\mu)\\
&\qquad=2\left(\eta^{\ell+\wp}\omega(\gamma)+\eta^{\hbar}\omega(\kappa)\right)
\end{aligned}
$$

Solution: The solution is trivial. Hence we conclude that ω is not a linear transformation.

4. Conclusions

In this study, a novel additive functional Equation (5) has been introduced. The Hyers–Ulam stability in Banach spaces is investigated using the direct and the fixed-point approach in Section 2. In Section 3, the Hyers–Ulam stability in quasi-beta normed spaces is investigated by using the direct method and fixed-point approach. Additionally, the counter-example for non-stable cases is provided. One more contribution is the investigation of our functional equation in relation to a linear transformation. In the future, Hyers–Ulam stability can be determined in various normed spaces like Fuzzy normed spaces, random normed spaces, and non-Archimedean normed spaces in our additive functional Equation (5).

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