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Higher-Order Delay Differential Equation with Distributed Deviating Arguments: Improving Monotonic Properties of Kneser Solutions

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Abstract: This study aims to investigate the oscillatory behavior of the solutions of an even-order delay differential equation with distributed deviating arguments. We first study the monotonic properties of positive decreasing solutions or the so-called Kneser solutions. Then, by iterative deduction, we improve these properties, which enables us to apply them more than once. Finally, depending on the symmetry between the positive and negative solutions of the studied equation and by combining the new condition for the exclusion of Kneser solutions with some well-known results in the literature, we establish a new standard for the oscillation of the investigated equation.

Keywords: differential equations; even-order; distributed deviating arguments; oscillatory theory

MSC: 34C10; 34K11

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1. Introduction

Engineering, physics, economics, and biology are just a few of the fields where differential equations (DE) are widely used. The study of DEs depends on finding their solutions or studying the properties of their solutions. However, many properties of the solutions of a particular DE may be specified without finding them exactly. The study of these differential equations is divided into two parts, the first is the qualitative study of solutions and the second is the study of numerical methods to find an approximate solution [1].

The oscillation of solutions to delay DEs has been the focus of extensive investigations in recent years. This is mostly because delay DEs are recognized as crucial in applications. Delay differential equations are being used more and more frequently in new applications for the modeling of many processes in physics, biology, ecology, and physiology. Therefore, a lot of authors have focused on solving DEs or determining some of their key properties, see [2].

One of the most crucial components of applied mathematics is the construction of mathematical models, which are essentially meant to solve real-world problems. For analysis and forecasting in many fields of life sciences, such as population dynamics, epidemiology, immunology, physiology, neural networks, etc., mathematical modeling, including delay DEs, is frequently utilized. In these cases, it is suggested that the delay plays a crucial role in representing the time needed to finish some hidden processes that are known to cause a time lag, such as the stages of the life cycle, the interval between a cell becoming infected and the production of new viruses, the length of the infectious period, the immune period, and so on, see [3].

Studying the qualitative properties, such as oscillation, symmetry, stability, periodicity, and others, of the solutions of differential equation models contributes to understanding and analyzing the phenomena that these models describe. Over the past 25 years, many researchers have become interested in the oscillation theory of functional DEs. Numerous books and hundreds of research papers in every major mathematical magazine have been created as a consequence of this; they have examined the oscillations of higher order DEs and more deeply studied methods for generating oscillatory criteria for higher in the literature.

Based on the general Riccati substitution, Moaaz and Muhib [4] introduced more efficient oscillation criteria to test the oscillation of a fourth-order half-linear delay DE. Elabbasy et al. [5] extended the results in [4] to delay DEs with multiple delays. For odd-order equations, Moaaz et al. [6] established criteria for the non-existence of kneser solutions of delay DEs. For second-order equations, Gui and Xu [7] introduced Kamenev-type oscillation criteria for delay DEs with distributed deviating arguments. Meanwhile, Wang [8] deduced criteria of Philos type for such equations. Elabbasy et al. [9] and Zhao and Meng [10] extended and improved the results in [7,8].

In this work, we consider the even-order delay DE:

$$\left(\varphi(t)u^{(m-1)}(t)\right)' + \int_a^b \mathcal{K}(t,\mathfrak{h})u(\rho(t,\mathfrak{h}))\mathrm{d}\mathfrak{h} = 0, \quad t \geq t_0, \tag{1}$$

where $m \geq 4$ and is an even integer. Throughout this work, we assume that:

(A$_1$) $\varphi \in C([t_0,\infty))$, $\varphi > 0$, $\varphi' \geq 0$, and

$$\int_{t_0}^{\infty} \frac{1}{\varphi(\mathfrak{h})}\mathrm{d}\mathfrak{h} < \infty; \tag{2}$$

(A$_2$) $\mathcal{K} \in C([t_0,\infty) \times (a,b),\mathbb{R})$ and $\mathcal{K}(t,\mathfrak{h}) \geq 0$;

(A$_3$) $\rho \in C([t_0,\infty) \times (a,b),\mathbb{R})$, $\rho(t,\mathfrak{h}) < t$ for $\mathfrak{h} \in [a,b]$, ρ has nonnegative partial derivatives with respect to t and nondecreasing with respect to $\mathfrak{h}$ and $\lim_{t\to\infty}\rho(t,\mathfrak{h}) = \infty$ for $\mathfrak{h} \in [a,b]$.

By solving (1), we purpose a function $u \in C^{m-1}([t_u,\infty),\mathbb{R})$ for some $t_u \geq t_0$ such that $\varphi u^{(m-1)} \in C^1([t_u,\infty),\mathbb{R})$ and satisfies (1) on $[t_u,\infty)$. We consider only those solutions of (1) which satisfy the condition $\sup\{|x(t)| : t \geq t_0\} > 0$ for all $t \geq t_x$. If u is neither positive nor negative eventually, then u is called oscillatory, or it will be nonoscillatory.

Some of the important papers that helped improve the oscillation theory of even-order DEs are reviewed in the following:

Grace et al. [11] considered the fourth-order delay DE:

$$\left(\varphi_3 \cdot \left(\varphi_2 \cdot \left(\varphi_1 \cdot u'\right)'\right)'\right)' + q \cdot [u \circ \rho] = 0,$$

where $[f \circ g](t) := f(g(t))$. They presented criteria for the oscillation of all solutions of this equation. By using an iterative technique, Moaaz and Cesarano [12] focused on the oscillatory behavior of the DE:

$$\left(\varphi \cdot u^{(m-1)}\right)' + q \cdot [u \circ \rho] = 0.$$

Muhib et al. [13] established comparison theorems for the delay DE:

$$\left(\varphi \cdot \left(u^{(m-1)}\right)^{\alpha}\right)' + \int_a^b \mathcal{K}(t,\mathfrak{h})f(u(\rho(t,\mathfrak{h})))\mathrm{d}\mathfrak{h} = 0.$$

For neutral DEs, Zhang et al. [14] considered the even-order DE:

$$(u + p \cdot [u \circ \sigma])^{(m)} + q \cdot [f \circ u \circ \rho] = 0,$$

where $uf(u) > 0$ for all $u \neq 0$ and $f'(u) \geq 0$, and studied the qualitative behavior of the solutions of this equation. Moaaz et al. [15] studied the asymptotic behavior of the DE

$$\left(\varphi \cdot \left((u + p \cdot [u \circ \sigma])^{(m-1)}\right)^{\alpha}\right)' + f(t, [u \circ \rho]) = 0,$$

where $|f(t,u)| \geq \mathcal{K}(t)|u|^{\beta}$. Moaaz et al. [16] considered the neutral DEs

$$\left(\varphi \cdot (u^{\alpha} + p \cdot [u \circ \sigma])^{(m-1)}\right)' + \int_a^b \mathcal{K}(t, \mathfrak{h}) f(u(\rho(t, \mathfrak{h}))) \mathrm{d}\mathfrak{h} = 0.$$

By using the theory of comparison and the technique of Riccati transformation, they obtained two different conditions that ensured the oscillation. By employing the generalized Riccati transformation, Tunc and Bazighifan [17] studied the oscillation of the DE

$$\left(\varphi \cdot (z''')^{\alpha}\right)' + \int_c^d \mathcal{K}(t, \mathfrak{h}) u^{\alpha}(\rho(t, \mathfrak{h})) \mathrm{d}\mathfrak{h} = 0,$$

where

$$z := u + \int_a^b p(t, \sigma) u(\rho(t, \mathfrak{h})) \mathrm{d}\mathfrak{h}.$$

In this article, in the noncanonical case, we examine the asymptotic and monotonic features of Kneser solutions to the higher-order delay DE (1). The nature of the new attributes is iterative. The property of symmetry between the positive and negative solutions of the studied differential equations plays a key role in the study of oscillation. As the exclusion of positive solutions necessarily means the exclusion of negative solutions, we additionally employ the theory of comparison in conjunction with these new properties to derive criteria for the oscillation of the studied equation. A variety of similar findings presented in the literature are expanded upon and supplemented by our theorems.

The paper is divided into the following parts: The introduction section, which reviews the most important relevant results. Then, we begin the main results section by deriving some monotonic properties of the positive solutions of the studied equation. Furthermore, we improve these properties iteratively. Then, using the comparison technique, we present criteria that ensure the oscillation of all solutions of the studied equation. Finally, we present in the conclusion the most important results and recommendations.

2. Main Results

We begin this section by showing some notations that make it easier to present the main results.

Notation 1. *The set of all eventually positive solutions to (1) which satisfy the property*

$$u^{(s)}(t) u^{(s+1)}(t) < 0 \text{ for } s = 0, 1, 2, \ldots, m-2, \tag{3}$$

by Ω^*. Furthermore, we define the functions π_i by

$$\pi_0(t) := \int_t^{\infty} \varphi^{-1}(\mathfrak{h}) \mathrm{d}\mathfrak{h}$$

and

$$\pi_i(t) := \int_t^{\infty} \pi_{i-1}(\mathfrak{h}) \mathrm{d}\mathfrak{h}, \ i = 1, 2, \ldots, m-2.$$

Lemma 1. *Assume that $u \in \Omega^*$. If*

$$\int_{t_0}^{\infty} \pi_{m-3}(\varsigma)\left(\int_{t_2}^{\varsigma}\left(\int_a^b \mathcal{K}(\varrho,\mathfrak{h})\mathrm{d}\mathfrak{h}\right)\mathrm{d}\varrho\right)\mathrm{d}\varsigma = \infty, \tag{4}$$

and there exists an $\ell_0 \in (0,1)$ such that

$$\frac{\pi_{m-2}^2(t)}{\pi_{m-3}(t)}\int_a^b \mathcal{K}(t,\mathfrak{h})\mathrm{d}\mathfrak{h} \geq \ell_0, \tag{5}$$

then,

(C$_1$) $(-1)^{i+1}u^{(m-i-2)}(t) \leq \varphi(t)u^{(m-1)}(t)\pi_i(t)$ for $i = 0,1,2,\ldots,m-2$;
(C$_2$) $\lim_{t\to\infty} u(t) = 0$;
(C$_3$) $(u(t)/\pi_{m-2}(t))' > 0$;
(C$_4$) $\left(u(t)/\pi_{m-2}^{\ell_0}(t)\right)' < 0$;
(C$_5$) $\lim_{t\to\infty} u(t)/\pi_{m-2}^{\ell_0}(t) = 0$.

Proof. Assume that $u \in \Omega^*$. Thus, for some $t_2 \geq t_1$, we have $u(\rho(t)) > 0$ for all $t \geq t_2$. Hence, from (1), we obtain

$$\left(\varphi(t)u^{(m-1)}(t)\right)'(t) = -\int_a^b \mathcal{K}(t,\mathfrak{h})u(\rho(t,\mathfrak{h}))\mathrm{d}\mathfrak{h} \leq 0.$$

(C$_1$) Using (3), we have that

$$\varphi(t)u^{(m-1)}(t)\pi_0(t) \geq \int_t^{\infty} \frac{\varphi(\mathfrak{h})u^{(m-1)}(\mathfrak{h})}{\varphi(\mathfrak{h})}\mathrm{d}\mathfrak{h} \geq -u^{(m-2)}(t),$$

or equivalently

$$u^{(m-2)} \geq -\varphi u^{(m-1)}\pi_{0.} \tag{6}$$

Integrating (6) $m-2$ times over $[t,\infty)$, we find

$$(-1)^{i+1}u^{(m-i-2)} \leq \varphi u^{(m-1)}\pi_{i.}$$

(C$_2$) Since $u(t) > 0$ and $u'(t) < 0$, we obtain that $\lim_{t\to\infty} u(t) = c \geq 0$.

Suppose that $c > 0$. Therefore, for some $t_2 \geq t_1$, we have $u(t) \geq c$ for $t \geq t_2$. Hence, (1) reduces to

$$\left(\varphi(t)u^{(m-1)}(t)\right)' + c\int_a^b \mathcal{K}(t,\mathfrak{h})\mathrm{d}\mathfrak{h} \leq 0. \tag{7}$$

Integrating (7) two times over $[t_2,t)$, we arrive at

$$\varphi(t)u^{(m-1)}(t) - \varphi(t_2)u^{(m-1)}(t_2) \leq -c\int_{t_2}^{t}\left(\int_a^b \mathcal{K}(\varrho,\mathfrak{h})\mathrm{d}\mathfrak{h}\right)\mathrm{d}\varrho.$$

From (3), we have $u^{(m-1)}(t) < 0$ for $t \geq t_1$. Then, $\varphi(t_2)u^{(m-1)}(t_2) < 0$, and so

$$\varphi(t)u^{(m-1)}(t) \leq -c\int_{t_2}^{t}\left(\int_a^b \mathcal{K}(\varrho,\mathfrak{h})\mathrm{d}\mathfrak{h}\right)\mathrm{d}\varrho. \tag{8}$$

From (C$_1$) with $i = m-3$, we get that

$$\frac{u'}{\pi_{m-3}} \leq \varphi u^{(m-1)},$$

which with (8) yields

$$u'(t) \leq -c\pi_{m-3}(t) \int_{t_2}^{t} \left(\int_a^b \mathcal{K}(\varrho, \mathfrak{h}) \mathrm{d}\mathfrak{h} \right) \mathrm{d}\varrho.$$

Then,

$$u(t) \leq u(t_2) - c \int_{t_2}^{t} \pi_{m-3}(\varsigma) \left(\int_{t_2}^{\varsigma} \left(\int_a^b \mathcal{K}(\varrho, \mathfrak{h}) \mathrm{d}\mathfrak{h} \right) \mathrm{d}\varrho \right) \mathrm{d}\varsigma,$$

which with (4) gives $u \to -\infty$ as $t \to \infty$, a contradiction. Then, $u \to 0$ as $t \to \infty$.

(C_3) Using (C_1) at $i = 0$, we get

$$\left(\frac{u^{m-2}}{\pi_0} \right)' = \frac{\left(\pi_0 u^{(m-1)} + \varphi^{-1} u^{(m-2)} \right)}{\pi_0^2} \geq 0,$$

which leads to

$$-u^{(m-3)}(t) \geq \int_t^{\infty} \pi_0(\varrho) \frac{u^{(m-2)}(\varrho)}{\pi_0(\varrho)} \mathrm{d}\varrho \geq \frac{u^{(m-2)}(t)}{\pi_0(t)} \pi_1(t).$$

This implies

$$\left(\frac{u^{(m-3)}}{\pi_1} \right)' = \frac{1}{\pi_1^2} \left(\pi_1 u^{(m-2)} + \pi_0 u^{(m-3)} \right) \leq 0.$$

By repeating with a similar approach, we obtain $(u/\pi_{m-2})' > 0$.

(C_4) Since $\rho(t, \mathfrak{h})$ is nondecreasing with respect to $\mathfrak{h}$, we get $\rho(t, \mathfrak{h}) \geq \rho(t, a)$ for $\mathfrak{h} \in (a, b)$. Integrating (1) over $[t_2, t)$ and using (5), we find

$$\begin{aligned} \varphi(t) u^{(m-1)}(t) &= \varphi(t_2) u^{(m-1)}(t_2) - \int_{t_2}^{t} \left(\int_a^b \mathcal{K}(\varrho, \mathfrak{h}) u(\rho(\varrho, \mathfrak{h})) \mathrm{d}\mathfrak{h} \right) \mathrm{d}\varrho \\ &\leq \varphi(t_2) u^{(m-1)}(t_2) - \int_{t_2}^{t} u(\rho(\varrho, b)) \left(\int_a^b \mathcal{K}(\varrho, \mathfrak{h}) \mathrm{d}\mathfrak{h} \right) \mathrm{d}\varrho, \end{aligned}$$

hence,

$$\begin{aligned} \varphi(t) u^{(m-1)}(t) &\leq \varphi(t_2) u^{(m-1)}(t_2) - u(t) \int_{t_2}^{t} \left(\int_a^b \mathcal{K}(\varrho, \mathfrak{h}) \mathrm{d}\mathfrak{h} \right) \mathrm{d}\varrho \\ &\leq \varphi(t_2) u^{(m-1)}(t_2) + \ell_0 \frac{u(t)}{\pi_{m-2}(t_2)} - \ell_0 \frac{u(t)}{\pi_{m-2}(t)}, \end{aligned}$$

which with (C_2) gives

$$\varphi u^{(m-1)} \leq -\ell_0 \frac{u}{\pi_{m-2}}. \tag{9}$$

Thus, from (C_1) at $i = m-3$, we obtain

$$\frac{u'}{\pi_{m-3}} \leq -\ell_0 \frac{u}{\pi_{m-2}}.$$

Consequently,

$$\left(\frac{u}{\pi_{m-2}^{\ell_0}} \right)' = \frac{1}{\pi_{m-2}^{\ell_0+1}} \left(\pi_{m-2} u' + \ell_0 \pi_{m-3} u \right) \leq 0.$$

(C_5) Now, since $u/\pi_{m-2}^{\ell_0}$ is positive and decreasing, we get that $\lim_{t\to\infty} u(t)/\pi_{m-2}^{\ell_0}(t) = l_0 \geq 0$.

Suppose that $l_0 > 0$. Thus, for some $t_2 \geq t_1$, we obtain that $u(t)/\pi_{m-2}^{\ell_0}(t) \geq l_0$ for $t \geq t_2$. Now, let

$$F := \frac{u + \varphi \cdot u^{(m-1)} \cdot \pi_{m-2}}{\pi_{m-2}^{\ell_0}.} \tag{10}$$

Therefore, $F(t) > 0$ for $t \geq t_2$. From (5) and (10), we obtain

$$\begin{aligned} F' &= \frac{1}{\pi_{m-2}^{2\ell_0}}\left[\pi_{m-2}^{\ell_0}\left(u' - \varphi u^{(m-1)}\pi_{m-3} + \left(\varphi u^{(m-1)}\right)'\pi_{m-2}\right)\right. \\ &\quad \left. + \ell_0 \pi_{m-2}^{\ell_0-1}\pi_{m-3}\left(u + \varphi u^{(m-1)}\pi_{m-2}\right)\right] \\ &\leq \frac{1}{\pi_{m-2}^{\ell_0+1}}\left[-\pi_{m-2}^2\left(\int_a^b \mathcal{K}(t,\mathfrak{h})u(\rho(t,\mathfrak{h}))\mathrm{d}\mathfrak{h}\right) + \ell_0\pi_{m-3}\left(u + \varphi u^{(m-1)}\pi_{m-2}\right)\right], \end{aligned}$$

hence,

$$\begin{aligned} F' &\leq \frac{1}{\pi_{m-2}^{\ell_0+1}}\left[-\ell_0\pi_{m-3}u + \ell_0\pi_{m-3}u + \ell_0\pi_{m-3}\varphi u^{(m-1)}\pi_{m-2}\right] \\ &\leq \frac{\ell_0}{\pi_{m-2}^{\ell_0}}\pi_{m-3}\varphi u^{(m-1)}. \end{aligned} \tag{11}$$

Using the fact that $u(t)/\pi_{m-2}^{\ell_0}(t) \geq l_0$ with (9), we obtain

$$\varphi u^{(m-1)} \leq -\ell_0 \frac{u}{\pi_{m-2}} \leq -\ell_0 l_0 \pi_{m-2}^{\ell_0-1}. \tag{12}$$

Combining (11) and (12) yields

$$F' \leq -\ell_0^2 l_0 \frac{\pi_{m-3}}{\pi_{m-2}} < 0.$$

Integrating the above inequality over $[t_2, t)$, we find

$$-F(t_2) \leq -\ell_0^2 l_0 \log \frac{\pi_{m-2}(t_2)}{\pi_{m-2}(t)} \to \infty \text{ as } t \to \infty,$$

a contradiction, and thus, $l_0 = 0$. The proof is complete. □

Lemma 2. *Assume that $u \in \Omega^*$. If (5) holds for some $\ell_0 \in (0,1)$, then (4) holds.*

Proof. Assume that $u \in \Omega^*$. Using (5), we get

$$\begin{aligned} \int_{t_0}^{\varsigma}\left(\int_a^b \mathcal{K}(\varrho,\mathfrak{h})\mathrm{d}\mathfrak{h}\right)\mathrm{d}\varrho &\geq \int_{t_0}^{\varsigma} \ell_0 \frac{\pi_{m-3}(\varrho)}{\pi_{m-2}^2(\varrho)}\mathrm{d}\varrho \\ &= \ell_0\left(\frac{1}{\pi_{m-2}(\varsigma)} - \frac{1}{\pi_{m-2}(t_0)}\right). \end{aligned}$$

Using the fact that $\pi_{n-2} \to 0$ as $t \to \infty$, we obtain eventually that

$$\frac{1}{\pi_{m-2}(t)} - \frac{1}{\pi_{m-2}(t_0)} \geq \frac{\mu}{\pi_{m-2}(t)},$$

for $\mu \in (0,1)$. Therefore,

$$\pi_{m-3}(\varsigma)\int_{t_0}^{\varsigma}\left(\int_a^b \mathcal{K}(\varrho,\mathfrak{h})\mathrm{d}\mathfrak{h}\right)\mathrm{d}\varrho \geq \ell_0\mu\frac{\pi_{m-3}(\varsigma)}{\pi_{m-2}(\varsigma).}$$

Thus,

$$\int_{t_0}^{t}\pi_{m-3}(\varsigma)\left(\int_{t_2}^{\varsigma}\left(\int_a^b \mathcal{K}(\varrho,\mathfrak{h})\mathrm{d}\mathfrak{h}\right)\mathrm{d}\varrho\right)\mathrm{d}\varsigma \geq \ell_0\mu\ln\frac{\pi_{m-3}(t_0)}{\pi_{m-2}(t)} \to \infty \text{ as } t\to\infty.$$

The proof is complete. □

Lemma 3. *Assume that $u \in \Omega^*$, (5) holds for some $\ell_0 \in (0,1)$. If there is an $n \in \mathbb{N}$ such that $\ell_i \leq \ell_{i+1} < 1$ for all $i = 0,1,2,\ldots,n-1$, then*

$$\begin{array}{ll}(\mathrm{C}_{1,n}) & \left(u(t)/\pi_{m-2}^{\ell_n}(t)\right)' < 0; \\ (\mathrm{C}_{2,n}) & \lim_{t\to\infty} u(t)/\pi_{m-2}^{\ell_n}(t) = 0,\end{array}$$

where

$$\ell_j = \ell_0\frac{\lambda^{\ell_{j-1}}}{1-\ell_{j-1}}, \quad j = 1,2,\ldots,n,$$

and

$$\frac{\pi_{m-2}(\rho(t,b))}{\pi_{m-2}(t)} \geq \lambda, \tag{13}$$

for some $\lambda \geq 1$.

Proof. Assume that $u \in \Omega^*$. Then, from Lemma 1, we have that $(\mathrm{C}_1) - (\mathrm{C}_5)$ hold. Using induction, we have from Lemma 1 that $(\mathrm{C}_{1,0})$ and $(\mathrm{C}_{2,0})$ hold. Now, we assume that $(\mathrm{C}_{1,s-1})$ and $(\mathrm{C}_{2,s-1})$ hold. Over $[t_1,t)$, integration (1) yields

$$\begin{aligned}\varphi(t)u^{(m-1)}(t) &= \varphi(t_2)u^{(m-1)}(t_2) - \int_{t_2}^{t}\left(\int_a^b \mathcal{K}(\varrho,\mathfrak{h})u(\rho(\varrho,\mathfrak{h}))\mathrm{d}\mathfrak{h}\right)\mathrm{d}\varrho \\ &\leq \varphi(t_2)u^{(m-1)}(t_2) - \int_{t_2}^{t}u(\rho(\varrho,b))\left(\int_a^b \mathcal{K}(\varrho,\mathfrak{h})\mathrm{d}\mathfrak{h}\right)\mathrm{d}\varrho.\end{aligned} \tag{14}$$

Using $(\mathrm{C}_{1,s-1})$, we obtain that

$$u(\rho(t,\mathfrak{h})) \geq \pi_{m-2}^{\ell_{s-1}}(\rho(t,\mathfrak{h}))\frac{u(t)}{\pi_{m-2}^{\ell_{s-1}}(t)},$$

then (14) becomes

$$\varphi(t)u^{(m-1)}(t) \leq \varphi(t_2)u^{(m-1)}(t_2) - \int_{t_2}^{t}\pi_{m-2}^{\ell_{s-1}}(\rho(\varrho,b))\frac{u(\varrho)}{\pi_{m-2}^{\ell_{s-1}}(\varrho)}\left(\int_a^b \mathcal{K}(\varrho,\mathfrak{h})\mathrm{d}\mathfrak{h}\right)\mathrm{d}\varrho.$$

Since $\left(u(t)/\pi_{m-2}^{\ell_{s-1}}(t)\right)' \leq 0$, we arrive at

$$\begin{aligned}\varphi(t)u^{(m-1)}(t) \leq\ & \varphi(t_2)u^{(m-1)}(t_2) \\ & -\frac{u(t)}{\pi_{m-2}^{\ell s-1}(t)}\int_{t_2}^{t}\pi_{m-2}^{\ell_{s-1}}(\varrho)\frac{\pi_{m-2}^{\ell_{s-1}}(\rho(\varrho,b))}{\pi_{m-2}^{\ell_{s-1}}(\varrho)}\left(\int_a^b \mathcal{K}(\varrho,\mathfrak{h})\mathrm{d}\mathfrak{h}\right)\mathrm{d}\varrho.\end{aligned}$$

Hence, from (5) and (13), we obtain

$$\begin{aligned}\varphi(t)u^{(m-1)}(t) &\leq \varphi(t_2)u^{(m-1)}(t_2) - \ell_0\lambda^{\ell_{s-1}}\frac{u(t)}{\pi_{m-2}^{\ell_{s-1}}(t)}\int_{t_2}^{t}\frac{\pi_{m-3}(\rho(\varrho))}{\pi_{m-2}^{2-\ell_{s-1}}(\varrho)}\mathrm{d}\varrho \\ &= \varphi(t_2)u^{(m-1)}(t_2) - \ell_0\frac{\lambda^{\ell_{s-1}}}{1-\ell_{s-1}}\frac{u(t)}{\pi_{m-2}^{\ell_{s-1}}(t)}\left(\frac{1}{\pi_{m-2}^{1-\ell_{s-1}}(t)} - \frac{1}{\pi_{m-2}^{1-\ell_{s-1}}(t_2)}\right),\end{aligned}$$

hence,

$$\varphi(t)u^{(m-1)}(t) \leq \varphi(t_2)u^{(m-1)}(t_2) - \ell_s\frac{u(t)}{\pi_{m-2}^{\ell_{s-1}}(t)}\frac{1}{\pi_{m-2}^{1-\ell_{s-1}}(t_2)} - \ell_s\frac{u(t)}{\pi_{m-2}(t).}$$

Using the property $\lim_{t\to\infty} u(t)/\pi_{m-2}^{\ell_{s-1}}(t) = 0$, we obtain

$$\varphi u^{(m-1)} \leq -\ell_s\frac{u}{\pi_{m-2}.} \tag{15}$$

Thus, form (C_1) at $k = m-3$, we obtain

$$\frac{u'}{\pi_{m-3}} \leq -\ell_s\frac{u}{\pi_{m-2}.}$$

Consequently,

$$\left(\frac{u}{\pi_{m-2}^{\ell_s}}\right)' = \frac{1}{\pi_{m-2}^{\ell_{s+1}}}(\pi_{m-2}u' + \ell_s\pi_{m-3}u) \leq 0.$$

The remainder of the proof is exactly the same as the proof of (C_5) in Lemma 1. Therefore, the proof is complete. □

Theorem 1. *Assume that $u \in \Omega^*$ and, for some $\ell_0 \in (0,1)$, (5) holds. If there is an $n \in \mathbb{N}$ such that $\ell_i \leq \ell_{i+1} < 1$ for all $i = 0,1,2,\ldots,n-1$, then the delay DE*

$$G'(t) + \frac{1}{1-\ell_n}\left(\int_a^b \mathcal{K}(t,\mathfrak{h})\mathrm{d}\mathfrak{h}\right)\pi_{m-2}(t)G(\rho(t,b)) = 0 \tag{16}$$

has a positive solution, where ℓ_j and λ are defined as in Lemma 3.

Proof. Assume that $u \in \Omega^*$. Then, from Lemma 3, we have that $(\mathrm{C}_{1,n})$ and $(\mathrm{C}_{2,n})$ hold. Next, let

$$G = \varphi u^{(m-1)}\pi_{m-2} + u. \tag{17}$$

Thus, from (C_1) at $i = m-2$, $G(t) > 0$ for $t \geq t_2$ and

$$G' = \left(\varphi u^{(m-1)}\right)'\pi_{m-2} - \varphi u^{(m-1)}\pi_{m-3} + u'.$$

Using (C_1) at $i = m-3$, we find that

$$G' \leq \left(\varphi u^{(m-1)}\right)'\pi_{m-2} \leq -u(\rho(t,b))\pi_{m-2}\int_a^b \mathcal{K}(t,\mathfrak{h})\mathrm{d}\mathfrak{h}. \tag{18}$$

From the proof of Lemma 3, we obtain that (15) holds. Combining (17) and (15), we obtain

$$G \leq (1-\ell_n)u.$$

Thus, (18) becomes

$$G'(t) + \frac{1}{1-\ell_n}\left(\int_a^b \mathcal{K}(t,\mathfrak{h})\mathrm{d}\mathfrak{h}\right)\pi_{m-2}(t)G(\rho(t,b)) \leq 0. \tag{19}$$

Hence, G is a positive solution of the differential inequality (19). Using [18] (Theorem 1), Equation (16) also has a positive solution and this completes the proof. □

Theorem 2. *Assume that, for some $\ell_0 \in (0,1)$, (5) holds. Assume also that there is an $n \in \mathbb{N}$ such that $\ell_i \leq \ell_{i+1} < 1$ for all $i = 0,1,2,\ldots,n-1$ and the delay DEs (16)*

$$\varpi'(t) + \frac{\epsilon_1 \rho^{m-1}(t,b)}{(m-1)!(\varphi(\rho(t,b)))}\left(\int_a^b \mathcal{K}(t,\mathfrak{h})\mathrm{d}\mathfrak{h}\right)\varpi(\rho(t,b)) = 0 \tag{20}$$

and

$$\varpi'(t) + \frac{\epsilon_2 \rho^{m-1}(t)}{(m-2)!\varphi(t)}\left(\int_{t_0}^t \left(\int_a^b \mathcal{K}(\varrho,\mathfrak{h})\mathrm{d}\mathfrak{h}\right)\rho^{m-2}(\varrho,b)\mathrm{d}\varrho\right)\varpi(\rho(t,b)) = 0, \tag{21}$$

are oscillatory for some $\epsilon_1, \epsilon_2, \ell_n \in (0,1)$, where ℓ_j and λ are defined as in Theorem 1. Then, every solution of (1) is oscillatory.

Proof. Assume the contrary that u is eventually a positive solution. Then, from [19] (Lemma 2.2.1), we have the following three cases, eventually

(1) u', $u^{(m-1)}$ are positive and $u^{(m)}$ is negative;
(2) u', $u^{(m-2)}$ are positive and $u^{(m-1)}$ is negative;
(3) $(-1)^i u^{(i)}$are positive, for $i = 1,2,\ldots,m-1$.

Proceeding with an approach quite similar to that used in [20] (Theorem 3), we can guarantee that cases (1) and (2) will not be fulfilled based on the assumption that Equations (20) and (21) oscillate.

Then, we have that (3) holds. Using Theorem 1, we obtain that (16) has a positive solution, a contradiction. Therefore, the proof is complete. □

Corollary 1. *Assume that, for some $\ell_0 \in (0,1)$, (5) holds. Assume also that there is an $n \in \mathbb{N}$ such that $\ell_i \leq \ell_{i+1} < 1$ for all $i = 0,1,2,\ldots,n-1$,*

$$\liminf_{t\to\infty} \int_{\rho(t,b)}^t \pi_{m-2}(\varrho)\left(\int_a^b \mathcal{K}(\varrho,\mathfrak{h})\mathrm{d}\mathfrak{h}\right)\mathrm{d}\varrho > \frac{1-\ell_n}{\mathrm{e}}, \tag{22}$$

$$\liminf_{t\to\infty} \int_{\rho(t,b)}^t \frac{1}{\varphi(\rho(\varrho,b))}\rho^{m-1}(\varrho,b)\left(\int_a^b \mathcal{K}(\varrho,\mathfrak{h})\mathrm{d}\mathfrak{h}\right)\mathrm{d}\varrho > \frac{(m-1)!}{\mathrm{e}}, \tag{23}$$

and

$$\liminf_{t\to\infty} \int_{\rho(t,b)}^t \frac{1}{\varphi(\varrho)}\left(\int_{t_0}^{\varrho} \rho^{m-2}(\eta,b)\left(\int_a^b \mathcal{K}(\eta,\mathfrak{h})\mathrm{d}\mathfrak{h}\right)\mathrm{d}\eta\right)\mathrm{d}\varrho > \frac{(m-2)!}{\mathrm{e}}, \tag{24}$$

for some $\epsilon, \ell_n \in (0,1)$. Then every solution of (1) is oscillatory.

Proof. In view of [21] (Corollary 2.1), Conditions (22)–(24) imply the oscillation of the solution of (16), (20), and (21), respectively. Therefore, from Theorem 2, every solution of (1) is oscillatory. □

Example 1. *We consider a special case of (1) of the form*

$$\left(\mathrm{e}^t\left(u^{(m-1)}(t)\right)\right)' + \int_{0.4}^1 k_0 \mathfrak{h} \mathrm{e}^t u(t-\mathfrak{h})\mathrm{d}\mathfrak{h} = 0, \tag{25}$$

where $t \geq 1$, $\varphi(t) = \mathrm{e}^t$, $\mathcal{K}(t, \mathfrak{h}) = k_0 \mathfrak{h} \mathrm{e}^t$, $\rho(t, \mathfrak{h}) = t - \mathfrak{h}$, $k_0 \in (0, 1)$, *and* $\mathfrak{h} \in (0.4, 1)$. *Note that* $\pi_i(t) = \mathrm{e}^{-t}$, $i = 0, 1, \ldots, m-2$. *By selecting* $\ell_0 = 0.42k_0$, *we have that (5) holds. Furthermore, it is simple to confirm that (23) and (24) are satisfied. It follows from Corollary 1 that all solutions of (25) are oscillatory if (22) holds; that is,*

$$0.42k_0 > \frac{1 - \ell_n}{\mathrm{e}}.$$

Example 2. *We consider a special case of (1) of the form*

$$\left(t^4 \left(u^{(m-1)}(t)\right)\right)' + \int_0^1 k_0 u\left(\frac{\mathfrak{h}}{2} t\right) \mathrm{d}\mathfrak{h} = 0, \tag{26}$$

where $t \geq 1$ *and* $k_0 > 0$. *It is easy to see that if we choose* $\ell_0 = \frac{1}{6} k_0$, *then condition (5) is satisfied. Moreover, we note that conditions (23) and (24) hold if*

$$k_0 > \frac{24}{\mathrm{e} \ln 2}.$$

Condition (22) reduces to

$$k_0 > \frac{(6 - k_0)}{\mathrm{e} \ln 2},$$

or

$$k_0 > \frac{6}{\mathrm{e} \ln 2 + 1}.$$

Using Corollary 1, we obtain that all solution of (26) are oscillatory if $k_0 > 12.738$.

3. Conclusions

Delay DEs have many applications in different sciences. This is not the only motive for studying such equations; studying these equations is full of interesting analytical issues. Among these interesting points is the study of the monotonic properties of the so-called Kneser solutions, whose signs differ from the signs of their first derivative.

In this article, we deduced some monotonic properties of a separation from positive solutions of noncanonical delay DEs of higher order. Then, we refined these properties by giving them an iterative feature. Moreover, we obtained a standard guarantee that there are no Kneser solutions. Finally, by combining our results with well-known results in the literature, we obtained a new oscillation criterion for the studied equation. Our results differ from previous results that focused on the study of equations with distributed deviating arguments in that they work to improve the monotonic properties, as well as the fact that our criteria are iterative, meaning they can be applied multiple times.

We propose, as a future research point, to obtain an oscillation criterion using the Riccati substitution technique for the studied equation by using the improved monotonic properties. Moreover, it would be interesting to extend the development of oscillatory theory of integer order differential equations to fractional order differential equations, see [22–25].

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