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Characterizations of Pointwise Hemi-Slant Warped Product Submanifolds in $\mathcal{LCK}$ Manifolds

Fatimah Alghamdi

Department of Mathematics and Statistics, College of Science, University of Jeddah, Jeddah 21589, Saudi Arabia; fmalghamdi@uj.edu.sa

Abstract: In this paper, we investigate the pointwise hemi-slant submanifolds of a locally conformal Kähler manifold and their warped products. Moreover, we derive the necessary and sufficient conditions for integrability and totally geodesic foliation. We establish characterization theorems for pointwise hemi-slant submanifolds. Several fundamental results that extend the CR submanifold warped product in Kähler manifolds are proven in this study. We also provide some non-trivial examples and applications.

Keywords: warped products; Kähler manifold; pointwise hemi-slant warped products; $\mathcal{LCK}$ manifolds; locally conformal Kähler manifold

MSC: 53C15; 53C40; 53C42; 53B25

Citation: Alghamdi, F. Characterizations of Pointwise Hemi-Slant Warped Product Submanifolds in $\mathcal{LCK}$ Manifolds. *Symmetry* **2024**, *16*, 281. https://doi.org/10.3390/sym16030281

Academic Editors: Yanlin Li and Tiehong Zhao

Received: 21 January 2024
Revised: 17 February 2024
Accepted: 20 February 2024
Published: 29 February 2024

1. Introduction

Slant submanifold geometry shows a growing development in differential geometry to study submanifolds that have particular geometric characteristics. The notion of slant submanifolds of an almost Hermitian manifold was introduced by Chen [1,2] as an extension of both totally real submanifolds and complex submanifolds. Then, many geometers have discussed the notion of these submanifolds in various ambient manifolds. As an extension of slant submanifolds, N. Papaghiuc [3] introduced the notion of semi-slant submanifolds of an almost Hermitian manifold, which includes the class of proper CR submanifolds and slant submanifolds (see also [4–8]).

Furthermore, as a generalization of slant submanifolds of an almost Hermitian manifold, F. Etayo [9] proposed the concept of pointwise slant submanifolds of almost Hermitian manifolds under the name of quasi-slant submanifolds. Later, Chen and Garay [10] studied pointwise slant submanifolds of almost Hermitian manifolds. They obtained many fundamental results of these submanifolds.

On the other hand, in the late 19th century, the notion of warped product manifolds was introduced by Bishop and O'Neill [11]. The concept of warped products stands out as an important extension of Riemannian products. Furthermore, warped products assume significant significance in differential geometry and physics, particularly within general relativity. Also, several fundamental solutions to the Einstein field equations can be characterized as warped products [12]. Recently, Chen [13] initiated the study of warped product CR submanifolds of Kähler manifolds. Since then, several researchers have been motivated to investigate the geometry of warped product submanifolds following Chen's work in this field (see, e.g., [14–17]). Sahin proved [18] that there exist no proper warped product semi-slant submanifolds of Kähler manifolds. Then, he introduced the notion of warped product hemi-slant submanifolds of Kähler manifolds [19]. He proved that the warped products of the form $\mathcal{N}^{\perp} \times_f \mathcal{N}^{\theta}$ in a Kähler manifold $\tilde{\mathcal{M}}$ do not exist and then he introduced hemi-slant warped products of the form $\mathcal{N}^{\theta} \times_f \mathcal{N}^{\perp}$, where $\mathcal{N}^{\perp}$ and $\mathcal{N}^{\theta}$ are totally real and proper slant submanifolds of $\tilde{\mathcal{M}}$. He provided many examples and proved a characterization theorem. Later, he investigated warped product pointwise slant submanifolds of Kähler manifolds [20], (see also [21–23]).

Further, Bonanzinga and Matsumoto [24] introduced the warped product CR-submanifold in $\mathcal{LCK}$ manifolds of the form $\mathcal{N}^{\perp} \times_f \mathcal{N}^{\mathcal{T}}$, where $\mathcal{N}^{\mathcal{T}}$ and $\mathcal{N}^{\perp}$ are holomorphic and totally real submanifolds, respectively (see also [25–30]).

Motivated by the above studies, we investigate pointwise hemi-slant warped products in a more general setting of almost Hermitian manifolds, namely $\mathcal{LCK}$ manifolds. The notion of pointwise slant submanifolds in $\mathcal{LCK}$ manifolds extends the several results regarding the Kähler manifold in a very natural way.

The structure of the paper is as follows: in Section 2, we provide the fundamental background required for this paper. In Section 3, we define pointwise hemi-slant submanifolds of $\mathcal{LCK}$ manifolds. Then, we investigate the geometry of the leaves of distributions and prove some preparatory results in $\mathcal{LCK}$ manifolds. Section 4 proves characterization theorems, while Section 5 shows various applications. We conclude with several non-trivial examples of pointwise hemi-slant warped products.

2. Preliminaries

Let $(\tilde{\mathcal{M}}, J)$ be an almost complex manifold, $dim\tilde{\mathcal{M}} \geq 2$, and g Riemannian metric consistent with the almost complex structure J such that

$$g(JY_1, JY_2) = g(Y_1, Y_2), \tag{1}$$

for all $Y_1, Y_2 \in \Gamma(T\tilde{\mathcal{M}})$, then g is called a *Hermitian metric* on $\tilde{\mathcal{M}}$. An almost complex manifold with a Hermitian metric $(\tilde{\mathcal{M}}, J, g)$ is called an *almost Hermitian manifold*. The vanishing of the Nijenhuis tensor field $[J, J] = 0$ on almost Hermitian manifolds leads to a special class called Hermitian manifolds.

Futhermore, the fundamental 2-form Ω on $\tilde{\mathcal{M}}$ defined as $\Omega(Y_1, Y_2) = g(Y_1, JY_2)$ for all $Y_1, Y_2 \in \Gamma(T\tilde{\mathcal{M}})$. This fundamental 2-form Ω is considered a closed form if $d\Omega = 0$ and an exact form if there exists a 1-form ω such that $d\omega = \Omega$.

Moreover, if the fundamental 2-form is closed on almost Hermitian manifold $\tilde{\mathcal{M}}$, then Hermitian metric g on $\tilde{\mathcal{M}}$ is called *Kähler metric*. Further, a complex manifold endowed with a Kähler metric is said to be *Kähler manifold*.

The complex manifold $(\tilde{\mathcal{M}}, J)$ is called a *locally conformally Kähler manifold* ($\mathcal{LCK}$ manifold) if it has a Hermitian metric g that is locally conformal to a Kähler metric.

Theorem 1 ([31]). *The Hermitian manifold is called an $\mathcal{LCK}$ manifold if and only if there is a closed 1-form α globally defined on $\tilde{\mathcal{M}}$ such that $d\Omega = \alpha \wedge \Omega$.*

In Theorem 1, Ω is the 2-form associated with (J, g) and α is closed 1-form called the *Lee form* of the $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$ such that the *Lee vector field* $\alpha^{\#}$ dual to α, (i.e., $\Omega(Y_1, Y_2) = g(Y_1 JY_2)$, $g(Y_1, \alpha^{\#}) = \alpha(Y_1)$ for $Y_1, Y_2 \in \mathcal{T}(\tilde{\mathcal{M}})$. If the 1-form α of the $\mathcal{LCK}$-manifold is exact, then an $\mathcal{LCK}$ manifold is called a *globally conformal Kähler manifold* ($\mathcal{GCK}$ manifold).

Let $\tilde{\nabla}$ be the Levi-Civita connection on an $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$ we have for any Y_1, Y_2 on $T\tilde{\mathcal{M}}$

$$(\tilde{\nabla}_{Y1} J)Y_2 = -g(\beta^{\#}, Y_2)Y_1 - g(\alpha^{\#}, Y_2)JY_1 + g(JY_1, Y_2)\alpha^{\#} + g(Y_1, Y_2)\beta^{\#}. \tag{2}$$

where β is the 1-form provided by $\beta(Y_1) = -\alpha(JY_1)$, $\beta^{\#}$ is the dual vector field of β, and $\alpha^{\#}$ is the Lee vector field [31,32].

Let $\mathcal{M}$ be a Riemannian manifold of dimension n isometrically immersed in an $\mathcal{LCK}$ manifold $(\tilde{\mathcal{M}}, J, g, \alpha)$ of dimension m, where g denotes the induced metric tensor on $\mathcal{M}$ and $n \leq m$. Then, for any $Y_1, Y_2 \in \Gamma(\mathcal{T}\mathcal{M})$ and $Z_1 \in \Gamma(\mathcal{T}^{\perp}\mathcal{M})$, we have

$$\tilde{\nabla}_{Y1} Y_2 = \nabla_{Y1} Y_2 + \mathfrak{h}(Y_1, Y_2), \tag{3}$$

$$\tilde{\nabla}_{Y1} Z_1 = -\mathfrak{A}_{Z1} Y_1 + \nabla^{\perp}_{Y1} Z_1, \tag{4}$$

where ∇ is the covariant differentiation concerning the induced metric on $\mathcal{M}$, $\nabla^{\perp}$ is the normal connection, $\mathfrak{h}$ is the second fundamental form, and $\mathfrak{A}_{Z1}$ is the shape operator. The shape operator and second fundamental form are related by

$$g(\mathfrak{h}(Y_1, Y_2), Z_1) = g(\mathfrak{A}_{Z1} Y_1, Y_2). \tag{5}$$

For a vector Y_1 tangent to $\mathcal{M}$ and a vector Z_1 normal to $\mathcal{M}$, we write

$$JY_1 = \mathcal{T} Y_1 + \mathcal{F} Y_1, \tag{6}$$

$$JZ_1 = tZ_1 + fZ_1, \tag{7}$$

where $\mathcal{T} Y_1$ and $\mathcal{F} Y_1$ (respectively, tZ_1 and fZ_1) are the tangential and normal components of JY_1 (respectively, JZ_1).

Let $\mathcal{M}$ be a submanifold of an $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$. Then, we can prove that $\mathcal{M}$ is *pointwise slant* if and only if

$$\mathcal{T}^2 = -(\cos^2\theta) I, \tag{8}$$

where θ is a real-valued function on $\mathcal{M}$ and I is the identity map of $\mathcal{TM}$.

Further, the following relations are straightforward consequences from (8) for any $Y_1, Y_2 \in \Gamma(\mathcal{TM})$

$$g(\mathcal{T} Y_1, \mathcal{T} Y_2) = (\cos^2\theta)\, g(Y_1, Y_2). \tag{9}$$

$$g(\mathcal{F} Y_1, \mathcal{F} Y_2) = (\sin^2\theta)\, g(Y_1, Y_2). \tag{10}$$

Clearly, for any $Y_1 \in \Gamma(\mathcal{TM})$, we have

$$t\mathcal{F} Y_1 = -\sin^2\theta\, Y_1, \quad f\mathcal{F} Y_1 = -\mathcal{F}\mathcal{T} Y_1. \tag{11}$$

3. Pointwise Hemi-Slant Submanifolds of an $\mathcal{LCK}$ Manifold

In this section, we define and study the proper pointwise hemi-slant submanifold of an $\mathcal{LCK}$ manifold. Moreover, we investigate the geometry of the leaves of distributions. We begin by recalling the following submanifolds:

Definition 1. *Let $\mathcal{N}$ be a submanifold of an almost Hermitian manifold $\tilde{\mathcal{M}}$. Then, the pointwise hemi-slant submanifold $\mathcal{N}$ is a submanifold with a tangent bundle that has orthogonal direct decomposition $\mathcal{TN} = \mathfrak{D}^{\perp} \oplus \mathfrak{D}^{\theta}$ such that $\mathfrak{D}^{\perp}$ is a totally real distribution and $\mathfrak{D}^{\theta}$ is a pointwise slant distribution with slant function θ.*

In the above definition, if we assume that the dimensions are $n_1 = \dim \mathfrak{D}^{\perp}$ and $n_2 = \dim \mathfrak{D}^{\theta}$, then we have

(i) $\mathcal{N}$ is a pointwise slant submanifold if $n_1 = 0$.
(ii) $\mathcal{N}$ is a totally real submanifold if $n_2 = 0$.
(iii) $\mathcal{N}$ is a holomorphic submanifold if $\theta = 0$ and $n_1 = 0$.
(iv) $\mathcal{N}$ is a slant submanifold if θ is globally constant and $n_1 = 0$.
(v) $\mathcal{N}$ is a hemi-slant submanifold with slant angle θ if θ is constant on $\mathcal{N}$ and $n_1 \neq 0$.
(vi) $\mathcal{N}$ is a CR submanifold if $\theta = 0$ and $n_1 \neq 0$, $n_2 \neq 0$.

We note that a pointwise hemi-slant submanifold is *proper* if neither $n_1 \neq 0$ nor $n_2 \neq 0$ and θ is not a constant. Otherwise, $\mathcal{N}$ is called *improper*.

Definition 2. Let $\mathcal{M}$ be an almost Hermitian manifold and $\mathcal{N}$ is a submanifold of $\mathcal{M}$. Then, $\mathcal{N}$ is said to be a *mixed totally geodesic* if $\mathfrak{h}(Y_1, Z_1) = 0$ for all $Y_1 \in \Gamma(\mathfrak{D}^\theta)$ and for all $Z_1 \in \Gamma(\mathfrak{D}^\perp)$.

Now, we provide the following useful results.

Lemma 1. *Let $\tilde{\mathcal{M}}$ be an $\mathcal{LCK}$ manifold and $\mathcal{M}$ is a proper pointwise hemi-slant submanifold of $\tilde{\mathcal{M}}$. Then, for any $Y_1 \in \Gamma(\mathfrak{D}^\theta)$ and $Z_1, Z_2 \in \Gamma(\mathfrak{D}^\perp)$, we have*

$$g(\nabla_{Z2} Z_1, Y_1) = (\sec^2\theta)\,[g(\mathfrak{A}_{\mathcal{F}\mathcal{T}Y_1} Z_1 - \mathfrak{A}_{JZ_1}\mathcal{T}Y_1, Z_2) - g(Z_1, Z_2) g(\alpha^\#, Y_1)].$$

Proof. For any $Z_1, Z_2 \in \Gamma(\mathfrak{D}^\perp)$ and $Y_1 \in \Gamma(\mathfrak{D}^\theta)$, we have

$$g(\nabla_{Z2} Z_1, Y_1) = g(J\tilde{\nabla}_{Z2} Z_1, JY_1) = g(\tilde{\nabla}_{Z2} JZ_1, JY_1) - g((\tilde{\nabla}_{Z2} J)Z_1, JY_1).$$

Using (2) and (6), we obtain

$$g(\nabla_{Z2} Z_1, Y_1) = g(\tilde{\nabla}_{Z2} JZ_1, \mathcal{F}Y_1) + g(\tilde{\nabla}_{Z2} JZ_1, \mathcal{T}Y_1) - g(Z_1, Z_2) g(\alpha^\#, Y_1).$$

Then, from (4), we derive

$$g(\nabla_{Z2} Z_1, Y_1) = g(Z_1, J\tilde{\nabla}_{Z2}\mathcal{F}Y_1) - g(\mathfrak{A}_{JZ_1} Z_2, \mathcal{T}Y_1) - g(Z_1, Z_2) g(\alpha^\#, Y_1).$$

From (2), (2.6) and (11), we derive

$$\begin{aligned} g(\nabla_{Z2} Z_1, Y_1) = & - g(\tilde{\nabla}_{Z2} \sin^2\theta Y_1, Z_1) - g(\tilde{\nabla}_{Z2}\mathcal{F}\mathcal{T}Y_1, Z_1) - g(\mathfrak{A}_{JZ_1} Z_2, \mathcal{T}Y_1) \\ & - g(Z_1, Z_2) g(\alpha^\#, Y_1). \end{aligned}$$

Since $\mathcal{M}$ is a proper pointwise hemi-slant submanifold, we obtain that

$$\begin{aligned} g(\nabla_{Z2} Z_1, Y_1) = & \sin^2\theta g(\tilde{\nabla}_{Z2} Z_1, Y_1) - \sin 2\theta Z_2(\theta) g(Z_1, Y_1) + g(\mathfrak{A}_{\mathcal{F}\mathcal{T}Y_1} Z_2, Z_1) \\ & - g(\mathfrak{A}_{JZ_1} Z_2, \mathcal{T}Y_1) - g(Z_1, Z_2) g(\alpha^\#, Y_1). \end{aligned}$$

By theorthogonality of two distributions and the symmetry of the shape operator, the above equation takes the form

$$\cos^2\theta\, g(\nabla_{Z2} Z_1, Y_1) = g(\mathfrak{A}_{\mathcal{F}\mathcal{T}Y_1} Z_1, Z_2) - g(\mathfrak{A}_{JZ_1}\mathcal{T}Y_1, Z_2) - g(Z_1, Z_2) g(\alpha^\#, Y_1),$$

Thus, the lemma follows from the above relation. □

Lemma 1 implies the following result.

Corollary 1. *The leaves of totally real distribution $\mathfrak{D}^\perp$ in a proper pointwise hemi-slant submanifold $\mathcal{M}$ of an $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$ are totally geodesic in $\mathcal{M}$ if and only if*

$$g(\mathfrak{A}_{\mathcal{F}\mathcal{T}Y_1} Z_1 - \mathfrak{A}_{JZ_1}\mathcal{T}Y_1, Z_2) = g(\alpha^\#, Y_1) g(Z_2, Z_1),$$

for any $Y_1 \in \Gamma(\mathfrak{D}^\theta)$ and $Z_2, Z_1 \in \Gamma(\mathfrak{D}^\perp)$.

Now, we have the following results for the pointwise slant distribution $\mathfrak{D}^\theta$.

Lemma 2. *Let $\tilde{\mathcal{M}}$ be an $\mathcal{LCK}$ manifold and $\mathcal{M}$ is a proper pointwise hemi-slant submanifold of $\tilde{\mathcal{M}}$ with proper pointwise slant distribution $\mathfrak{D}^\theta$. Then, we have*

$$\begin{aligned} g(\nabla_{Y1} Y_2, Z_1) = & (\sec^2\theta)\,[g(\mathfrak{A}_{JZ_1}\mathcal{T}Y_2 - \mathfrak{A}_{\mathcal{F}\mathcal{T}Y_2} Z_1, Y_1) - g(Y_1, \mathcal{T}Y_2) g(\beta^\#, Z_1)] \\ & - g(Y_1, Y_2) g(\alpha^\#, Z_1). \end{aligned}$$

Proof. For any $Y_1, Y_2 \in \Gamma(\mathfrak{D}^\theta)$ and $Z_1 \in \Gamma(\mathfrak{D}^\perp)$, we have

$$g(\nabla_{Y1} Y_2, Z_1) = g(J\tilde{\nabla}_{Y1} Y_2, JZ_1).$$

From the covariant derivative formula of J, we obtain

$$g(\nabla_{Y1} Y_2, Z_1) = g(\tilde{\nabla}_{Y1} JY_2, JZ_1) - g((\tilde{\nabla}_{Y1} J)Y_2, JZ_1).$$

Then, using (2), (5) and (6), the above equation takes the form

$$\begin{aligned} g(\nabla_{Y1} Y_2, Z_1) = {} & g(\mathfrak{A}_{JZ_1} \mathcal{T} Y_2, Y_1) + g(\tilde{\nabla}_{Y1} \mathcal{F} Y_2, JZ_1) - g(\mathcal{T} Y_1, Y_2) g(\alpha^\#, JZ_1) \\ & - g(Y_1, Y_2) g(\alpha^\#, Z_1). \end{aligned}$$

Now, from (2), (5) and (11), we derive

$$\begin{aligned} g(\nabla_{Y1} Y_2, Z_1) = {} & g(\mathfrak{A}_{JZ_1} \mathcal{T} Y_2, Y_1) + g(\tilde{\nabla}_{Y1} \sin^2\theta Y_2, Z_1) - g(\mathfrak{A}_{\mathcal{F}\mathcal{T}Y_2} Z_1, Y_1) \\ & + \sin^2\theta g(Y_1, Y_2) g(\alpha^\#, Z_1) - g(\mathcal{T} Y_1, Y_2) g(\alpha^\#, JZ_1) - g(Y_1, Y_2) g(\alpha^\#, Z_1). \end{aligned}$$

As $\mathcal{M}$ is a proper pointwise hemi-slant submanifold, we have

$$\begin{aligned} g(\nabla_{Y1} Y_2, Z_1) = {} & g(\mathfrak{A}_{JZ_1} \mathcal{T} Y_2 - \mathfrak{A}_{\mathcal{F}\mathcal{T}Y_2} Z_1, Y_1) + \sin^2\theta g(\tilde{\nabla}_{Y1} Y_2, Z_1) + \sin 2\theta Y_1(\theta) g(Y_2, Z_1) \\ & - g(\mathcal{T} Y_1, Y_2) g(\alpha^\#, JZ_1) - \cos^2\theta g(Y_1, Y_2) g(\alpha^\#, Z_1). \end{aligned}$$

By using the orthogonality of the two distributions, the lemma is derived from the relations stated above. □

Lemma 3. *Let $\tilde{\mathcal{M}}$ be an $\mathcal{LCK}$ manifold and $\mathcal{M}$ is a proper pointwise hemi-slant submanifold of $\tilde{\mathcal{M}}$. Then, we have*

$$\begin{aligned} \cos^2\theta g([Y_1, Y_2], Z_1) = {} & g(\mathfrak{A}_{JZ_1} \mathcal{T} Y_2 - \mathfrak{A}_{\mathcal{F}\mathcal{T}Y_2} Z_1, Y_1) - g(\mathfrak{A}_{JZ_1} \mathcal{T} Y_1 - \mathfrak{A}_{\mathcal{F}\mathcal{T}Y_1} Z_1, Y_2) \\ & - 2g(Y_1, \mathcal{T} Y_2) g(\beta^\#, Z_1), \end{aligned}$$

for any $Y_1, Y_2 \in \Gamma(\mathfrak{D}^\theta)$, and $Z_1 \in \Gamma(\mathfrak{D}^\perp)$.

Proof. Let $\mathcal{M}$ be a proper pointwise hemi-slant submanifold of an $\mathcal{LCK}$ manifold. Then, from Lemma 3, we have

$$\begin{aligned} \cos^2\theta\, g(\nabla_{Y1} Y_2, Z_1) = {} & [g(\mathfrak{A}_{JZ_1} \mathcal{T} Y_2 - \mathfrak{A}_{\mathcal{F}\mathcal{T}Y_2} Z_1, Y_1) - g(Y_1, \mathcal{T} Y_2) g(\beta^\#, Z_1)] \\ & - \cos^2\theta\, g(Y_1, Y_2) g(\alpha^\#, Z_1), \end{aligned} \tag{12}$$

for any $Y_1 \in \Gamma(\mathfrak{D}^\theta)$ and $Z_1, Z_2 \in \Gamma(\mathfrak{D}^\perp)$. Then, using polarization identity and using symmetry of g, we obtain

$$\begin{aligned} \cos^2\theta\, g(\nabla_{Y2} Y_1, Z_1) = {} & [g(\mathfrak{A}_{JZ_1} \mathcal{T} Y_1 - \mathfrak{A}_{\mathcal{F}\mathcal{T}Y_1} Z_1, Y_2) - g(Y_2, \mathcal{T} Y_1) g(\beta^\#, Z_1)] \\ & - \cos^2\theta\, g(Y_1, Y_2) g(\alpha^\#, Z_1). \end{aligned} \tag{13}$$

Subtracting (13) from (12), as a result, the lemma is completely proven. □

The following result is a consequence of Lemma 2 if $\mathfrak{D}^\theta$ is a totally geodesic distribution in $\mathcal{M}$.

Lemma 4. *Let $\tilde{\mathcal{M}}$ be an $\mathcal{LCK}$ manifold and $\mathcal{M}$ is a proper pointwise hemi-slant submanifold of $\tilde{\mathcal{M}}$. Then, the proper slant distribution $\mathfrak{D}^\theta$ defines a totally geodesic foliation if and only if*

$$g(\mathfrak{A}_{JZ_1} \mathcal{T} Y_1 - \mathfrak{A}_{\mathcal{F}\mathcal{T}Y_1} Z_1, Y_2) = \cos^2\theta g(Y_1, Y_2) g(\alpha^\#, Z_1) + g(\mathcal{T} Y_1, Y_2) g(\beta^\#, Z_1),$$

for any $Y_1, Y_2 \in \Gamma(\mathfrak{D}^\theta)$ *and* $Z_1 \in \Gamma(\mathfrak{D}^\perp)$.

Next, we have the following theorems.

Theorem 2. *Let* $(\tilde{\mathcal{M}}, J, g)$ *be an* $\mathcal{LCK}$ *manifold and* $\mathcal{M}$ *is a proper pointwise hemi-slant submanifold of* $\tilde{\mathcal{M}}$. *Then, for any* $Z_1 \in \Gamma(\mathfrak{D}^\perp)$, $Y_1 \in \Gamma(\mathfrak{D}^\theta)$, *we have*

(i) *The totally real distribution* $\mathfrak{D}^\perp$ *defines a totally geodesic foliation in* $\mathcal{M}$ *if and only if*

$$\mathfrak{A}_{\mathcal{F}\mathcal{T}Y_1} Z_1 - \mathfrak{A}_{JZ_1} \mathcal{T} Y_1 = g(\alpha^\#, Y_1) Z_1.$$

(ii) *The proper pointwise slant distribution* $\mathfrak{D}^\theta$ *defines a totally geodesic foliation if and only if*

$$\mathfrak{A}_{JZ_1} \mathcal{T} Y_1 - \mathfrak{A}_{\mathcal{F}\mathcal{T}Y_1} Z_1 = \cos^2\theta g(\alpha^\#, Z_1) Y_1 + g(\beta^\#, Z_1) \mathcal{T} Y_1.$$

Proof. The first part (i) of the theorem follows from Lemma 1 and the second part (ii) follows from Lemma 3. □

Now, we provide the following integrability theorem for a totally real distribution $\mathfrak{D}^\perp$ and slant distribution $\mathfrak{D}^\theta$.

Theorem 3. *Let* $(\tilde{\mathcal{M}}, J, g)$ *be an* $\mathcal{LCK}$ *manifold and* $\mathcal{M}$ *is a proper pointwise hemi-slant submanifold of* $\tilde{\mathcal{M}}$. *Then, for any* $Z_1, Z_2 \in \Gamma(\mathfrak{D}^\perp)$, $Y_1, Y_2 \in \Gamma(\mathfrak{D}^\theta)$, *we have*

(i) *The totally real distribution* $\mathfrak{D}^\perp$ *of* $\mathcal{M}$ *is integrable if and only if*

$$\mathfrak{A}_{JZ_2} Z_1 = \mathfrak{A}_{JZ_1} Z_2.$$

(ii) *The pointwise slant distribution* $\mathfrak{D}^\theta$ *of* $\mathcal{M}$ *is integrable if and only if*

$$g(\mathfrak{A}_{JZ_1} \mathcal{T} Y_2 - \mathfrak{A}_{\mathcal{F}\mathcal{T}Y_2} Z_1, Y_1) = g(\mathfrak{A}_{JZ_1} \mathcal{T} Y_1 - \mathfrak{A}_{\mathcal{F}\mathcal{T}Y_1} Z_1, Y_2) + 2g(Y_1, \mathcal{T} Y_2) g(\beta^\#, Z_1).$$

Proof. We prove (i) as well as (ii) in the same way. We deduce from Lemma 1 by interchanging Z_1 and Z_2 and applying the symmetry of $\mathfrak{A}$, such that

$$\cos^2\theta\, g([Z_1, Z_2], Y_1) = g(\mathfrak{A}_{JZ_1} Z_2, \mathcal{T} Y_1) - g(\mathfrak{A}_{JZ_2} Z_1, \mathcal{T} Y_1),$$

for any $Y_1, Y_2 \in \Gamma(\mathfrak{D}^\theta)$ and $Z_1 \in \Gamma(\mathfrak{D}^\perp)$. Thus, the distribution $\mathfrak{D}^\perp$ is integrable if and only if $g([Z_1, Z_2], Y_1) = 0$ for all $Y_1 \in \Gamma(\mathfrak{D}^\theta)$ and $Z_1, Z_2 \in \Gamma(\mathfrak{D}^\perp)$; i.e.,

$$g(\mathfrak{A}_{JZ_1} Z_2, \mathcal{T} Y_1) = g(\mathfrak{A}_{JZ_2} Z_1, \mathcal{T} Y_1).$$

Hence,the statement (i) follows from the above relation. Similarly, we can prove (ii). □

4. Pointwise Hemi-Slant Warped Products: $\mathcal{N}^\perp \times_f \mathcal{N}^\theta$

Sahin studied hemi-slant submanifolds of Kähler manifolds [19] as a generalized class of CR submanifolds. He investigated their warped products in Kähler manifolds in the same paper. Also, Sahin proved that there are no proper warped products of the type $\mathcal{N}^\perp \times_f \mathcal{N}^\theta$ in a Kähler manifold $\tilde{\mathcal{M}}$, where $\mathcal{N}^\perp$ and $\mathcal{N}^\theta$ are totally real and proper slant submanifolds of $\tilde{\mathcal{M}}$.

Lately, Srivastava et al [33] introduced pointwise hemi-slant warped products in a Kähler manifold of the form $\mathcal{N}^\perp \times_f \mathcal{N}^\theta$ and $\mathcal{N}^\theta \times_f \mathcal{N}^\perp$. They obtained fundamental results.

In this section, we study the pointwise hemi-slant warped product of the form $\mathcal{N}^\perp \times_f \mathcal{N}^\theta$ in a locally conformally Kähler manifold $\tilde{\mathcal{M}}$ under the assumption that the Lee vector field $\alpha^\#$ is tangent to $\mathcal{N}$.

Now, we provide a brief introduction to warped product manifolds: consider the Riemannian manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ endowed with Riemannian metrics g_1 and g_2, respectively, and let f be a positive differential function on $\mathcal{M}_1$. Then, the product manifold is $\mathcal{M}_1 \times \mathcal{M}_2$ with its natural projections $\pi_1 : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_1$ and $\pi_2 : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_2$. Then, the *warped product manifold* $\mathcal{M}_1 \times_f \mathcal{M}_2$ is the product manifold $\mathcal{M}_1 \times \mathcal{M}_2$ and the function f is called the *warping function* on $\mathcal{M}$. It is equipped with the warped product metric g defined by

$$g(Y_1, Y_2) = g_1(\pi_{1\star}Y_1, \pi_{1\star}Y_2) + (f \circ \pi_1)^2 g_2(\pi_{2\star}Y_1, \pi_{2\star}Y_2)$$

for $Y_1, Y_2 \in \Gamma(\mathcal{TM})$, where $\pi_{i\star}$ is the tangent map of π_i.

If the warping function f of the warped products manifolds $\mathcal{N}_1 \times_f \mathcal{N}_2$ is constant, then they are *trivial*.

First, we recall the following result.

Lemma 5 ([11])**.** *Let $\mathcal{M} = \mathcal{N}_1 \times {}_f\mathcal{N}_2$ be a warped product manifold with the warping function f; then, for any $Y_1, Y_2 \in \mathcal{T}(\mathcal{N}_1)$ and $Z, Z_2 \in \mathcal{T}(N_2)$, we have*

(i) $\nabla_{Y1}Y_2 \in \mathcal{T}(\mathcal{N}_1)$,
(ii) $\nabla_{Y1}Z = \nabla_Z Y_1 = Y_1(\ln f)Z$,
(iii) $\nabla_Z Z_2 = \nabla^{\mathcal{N}_2}_Z Z_2 - g(Z, Z_2)\vec{\nabla} \ln f$,

where $\vec{\nabla} \ln f$ is the gradient of the function $\ln f$ and ∇, $\nabla^{\mathcal{N}_2}$ are the Levi-Civita connections on $\mathcal{M}$, $\mathcal{N}_2$, respectively.

Definition 3. *A warped product $\mathcal{N}^\perp \times_f \mathcal{N}^\theta$ of an $\mathcal{LCK}$ manifold $(\tilde{\mathcal{M}}, J, g, \alpha)$ such that $\mathcal{N}^\perp$ a totally real submanifold and $\mathcal{N}^\theta$ a pointwise slant submanifold is called a warped product pointwise hemi-slant submanifold.*

If $\mathcal{N}^\theta$ is proper pointwise slant and $\mathcal{N}^\perp$ is totally real in $\tilde{\mathcal{M}}$, then a warped product $\mathcal{N}^\perp \times_f \mathcal{N}^\theta$ is said to be proper pointwise hemi-slant submanifold. Otherwise, it is called non-proper.

For simplicity, we denote the tangent spaces of $\mathcal{N}^\perp$ and $\mathcal{N}^\theta$ by $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$, respectively. It is also important to note that, for a warped product $\mathcal{N}_1 \times_f \mathcal{N}_2$, $\mathcal{N}_1$ is totally geodesic and $\mathcal{N}_2$ is totally umbilical in $\mathcal{M}$ [11].

Now, we prove the following useful lemmas.

Lemma 6. *On a proper pointwise hemi-slant warped product $\mathcal{M} = \mathcal{N}^\perp \times {}_f\mathcal{N}^\theta$ in an $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$, where the Lee vector field $\alpha^\#$ is tangent to $\mathcal{M}$, we have*

(i) $g(\mathfrak{h}(Y_1, Y_2), JZ_1) = [g(\alpha^\#, Z_1) - Z_1(\ln f)]g(\mathcal{T}Y_1, Y_2) + g(\mathfrak{h}(Y_1, Z_1), \mathcal{F}Y_2)$,
(ii) $g(\mathfrak{h}(Z_1, Z_2), \mathcal{F}Y_1) = g(\beta^\#, Y_1)g(Z_1, Z_2) + g(\mathfrak{h}(Z_1, Y_1), JZ_2)$, *for any $Z_1 \in \Gamma(\mathfrak{D}^\perp)$ and $Y_1, Y_2 \in \Gamma(\mathfrak{D}^\theta)$.*

Proof. We have for any $Z_1 \in \Gamma(\mathfrak{D}^\perp)$ and $Y_1, Y_2 \in \Gamma(\mathfrak{D}^\theta)$

$$g(\mathfrak{h}(Y_1, Y_2), JZ_1) = g(\tilde{\nabla}_{Y1}Y_2, JZ_1).$$

Hence, we obtain from the covariant derivative property of J

$$g(\mathfrak{h}(Y_1, Y_2), JZ_1) = -g(J\tilde{\nabla}_{Y1}Y_2, Z_1) = g((\tilde{\nabla}_{Y1}J)Y_2, Z_1) - g(\tilde{\nabla}_{Y1}JY_2, Z_1).$$

Thus, from (2) and (6), we obtain

$$\begin{aligned} g(\mathfrak{h}(Y_1, Y_2), JZ_1) =& g(JY_1, Y_2)g(\alpha^\#, Z_1) - g(Y_1, Y_2)g(\alpha^\#, JZ_1) - g(\tilde{\nabla}_{Y1}\mathcal{T}Y_2, Z_1) \\ &- g(\tilde{\nabla}_{Y1}\mathcal{F}Y_2, Z_1). \end{aligned}$$

Then, it follows from (4) and the fact that $\alpha^\#$ is tangent to $\mathcal{M}$

$$g(\mathfrak{h}(Y_1, Y_2), JZ_1) = g(\mathcal{T}Y_1, Y_2)g(\alpha^{\#}, Z_1) + g(\mathcal{T}Y_2, \tilde{\nabla}_{Y1}Z_1) + g(\mathfrak{A}_{\mathcal{F}_{Y_2}Y_1, Z_1}).$$

Applying Lemma 5 (ii) and (5), we obtain

$$g(\mathfrak{h}(Y_1, Y_2), JZ_1) = g(\mathcal{T}Y_1, Y_2)g(\alpha^{\#}, Z_1) + Z_1()g(\mathcal{T}Y_2, Y_1) + g(\mathfrak{h}(Y_1, Z_1), \mathcal{F}Y_2).$$

As a result, the above relation leads to the lemma's first relation. For the second part, we have

$$g(\mathfrak{h}(Z_1, Z_2), \mathcal{F}Y_1) = g(\tilde{\nabla}_{Z2}Z_1, \mathcal{F}Y_1) = g(\tilde{\nabla}_{Z2}Z_1, JY_1) - g(\tilde{\nabla}_{Z2}Z_1, \mathcal{T}Y_1),$$

Applying Lemma 5 (ii) and the covariant derivative property of J, we derive

$$g(\mathfrak{h}(Z_1, Z_2), \mathcal{F}Y_1) = g((\tilde{\nabla}_{Z2}J)Z_1, Y_1) - g(\tilde{\nabla}_{Z2}JZ_1, Y_1).$$

Now, from (2) and (4), we find

$$g(\mathfrak{h}(Z_1, Z_2), \mathcal{F}Y_1) = g(Z_2, Z_1)g(J\alpha^{\#}, Y_1) + g(\mathfrak{A}_{JZ1}Z_2, Y_1).$$

Thus, the last equation provides us the second part of the lemma. □

Now, if we interchange Y_1 with $\mathcal{T}Y_1$ in (6) (i), and then using (8), we can easily obtain the following relation:

$$\begin{aligned} g(\mathfrak{h}(\mathcal{T}Y_1, Y_2), JZ_1) = \cos^2\theta[Z_1(\ln f) - g(\alpha^{\#}, Z_1)]g(Y_1, Y_2) + \\ g(\mathfrak{h}(Z_1, \mathcal{T}Y_1), \mathcal{F}Y_2). \end{aligned} \tag{14}$$

Next, we provide the following result for later use.

Lemma 7. *Let $\mathcal{M} = \mathcal{N}^{\perp} \times {}_f\mathcal{N}^{\theta}$ be a proper pointwise hemi-slant warped product submanifold of an $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$ and the Lee vector field $\alpha^{\#}$ is tangent to $\mathcal{M}$. Then, the following holds:*

$$g(\mathfrak{h}(\mathcal{T}Y_1, Z_1), \mathcal{F}Y_2) - g(\mathfrak{h}(Y_2, Z_1), \mathcal{F}\mathcal{T}Y_1) = 2\cos^2\theta[g(\alpha^{\#}, Z_1) - Z_1 lnf]g(Y_1, Y_2),$$

for any $Y_1, Y_2 \in \Gamma(\mathfrak{D}^{\theta})$, and $Z_1 \in \Gamma(\mathfrak{D}^{\perp})$.

Proof. From Lemma 6, we have

$$g(\mathfrak{h}(Y_1, Y_2), JZ_1) = [g(\alpha^{\#}, Z_1) - Z_1(\ln f)]g(\mathcal{T}Y_1, Y_2) + g(\mathfrak{h}(Y_1, Z_1), \mathcal{F}Y_2) \tag{15}$$

for any $Y_1, Y_2 \in \Gamma(\mathfrak{D}^{\theta})$, and $Z_1 \in \Gamma(\mathfrak{D}^{\perp})$. By interchanging Y_1 and Y_2 in (15), we find

$$g(\mathfrak{h}(Y_1, Y_2), JZ_1) = [g(\alpha^{\#}, Z_1) - Z_1(\ln f)]g(\mathcal{T}Y_2, Y_1) + g(\mathfrak{h}(Y_2, Z_1), \mathcal{F}Y_1). \tag{16}$$

Subtracting (16) from (15), we obtain

$$g(\mathfrak{h}(Z_1, Y_2), \mathcal{F}Y_1) - g(\mathfrak{h}(Z_1, Y_1), \mathcal{F}Y_2) = 2[g(\alpha^{\#}, Z_1) - Z_1(\ln f)]g(\mathcal{T}Y_1, Y_2). \tag{17}$$

Now, interchange Y_1 by $\mathcal{T}Y_1$ in the above relation and use (6). This completes the proof of the lemma. □

Let $\mathcal{M} = \mathcal{N}^{\perp} \times {}_f\mathcal{N}^{\theta}$ be a proper pointwise hemi-slant warped product in an $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$. Then, the normal bundle can be decomposed by

$$\mathcal{T}^{\perp}\mathcal{M} = J\mathfrak{D}^{\perp} \oplus F\mathfrak{D}^{\theta} \oplus \nu, \;\; J\mathfrak{D}^{\perp} \perp F\mathfrak{D}^{\theta}, \tag{18}$$

where ν is the invariant normal subbundle of $\mathcal{T}^{\perp}\mathcal{M}$.

Theorem 4. *Let $\mathcal{M} = \mathcal{N}^{\perp} \times {}_f\mathcal{N}^{\theta}$ be a pointwise hemi-slant warped product submanifold of an $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$. If $\mathfrak{h}(Y_1, Z_1) \in \nu$ for any $Z_1 \in \Gamma(\mathfrak{D}^{\perp})$ and $Y_1 \in \Gamma(\mathfrak{D}^{\theta})$, then we have $\alpha(Z_1) = Z_1(\ln f)$.*

Proof. By the fact of Lemma 7 and the hypothesis of the theorem, we have

$$2\cos^2\theta[g(\alpha^{\#}, Z_1) - Z_1(\ln f)]g(Y_1, Y_2) = 0, \tag{19}$$

for any $Z_1 \in \Gamma(\mathfrak{D}^{\perp})$ and $Y_1, Y_2 \in \Gamma(\mathfrak{D}^{\theta})$. Equation (19) leads to a required result as $\mathcal{M}$ is proper pointwise hemi-slant and g is the Riemannian metric. □

The provided theorem immediately results in the following corollary.

Corollary 2. *Let $\mathcal{N}^{\perp} \times {}_f\mathcal{N}^{\theta}$ be a mixed totally geodesic pointwise hemi-slant warped product in an $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$. Then, $\alpha(Z_1) = Z_1(\ln f)$ for any $Z_1 \in \Gamma(\mathfrak{D}^{\perp})$.*

Theorem 5. *Let $\mathcal{N}^{\perp} \times_f \mathcal{N}^{\theta}$ be a warped product pointwise hemi-slant submanifold in an $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$ such that $\mathcal{M}$ is mixed totally geodesic. Then, $\mathcal{M}$ is a locally direct product submanifold of the form $\mathcal{N}^{\perp} \times_f \mathcal{N}^{\theta}$ if and only if the Lee form α normal to for any $(\mathfrak{D}^{\perp})$.*

The following result is an immediate consequence of Lemma 6.

Theorem 6. *Let $\mathcal{M} = \mathcal{N}^{\perp} \times {}_f\mathcal{N}^{\theta}$ a proper pointwise hemi-slant warped product in an $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$ and the Lee vector field $\alpha^{\#}$ is tangent to $\mathcal{M}$. Then,*

$$g(\mathfrak{A}_{\mathcal{F}\mathcal{T}Y_1} Z_1 - \mathfrak{A}_{JZ_1}\mathcal{T}Y_1, Y_2) = \cos^2\theta[Z_1(\ln f) - g(\alpha^{\#}, Z_1)]g(Y_1, Y_2)$$

for any $Z_1 \in \Gamma(\mathfrak{D}^{\perp})$ and $Y_1, Y_2 \in \Gamma(\mathfrak{D}^{\theta})$.

Proof. Follows from Lemma 6 (i) and using (6). □

From the above lemma, we have

Corollary 3. *There does not exist a mixed totally geodesic warped product CR submanifold of the form $\mathcal{N}^{\perp} \times {}_f\mathcal{N}^{\mathcal{T}}$ in a Kähler manifold $\tilde{\mathcal{M}}$.*

Proof. Follows from Theorem 6. □

5. Characterizations Theorems

In this section, we first provide some important lemmas. Then, we derive the characterization results for proper pointwise hemi-slant warped product submanifolds of an $\mathcal{LCK}$ manifold and then deduce the necessary and sufficient conditions for a pointwise hemi-slant submanifold to be a warped product.

Lemma 8. *Let $\mathcal{M} = \mathcal{N}^{\perp} \times {}_f\mathcal{N}^{\theta}$ be a pointwise hemi-slant warped product submanifold of an $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$, where the Lee vector field $\alpha^{\#}$ is tangent to $\mathcal{M}$. Then, we have*

$$g(\mathfrak{h}(Z_1, Y_1), \mathcal{F}Y_2) - g(\mathfrak{h}(Z_1, Y_2), \mathcal{F}Y_1) = 2\tan\theta Z_1(\theta)g(\mathcal{T}Y_1, Y_2), \tag{20}$$

for any $Y_1, Y_2 \in \Gamma(\mathfrak{D}^{\theta})$ and $Z_1 \in \Gamma(\mathfrak{D}^{\perp})$.

Proof. For any $Y_1, Y_2 \in \Gamma(\mathfrak{D}^{\theta})$, and $Z_1 \in \Gamma(\mathfrak{D}^{\perp})$, we have

$$g(\tilde{\nabla}_{Z1} Y_1, Y_2) = Z_1(\ln f)g(Y_1, Y_2). \tag{21}$$

On other hand,

$$g(\tilde{\nabla}_{Z1}Y_1, Y_2) = g(J\tilde{\nabla}_{Z1}Y_1, JY_2).$$

From the covariant derivative formula of J, we derive

$$g(\nabla_{Z1}Y_1, Y_2) = g(\tilde{\nabla}_{Z1}JY_1, JY_2) - g((\tilde{\nabla}_{Z1}J)Y_1, JY_2).$$

Then, from (2), (3), (5) and (6), we arrive at

$$g(\nabla_{Z1}Y_1, Y_2) = g(\mathfrak{A}_{\mathcal{F}Y_2}\mathcal{T}Y_2, Z_1) + Z_1(\ln f)\cos^2\theta g(Y_1, Y_2) + g(\tilde{\nabla}_{Z1}\mathcal{F}Y_1, JY_2),$$

which implies

$$g(\nabla_{Z1}Y_1, Y_2) = g(\mathfrak{A}_{\mathcal{F}Y_2}\mathcal{T}Y_1, Z_1) + Z_1(\ln f)\cos^2\theta g(Y_1, Y_2) - g(J\tilde{\nabla}_{Z1}\mathcal{F}Y_1, Y_2).$$

Using the covariant derivative formula of J again and (5), we derive

$$\begin{aligned} g(\nabla_{Z1}Y_1, Y_2) = g(\mathfrak{A}_{\mathcal{F}Y_2}\mathcal{T}Y_1, Z_1) + Z_1(\ln f)\cos^2\theta g(Y_1, Y_2) - g(\tilde{\nabla}_{Z1}t\mathcal{F}Y_1, Y_2) \\ - g(\tilde{\nabla}_{Z1}f\mathcal{F}Y_1, Y_2). \end{aligned}$$

Using (11), we derive that

$$\begin{aligned} g(\nabla_{Z1}Y_1, Y_2) = g(\mathfrak{A}_{\mathcal{F}Y_2}\mathcal{T}Y_1, Z_1) + Z_1(\ln f)\cos^2\theta g(Y_1, Y_2) + g(\tilde{\nabla}_{Z1}\sin^2\theta Y_1, Y_2) \\ + g(\tilde{\nabla}_{Z1}\mathcal{F}\mathcal{T}Y_1, Y_2). \end{aligned}$$

Since $\mathcal{M}$ is a proper pointwise hemi-slant submanifold and from (5), we have

$$\begin{aligned} g(\nabla_{Z1}Y_1, Y_2) = g(\mathfrak{A}_{\mathcal{F}Y_2}\mathcal{T}Y_1, Z_1) + Z_1(\ln f)\cos^2\theta g(Y_1, Y_2) + \sin^2\theta g(\nabla_{Z1}Y_1, Y_2) \\ + \sin 2\theta Z_1(\theta)g(Y_1, Y_2) - g(\mathfrak{A}_{\mathcal{F}\mathcal{T}Y_1}Y_2, Z_1). \end{aligned}$$

which implies that

$$\begin{aligned} \cos^2\theta g(\nabla_{Z1}Y_1, Y_2) = g(\mathfrak{A}_{\mathcal{F}Y_2}\mathcal{T}Y_1, Z_1) + Z_1(\ln f)\cos^2\theta g(Y_1, Y_2) \\ + \sin 2\theta Z_1(\theta)g(Y_1, Y_2) - g(\mathfrak{A}_{\mathcal{F}\mathcal{T}Y_1}Y_2, Z_1). \end{aligned} \tag{22}$$

Thus, it follows from (21) and (22) that

$$g(\mathfrak{h}(Z_1, Y_2), \mathcal{F}\mathcal{T}Y_1) - g(\mathfrak{h}(Z_1, \mathcal{T}Y_1), \mathcal{F}Y_2) = \sin 2\theta Z_1(\theta)g(Y_1, Y_2). \tag{23}$$

Thus, the lemma follows from the above relations by interchanging Y_1 by $\mathcal{T}Y_1$. □

Theorem 7. *Let $\mathcal{M} = \mathcal{N}^{\perp} \times {}_f\mathcal{N}^{\theta}$ a proper pointwise hemi-slant warped product in an $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$ with its Lee vector field $\alpha^{\#}$ tangent to $\mathcal{M}$. Then, we have*

$$Z_1(\ln f) = \tan\theta\, Z_1(\theta) + \alpha(Z_1), \tag{24}$$

for any $Z_1 \in \Gamma(\mathfrak{D}^{\perp})$.

Proof. From (17) and Lemma 8, we have

$$[\tan\theta\, Z_1(\theta) + g(\alpha^{\#}, Z_1) - Z_1(\ln f)]g(\mathcal{T}Y_1, Y_2) = 0. \tag{25}$$

Now, by interchanging Y_1 by $\mathcal{T}Y_1$ in Equation (25) and using relation (9), we obtain

$$\cos^2\theta\, [\tan\theta\, Z_1(\theta) + g(\alpha^{\#}, Z_1) - Z_1(\ln f)]g(Y_1, Y_2) = 0, \tag{26}$$

for any $Z_1 \in \Gamma(\mathfrak{D}^\perp)$ and $Y_1, Y_2 \in \Gamma(\mathfrak{D}^\theta)$.

Since $\mathcal{M}$ is a proper pointwise hemi-slant and g is the Riemannian metric, the desired result follows from Equation (26). □

Now, recall Hiepko's Theorem to establish the main theorem characterization for pointwise hemi-slant warped products.

Theorem 8 ([34])**.** *Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be two orthogonal distributions on a Riemannian manifold $\mathcal{M}$. Suppose that both $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are involutive such that $\mathfrak{D}_1$ is a totally geodesic foliation and $\mathfrak{D}_2$ is a spherical foliation. Then, $\mathcal{M}$ is locally isometric to a non-trivial warped product $\mathcal{M}_1 \times_f \mathcal{M}_2$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are integral manifolds of $\mathfrak{D}_1$ and $\mathfrak{D}_2$, respectively.*

Now, we can prove the main characterization theorem of proper pointwise hemi-slant warped product submanifolds of the form $\mathcal{N}^\perp \times_f \mathcal{N}^\theta$ in an $\mathcal{LCK}$ manifold.

Theorem 9. *Let $\mathcal{M}$ be a proper pointwise hemi-slant submanifold of an $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$ with the Lee vector field $\alpha^\#$ tangent to $\mathcal{M}$. Then, $\mathcal{M}$ is locally a warped product submanifold of the form $\mathcal{N}^\perp \times_f \mathcal{N}^\theta$ if and only if the shape operator for any $Y_1 \in \Gamma(\mathfrak{D}^\theta)$ and $Z_1 \in \Gamma(\mathfrak{D}^\perp)$ satisfies*

$$\mathfrak{A}_{\mathcal{F}\mathcal{T}Y_1} Z_1 - \mathfrak{A}_{JZ_1} \mathcal{T} Y_1 = \cos^2\theta (Z_1(\mu) - \alpha(Z_1)) Y_1, \tag{27}$$

for some smooth function μ on $\mathcal{M}$ satisfying $Y_2(\mu) = 0$ for any $Y_2 \in \Gamma(\mathfrak{D}^\theta)$.

Proof. Let $\mathcal{M}$ be a pointwise hemi-slant warped product submanifold of an $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$. Then, by Theorem 6, we derive condition (27) for any $Z_1 \in \Gamma(\mathfrak{D}^\perp)$ and $Y_1 \in \Gamma(\mathfrak{D}^\theta)$ with $\mu = \ln f$ and $\alpha(Z_1) = g(\alpha^\#, Z_1)$.

In contrast, consider $\mathcal{M}$ to be a proper pointwise hemi-slant submanifold of an $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$, where $\mathcal{M}$ satisfies the condition (27).

Consequently, from the given condition (27) and Lemma 1, we have that $\cos^2\theta\, g(\nabla_{Z1} Z_2, Y_1) = 0$ for $Z_1, Z_2 \in \Gamma(\mathfrak{D}^\perp)$ and $Y_1 \in \Gamma(\mathfrak{D}^\theta)$. Since $\mathcal{M}$ is a proper pointwise hemi-slant submanifold, $g(\nabla_{Z1} Z_2, Y_1) = 0$ holds. Hence, the leaves of the distribution $\mathfrak{D}^\perp$ are totally geodesic in $\mathcal{M}$. Conversely, condition (27) and Lemma 4 indicate that $\cos^2\theta\, g([Y_1, Y_2], Z_1) = 0$ holds for any $Z_1 \in \Gamma(\mathfrak{D}^\perp)$ and $Y_1, Y_2 \in \Gamma(\mathfrak{D}^\theta)$. Since $\mathcal{M}$ is a proper pointwise hemi-slant submanifold, then $\cos^2\theta \neq 0$; thus, we find that the pointwise slant distribution $\mathfrak{D}^\theta$ is integrable.

Moreover, let $\mathfrak{h}^\theta$ be a second fundamental form of a leaf $\mathcal{N}^\theta$ of $\mathfrak{D}^\theta$ in $\mathcal{M}$. Then, for any $Y_1, Y_2 \in \Gamma(\mathfrak{D}^\theta)$, and $Z_1 \in \Gamma(\mathfrak{D}^\perp)$, we have

$$g(\mathfrak{h}^\theta(Y_1, Y_2), Z_1) = g(\nabla_{Y1} Y_2, Z_1) = g(\tilde{\nabla}_{Y1} Y_2, Z_1) = g(J\tilde{\nabla}_{Y1} Y_2, JZ_1).$$

Using (2) and (6), we have

$$\begin{aligned} g(\mathfrak{h}^\theta(Y_1, Y_2), Z_1) &= g(\tilde{\nabla}_{Y1} \mathcal{T} Y_2, JZ_1) + g(\tilde{\nabla}_{Y1} \mathcal{F} Y_2, JZ_1) - g((\tilde{\nabla}_{Y1} J) Y_2, JZ_1) \\ &= g(\tilde{\nabla}_{Y1} \mathcal{T} Y_2, JZ_1) + g(\tilde{\nabla}_{Y1} \mathcal{F} Y_2, JZ_1) - g(JY_1, Y_2) g(\alpha^\#, JZ_1) \\ &\quad - g(Y_1, Y_2) g(\alpha^\#, Z_1). \end{aligned}$$

By the hypothesis of the theorem and applying the covariant derivative property of J, we find

$$\begin{aligned} g(\mathfrak{h}^\theta(Y_1, Y_2), Z_1) &= g(\mathfrak{h}(Y_1, \mathcal{T} Y_2), JZ_1) - g(\tilde{\nabla}_{Y1} J\mathcal{F} Y_2, Z_1) + g((\tilde{\nabla}_{Y1} J)\mathcal{F} Y_2, Z_1) \\ &\quad - g(\alpha^\#, Z_1) g(Y_1, Y_2). \end{aligned}$$

Therefore, by (2), (5), (6) and (11), we derive that

$$g(\mathfrak{h}^\theta(Y_1,Y_2),Z_1) = g(\mathfrak{A}_{JZ_1}\mathcal{T}Y_2,Y_1) + g(\tilde{\nabla}_{Y1}\sin^2\theta Y_2,Z_1) - g(\mathfrak{A}_{\mathcal{F}\mathcal{T}Y_2}Y_1,Z_1) \\ - g(\alpha^\#,Z_1)g(Y_1,Y_2).$$

Since $\mathcal{M}$ is a proper pointwise hemi-slant submanifold, we obtain

$$\begin{aligned} g(\mathfrak{h}^\theta(Y_1,Y_2),Z_1) &= g(\mathfrak{A}_{JZ_1}\mathcal{T}Y_2 - \mathfrak{A}_{\mathcal{F}\mathcal{T}Y_2}Z_1,Y_1) + \sin^2\theta\, g(\tilde{\nabla}_{Y1}Y_2,Z_1) \\ &\quad + \sin 2\theta\, Y_1(\theta)g(Y_2,Z_1) - g(\alpha^\#,Z_1)g(Y_1,Y_2), \\ &= g(\mathfrak{A}_{JZ_1}\mathcal{T}Y_2 - \mathfrak{A}_{\mathcal{F}\mathcal{T}Y_2}Z_1,Y_1) + \sin^2\theta\, g(\nabla_{Y1}Y_2,Z_1) \\ &\quad - g(\alpha^\#,Z_1)g(Y_1,Y_2). \end{aligned}$$

From the condition (27), we obtain $\cos^2\theta\, g(\mathfrak{h}^\theta(Y_1,Y_2),Z_1) = -\cos^2\theta\, Z_1(\mu)g(Y_1,Y_2)$. Hence, we arrive at $\mathfrak{h}^\theta(Y_1,Y_2) = -\vec{\nabla}\mu g(Y_1,Y_2)$, from the definition of gradient. Then, $\mathcal{N}^\theta$ is totally umbilical in $\mathcal{M}$ with the mean curvature vector provided by $\mathcal{H}^\theta = -\vec{\nabla}\mu$. Since $Y_2(\mu) = 0$, for all $Y_2 \in \mathfrak{D}^\theta$, then we can prove that the mean curvature is parallel concerning the normal connection. Hence, $\mathcal{N}^\theta$ is an extrinsic sphere in $\mathcal{M}$. Therefore, we conclude that $\mathcal{M}$ is a warped product submanifold $\mathcal{N}^\perp \times_f \mathcal{N}^\theta$ with the warping function μ according to Theorem 8. Thus, the theorem is proved complete. □

6. Some Applications

In this section, we introduce various special cases derived from our prior results; some of them represent significant theorems established in earlier works. This signifies that the outcomes delineated in this paper serve as expansions and generalizations of fundamental theorems. Now, we provide the following consequences:

The warped product in Theorem 7 would be a hemi-slant warped product in an $\mathcal{LCK}$ manifold if we assume θ is constant. Then, we have the following theorem for the hemi-slant warped product submanifold of an $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$.

Theorem 10. *Let $\mathcal{N}^\perp \times_f \mathcal{N}^\theta$ be a proper hemi-slant warped product submanifold of an $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$ with its Lee vector field $\alpha^\#$ tangent to $\mathcal{M}$, where $\mathcal{N}^\perp$ and $\mathcal{N}^\theta$ are totally real and proper slant submanifolds of $\tilde{\mathcal{M}}$, respectively. Then, we have*

$$Z_1(\ln f) = \alpha(Z_1)\ \forall\ Z_1 \in \Gamma(\mathfrak{D}^\perp). \tag{28}$$

Moreover, the warped product in Theorem 7 would be a warped product CR submanifold in an $\mathcal{LCK}$ manifold if we assume $\theta = 0$. In this particular case, Theorem 7 implies the following result for the warped product CR submanifold in an $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$.

Theorem 11 ([24])**.** *A proper warped product CR submanifold $\mathcal{M}$ of an $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$ such that the Lee vector field $\alpha^\#$ orthogonal to $\mathfrak{D}^\perp$ is a CR product.*

It is clear that Theorem 11 is Theorem 2.2 in [24]. Thus, the fundamental result of [24] is generalized by Theorem 7.

Now, if we consider $\alpha^\# = 0$ in Theorem 7, i.e., $\tilde{\mathcal{M}}$ is Kählerian, Theorem 7 also implies the following.

Theorem 12 ([35])**.** *Let $\mathcal{N}^\perp \times_f \mathcal{N}^\theta$ be a warped product pointwise hemi-slant submanifold of a Kähler manifold $\tilde{\mathcal{M}}$, such that $\mathcal{N}^\perp$ and $\mathcal{N}^\theta$ are totally real and proper pointwise slant submanifolds of $\tilde{\mathcal{M}}$, respectively*

$$Z(\ln f) = \tan\theta Z(\theta)\ \forall\ Z \in \Gamma(\mathfrak{D}^\perp). \tag{29}$$

Clearly, Theorem 12 is Theorem 4.7 of [35]. Thus, Theorem 7 also generalizes the main result in [35].

Moreover, in Theorem 7, if we consider that $\alpha^{\#} = 0$ and θ is a constant, then the warped product will be a hemi-slant warped product submanifold of a Kähler manifold in the form $\mathcal{M} = \mathcal{N}^{\perp} \times_f \mathcal{N}^{\theta}$, where $\mathcal{N}^{\perp}$ and $\mathcal{N}^{\theta}$ are the totally real and proper slant of $\mathcal{M}$, respectively [19].

Theorem 13 ([19])**.** *Let $\tilde{\mathcal{M}}$ be a Kähler manifold. Then, there does not exist any proper hemi-slant warped product submanifold of the form $\mathcal{M} = \mathcal{N}^{\perp} \times_f \mathcal{N}^{\theta}$, where $\mathcal{N}^{\perp}$ and $\mathcal{N}^{\theta}$ are the totally real and proper slant of $\mathcal{M}$, respectively.*

Theorem 13 is the main result (Theorem 4.2) of [19]. As a consequence, Theorem 4.2 of [19] is a special case of Theorem 7.

Now, assume that $\alpha^{\#} = 0$ and the slant function $\theta = 0$ in Theorem 7. Then, the submanifold $\mathcal{M}$ in Theorem 7 is a CR submanifold of a Kähler manifold.

Theorem 14 ([36])**.** *Let $\tilde{\mathcal{M}}$ be a Kähler manifold. Then, there does not exist any proper warped product CR submanifold of the form $\mathcal{M} = \mathcal{N}^{\perp} \times_f \mathcal{N}^{\mathcal{T}}$, where $\mathcal{N}^{\perp}$ and $\mathcal{N}^{\mathcal{T}}$ are the totally real and holomorphic submanifolds of $\mathcal{M}$, respectively.*

Theorem 14 is the main result (Theorem 3.1) of [36]. Therefore, Theorem 3.1 of [36] is a special case of Theorem 7.

A characterization theorem for the hemi-slant submanifold of an $\mathcal{LCK}$ manifold manifold is provided in the following.

If θ is constant on $\mathcal{M}$ in Theorem 9, then the warped product in Theorem 9 would be a hemi-slant warped product in an $\mathcal{LCK}$ manifold. Hence, the following theorem is an immediate consequence of Theorem 9.

Theorem 15. *A hemi-slant submanifold $\mathcal{M}$ of an $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$ with its Lee vector field $\alpha^{\#}$ tangent to $\mathcal{M}$ is locally a non-trivial warped product manifold of the form $\mathcal{M} = \mathcal{N}^{\perp} \times_f \mathcal{N}^{\theta}$ such that $\mathcal{N}^{\perp}$ is a totally real submanifold and $\mathcal{N}^{\theta}$ is a proper slant submanifold in $\tilde{\mathcal{M}}$ if and only if the shape operator $\mathfrak{A}$ for any $Y_1 \in \Gamma(\mathfrak{D}^{\theta})$ and $Z_1 \in \Gamma(\mathfrak{D}^{\perp})$ satisfies*

$$\mathfrak{A}_{\mathcal{F}\mathcal{T}Y_1} Z_1 - \mathfrak{A}_{JZ_1} \mathcal{T} Y_1 = \cos^2\theta (Z_1(\mu) - \alpha(Z_1)) Y_1, \tag{30}$$

for some smooth function μ on $\mathcal{M}$ satisfying $Y_2(\mu) = 0$ for any $Y_2 \in \Gamma(\mathfrak{D}^{\theta})$.

Furthermore, the characterization theorem for the pointwise hemi-slant submanifold of Kähler manifolds is provided in the following.

Hence, Theorem 9 implies the following characterization theorems (Theorem 4.1) of [33] and (Theorem 4.10) of [35] if $\alpha^{\#} = 0$ and θ is a slant function in Theorem 9.

Theorem 16 ([35])**.** *Let $\mathcal{M}$ be a pointwise hemi-slant submanifold of a Kähler manifold $\tilde{\mathcal{M}}$. Then, $\mathcal{M}$ is locally a non-trivial warped product manifold of the form $\mathcal{M} = \mathcal{N}^{\perp} \times_f \mathcal{N}^{\theta}$ such that $\mathcal{N}^{\perp}$ is a totally real submanifold and $\mathcal{N}^{\theta}$ is a proper pointwise slant submanifold in $\tilde{\mathcal{M}}$ if the following condition is satisfied*

$$\mathfrak{A}_{\mathcal{F}\mathcal{T}Y_1} Z_1 - \mathfrak{A}_{JZ_1} \mathcal{T} Y_1 = (\cos^2\theta) Z_1(\mu) Y_1,\ \forall\ Z_1 \in \Gamma(\mathfrak{D}^{\perp}),\ Y_1 \in \Gamma(\mathfrak{D}^{\theta}), \tag{31}$$

where μ is a function on $\mathcal{M}$ such that $Y_2(\mu) = 0$, for every $Y_2 \in \Gamma(\mathfrak{D}^{\theta})$.

7. Non-Trivial Examples

In this section, we construct some examples that guarantee the existence of a pointwise hemi-slant warped product submanifold of form $\mathcal{M} = \mathcal{N}^{\perp} \times_f \mathcal{N}^{\theta}$ of an $\mathcal{LCK}$ manifold $\tilde{\mathcal{M}}$. Now, we consider the Euclidean $2n$-space $\mathbb{E}^{2n}$ equipped with the Euclidean metric

g_0 and the Cartesian coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n)$. Then, the flat Kähler manifold $\mathbb{C}^n = (\mathbb{E}^{2n}, J, g_0)$ equipped with the canonical almost complex structure J is provided by

$$J(x_1, \cdots, x_n, y_1, \cdots, y_n) = (-y_1, \cdots, -y_n, x_1, \cdots, x_n). \tag{32}$$

The next proposition can be proven similarly to Proposition 2.2 of [10].

Proposition 1. *Let $\mathcal{M} = \mathcal{N}^\perp \times_f \mathcal{N}^\theta$ be a pointwise hemi-slant warped product of submanifolds in a Kähler manifold $\tilde{\mathcal{M}}$. Then, $\mathcal{M}$ is a warped product pointwise hemi-slant submanifold with the same slant function in an $\mathcal{LCK}$ manifold $(\tilde{\mathcal{M}}, J, \tilde{g})$ with $\tilde{g} = e^{-f} g$, where f is any smooth function on $\tilde{\mathcal{M}}$.*

Example 1. Let $\mathbb{C}^3 = (\mathbb{E}^6, J, g_0)$ be a flat Kähler manifold defined above. Consider submanifold $\mathcal{M}$ of $\mathbb{C}^3$ provided by

$$x_1 = u,\ x_2 = kw\cos\varphi,\ x_3 = \cos w,\ y_1 = u,\ y_2 = kw\sin\varphi,\ y_3 = \sin w, \tag{33}$$

where k is a positive number and u, φ, and w are non-vanishing functions on $\mathcal{M}$.

Thus, the tangent bundle $\mathcal{TM}$ of $\mathcal{M}$ is spanned by the vectors

$$U_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1},\ U_2 = -kw\sin\varphi\frac{\partial}{\partial x_2} + kw\cos\varphi\frac{\partial}{\partial y_2},$$
$$U_3 = k\cos\varphi\frac{\partial}{\partial x_2} - \sin w\frac{\partial}{\partial x_3} + k\sin\varphi\frac{\partial}{\partial y_2} + \cos w\frac{\partial}{\partial y_3}.$$

Obviously, JU_1 is orthogonal to $\mathcal{TM}$. Hence $\mathcal{M}$ is a proper hemi-slant submanifold such that the totally real distribution $\mathfrak{D}^\perp = \text{Span}\{U_1\}$ and the slant distribution $\mathfrak{D}^\theta = \text{Span}\{U_2, U_3\}$. Thus, the slant angle provided by $\theta = \cos^{-1}(k/\sqrt{k^2+1})$. Moreover, it is easy to verify that both $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$ are integrable and totally geodesic in $\mathcal{M}$. The metric tensor $\hat{g}$ on $\mathcal{M} = \mathcal{M}^\perp \times \mathcal{M}^\theta$, where $\mathcal{M}^\perp$ and $\mathcal{M}^\theta$ are the integral manifolds of $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$, respectively, is provided by

$$\hat{g} = g_\perp + g_{\mathcal{M}^\theta},\quad g_\perp = 2du^2,\ g_{\mathcal{M}^\theta} = k^2w^2d\varphi^2 + (1+k^2)dw^2. \tag{34}$$

Consider that $f = f(x_1, y_1)$ is a non-constant smooth function on $\mathbb{C}^3$ that depends on coordinates x_1, y_1. Moreover, $\tilde{\mathcal{M}} = (\mathbb{E}^6, J, \tilde{g})$ is an $\mathcal{GCK}$ manifold since the Riemannian metric $\tilde{g} = e^{-f} g_0$ on $\mathbb{C}^3$ is conformal to the standard metric g_0. Thus, the warped product metric is the metric on $\mathcal{M}$ induced from the $\mathcal{GCK}$ manifold:

$$g_{\mathcal{M}} = g_{\mathcal{M}^\perp} + e^{-f} g_{\mathcal{M}^\theta},\quad g_{\mathcal{M}^\perp} = e^{-f} g_\perp. \tag{35}$$

Furthermore, we conclude that $(\mathcal{M}, g_M)$ is a proper warped product hemi-slant submanifold in $\tilde{\mathcal{M}}$ by employing Proposition 1. Moreover, the Lee form is provided by

$$\alpha = df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial y_1}dy_1, \tag{36}$$

since $f = f(x_1, y_1)$ is a non-constant smooth function on $\mathbb{C}^3$ that depends only on coordinates x_1, y_1.

According to (35) and (36), the Lee vector field $\alpha^\#$ is tangent to $\mathcal{M}^\perp$; therefore, it is tangent to $\mathcal{M}$.

Example 2. Let $\mathcal{M}$ be a submanifold of $\mathbb{C}^4$ provided by the equations:

$$\begin{aligned} &x_1 = v,\ \ x_2 = ks\cos s^*,\ \ x_3 = h_1(s), x_4 = g_1(s^*),\\ &y_1 = v,\ \ y_2 = ks\sin s^*,\ \ y_3 = h_2(s),\ \ y_4 = g_2(s^*), \end{aligned} \tag{37}$$

defined on an open subset of $\mathbb{E}^8$ with a positive number k and non-vanishing functions v, s and s^* on $\mathcal{M}$. Also, the curves γ and δ are unit speed planar curves on $\mathcal{M}$, where $\gamma(s) = (h_1(s), h_2(s))$ and $\delta(s^*) = (g_1(s^*), g_2(s^*))$. Then, the tangent bundle $\mathcal{TM}$ is spanned by Z_1, Z_2 and Z_3, where

$$Z_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}, \quad Z_2 = k\cos s^* \frac{\partial}{\partial x_2} + k\sin s^* \frac{\partial}{\partial y_2} + h'_1(s)\frac{\partial}{\partial x_3} + h'_2(s)\frac{\partial}{\partial y_3},$$
$$Z_3 = -ks\sin s^* \frac{\partial}{\partial x_2} + ks\cos s^* \frac{\partial}{\partial y_2} + g'_1(s^*)\frac{\partial}{\partial x_4} + g'_2(s^*)\frac{\partial}{\partial y_4}.$$

Further, $\mathcal{M}$ is a proper pointwise hemi-slant submanifold such that the totally real distribution is provided by $\mathfrak{D}^\perp = \text{Span}\{Z_1\}$ and the proper pointwise slant distribution is $\mathfrak{D}^\theta = \text{Span}\{Z_2, Z_3\}$. Clearly, the Wirtinger function θ of $\mathfrak{D}^\theta$ satisfies

$$\cos\theta = \frac{k^2 s}{\sqrt{(k^2s^2+1)(k^2+1)}}.$$

Moreover, both $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$ are integrable and totally geodesic in $\mathcal{M}$. It easy to see that the metric $\hat{g}$ on $\mathcal{M} = \mathcal{M}^\perp \times \mathcal{M}^\theta$ such that $\mathcal{M}^\perp$ and $\mathcal{M}^\theta$ are integral manifolds of $\mathfrak{D}^\perp$, and $\mathfrak{D}^\theta$, respectively, is provided by

$$\hat{g} = g_\perp + g_{\mathcal{M}^\theta}, \tag{38}$$

where

$$g_\perp = 2dv^2, \quad g_{\mathcal{M}^\theta} = (1+k^2)ds^2 + (1+(ks)^2)ds^{*2}. \tag{39}$$

As in Example 1, we consider the Riemannian metric $\tilde{g} = e^{-f} g_0$ on $\mathbb{C}^4$ such that $f = f(x_1, y_1)$ is a non-constant smooth function on $\mathbb{C}^4$ that depends only on coordinates x_1, y_1. Thus, the warped product metric is the induced metric on $\mathcal{M}$:

$$g_{\mathcal{M}} = g_{\mathcal{M}^\perp} + e^{-f} g_{\mathcal{M}^\theta}, \quad g_{\mathcal{M}^\perp} = e^{-f} g_\perp. \tag{40}$$

Moreover, we apply Proposition 1 to show that $(\mathcal{M}, g_M)$ is a proper pointwise hemi-slant warped product submanifold in $\tilde{\mathcal{M}}$.

Thus, the Lee form of $\tilde{\mathcal{M}}$ is provided by (36) since $f = f(x_1, y_1)$ is a smooth function on $\mathbb{C}^4$. It is clear that the Lee vector field $\alpha^\#$ is tangent to $\mathcal{M}$ from (36), (39) and (40).

8. Conclusions

The study of warped product submanifolds has recently garnered heightened interest owing to their importance in mathematics and their application in diverse fields such as mathematical physics. The research introduces a significant contribution to the warped product submanifolds field as it defines pointwise hemi-slant submanifolds in locally conformal Kahler manifolds. It explores the properties of these submanifolds, particularly focusing on their integrability conditions and totally geodesic nature. Additionally, the research has extended to include warped product pointwise hemi-slant submanifolds and has established sufficient and necessary conditions for the classification of pointwise submanifolds as warped products of the form $\mathcal{N}^\perp \times_f \mathcal{N}^\theta$. Moreover, the research provides non-trivial examples to illustrate and support the results by elucidating the relationships and properties of these submanifolds. It is also crucial to highlight that some of the results obtained in this study serve as a generalization of the previously established results in the following papers [19,24,35,36]. Overall, the study represents a significant advancement in understanding these submanifolds and their warped products, paving the way for further research in the field of differential geometry.

Funding: We would like to express our gratitude to the University of Jeddah, Jeddah, Saudi Arabia, for providing funding under grant No. UJ-22-DR-68. Therefore, we sincerely thank the University of Jeddah for its technical and financial support.

Data Availability Statement: In this research, no external data are used.

Acknowledgments: I would like to express my sincere gratitude to all those who provided valuable feedback and reviewed the research.

Conflicts of Interest: The authors declare no conflicts of interest.

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