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Efficient Multistep Algorithms for First-Order IVPs with Oscillating Solutions: II Implicit and Predictor–Corrector Algorithms

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Abstract: This research introduces a fresh methodology for creating efficient numerical algorithms to solve first-order Initial Value Problems (IVPs). The study delves into the theoretical foundations of these methods and demonstrates their application to the Adams–Moulton technique in a five-step process. We focus on developing amplification-fitted algorithms with minimal phase-lagor phase-lag equal to zero (phase-fitted). The request of amplification-fitted (zero dissipation) is to ensure behavior like symmetric multistep methods (symmetric multistep methods are methods with zero dissipation). Additionally, the stability of the innovative algorithms is examined. Comparisons between our new algorithm and traditional methods reveal its superior performance. Numerical tests corroborate that our approach is considerably more effective than standard methods for solving IVPs, especially those with oscillatory solutions.

Keywords: numerical solution; initial value problems (IVPs); Adams–Bashforth methods; trigonometric fitting; multistep methods

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1. Introduction

Equations or systems of Equations of the form

$$\mathbf{s}'(\mathbf{t}) = \mathbf{q}(\mathbf{t}, \mathbf{s}), \ \mathbf{s}(\mathbf{t_0}) = \mathbf{s_0} \tag{1}$$

are utilized across a range of fields such as astrophysics, chemistry, physics, electronics, nanotechnology, materials science, and more. Equations with oscillatory or periodic solutions are of particular interest (refer to [1,2]).

Extensive research has been dedicated to studying the numerical solutions for the aforementioned equation or system of equations over the past two decades (see, for instance [3–11], and references therein). For a detailed examination of techniques for solving (1) with oscillating solutions, refer to [3,7,12], as well as the works of Quinlan and Tremaine [5,6,8,13], among others. Most existing numerical methods for solving (1) share a common feature of being multistep or hybrid approaches and were primarily developed for second-order differential equations. Some of the key method categories and their bibliography include:

- Exponentially-fitted, Trigonometrically-Fitted, Phase-Fitted, and Amplification-Fitted Runge–Kutta and Runge–Kutta-Nyström Methods with minimal phase-lag (refer to [14–49]);

- Exponentially-fitted and Trigonometrically-Fitted Phase-Fitted and Amplification-Fitted Multistep Methods and Multistep Methods with minimal phase-lag (refer to [50–114]).

Recently, Simos [115] developed the theory for the development of multistep methods with minimal phase-lag or phase-fitted multistep methods for first-order IVPs. More specifically, he developed the theory for computing the phase-lag and amplification error of multistep methods for first-order IVPs. In this paper, we will extend this theory to the implicit multistep methods for first-order IVPs This paper introduces the theory.

The rest of the paper is structured as follows:

- Section 2 presents the general theory for calculating the phase-lag and amplification error of implicit multistep methods for first-order IVPs. In this section, we produce the direct formulae for the calculation of the phase-lag and amplification factor;
- Section 3 introduces the methodologies and the methods which that will be developed in Sections 4–14. In this section, we present the methodologies for the development of efficient multistep methods for first-order initial value problems;
- Section 4 introduces the Adams–Bashforth five-step method and presents the methodology for the minimization of the phase-lag. In this section, we present the explicit Adams–Bashforth five-step method and we study its phase-lag and amplification error.;
- Section 5 presents the development of the amplification-fitted Adams–Bashforth five-step method of fourth algebraic order with phase-lag of order four. Based on the theory developed in Section 2, we eliminate the amplification error and we calculate the coefficients of the method in order for the method to have phase-lag of order four;
- Section 6 presents the development of the amplification-fitted Adams–Bashforth five-step method of third algebraic order with phase-lag of order six. Based on the theory presented in Section 2, we eliminate the amplification error and we calculate the coefficients of the method in order for the method to have phase-lag of order six.
- Section 7 presents the development of the amplification-fitted Adams–Bashforth five-step method of fourth algebraic order. We calculate the coefficients of the method, in order for its amplification error to be equal to zero;
- Section 8 presents the development of the amplification-fitted and phase-fitted Adams–Bashforth five-step method of fourth algebraic order. We calculate the coefficients of the method, in order for its phase-lag and amplification error to be eliminated;
- Section 9 introduces the Adams–Moulton five-step method and presents the methodology for the minimization of the phase-lag. In this section, we present the implicit Adams–Moulton five-step method and we study its phase-lag and amplification error;
- Section 10 presents the development of the amplification-fitted Adams–Moulton five-step method of fifth algebraic order with phase-lag of order four. Based on the theory developed in Section 2, we eliminate the amplification error and we calculate the coefficients of the method in order for the method to have phase-lag of order four;
- Section 11 presents the development of the amplification-fitted Adams–Moulton five-step method of second algebraic order with phase-lag of order six. Based on the theory presented in Section 2, we vanish the amplification error and we calculate the coefficients of the method in order for the method to have phase-lag of order six;
- Section 12 presents the development of the amplification-fitted Adams–Moulton five-step method of second algebraic order with phase-lag of order eight. Based on the theory presented in Section 2, we demand the amplification error to be equal to zero and we calculate the coefficients of the method in order for the method to have phase-lag of order eight;
- Section 13 presents the development of the amplification-fitted Adams–Moulton five-step method of fifth algebraic order.We calculate the coefficients of the method, in order for its amplification error to be equal to zero;
- Section 14 presents the development of the amplification-fitted and phase-fitted Adams–Moulton five-step method of fifth algebraic order. We calculate the coefficients of the method, in order for its phase-lag and amplification error to be eliminated;

- Section 15 discusses a stability analysis for the newly proposed methods in Sections 4–14. We examine the stability of the developed methods for several values of v.
- Section 16 presents numerical results. We examine the efficiency of the proposed methods in their application on seven well-known problems. For each problem, we give conclusion for the behavior of the developed methods;
- Finally, Section 17 presents the conclusions of this research.

The numerical results demonstrate that the methodology for developing phase-fitted and amplification-fitted multistep methods has yielded the most effective solutions for problems with oscillating behavior.

2. The Theory

Using the following scalar test equation, we can examine the phase-lag of multistep approaches for the problems (1).

$$s'(t) = I\,\omega\,s(t). \tag{2}$$

This problem is solved by using the following formula:

$$s(t) = \exp(I\,\omega\,t). \tag{3}$$

Taking into consideration the multistep approaches that enable the numerical solution of the problem that was discussed before (1):

$$s_{n+k} - s_{n+k-1} = h \sum_{j=0}^{k-1} \Big[A_{n+k-j}(\omega\,h)\, q_{n+k-j} \Big], \tag{4}$$

where $A_{n+k-j}(\omega\,h)$, $j = 1, 2, \ldots, k$ are polynomials of $\omega\,h$ and h is the step length of the integration.

The following result is obtained by applying Equations (2)–(4):

$$s_{n+k} - s_{n+k-1} = I\,\omega\,h \sum_{j=0}^{k-1} \Big[A_{n+k-j}(\omega\,h)\, s_{n+k-j} \Big]. \tag{5}$$

While considering:

$$vs. = \omega\,h, \tag{6}$$

(5) gives:

$$s_{n+k} - s_{n+k-1} = I\,vs. \sum_{j=0}^{k-1} \Big[A_{n+k-j}(v)\, s_{n+k-j} \Big], \tag{7}$$

and

$$(1 - I\,v\,A_{n+k}(v))\, s_{n+k} - (1 + I\,v\,A_{n+k-1}(v))\, s_{n+k-1} - I\,vs. \sum_{j=2}^{k-1} \Big[A_{n+k-j}(v)\, s_{n+k-j} \Big] = 0. \tag{8}$$

What follows is the characteristic equation of the difference equation that was mentioned in (8):

$$[1 - I\,v\,A_{n+k}(v)]\, \lambda^{k} - [1 + I\,v\,A_{n+k-1}(v)]\, \lambda^{k-1} - I\,vs. \sum_{j=2}^{k-1} \Big[A_{n+k-j}(v)\, \lambda^{k-j} \Big] = 0. \tag{9}$$

Definition 1. *Considering that the theoretical solution of the scalar test Equation (2) for $t = h$ is $\exp(I\,\omega\,h)$, which can also be written as $\exp(I\,v)$ (refer to (6)), and the numerical solution of the scalar test Equation (2) for $t = h$ is $\exp(I\,\theta(vs.))$, we can define the phase-lag as follows:*

$$\Phi = vs. - \theta(vs.). \tag{10}$$

Assuming $v \longrightarrow 0$, the phase-lag order is q if and only if $\Phi = O\left(v^{q+1}\right)$.

Considering the following:

$$\lambda^n = \exp^{n\,I\,\theta(v)} = \cos[n\,\theta(v)] + I\,\sin[n\,\theta(v)]\; n = 1, 2, \ldots, \tag{11}$$

we obtain:

$$\begin{aligned}
&\left(1 - I\,v\,A_{n+k}\right)\left\{\cos[k\,\theta(v)] + I\,\sin[k\,\theta(v)]\right\} \\
&-\left(1 + I\,v\,A_{n+k-1}\right)\left\{\cos[(k-1)\,\theta(v)] + I\,\sin[(k-1)\,\theta(v)]\right\} \\
&-\sum_{j=2}^{k-1} I\,v\,A_{n+k-j}(v)\left\{\cos[(k-j)\,\theta(v)] + I\,\sin[(k-j)\,\theta(v)]\right\} = 0.
\end{aligned} \tag{12}$$

To understand the connection mentioned above (12), you must use the next lemmas.

Lemma 1. *The following relations hold:*

$$\cos[\theta(v)] = cos(vs.) + c\,v^{q+2} + O\left(v^{q+4}\right). \tag{13}$$

$$\sin[\theta(v)] = sin(vs.) - c\,v^{q+1} + O\left(v^{q+3}\right). \tag{14}$$

For the proof, see [115].

Lemma 2. *This relation holds:*

$$\cos[j\,\theta(v)] = cos(j\,vs.) + c\,j^2\,v^{q+2} + O\left(v^{q+4}\right). \tag{15}$$

$$\sin[j\,\theta(v)] = sin(j\,vs.) - c\,j\,v^{q+1} + O\left(v^{q+3}\right). \tag{16}$$

For the proof, see [115].

When the relations (15) and (16) are considered, relation (12) shifts to:

$$\begin{aligned}
&\left(1 - I\,v\,A_{n+k}\right)\left\{\cos[k\,vs.] + c\,k^2\,v^{q+2} + I\left[\sin[k\,vs.] - c\,k\,v^{q+1}\right]\right\} \\
&-\left(1 + I\,v\,A_{n+k-1}\right)\left\{\left[\cos[(k-1)\,vs.] + c\,(k-1)^2\,v^{q+2}\right]\right. \\
&\left. + I\left[\sin[(k-1)\,vs.] - c\,(k-1)\,v^{q+1}\right]\right\} \\
&-\sum_{j=2}^{k-1} I\,v\,A_{n+k-j}(v)\left\{\left[\cos[(k-j)\,vs.] + c\,(k-j)^2\,v^{q+2}\right]\right. \\
&\left. + I\left[\sin[(k-j)\,vs.] - c\,(k-j)\,v^{q+1}\right]\right\} = 0.
\end{aligned} \tag{17}$$

It is possible to split connection (17) into two halves, the real and the imaginary.

The Real Part

The real part gives:

$$\begin{aligned}
&\cos[k\,vs.] + c\,k^2\,v^{q+2} + v\,A_{n+k}\left[\sin[k\,vs.] - c\,k\,v^{q+1}\right] \\
&- \cos[(k-1)\,vs.] - c\,(k-1)^2\,v^{q+2} \\
&+ v\,A_{n+k-1}\left[\sin[(k-1)\,vs.] - c\,(k-1)\,v^{q+1}\right] \\
&+ \sum_{j=0}^{k-1} v\,A_{n+k-j}(v)\left[\sin[(k-j)\,vs.] - c\,(k-j)\,v^{q+1}\right] = 0.
\end{aligned} \tag{18}$$

Relation (18) gives:

$$
\boxed{
\begin{aligned}
&\cos[k\,vs.] - \cos[(k-1)\,vs.] + v\sum_{j=0}^{k-1} A_{n+k-j}(v)\,\sin[(k-j)\,vs.] \\
&\quad = -c\,v^{q+2}\Big[k^2 - (k-1)^2 - \sum_{j=0}^{k-1}(k-j)\,A_{n+k-j}(v)\Big] \Longrightarrow \\
-c\,v^{q+2} &= \frac{\cos[k\,vs.] - \cos[(k-1)\,vs.] + v\sum_{j=0}^{k-1} A_{n+k-j}(v)\,\sin[(k-j)\,vs.]}{2\,k - 1 - \sum_{j=0}^{k-1} A_{n+k-j}(v)\,(k-j)}. \qquad (19)
\end{aligned}
}
$$

According to technique(4), this is the direct formula for calculating the phase-lag of the multistep approach. We will outline the steps to calculate the phase-lag of technique (4) below.

The Imaginary Part

The imaginary part gives:

$$
\begin{aligned}
&\sin[k\,vs.] - c\,k\,v^{q+1} - v\,A_{n+k}\Big[cos[k\,vs.] + c\,k^2\,v^{q+2}\Big] \\
&\quad - \sin[(k-1)\,vs.] + c\,(k-1)\,v^{q+1} \\
&\quad - v\,A_{n+k-1}\Big[cos[(k-1)\,vs.] + c\,(k-1)^2\,v^{q+2}\Big] \\
&- \sum_{j=0}^{k-1} v\,A_{n+k-j}(v)\Big[\cos[(k-j)\,vs.] + c\,(k-j)^2\,v^{q+2}\Big] = 0.
\end{aligned}
\tag{20}
$$

Relation (20) gives:

$$
\boxed{
\begin{aligned}
&\sin[k\,vs.] - \sin[(k-1)\,vs.] - v\sum_{j=0}^{k-1} A_{n+k-j}(v)\,\cos[(k-j)\,vs.] \\
&\quad = -c\,v^{q+1}\Big[-1 - v^2\sum_{j=0}^{k-1} A_{n+k-j}(v)(k-j)^2\Big] \Longrightarrow \\
-c\,v^{q+1} &= \frac{\sin[k\,vs.] - \sin[(k-1)\,vs.] - v\sum_{j=0}^{k-1} A_{n+k-j}(v)\,\cos[(k-j)\,vs.]}{-1 - v^2\sum_{j=0}^{k-1} A_{n+k-j}(v)(k-j)^2}. \qquad (21)
\end{aligned}
}
$$

In the multistep technique (4), this is the straightforward approach to calculating the amplification factor.

Definition 2. *We refer to the method with eliminated phase-lag as the* ***phase-fitted*** *method.*

Definition 3. *We refer the method with eliminated amplification factor as the* ***amplification-fitted*** *method.*

3. Procedures for the Methodologies for Achieving the Minimum Phase-Lag, Minimum Amplification Factor, Phase-Fitted, and Amplification-Fitted

In the following sections, we will present several procedure for:

- Procedures for the methodologies for achieving the minimum phase-lag;
- Procedures for the methodologies for achieving the minimum amplification factor;
- Procedures for the methodologies for achieving phase-fitted and amplification-fitted algorithms.

Methodologies for the Development of the Newly Introduced Methods

We can divided the methodologies for the development of efficient multistep methods into the following categories:

- Methods with minimization of the phase-lag (see Sections 5, 6, 10–12);
- Amplification-fitted methods (see Sections 7 and 13);
- Phase-fitted and amplification-fitted methods (see Sections 8 and 14).

4. Explicit Method: Adams–Bashforth Five-Step Method

In particular, we shall illustrate the famous Adams–Bashforth approach of fourth algebraic order, which is the following:

$$s_{n+1} - s_n = \frac{h}{24}\left(55\, s'_n - 59\, s'_{n-1} + 37\, s'_{n-2} - 9\, s'_{n-3}\right), \tag{22}$$

together with the local truncation error (LTE) provided by:

$$LTE = \frac{251}{720}\, h^5\, s^{\{(5)\}}(t) + O\left(h^6\right). \tag{23}$$

We use the theory from Section 2 to obtain the method's phase-lag and amplification error.

The difference Equation (7) with $k = 4$ is obtained by applying algorithm (22) to the test Equation (2) with:

$$A_4(vs.) = 0,\ A_3(vs.) = \frac{55}{24},\ A_2(vs.) = -\frac{59}{24},\ A_1(vs.) = \frac{37}{24},\ A_0(vs.) = -\frac{3}{8}. \tag{24}$$

By applying the Taylor series expansion to the above Equation (19) and setting $m = 1(1)4$, we can obtain the following:

$$\frac{\cos(4\,vs.) - \cos(3\,vs.) + v\,\sum_{j=0}^{4} A_{k-j}(v)\,\sin[(k-j)\,vs.]}{2\,k - 1 - \sum_{j=0}^{4} A_{k-j}(v)\,(k-j)} = -\frac{977}{5040}\,v^6 + \frac{611}{4032}\,v^8 + \ldots. \tag{25}$$

Consequently, $q = 4$ and $c = -\frac{977}{5040}$. The fourth algebraic order Adams–Bashforth method is of fourth order phase-lag.

By applying the Taylor series expansion to the above Equation (21) and setting $m = 1(1)4$, we can obtain the following:

$$\frac{\sin(4\,vs.) - \sin(3\,vs.) - v\,\sum_{j=0}^{4} A_{k-j}(v)\,\cos[(k-j)\,vs.]}{-1 - v^2\,\sum_{j=0}^{4} A_{k-j}(v)(k-j)^2} = -\frac{251}{720}\,v^5 + \frac{151{,}577}{30{,}240}\,v^7 + \ldots. \tag{26}$$

Consequently, $q = 4$ and $c = -\frac{251}{720}$. The Adams–Bashforth approach, which is of fourth order algebraic, has an amplification error of the same order. We will refer to the fourth algebraic order Adams–Bashforth algorithm as **Algorithm I** for our computing purposes.

4.1. Minimal Phase-Lag

We examine the following basic five-step algorithm to learn more about the methodologies to minimize the phase-lag:

$$s_{n+1} - s_n = h\left(K_0(vs.), s'_n + K_1(vs.)\,s'_{n-1} + K_2(vs.)\,s'_{n-2} + K_3(vs.)\,s'_{n-3}\right). \tag{27}$$

Procedure to Minimize the Phase-Lag

Below is the procedure that minimizes the phase-lag:

- Eliminating the amplification factor;
- Phase-lag calculation using the coefficient acquired in the preceding stage;
- Expanding the phase-lag calculated before using a Taylor series;
- Defining the set of equations that minimizes the phase-lag;
- Calculation of the updated coefficients.

The following two phase-lag-minimizing algorithms are derived from the aforementioned procedure.

5. Amplification-Fitted Method of Fourth Algebraic Order with Phase-Lag of Order Four

Let us consider method (27) with $K_1(vs.) = -\frac{59}{24}$, $K_3(vs.) = -\frac{9}{24}$.

5.1. Eliminating the Amplification Factor

We obtain the following result when we use the straightforward approach for computing the amplification factor (21):

$$AF = \frac{\sin(4\,v) - \sin(3\,v) - K_0(vs.)\,v\,\cos(3\,v) + \frac{59}{24}\cos(2\,v)\,v - K_2(vs.)\,v\,\cos(vs.) + \frac{3}{8}\,v}{-v^2\,K_2(vs.) - 9\,v^2\,K_0(vs.) + \frac{59}{6}\,v^2 - 1}, \tag{28}$$

where AF denotes the amplification factor.

Assuming that the amplification factor must be eliminated, or that $AF = 0$, we derive:

$$K_0(vs.) = -\frac{1}{24}\,\frac{24\,K_2\,v\,\cos(vs.) - 59\,v\,\cos(2\,v) - 24\,\sin(4\,v) + 24\,\sin(3\,v) - 9\,v}{v\,\cos(3\,v)}. \tag{29}$$

5.2. Procedure for Minimizing the Phase-Lag

We can obtain the phase-lag by plugging the values of $K_0(vs.)$ and $K_2(vs.)$ that were previously provided into the direct formula for calculating it (19):

$$PhErr = -\frac{v\left(18\,\sin(vs.)\,(\cos(vs.))^2\,v - 24\,\sin(vs.)\,\cos(vs.)\,v\,K_2(vs.) + 25\,v\,\sin(vs.) + 12\,\cos(vs.) - 12\right)}{\Xi_1(vs.)}, \tag{30}$$

where

$$\begin{aligned} \Xi_1(vs.) &= 48\,(\cos(vs.))^3\,v\,K_2(vs.) + 288\,\sin(vs.)\,(\cos(vs.))^3 - 572\,(\cos(vs.))^3\,vs. \\ &- 144\,\sin(vs.)\,(\cos(vs.))^2 + 177\,v\,(\cos(vs.))^2 - 72\,K_2(vs.)\,v\,\cos(vs.) \\ &- 144\,\sin(vs.)\,\cos(vs.) + 429\,v\,\cos(vs.) + 36\,\sin(vs.) - 75\,v, \end{aligned} \tag{31}$$

and $PhErr$ denotes the phase-lag.

By applying the Taylor series expansion to the Formula (30), we are able to retrieve:

$$\begin{aligned} PhErr &= -\frac{(37 - 24\,K_2(vs.))v^2}{-24\,K_2(vs.) - 5} \\ &- \frac{v^4}{-24\,K_2(vs.) - 5}\left[-\frac{74}{3} + 16\,K_2(vs.) - \frac{-37 + 24\,K_2(vs.)}{24\,K_2(vs.) + 5}\left(-36\,K_2(vs.) + \frac{489}{2}\right)\right] \\ &- \frac{v^6}{-24\,K_2(vs.) - 5}\left[\frac{1121}{120} - \frac{16\,K_2(vs.)}{5} - \frac{-37 + 24\,K_2(vs.)}{24\,K_2(vs.) + 5}\left(39\,K_2(vs.) - \frac{7573}{40}\right)\right. \\ &+ \left.\frac{1}{6}\,\frac{7488\,K_2(vs.)^2 - 46{,}272\,K_2(vs.) + 53{,}539}{(24\,K_2(vs.) + 5)^2}\left(-36\,K_2(vs.) + \frac{489}{2}\right)\right] + \ldots. \end{aligned} \tag{32}$$

Requiring the phase-lag to be minimized, we obtain the equation mentioned below:

$$-\frac{(37-24\,K_2(vs.))v^2}{-24\,K_2(vs.)-5}=0 \Longrightarrow K_2(vs.)=\frac{37}{24}. \tag{33}$$

This novel algorithm has the following features:

$$\begin{aligned}
K_0(vs.) &= \frac{1}{24}\,\frac{\Xi_2(vs.)}{v\cos(3\,v)}, \\
K_1(vs.) &= -\frac{59}{24}, \\
K_2(vs.) &= \frac{37}{24}, \\
K_3(vs.) &= -\frac{9}{24}, \\
LTE &= \frac{251}{720}\,h^5\left(s^{\{(5)\}}(t)-\omega^4\,s'(t)\right)+O\left(h^6\right) \\
PhErr &= \frac{529}{5040}\,v^6+\frac{20{,}551}{47{,}040}\,v^8+\ldots, \\
AF &= 0,
\end{aligned} \tag{34}$$

where

$$\begin{aligned}
\Xi_2(vs.) &= 192\,\sin(vs.)\,(\cos(vs.))^3-96\,\sin(vs.)\,(\cos(vs.))^2 \\
&+ 118\,v\,(\cos(vs.))^2-96\,\sin(vs.)\,\cos(vs.) \\
&- 37\,v\cos(vs.)+24\,\sin(vs.)-50\,v.
\end{aligned} \tag{35}$$

$K_0(vs.)$ may be expressed as a Taylor series expansion:

$$K_0(vs.)=\frac{55}{24}+\frac{251\,v^4}{720}+\frac{647\,v^6}{756}+\ldots. \tag{36}$$

From a computational standpoint, we shall refer to the aforementioned new technique as **Algorithm II**.

Remark 1. *If we chose three free parameters (for example K_j, $j=0(1)2$), the resulting algorithm will be the same as above.*

6. Amplification-Fitted Method of Third Algebraic Order with Phase-Lag of Order Six

Let us consider method (27) with the parameter $K_j(vs.)$, $j=0(1)3$ free.

For the development of the method, see Appendix A.

This novel algorithm has the following features:

$$\begin{aligned}
K_0(vs.)(vs.) &= \frac{\Psi_{12}(vs.)}{720\,v\cos(3\,v)}, \\
K_1(vs.) &= -\frac{179}{288}, \\
K_2(vs.) &= \frac{13}{180}, \\
K_3(vs.) &= -\frac{11}{1440}, \\
LTE &= \frac{529}{1440}\,h^4\left(s^{\{(3)\}}(t)-\omega^2\,s'(t)\right)+O\left(h^5\right), \\
PhErr &= -\frac{191}{423{,}360}\,v^8-\frac{426{,}379}{237{,}081{,}600}\,v^{10}+\ldots, \\
AF &= 0,
\end{aligned} \tag{37}$$

where

$$\begin{aligned}\Psi_{12}(vs.) &= 5760\,\sin(vs.)(\cos(vs.))^3 - 2880\,\sin(vs.)(\cos(vs.))^2 \\ &+ 895\,v(\cos(vs.))^2 - 2880\,\sin(vs.)\cos(vs.) \\ &- 52\,v\cos(vs.) + 720\,\sin(vs.) - 442\,v.\end{aligned} \tag{38}$$

$K_0(vs.)(vs.)$ may be expressed as a Taylor series expansion:

$$K_0(vs.)(vs.) = \frac{1121}{720} - \frac{529}{1440}\,v^2 + \frac{41}{3456}\,v^4 - \frac{8623}{3{,}628{,}800}\,v^6 + \ldots. \tag{39}$$

From a computational standpoint, we shall refer to the aforementioned new technique as **Algorithm III**.

7. Amplification-Fitted Method of Fourth Algebraic Order

Let us consider method (27) with $K_1(vs.) = -\frac{59}{24}$, $K_2(vs.) = \frac{37}{24}$, $K_3(vs.) = -\frac{9}{24}$

7.1. Eliminating the Amplification Factor

We obtain the following result when we use the straightforward approach for computing the amplification factor (21):

$$AF = \frac{\sin(4\,v) - \sin(3\,v) - K_0(vs.)\,v\cos(3\,v) + \frac{59}{24}\,v\cos(2\,v) - \frac{37}{24}\,v\cos(vs.) + \frac{3}{8}\,v}{-9\,v^2\,K_0(vs.) + \frac{199}{24}\,v^2 - 1}, \tag{40}$$

where AF denotes the amplification factor.

Assuming that the amplification factor must be eliminated, or that $AF = 0$, we derive:

$$K_0(vs.) = \frac{1}{24}\,\frac{59\,v\cos(2\,v) - 37\,v\cos(vs.) + 24\,\sin(4\,v) - 24\,\sin(3\,v) + 9\,v}{v\cos(3\,v)}. \tag{41}$$

Phase-Lag of the Method

We can obtain the phase-lag by plugging the value of $K_0(vs.)$ that was previously provided into the direct formula for calculating it (19):

$$PhErr = -\frac{1}{3}\,\frac{v\,\Psi_{13}(vs.)}{\Psi_{14}(vs.)}. \tag{42}$$

where

$$\begin{aligned}\Psi_{13}(vs.) &= 18\,(\cos(vs.))^2\sin(vs.)v - 37\,\cos(vs.)\sin(vs.)v \\ &+ 25\,v\sin(vs.) + 12\,\cos(vs.) - 12, \\ \Psi_{14}(vs.) &= 96\,\sin(vs.)(\cos(vs.))^3 - 166\,v(\cos(vs.))^3 \\ &- 48\,(\cos(vs.))^2\sin(vs.) + 59\,v(\cos(vs.))^2 \\ &- 48\,\cos(vs.)\sin(vs.) + 106\,v\cos(vs.) + 12\,\sin(vs.) - 25\,v,\end{aligned} \tag{43}$$

and $PhErr$ denotes the phase-lag.

By applying the Taylor series expansion to the Formula (42), we are able to retrieve:

$$PhErr = \frac{529}{5040}\,v^6 + \frac{20{,}551}{47{,}040}\,v^8 + \ldots. \tag{44}$$

This novel algorithm has the following features:

$$\begin{aligned}
K_0(vs.) &= \frac{1}{24}\,\frac{59\,v\cos(2\,v) - 37\,v\cos(vs.) + 24\,\sin(4\,v) - 24\,\sin(3\,v) + 9\,v}{v\cos(3\,v)}, \\
K_1(vs.) &= -\frac{59}{24}, \\
K_2(vs.) &= \frac{37}{24}, \\
K_3(vs.) &= -\frac{9}{24}, \\
LTE &= \frac{251}{720}\,h^5\left(s^{\{(5)\}}(t) - \omega^4\,s'(t)\right) + O\left(h^6\right), \\
PhErr &= \frac{529}{5040}\,v^6 + \frac{20{,}551}{47{,}040}\,v^8 + \ldots, \\
AF &= 0.
\end{aligned} \tag{45}$$

$K_0(vs.)$ may be expressed as a Taylor series expansion:

$$K_0(vs.) = \frac{55}{24} + \frac{251\,v^4}{720} + \frac{647\,v^6}{756} + \ldots. \tag{46}$$

From a computational standpoint, we shall refer to the aforementioned new technique as **Algorithm IV**.

8. Phase-Fitted and Amplification-Fitted Fourth Order Adams–Bashforth Method

Procedure (27) is taken into account, with $K_1(vs.) = -\frac{59}{24}$, and $K_3(vs.) = -\frac{9}{24}$

The following is the result that we obtain when we use the straightforward approach for calculating the phase-lag and the amplification factor:

$$PhErr = \frac{\cos(4\,v) - \cos(3\,v) + K_0(vs.)\,v\,\sin(3\,v) - \frac{59\,v\,\sin(2\,v)}{24} + K_2(vs.)\,v\,\sin(vs.)}{\frac{143}{12} - 3\,K_0(vs.) - K_2(vs.)}, \tag{47}$$

$$AF = \frac{\sin(4\,v) - \sin(3\,v) - K_0(vs.)\,v\,\cos(3\,v) + \frac{59\,v\cos(2\,v)}{24} - K_2(vs.)\,v\,\cos(vs.) + \frac{3}{8}\,v}{-v^2\,K_2(vs.) - 9\,v^2\,K_0(vs.) + \frac{59}{6}\,v^2 - 1}, \tag{48}$$

with $PhErr$ denoting the phase-lag and AF denoting the amplification factor.

After the phase-lag and amplification factors have been eliminated, or $PhErr = 0$ and $AF = 0$, the following result is obtained:

$$K_0(vs.) = \frac{1}{24}\,\frac{48\,(\sin(vs.))^2\,\cos(vs.) + 25\,v\,\sin(vs.) - 24\,(\sin(vs.))^2 - 12\,\cos(vs.) + 12}{v\,\cos(vs.)\,\sin(vs.)}, \tag{49}$$

$$K_2(vs.) = -\frac{1}{24}\,\frac{18\,v\,(\sin(vs.))^3 - 43\,v\,\sin(vs.) - 12\,\cos(vs.) + 12}{v\,\cos(vs.)\,\sin(vs.)}. \tag{50}$$

By expanding the aforementioned formulas using the Taylor series, we obtain:

$$\begin{aligned}
K_0(vs.) &= \frac{55}{24} + \frac{95}{576}\,v^4 + \frac{2935}{48{,}384}\,v^6 + \frac{14{,}417}{580{,}608}\,v^8 \\
&+ \frac{27{,}559}{2{,}737{,}152}\,v^{10} + \frac{121{,}989{,}367}{29{,}889{,}699{,}840}\,v^{12} + \ldots,
\end{aligned} \tag{51}$$

$$\begin{aligned}
K_2(vs.) &= \frac{37}{24} + \frac{529}{2880}\,v^4 + \frac{14{,}621}{241{,}920}\,v^6 + \frac{10{,}321}{414{,}720}\,v^8 \\
&+ \frac{4{,}824{,}823}{479{,}001{,}600}\,v^{10} + \frac{610{,}012{,}553}{149{,}448{,}499{,}200}\,v^{12} + \ldots.
\end{aligned} \tag{52}$$

The characteristics of this new method are:

$$\begin{aligned} K_0(vs.) &\text{ see (49)},\\ K_1(vs.) &= -\frac{59}{24},\\ K_2(vs.) &\text{ see (50)},\\ K_3(vs.) &= -\frac{9}{24},\\ LTE &= \frac{251}{720}\, h^5 \left(s^{\{(5)\}}(t) - \omega^4\, s'(t)\right) + O\left(h^6\right),\\ PhErr &= 0,\\ AF &= 0. \end{aligned} \tag{53}$$

From a computational standpoint, we shall refer to the aforementioned new technique as **Algorithm V**.

9. Implicit Method: Adams–Moulton Five-Step Method

In particular, we shall illustrate the famous Adams–Moulton approach of fifth algebraic order, which is the following:

$$s_{n+1} - s_n = \frac{h}{720}\left(251\, s'_{n+1} + 646\, s'_n - 264\, s'_{n-1} + 106\, s'_{n-2} - 19\, s'_{n-3}\right). \tag{54}$$

together with the local truncation error (LTE) provided by:

$$LTE = -\frac{3}{360}\, h^6\, s^{\{(6)\}}(t) + O\left(h^7\right). \tag{55}$$

We use the theory from Section 2 to obtain the method's phase-lag and amplification error.

Difference Equation (7) with $k = 4$ is obtained by applying the algorithm (54) to the test Equation (2) with:

$$A_4(vs.) = \frac{251}{720},\ A_3(vs.) = \frac{323}{360},\ A_2(vs.) = -\frac{11}{30},\ A_1(vs.) = \frac{53}{360},\ A_0(vs.) = -\frac{19}{720}. \tag{56}$$

By applying the Taylor series expansion to the above Equation (19) and setting $m = 0(1)4$, we can obtain:

$$\frac{\cos(4\, vs.) - \cos(3\, vs.) + v\, \sum_{j=0}^{4} A_{k-j}(v)\, \sin[(k-j)\, vs.]}{2\, k - 1 - \sum_{j=0}^{4} A_{k-j}(v)\, (k-j)} = \frac{3}{560}\, v^6 - \frac{25}{1728}\, v^8 + \ldots. \tag{57}$$

Consequently, $q = 4$ and $c = \frac{3}{560}$. The fifth algebraic order Adams–Moulton method is of fourth order phase-lag.

By applying the Taylor series expansion to the above Equation (21) and setting $m = 1(1)4$, we can obtain the following:

$$\frac{\sin(4\, vs.) - \sin(3\, vs.) - v\, \sum_{j=0}^{4} A_{k-j}(v)\, \cos[(k-j)\, vs.]}{-1 - v^2\, \sum_{j=0}^{4} A_{k-j}(v)(k-j)^2} = -\frac{641}{15{,}120}\, v^7 + \frac{56{,}953}{100{,}800}\, v^9 + \ldots. \tag{58}$$

Consequently, $q = 6$ and $c = -\frac{641}{15{,}120}$. The Adams–Moulton approach, which is of fifth order algebraic, has an amplification error of sixth order. We will refer to the fifth algebraic order Adams–Moulton algorithm as **Algorithm VI** for our computing purposes.

9.1. Minimal Phase-Lag

We examine the following basic five-step algorithm to learn more about the methodologies to minimize the phase-lag:

$$s_{n+1} - s_n = h\left(Q_0(vs.), s'_{n+1} + Q_1(vs.), s'_n + Q_2(vs.)\, s'_{n-1} + Q_3(vs.)\, s'_{n-2} + Q_4(vs.)\, s'_{n-3}\right). \tag{59}$$

Procedure to Minimize the Phase-Lag

Below is the procedure that minimizes the phase-lag:

- Eliminating the amplification factor;
- Phase-lag calculation using the coefficient acquired in the preceding stage;
- Expanding the phase-lag calculated before using a Taylor series;
- Defining the set of equations that minimizes the phase-lag;
- Calculation of the updated coefficients.

The following three phase-lag-minimizing algorithms are derived from the aforementioned procedure.

10. Amplification-Fitted Adams–Moulton Five-Step Method of Fifth Algebraic Order with Phase-Lag of Order Four

Let us consider method (59) with $Q_1(vs.) = \frac{323}{360}$, $Q_2(vs.) = -\frac{11}{30}$.
For the development of this algorithm, see Appendix B.

This novel algorithm has the following features:

$$\begin{aligned}
Q_0(vs.) &= \frac{\Psi_{21}(vs.)}{360\, v \cos(4\, v)}, \\
Q_1(vs.) &= \frac{323}{360}, \\
Q_2(vs.) &= -\frac{11}{30}, \\
Q_3(vs.) &= \frac{53}{360}, \\
Q_4(vs.) &= -\frac{19}{720}, \\
LTE &= -\frac{3}{160}\, h^6 \left(s^{\{(6)\}}(t)\right) + +O\left(h^7\right), \\
PhErr &= \frac{3}{560}\, v^6 + \dots, \\
AF &= 0,
\end{aligned} \tag{60}$$

where

$$\begin{aligned}
\Psi_{21}(vs.) &= -53\, v\, \cos(vs.) - 323\, v\, \cos(3\, v) + 132\, v\, \cos(2\, v) \\
&+ \frac{19}{2}\, v + 360\, \sin(4\, v) - 360\, \sin(3\, v).
\end{aligned} \tag{61}$$

$Q_0(vs.)$ may be expressed as a Taylor series expansion:

$$Q_0(vs.) = \frac{251}{720} + \frac{641\, v^6}{15{,}120} + \frac{269{,}443\, v^8}{907{,}200} + \frac{77{,}375{,}869\, v^{10}}{39{,}916{,}800} + \dots. \tag{62}$$

From a computational standpoint, we shall refer to the aforementioned new technique as **Algorithm VII**.

11. Amplification-Fitted Adams–Moulton Five-Step Method of Second Algebraic Order with Phase-Lag of Order Six

Let us consider method (59) with $Q_2(vs.) = \frac{323}{360}$.

For the development of this algorithm, see Appendix C.

This novel algorithm has the following features:

$$\begin{aligned}
Q_0(vs.) &= \frac{\Psi_{34}(vs.)}{20{,}160\, v\, \cos(4\,v)}, \\
Q_1(vs.) &= \frac{323}{360}, \\
Q_2(vs.) &= -\frac{167}{480}, \\
Q_3(vs.) &= \frac{317}{2520}, \\
Q_4(vs.) &= -\frac{397}{20{,}160}, \\
LTE &= -\frac{3}{560}\, h^3 \left(s^{\{(3)\}}(t) + \omega^2\, s'(t)\right) + + O\left(h^4\right), \\
PhErr &= -\frac{313}{60{,}480}\, v^8 + \ldots, \\
AF &= 0,
\end{aligned} \tag{63}$$

where

$$\begin{aligned}
\Psi_{34}(vs.) &= 7014\, v\, \cos(2\,v) - 2536\, v\, \cos(vs.) - 18{,}088\, v\, \cos(3\,v) \\
&+ 397\, v + 20{,}160\, \sin(4\,v) - 20{,}160\, \sin(3\,v).
\end{aligned} \tag{64}$$

$Q_0(vs.)$ may be expressed as a Taylor series expansion:

$$Q_0(vs.) = \frac{6947}{20{,}160} - \frac{3}{560}\, v^2 - \frac{13}{1120}\, v^4 - \frac{4397}{302{,}400}\, v^6 - \frac{2{,}983{,}453}{50{,}803{,}200}\, v^8 + \ldots. \tag{65}$$

From a computational standpoint, we shall refer to the aforementioned new technique as **Algorithm VIII**.

12. Amplification-Fitted Adams–Moulton Five-Step Method of Second Algebraic Order with Phase-Lag of Order Eight

Let us consider method (59).

For the development of this algorithm, see Appendix D.

This novel algorithm has the following features:

$$\begin{aligned}
Q_0(vs.) &= \frac{1}{120{,}960}\, \frac{\Psi_{59}(vs.)}{v\, \cos(4\,v)}, \\
Q_1(vs.) &= \frac{5561}{8640}, \\
Q_2(vs.) &= -\frac{163}{1728}, \\
Q_3(vs.) &= \frac{23}{1344}, \\
Q_4(vs.) &= -\frac{191}{120{,}960}, \\
LTE &= -\frac{1867}{60{,}480}\, h^3 \left(s^{\{(3)\}}(t) + \omega^2\, s'(t)\right) + + O\left(h^4\right), \\
PhErr &= -\frac{2497}{25{,}401{,}600}\, v^{10} + \ldots, \\
AF &= 0,
\end{aligned} \tag{66}$$

where

$$\begin{aligned}\Psi_{59}(vs.) &= -77{,}854\, v\, \cos(3\, v) + 11{,}410\, v\, \cos(2\, v) + 191\, v \\ &- 2070\, v \cos(vs.) + 120{,}960\, \sin(4\, v) - 120{,}960\, \sin(3\, v). \end{aligned} \quad (67)$$

$Q_0(vs.)$ may be expressed as a Taylor series expansion:

$$Q_0(vs.) = \frac{52{,}637}{120{,}960} + \frac{1867}{60{,}480}\, v^2 + \frac{2531}{725{,}760}\, v^4 + \frac{14{,}257}{21{,}772{,}800}\, v^6 + \frac{448{,}163}{1{,}219{,}276{,}800}\, v^8 + \ldots. \quad (68)$$

From a computational standpoint, we shall refer to the aforementioned new technique as **Algorithm IX**.

13. Amplification-Fitted Adams–Moulton Five-Step Method of Fifth Algebraic Order

Let us consider method (59) with with $Q_1(vs.) = \frac{323}{360}$, $Q_2(vs.) = -\frac{11}{30}$, $Q_3(vs.) = \frac{53}{360}$, $Q_4(vs.) = -\frac{19}{720}$.

13.1. Eliminating the Amplification Factor

We obtain the following result when we use the straightforward approach for computing the amplification factor (21):

$$AF = \frac{\Psi_{60}(vs.)}{-16\, v^2\, Q_0(vs.) - \frac{304\, v^2}{45} - 1}. \quad (69)$$

where AF denotes the amplification factor, and

$$\begin{aligned}\Psi_{60}(vs.) &= \sin(4\, v) - \sin(3\, v) - Q_0(vs.)\, v\, \cos(4\, v) \\ &- \frac{323}{360}\, v\, \cos(3\, v) + \frac{11}{30}\, v\, \cos(2\, v) \\ &- \frac{53}{360}\, v\, \cos(vs.) + \frac{19\, v}{720}. \end{aligned} \quad (70)$$

Assuming that the amplification factor must be eliminated, or that $AF = 0$, we derive:

$$Q_0(vs.) = \frac{\Psi_{61}(vs.)}{720\, v \cos(4\, v)}. \quad (71)$$

where

$$\begin{aligned}\Psi_{61}(vs.) &= -646\, v\, \cos(3\, v) + 264\, v\, \cos(2\, v) - 106\, v\, \cos(vs.) \\ &+ 720\, \sin(4\, v) - 720\, \sin(3\, v) + 19\, v. \end{aligned} \quad (72)$$

Phase-Lag of the Method

We can obtain the phase-lag by plugging the value of $Q_0(vs.)$ that was previously provided into the direct formula for calculating it (19):

$$PhErr = \frac{1}{2}\, \frac{v\, \Psi_{62}(vs.)}{\Psi_{63}(vs.)}. \quad (73)$$

where

$$\begin{aligned}\Psi_{62}(vs.) &= 38\, (\cos(vs.))^3\, \sin(vs.) vs. - 106\, (\cos(vs.))^2\, \sin(vs.)\, v \\ &+ 113\, \cos(vs.)\, \sin(vs.) v - 135\, v \sin(vs.) \\ &- 180\, \cos(vs.) + 180, \\ \Psi_{63}(vs.) &= 3524\, v\, (\cos(vs.))^4 - 2880\, (\cos(vs.))^3\, \sin(vs.) \\ &+ 1292\, v\, (\cos(vs.))^3 + 1440\, (\cos(vs.))^2\, \sin(vs.) \\ &- 3788\, v\, (\cos(vs.))^2 + 1440\, \cos(vs.)\, \sin(vs.) \\ &- 916\, v\, \cos(vs.) - 360\, \sin(vs.) + 563\, v. \end{aligned} \quad (74)$$

and $PhErr$ denotes the phase-lag.

By applying the Taylor series expansion to Formula (42), we are able to retrieve:

$$PhErr = \frac{3}{560}\, v^{6} + \frac{14{,}387}{423{,}360}\, v^{8} + \ldots. \tag{75}$$

This novel algorithm has the following features:

$$\begin{aligned}
Q_0(vs.) &= \frac{\Psi_{61}(vs.)}{720\, v \cos(4\, v)}, \\
Q_1(vs.) &= \frac{323}{360}, \\
Q_2(vs.) &= -\frac{11}{30}, \\
Q_3(vs.) &= \frac{53}{360}, \\
Q_4(vs.) &= -\frac{19}{720}, \\
LTE &= -\frac{3}{160}\, h^{6}\, s^{\{(6)\}}(t) + O\left(h^{7}\right), \\
PhErr &= \frac{3}{560}\, v^{6} + \frac{14{,}387}{423{,}360}\, v^{8} + \ldots, \\
AF &= 0.
\end{aligned} \tag{76}$$

$Q_0(vs.)$ may be expressed as a Taylor series expansion:

$$Q_0(vs.) = \frac{251}{720} + \frac{641}{15{,}120}\, v^{6} + \frac{269{,}443}{907{,}200}\, v^{8} + \frac{77{,}375{,}869}{39{,}916{,}800}\, v^{10} + \ldots. \tag{77}$$

From a computational standpoint, we shall refer to the aforementioned new technique as **Algorithm X**.

14. Phase-Fitted and Amplification-Fitted Fifth Order Adams–Moulton Method

Let us considering the method (59) with $Q_1(vs.) = \frac{323}{360}$, $Q_2(vs.) = -\frac{11}{30}$, and $Q_4(vs.) = -\frac{19}{720}$.

The following is the result that we obtain when we use the straightforward approach for calculating the phase-lag and the amplification factor:

$$\begin{aligned}
PhErr &= \frac{\Psi_{64}(vs.)}{-1815 + 1440\, Q_0(vs.) + 360\, Q_3(vs.)}, \\
AF &= \frac{\Psi_{65}(vs.)}{11{,}520\, v^2\, Q_0(vs.) + 720\, v^2\, Q_3(vs.) + 4758\, v^2 + 720},
\end{aligned} \tag{78}$$

with $PhErr$ denoting the phase-lag and AF denoting the amplification factor, and

$$\begin{aligned}
\Psi_{64}(vs.) &= -360\, Q_0(vs.)\, v\, \sin(4\, v) - 323\, v\, \sin(3\, v) \\
&- 360\, \cos(4\, v) + 132\, v\, \sin(2\, v) \\
&- 360\, Q_3(vs.)\, v\, \sin(vs.) + 360\, \cos(3\, v), \\
\Psi_{65}(vs.) &= 5760\, v\, Q_0(vs.)\, (\cos(vs.))^4 - 5760\, \sin(vs.)\, (\cos(vs.))^3 \\
&+ 2584\, v\, (\cos(vs.))^3 - 5760\, (\cos(vs.))^2\, v\, Q_0(vs.) \\
&+ 2880\, \sin(vs.)\, (\cos(vs.))^2 - 528\, (\cos(vs.))^2\, v \\
&+ 720\, Q_3(vs.)\, v\, \cos(vs.) + 2880\, \sin(vs.)\, \cos(vs.) \\
&- 1938\, v\, \cos(vs.) + 720\, v\, Q_0(vs.) \\
&- 720\, \sin(vs.) + 245\, v.
\end{aligned} \tag{79}$$

After the phase-lag and amplification factors have been eliminated, or $PhErr = 0$ and $AF = 0$, the following result is obtained:

$$Q_0(vs.) = \frac{1}{720} \frac{\Psi_{66}(vs.)}{v\left(4\,(\cos(vs.))^3 + 4\,(\cos(vs.))^2 - \cos(vs.) - 1\right)}. \tag{80}$$

$$Q_3(vs.) = \frac{1}{360} \frac{\Psi_{67}(vs.)}{v\left(4\,(\cos(vs.))^3 + 4\,(\cos(vs.))^2 - \cos(vs.) - 1\right)}. \tag{81}$$

where

$$\begin{aligned}
\Psi_{66}(vs.) &= 2880\,\sin(vs.)\,(\cos(vs.))^2 - 1292\,(\cos(vs.))^2\,v \\
&+ 1440\,\sin(vs.)\,\cos(vs.) - 1047\,v\,\cos(vs.) \\
&- 720\,\sin(vs.) + 245\,v, \\
\Psi_{67}(vs.) &= 76\,v\,(\cos(vs.))^4 + 76\,v(\cos(vs.))^3 \\
&+ 226\,(\cos(vs.))^2\,v - 97\,v\cos(vs.) \\
&+ 360\,\sin(vs.) - 323\,v.
\end{aligned} \tag{82}$$

By expanding the aforementioned formulas using the Taylor series, we obtain:

$$\begin{aligned}
Q_0(vs.) &= \frac{251}{720} - \frac{1}{160}\,v^4 - \frac{1271}{362{,}880}\,v^6 - \frac{52{,}901}{21{,}772{,}800}\,v^8 \\
&- \frac{362{,}891}{179{,}625{,}600}\,v^{10} - \frac{2{,}012{,}951{,}791}{1{,}120{,}863{,}744{,}000}\,v^{12} + \ldots.
\end{aligned} \tag{83}$$

$$\begin{aligned}
Q_3(vs.) &= \frac{53}{360} + \frac{1}{160}\,v^4 - \frac{71}{72{,}576}\,v^6 - \frac{39{,}667}{21{,}772{,}800}\,v^8 \\
&- \frac{1{,}351{,}639}{718{,}502{,}400}\,v^{10} - \frac{1{,}977{,}358{,}457}{1{,}120{,}863{,}744{,}000}\,v^{12} + \ldots.
\end{aligned} \tag{84}$$

The characteristics of this new method are:

$$\begin{aligned}
Q_0(vs.) &\text{ see (80)}, \\
Q_1(vs.) &= \frac{323}{360}, \\
Q_2(vs.) &= -\frac{11}{30}, \\
Q_3(vs.) &\text{ see (81)}, \\
Q_4(vs.) &= -\frac{19}{720}, \\
LTE &= -\frac{3}{160}\,h^6\left(s^{\{(6)\}}(t) - \omega^4\,s^{\{(2)\}}(t)\right) + O\left(h^7\right), \\
PhErr &= 0, \\
AF &= 0.
\end{aligned} \tag{85}$$

From a computational standpoint, we shall refer to the aforementioned new technique as **Algorithm XI**.

15. Stability Analysis

In this section, we will study the stability of the methods developed in Sections 4–14.

15.1. Adams–Bashforth Algorithm

A general description of the five-step methods proposed by Adams–Bashforth (Explicit) and Adams–Moulton (Implicit) is as follows:

$$s_{n+1} - s_n = h\left(T_3(vs.), s'_n + T_2(vs.)\, s'_{n-1} + T_1(vs.)\, s'_{n-2} + T_0(vs.)\, s'_{n-3}\right). \tag{86}$$

Algorithms (22), (34), (37), (45), and (53) that were studied in Sections 4–8 constitute the general algorithm (86).

By combining the scalar test equation:

$$s' = \lambda\, s \quad \text{where } \lambda \in \mathcal{C}, \tag{87}$$

with the scheme (86), we can obtain the subsequent difference equation

$$s_{n+1} - S_3(H)\, s_n - S_2(H)\, s_{n-1} - S_1(H)\, s_{n-2} - S_0(H)\, s_{n-3} = 0, \tag{88}$$

with $H = \lambda\, h$ and

$$S_3(H) = 1 + T_3\, H,\ S_2(H) = T_2\, H,\ S_1(H)(H) = T_1\, H,\ S_0(H)(H) = T_0\, H. \tag{89}$$

Presenting the characteristic equation of (88), we have:

$$r^4 - S_3(H)\, r^3 - S_2(H)\, r^2 - S_1(H)\, r - S_0(H) = 0. \tag{90}$$

15.2. Adams–Moulton Five-Step Algorithm

A general description of the five-step methods proposed by Adams–Moulton (Implicit) is as follows:

$$s_{n+1} - s_n = h\left(G_4(vs.), s'_{n+1} + G_3(vs.), s'_n + G_2(vs.)\, s'_{n-1} + G_1(vs.)\, s'_{n-2} + G_0(vs.)\, s'_{n-3}\right). \tag{91}$$

Algorithms (54), (60), (66), (76), (85), and (A31) that were studied in Sections 4–8 constitute the general algorithm (91).

By combining scalar test Equation (87) with scheme (91), we can obtain the subsequent difference equation

$$W_4(H)\, s_{n+1} - W_3(H)\, s_n - W_2(H)\, s_{n-1} - W_1(H)\, s_{n-2} - W_0(H)\, s_{n-3} = 0, \tag{92}$$

with $H = \lambda\, h$ and

$$\begin{aligned} W_4(H) &= 1 - G_4\, H,\ W_3(H) = 1 + G_3\, H, \\ W_2(H) &= G_2\, H,\ W_1(H) = G_1\, H, \\ W_0(H) &= G_0\, H. \end{aligned} \tag{93}$$

Presenting the characteristic equation of (92), we have:

$$W_4(H)\, r^4 - W_3(H)\, r^3 - W_2(H)\, r^2 - W_1(H)\, r - W_0(H) = 0. \tag{94}$$

15.3. Stabilities of Adams–Bashforth and Adams–Moulton Algorithms

We can visualize the stability areas for $\theta \in [0, 2\,\pi]$ by solving the original Equations (90) and (94) in H and inserting $r = \exp(i\,\theta)$, where $i = \sqrt{-1}$. We display the stability areas for the accomplished **Methods I–V** in Figures 1–11. We show the stability areas for $v = 1$, $v = 30$, and $v = 1000$ for the instances of Methods II–V.

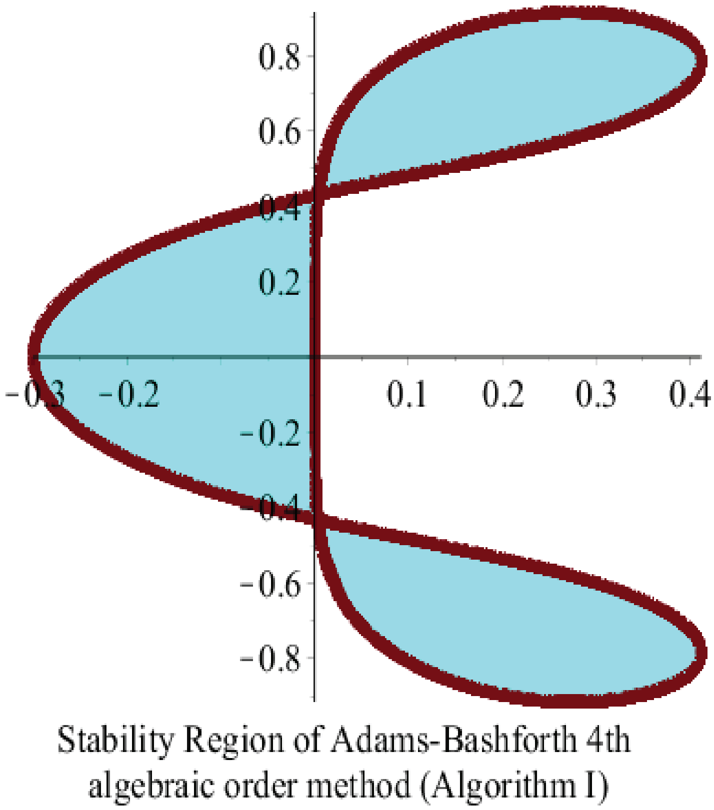


Figure 1. Stability Region for the classical Fourth-Order Adams–Bashforth Method (Algorithm I). The axes of the stability region are H and θ.

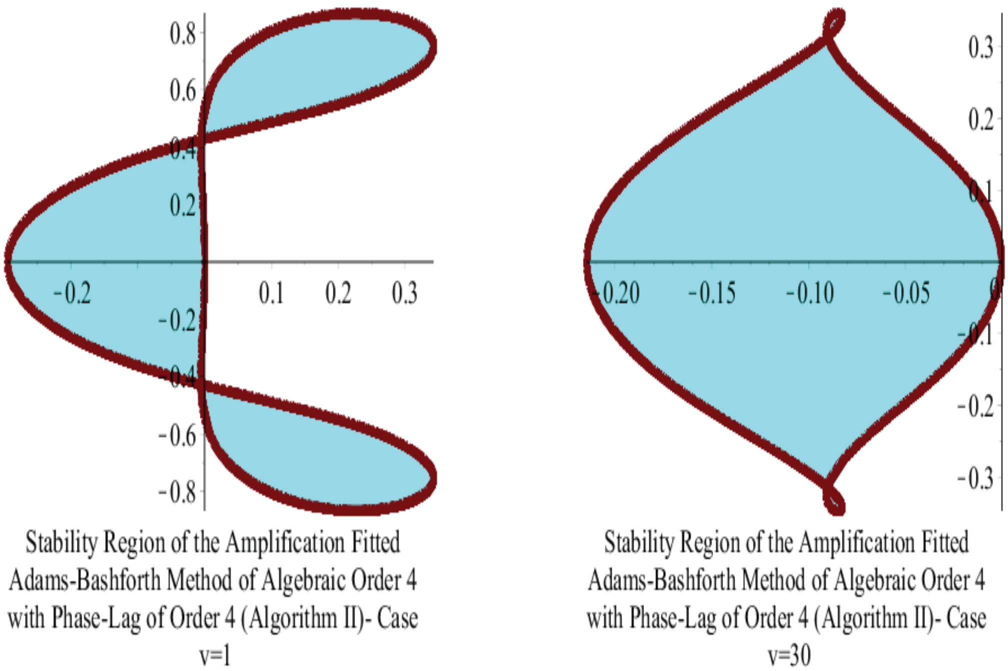


Figure 2. *Cont*.

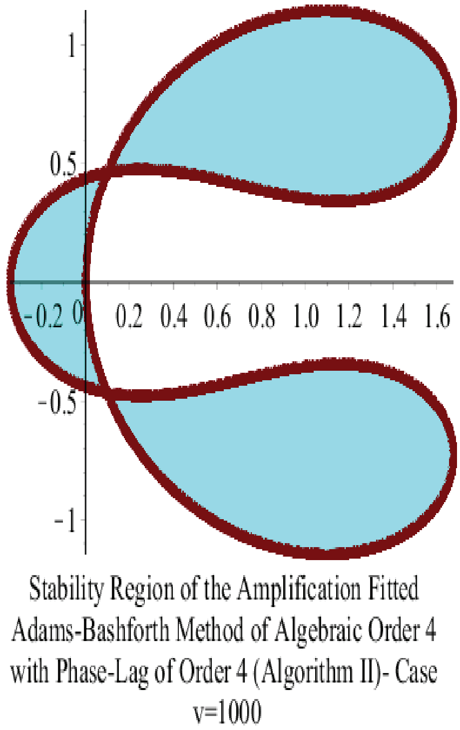


Figure 2. Stability Region of the Amplification-Fitted Adams–Bashforth Method of Algebraic Order Four with Phase-Lag of Order Four (Algorithm II). The axes of the stability region are H and θ.

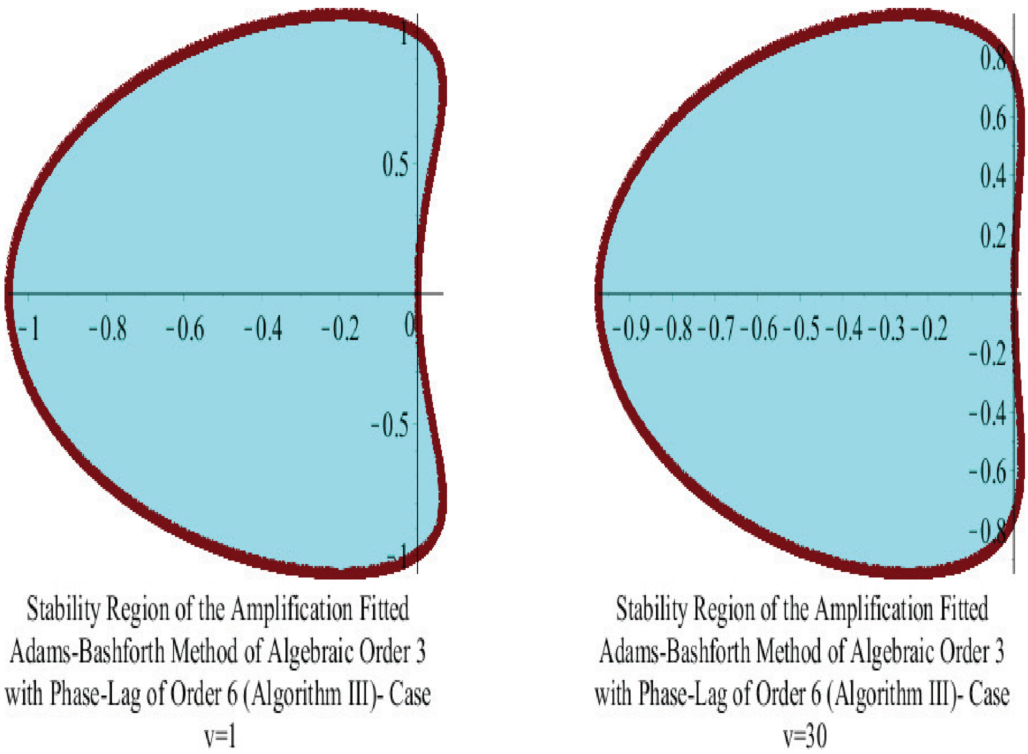


Figure 3. *Cont*.

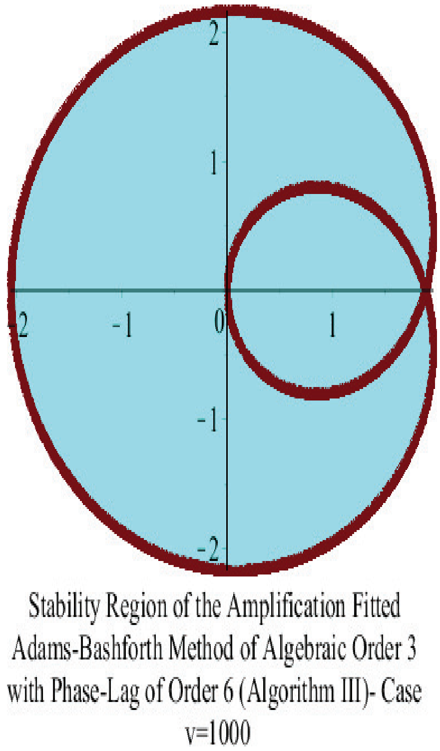


Figure 3. Stability Region of the Amplification-Fitted Adams–Bashforth Method of Algebraic Order Three with Phase-Lag of Order Six (Algorithm III). The axes of the stability region are H and θ.

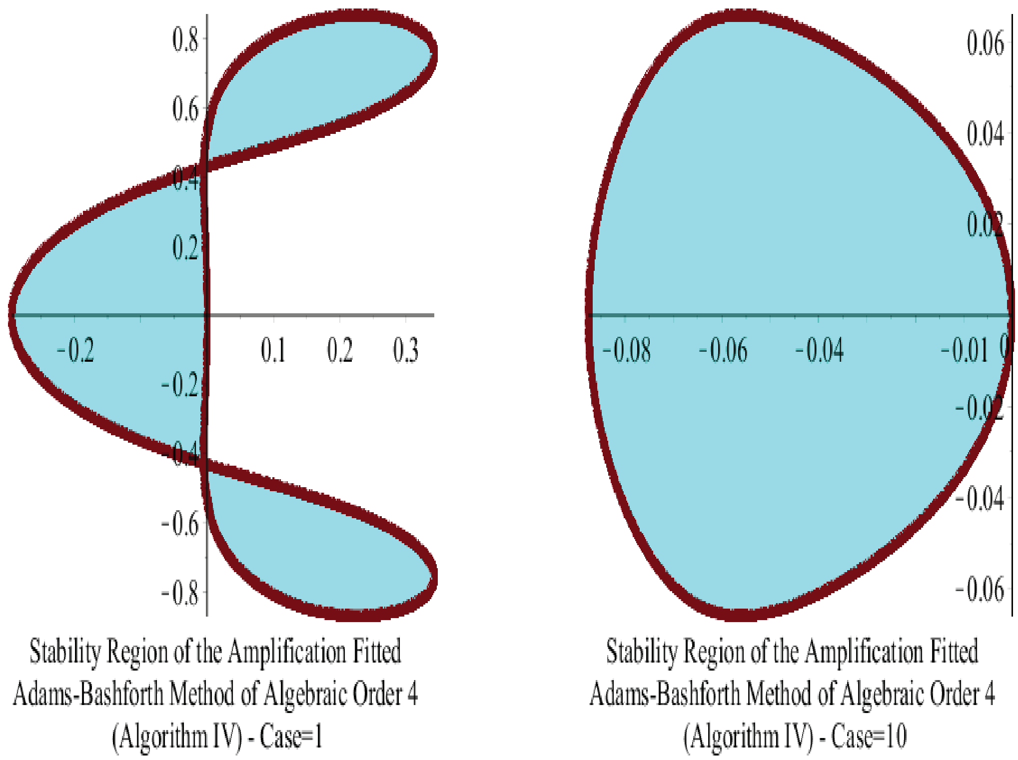


Figure 4. *Cont.*

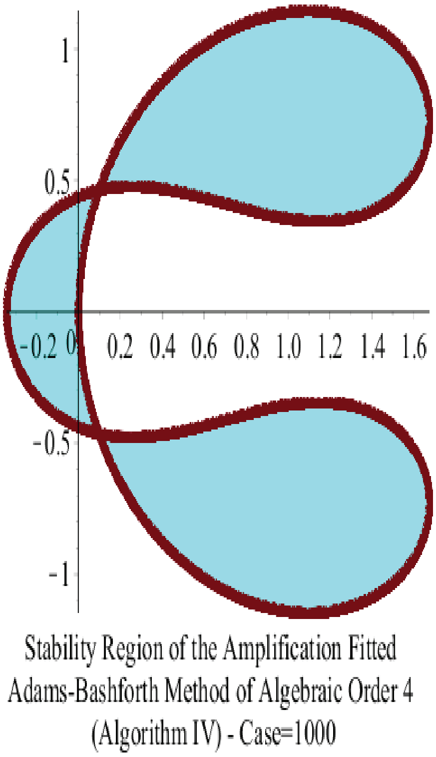
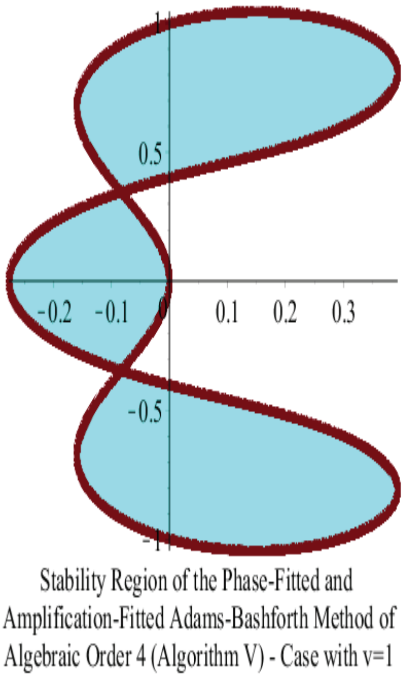


Figure 4. Stability Region of the Amplification-Fitted Adams–Bashforth Method of Algebraic Order Four (Algorithm IV). The axes of the stability region are H and θ.



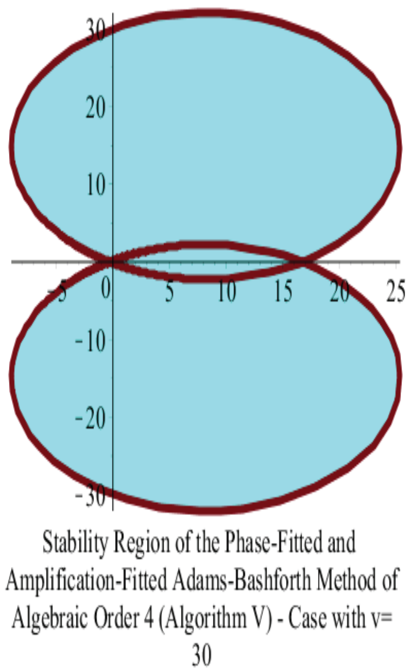


Figure 5. *Cont*.

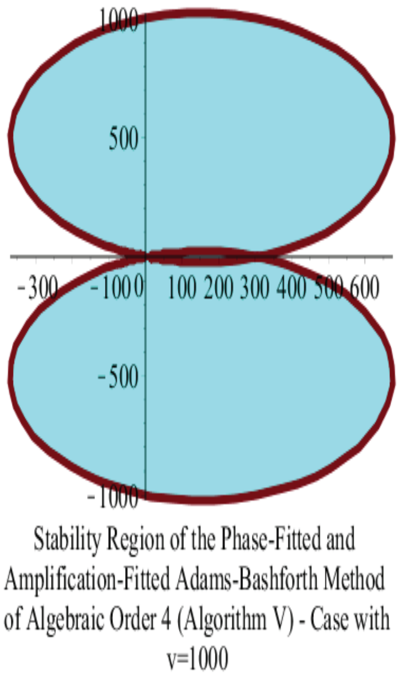


Figure 5. Stability Region of the Phase-Fitted and Amplification-Fitted Adams–Bashforth Method of Algebraic Order Four (Algorithm V). The axes of the stability region are H and θ.

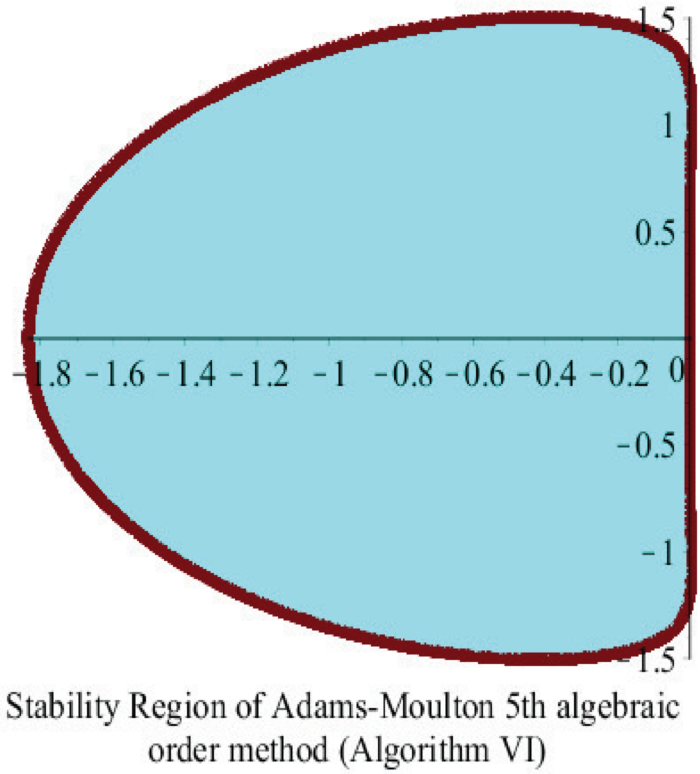


Figure 6. Stability Region of Adams–Moulton Fifth Algebraic Order Method (Algorithm VI). The axes of the stability region are H and θ.

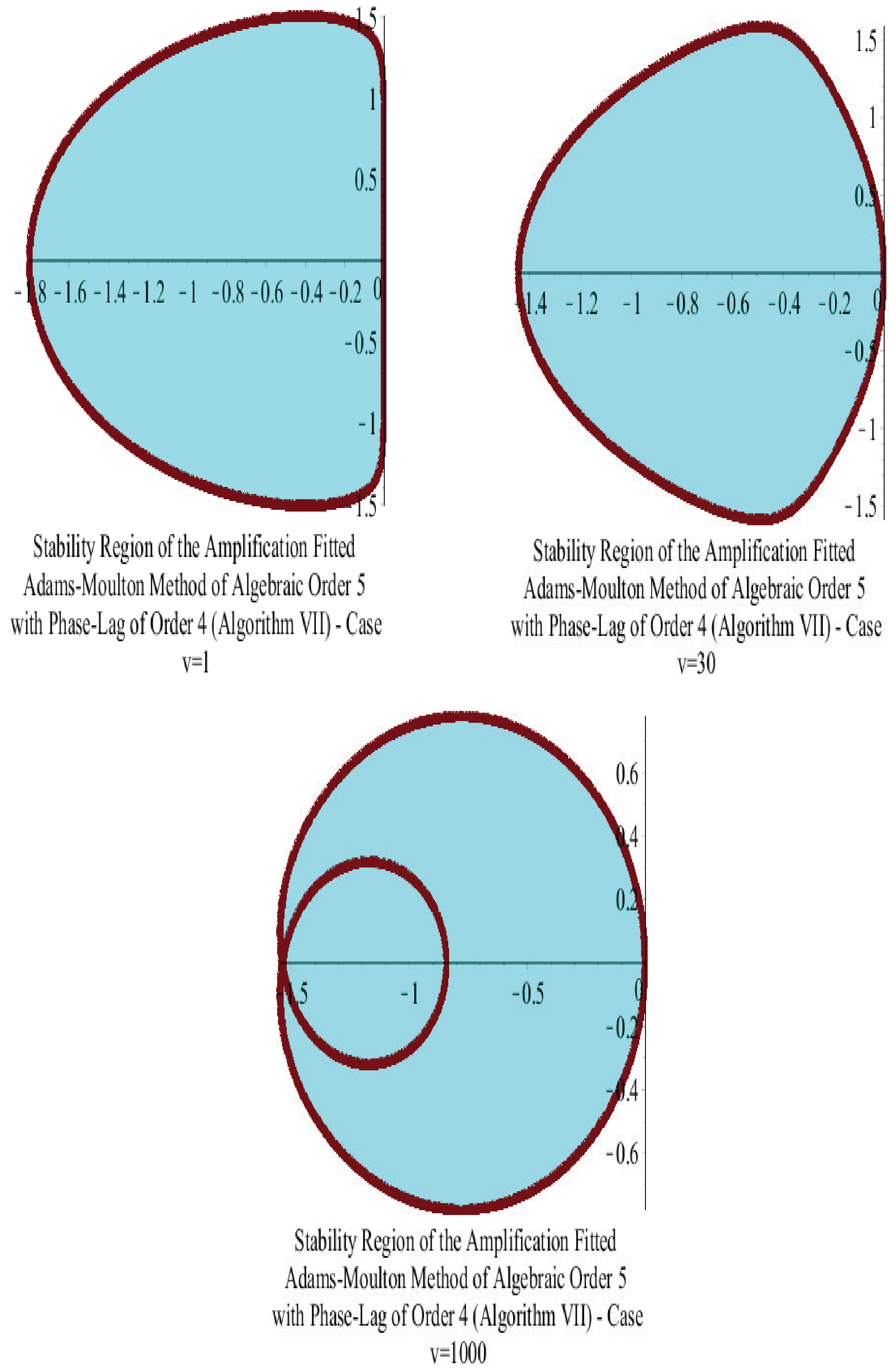


Figure 7. Stability Region of the Amplification-Fitted Adams–Moulton Method of Algebraic Order Five with Phase-Lag of Order Four (Algorithm VII). The axes of the stability region are H and θ.

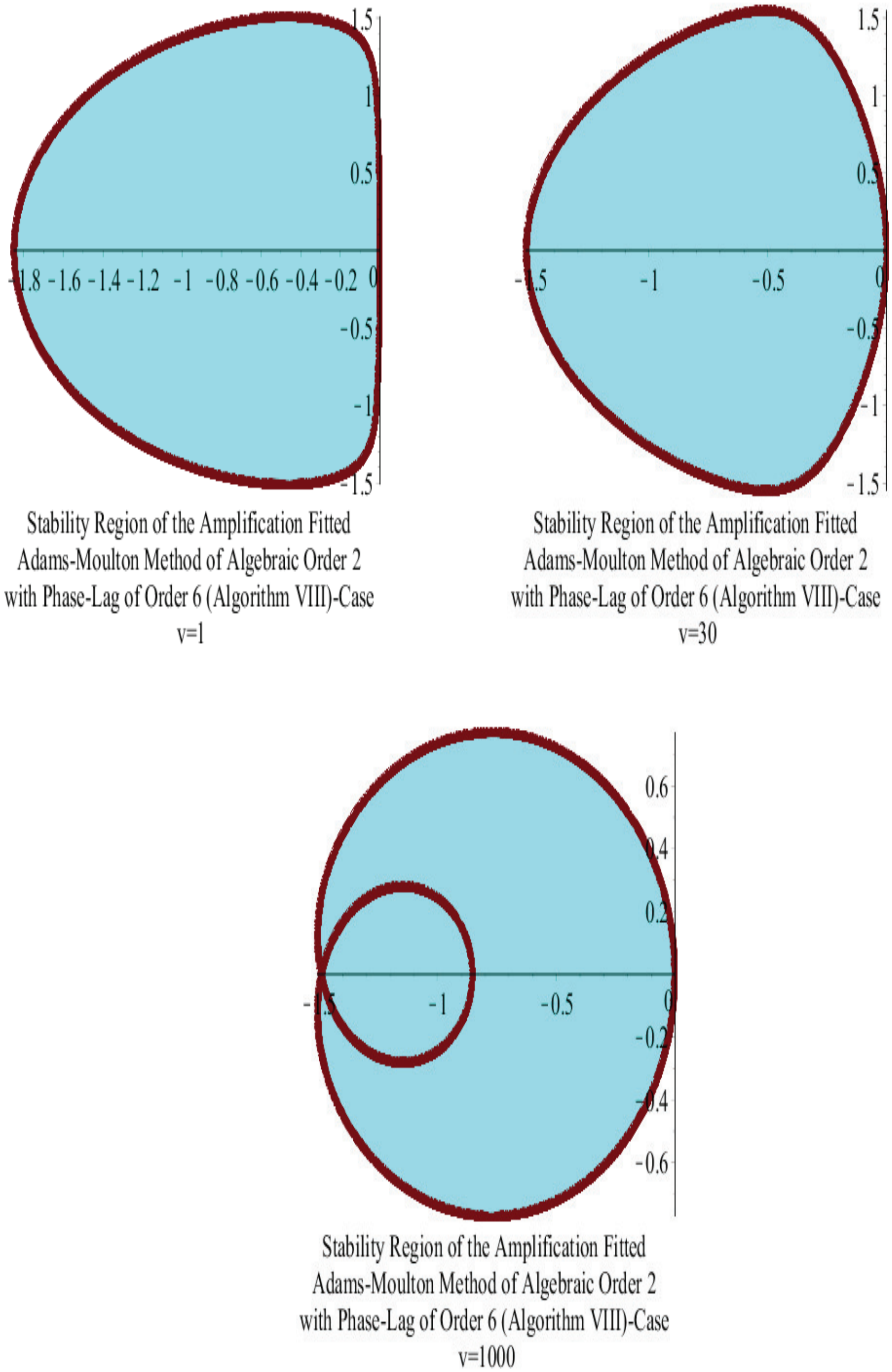


Figure 8. Stability Region of the Amplification-Fitted Adams–Moulton Method of Algebraic Order Two with Phase-Lag of Order Six (Algorithm VIII). The axes of the stability region are H and θ.

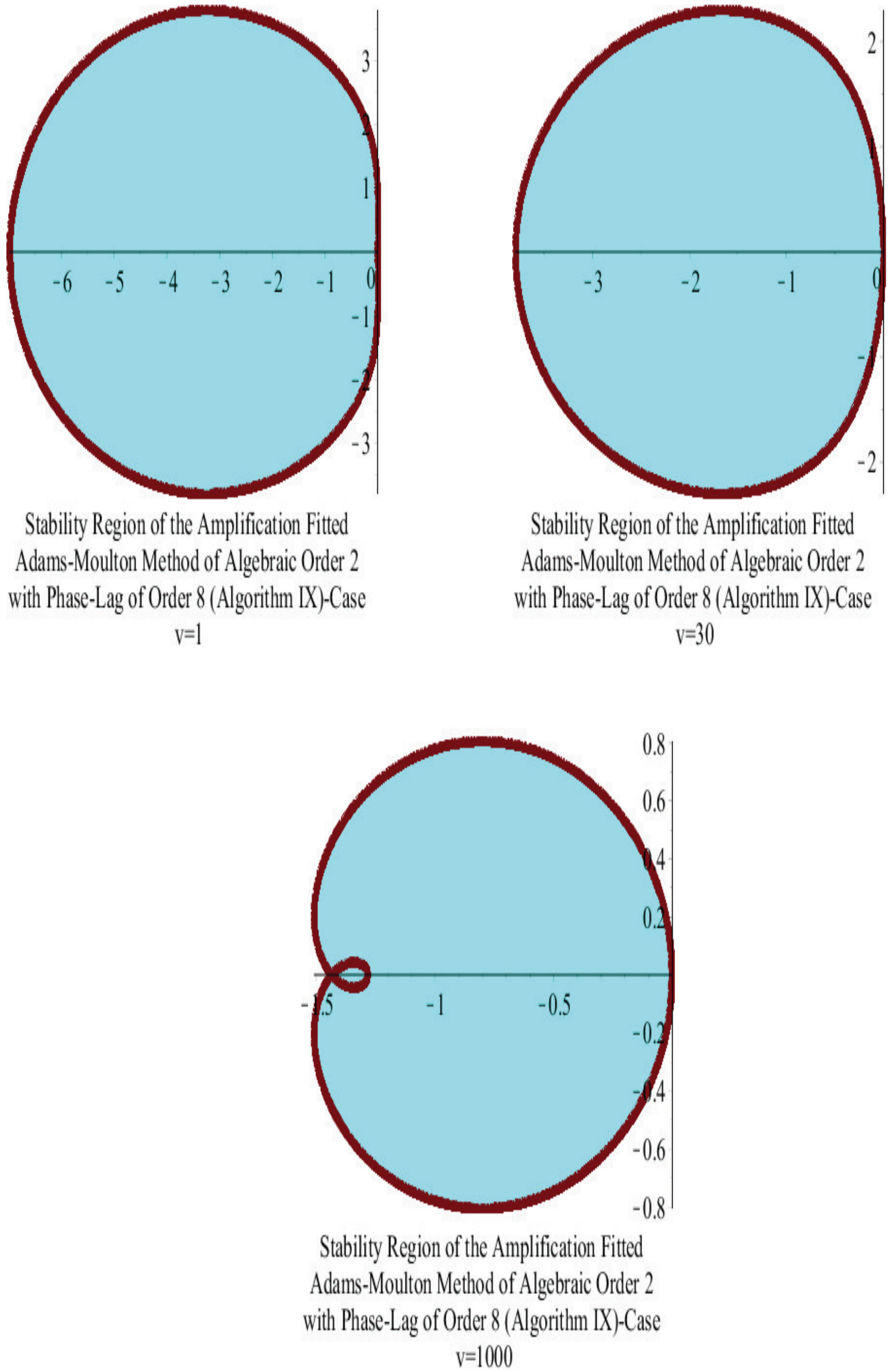


Figure 9. Stability Region of the Amplification-Fitted Adams–Moulton Method of Algebraic Order Two with Phase-Lag of Order Eight (Algorithm IX). The axes of the stability region are H and θ.

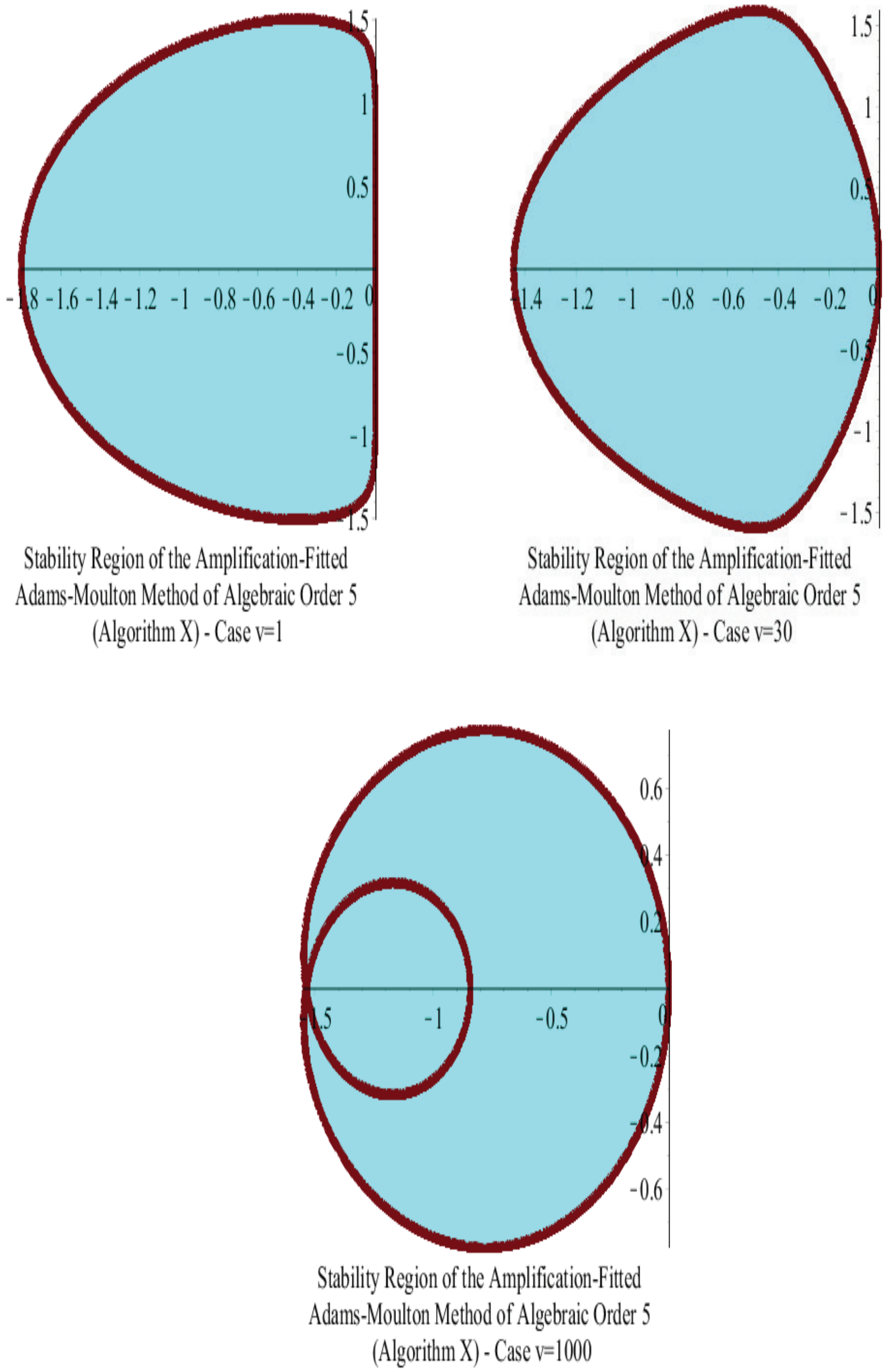


Figure 10. Stability Region of the Amplification-Fitted Adams–Moulton Method of Algebraic Order Five (Algorithm X). The axes of the stability region are H and θ.

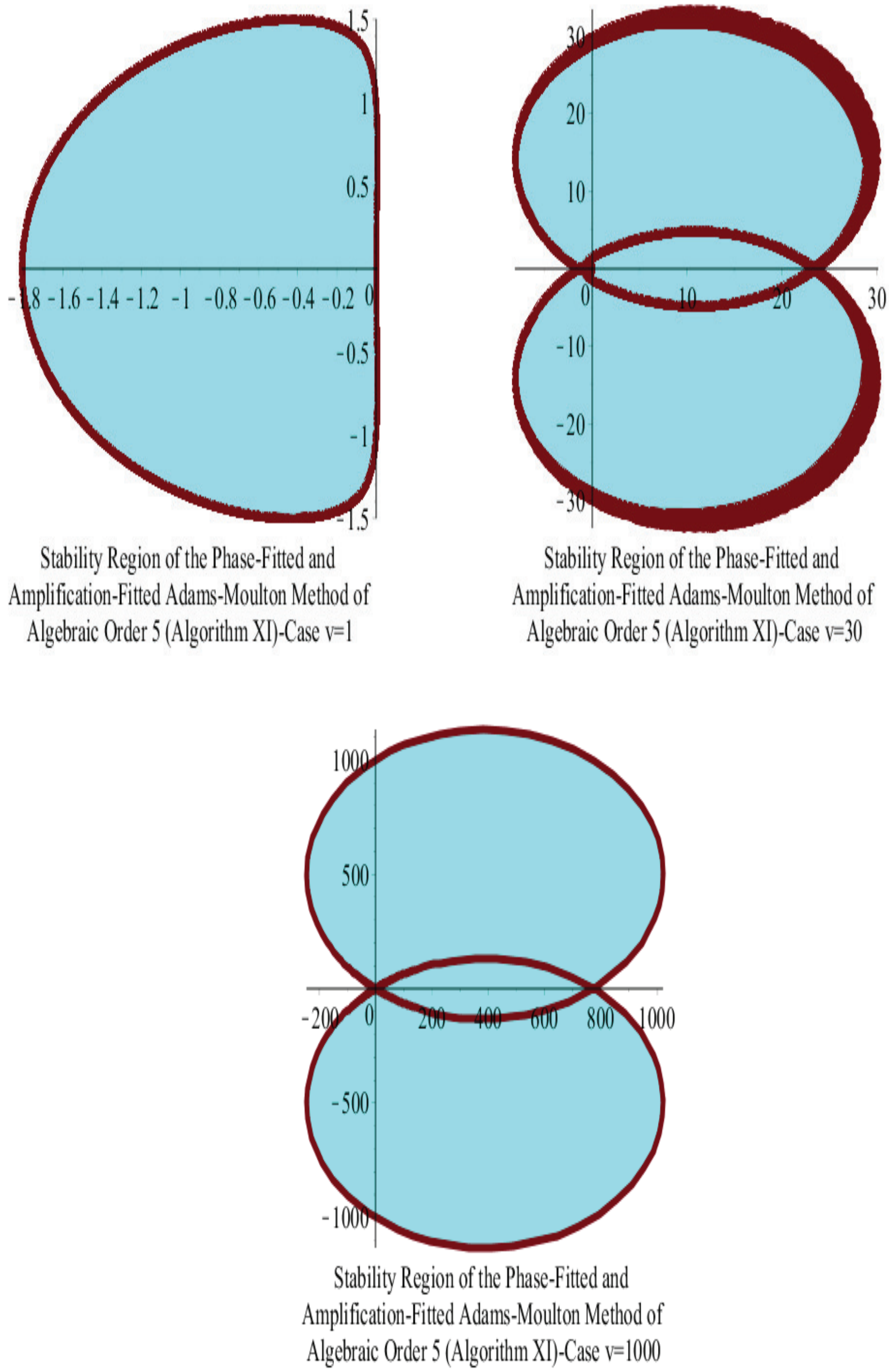


Figure 11. Stability Region of the Phase-Fitted and Amplification-Fitted Adams–Moulton Method of Algebraic Order Five (Algorithm XI). The axes of the stability region are H and θ.

16. Numerical Results

In this section, we will investigate the efficiency of the methods developed in Sections 4–14 comparing them with very well-known methods in the literature. The comparison will take place for well-known problems in the literature.

The newly developed methods are applied in the form of predictor–corrector. More specifically, for each problem, and for the initial four steps, we use a high-order Runge–

Kutta method. Then, we apply the Adams–Bashforth methods developed above as a predictor, and finally, we apply the Adams–Moulton methods developed above as a corrector.

16.1. Problem of Stiefel and Bettis

Stiefel and Bettis [116] investigated the following nearly periodic orbit issue, which we take into consideration.

$$\begin{aligned} s_1''(x) &= -s_1(x) + 0.001\cos(x), \quad s_1(0) = 1, \quad s_1'(0) = 0, \\ s_2''(x) &= -s_2(x) + 0.001\sin(x), \quad s_2(0) = 0, \quad s_2'(0) = 0.9995. \end{aligned} \tag{95}$$

Here is the exact solution:

$$\begin{aligned} s_1(x) = \cos(x) + 0.0005x\sin(x), \\ s_2(x) = \sin(x) - 0.0005x\cos(x). \end{aligned} \tag{96}$$

We apply the parameter $\omega = 1$ to this problem.

For values of $0 \leq x \leq 100{,}000$, the following numerical approaches have been used to solve Equation (95):

- The Classical Adams–Bashforth–Moulton Algorithm of the fifth order (Algorithms (22)–(54)), which is denoted as **Numer. Algor. I**;
- The amplification-fitted Adams–Bashforth–Moulton Algorithm of the fifth order (algorithms (45)–(76)), which is denoted as **Numer. Algor. II**;
- The Runge–Kutta–Dormand–Prince fourth-order method [48], which is denoted as **Numer. Algor. III**;
- The Runge–Kutta–Dormand–Prince fifth-order method [48], which is denoted as **Numer. Algor. IV**;
- The Runge–Kutta–Fehlberg fourth-order method [117], which is denoted as **Numer. Algor. V**;
- The Runge–Kutta–Fehlberg fifth-order method [117], which is denoted as **Numer. Algor. VI**;
- The Runge–Kutta–Cash–Karp fifth-order method [118], which is denoted as **Numer. Algor. VII**;
- The amplification-fitted Adams–Bashforth–Moulton Algorithm of the fifth algebraic order with phase-lag of order four (Algorithms (34)–(60)), which is denoted as **Numer. Algor. VIII**;
- The amplification-fitted Adams–Bashforth–Moulton Algorithm of the second algebraic order with phase-lag of order six (Algorithms (37)–(A31)), which is denoted as **Numer. Algor. IX**;
- The amplification-fitted Adams–Bashforth–Moulton Algorithm of the second algebraic order with phase-lag of order eight (Algorithms (37)–(66)), which is denoted as **Numer. Algor. X**;
- The amplification-fitted and phase-fitted Adams–Bashforth–Moulton Algorithm of the fifth order (Algorithms (53)–(85)), which is denoted as **Numer. Algor. XI**.

We show the greatest absolute error of the solutions obtained by each of the numerical approaches outlined earlier in Figure 6, which pertains to the Stiefel and Bettis problem [116].

The following may be seen in Figure 12:

- Numer. Algor. VII is more efficient than Numer. Algor. IV;
- Numer. Algor. V is more efficient than Numer. Algor. VII;
- Numer. Algor. VI is more efficient than Numer. Algor. V;
- Numer. Algor. III is more efficient than Numer. Algor. VI for the most step sizes but for small step sizes has approximately the same efficiency as Numer. Algor. VI;
- Numer. Algor. I is more efficient than Numer. Algor. VI;
- Numer. Algor. II and Numer. Algor. VIII are more efficient than Numer. Algor. I;

- Numer. Algor. IX has mixed behavior. For big step sizes, it has approximately the same efficiency as Numer. Algor. II and Numer. Algor. VIII. For middle step sizes, it is more efficient than Numer. Algor. III but less efficient than Numer. Algor. I. For small step sizes, it has approximately the same efficiency as Numer. Algor. II and Numer. Algor. VI;
- Numer. Algor. X has mixed behavior. For big step sizes, it is more efficient than Numer. Algor. II. For middle step sizes, it is more efficient than Numer. Algor. III but is less efficient than Numer. Algor. I. For small step sizes, it has approximately the same efficiency as Numer. Algor. III;
- Numer. Algor. XI gives the most efficient results.

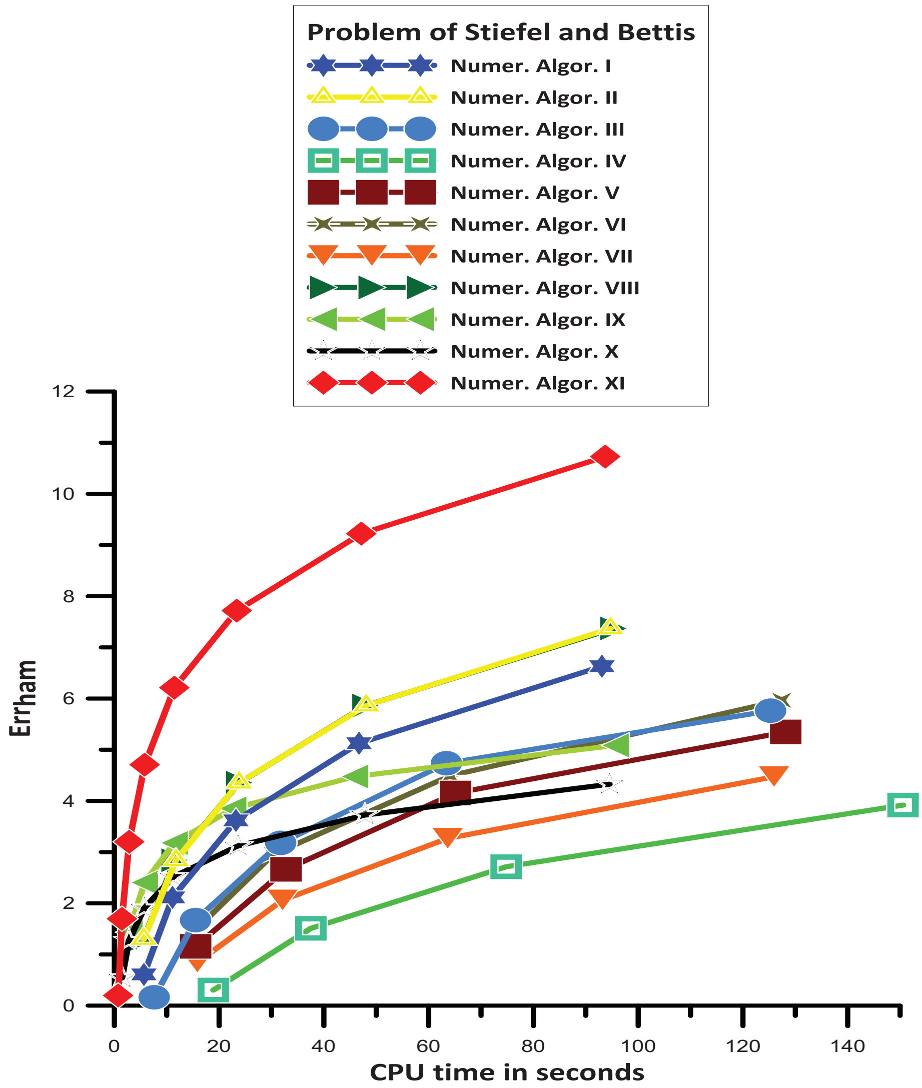


Figure 12. Numerical results for the problem of Stiefel and Bettis [116].

16.2. *Problem of Franco et al. [119]*

The inhomogeneous linear problem that Franco et al. [119] examined is taken into consideration here:

$$\begin{aligned} s_1''(x) &= -\frac{1}{2}\left(\mu^2+1\right)s_1(x) - \frac{1}{2}\left(\mu^2-1\right)s_2(x), \quad s_1(0)=1, \quad s_1'(0)=1, \\ s_2''(x) &= -\frac{1}{2}\left(\mu^2-1\right)s_1(x) - \frac{1}{2}\left(\mu^2+1\right)s_2(x), \quad s_2(0)=-1, \quad s_2'(0)=-1. \end{aligned} \tag{97}$$

The exact solution is

$$s_1(x) = \cos(x) + \sin(x),$$
$$s_2(x) = -\cos(x) - \sin(x). \tag{98}$$

where $\mu = 10^4$. For this problem, we use $\omega = 1$.

For $0 \le x \le 100{,}000$, the numerical solution of the system of Equation (97) has been found using the techniques outlined in Section 16.1.

The following may be seen in Figure 13:

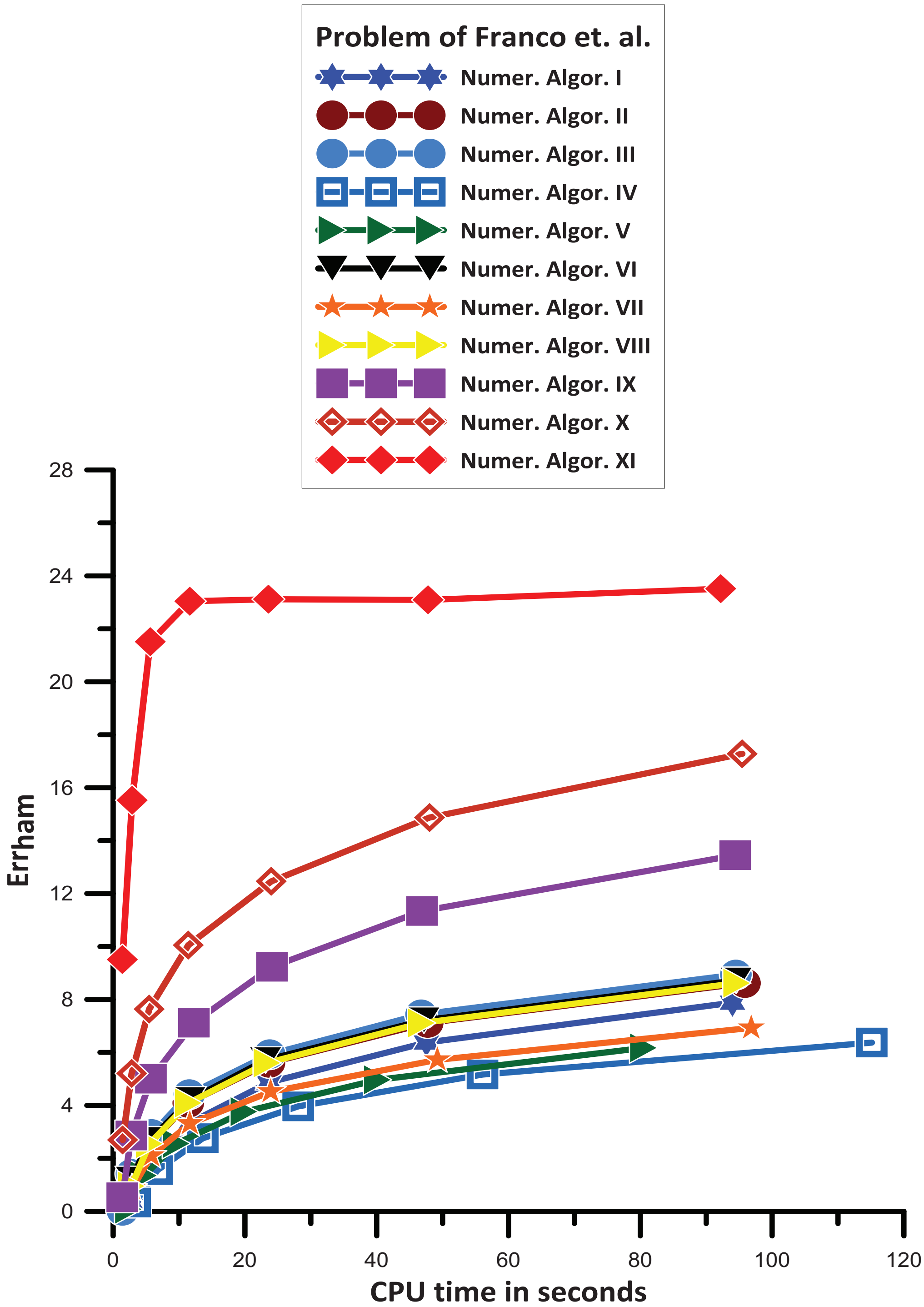


Figure 13. Numerical results for the problem of Franco et al. [119].

- Numer. Algor. V is more efficient than Numer. Algor. IV;
- Numer. Algor. VII is more efficient than Numer. Algor. V;
- Numer. Algor. I is more efficient than Numer. Algor. VII;
- Numer. Algor. VIII is more efficient than Numer. Algor. I;
- Numer. Algor. VIII has approximately the same efficiency as Numer. Algor. VI, Numer. Algor. III, and Numer. Algor. II;
- Numer. Algor. IX is more efficient than Numer. Algor. VIII;
- Numer. Algor. X is more efficient than Numer. Algor. IX;
- Numer. Algor. XI gives the most efficient results.

16.3. Problem of Franco and Palacios [120]

The problem that Franco and Palacios [120] investigated is taken into account here:

$$\begin{aligned} s_1''(x) &= -s_1(x) + \varepsilon\,\cos(\vartheta\,x), \quad s_1(0) = 1, \quad s_1'(0) = 0, \\ s_2''(x) &= -s_2(x) + \varepsilon\,\sin(\vartheta\,x), \quad s_2(0) = 0, \quad s_2'(0) = 1. \end{aligned} \tag{99}$$

The exact solution is

$$\begin{aligned} s_1(x) &= \frac{1-\varepsilon-\vartheta^2}{1-\vartheta^2}\,\cos(x) + \frac{\varepsilon}{1-\vartheta^2}\,\cos(\vartheta\,x), \\ s_2(x) &= \frac{1-\varepsilon\,\vartheta-\vartheta^2}{1-\vartheta^2}\,\sin(x) + \frac{\varepsilon}{1-\vartheta^2}\,\sin(\vartheta\,x). \end{aligned} \tag{100}$$

where $\varepsilon = 0.001$ and $\vartheta = 0.01$. For this problem, we use $\omega = \max\Big(1, |\vartheta|\Big)$.

Using the techniques outlined in Section 16.1, the numerical solution to the system of Equation (99) has been obtained for $0 \le x \le 100{,}000$.

The following may be seen in Figure 14:

- Numer. Algor. V is more efficient than Numer. Algor. IV;
- Numer. Algor. VII is more efficient than Numer. Algor. V;
- Numer. Algor. VI is more efficient than Numer. Algor. VII;
- Numer. Algor. VI has approximately the same efficiency as Numer. Algor. III;
- Numer. Algor. I is more efficient than Numer. Algor. VI;
- Numer. Algor. VIII is more efficient than Numer. Algor. II;
- Numer. Algor. IX is more efficient than Numer. Algor. VIII;
- Numer. Algor. X is more efficient than Numer. Algor. IX;
- Numer. Algor. XI gives the most efficient results.

16.4. A Nonlinear Orbital Problem [121]

The nonlinear orbital problem that Simos investigated in [121] is taken into consideration here:

$$\begin{aligned} s_1''(x) &= -s^2\,s_1(x) + \frac{2\,s_1(x)\,s_2(x) - \sin(2\,s\,x)}{\Big(s_1(x)^2 + s_2(x)^2\Big)^{\frac{3}{2}}}, \quad s_1(0) = 1, \quad s_1'(0) = 0, \\ s_2''(x) &= -s^2\,s_2(x) + \frac{s_1(x)^2 - s_2(x)^2 - \cos(2\,s\,x)}{\Big(s_1(x)^2 + s_2(x)^2\Big)^{\frac{3}{2}}}, \quad s_2(0) = 0, \quad s_2'(0) = s. \end{aligned} \tag{101}$$

The exact solution is

$$s_1(x) = \cos(s\,x), \quad s_2(x) = \sin(s\,x), \tag{102}$$

where $s = 10$. For this problem, we use $\omega = 10$.

The numerical solution of the system of Equation (101) has been achieved for $0 \le x \le 100{,}000$ by using the techniques outlined in Section 16.1.

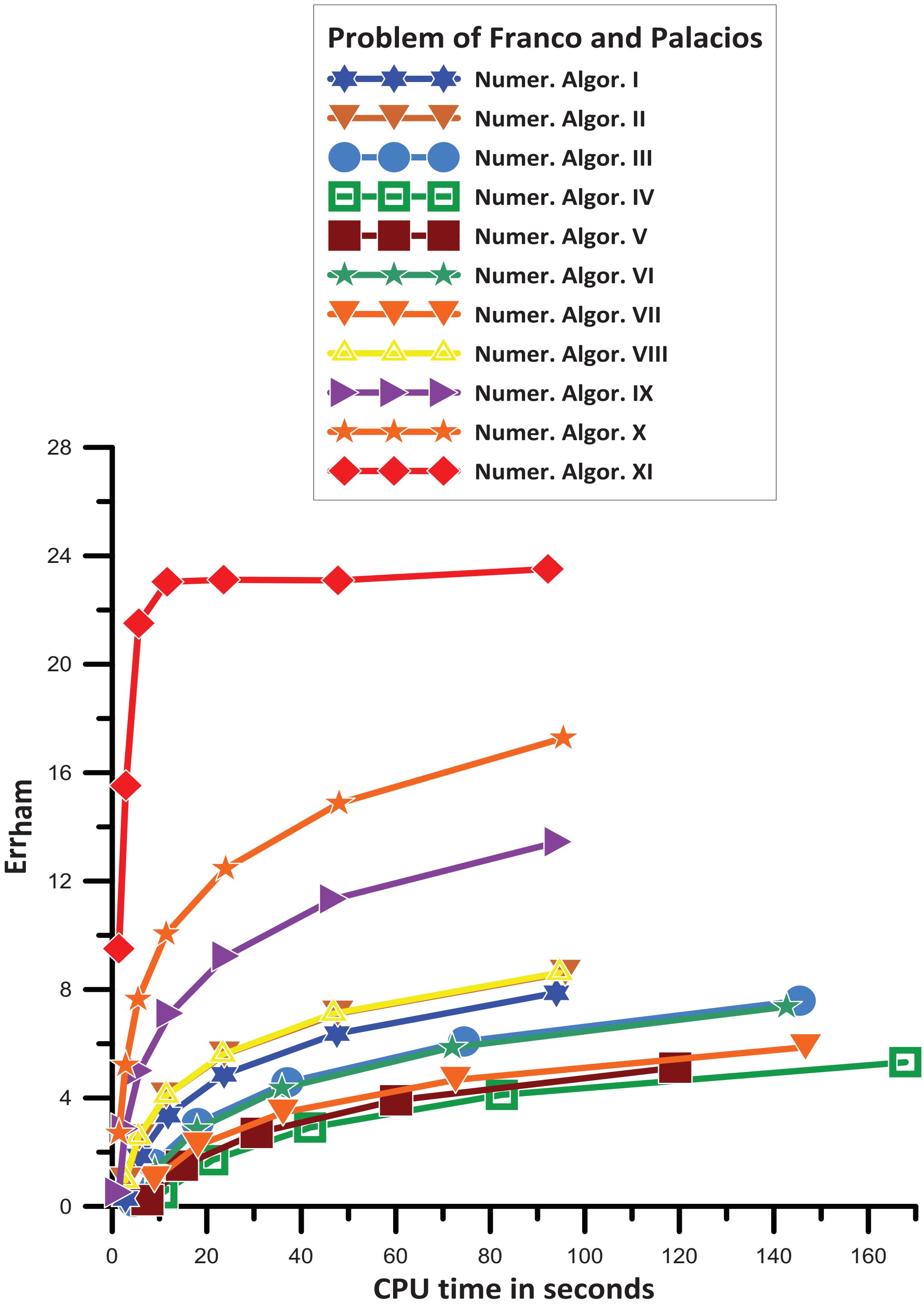


Figure 14. Numerical results for the problem of Franco and Palacios [120].

The following may be seen in Figure 15:

- Numer. Algor. IV has approximately the same efficiency as Numer. Algor. V;
- Numer. Algor. VII is more efficient than Numer. Algor. V;
- Numer. Algor. VI is more efficient than Numer. Algor. VII;
- Numer. Algor. III has approximately the same efficiency as Numer. Algor. VI;
- Numer. Algor. I is more efficient than Numer. Algor. III;
- Numer. Algor. VIII is more efficient than Numer. Algor. I;
- Numer. Algor. II has approximately the same efficiency as Numer. Algor. VIII;

- Numer. Algor. IX is more efficient than Numer. Algor. VIII;
- Numer. Algor. X is more efficient than Numer. Algor. IX;
- Numer. Algor. XI gives the most efficient results.

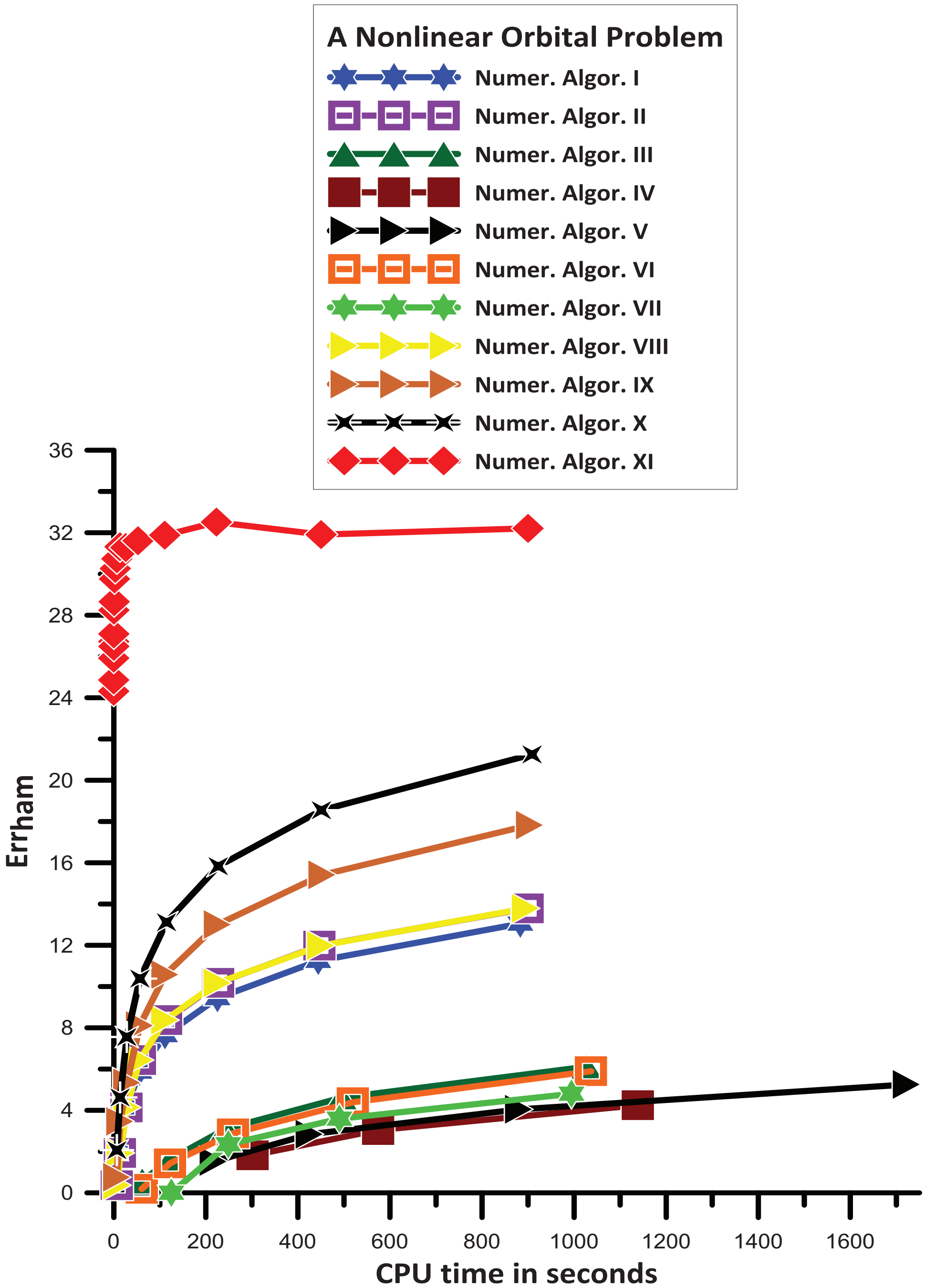


Figure 15. Numerical results for the Nonlinear Orbital problem of [121].

16.5. Nonlinear Problem of Petzold [122]

Petzold [122] investigated the following nonlinear problem, which we consider here:

$$
\begin{aligned}
s_1'(x) &= \lambda\, s_2(x), \quad s_1(0) = 1, \\
s_2'(x) &= -\lambda\, s_1(x) + \frac{\alpha}{\lambda}\,\sin(\lambda\, x), \quad s_2(0) = -\frac{\alpha}{2\,\lambda^2}.
\end{aligned}
\tag{103}
$$

The exact solution is

$$s_1(x) = \left(1 - \frac{\alpha}{2\,\lambda}\,x\right)\,\cos(\lambda\,x),$$
$$s_2(x) = -\left(1 - \frac{\alpha}{2\,\lambda}\,x\right)\,\sin(\lambda\,x) - \frac{\alpha}{2\,\lambda^2}\,\cos(\lambda\,x), \tag{104}$$

where $\lambda = 1000$, $\alpha = 100$. For this problem, we use $\omega = 1000$.

The numerical solution to the system of Equation (103) for $0 \leq x \leq 1000$ has been achieved by using the techniques outlined in Section 16.1.

The following may be seen in Figure 16:

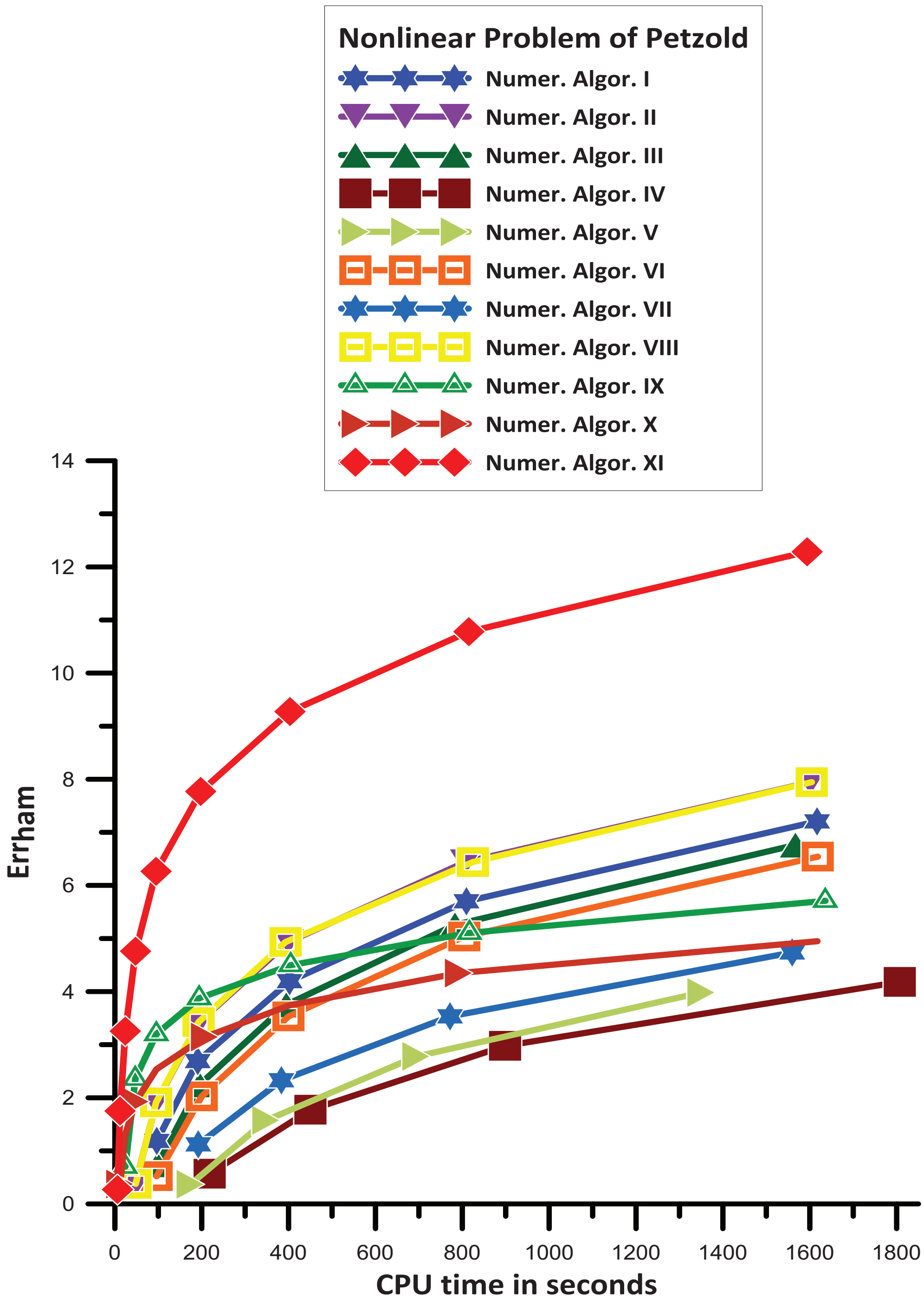


Figure 16. Numerical results for the nonlinear problem of [122].

- Numer. Algor. V is more efficient than Numer. Algor. IV;
- Numer. Algor. VII is more efficient than Numer. Algor. V;
- Numer. Algor. VI is more efficient than Numer. Algor. VII;
- Numer. Algor. III is more efficient than Numer. Algor. VI;
- Numer. Algor. I is more efficient than Numer. Algor. III;
- Numer. Algor. VIII is more efficient than Numer. Algor. I;
- Numer. Algor. II has approximately the same efficiency as Numer. Algor. VIII;
- Numer. Algor. IX has mixed behavior. For big step sizes, it is more efficient than Numer. Algor. VIII. For middle step sizes, it is more efficient than Numer. Algor. VI. For small step sizes, it is more efficient than Numer. Algor. VII;
- Numer. Algor. X has mixed behavior. For big step sizes, it is more efficient than Numer. Algor. VIII but less efficient than Numer. Algor. IX. For middle step sizes, it is more efficient than Numer. Algor. VII but less efficient than Numer. Algor. VI. For small step sizes, it is more efficient than Numer. Algor. VII;
- Numer. Algor. XI gives the most efficient results.

16.6. Two-Body Gravitational Problem

The two-body gravitational issue is under our consideration.

$$\begin{aligned} s_1''(x) &= -\frac{s_1(x)}{\left(s_1(x)^2+s_2(x)^2\right)^{\frac{3}{2}}}, \qquad s_1(0)=1, \quad s_1'(0)=0, \\ s_2''(x) &= -\frac{s_2(x)}{\left(s_1(x)^2+s_2(x)^2\right)^{\frac{3}{2}}}, \qquad s_2(0)=0, \quad s_2'(0)=1. \end{aligned} \tag{105}$$

The exact solution is

$$\begin{aligned} s_1(x) &= \cos(x), \\ s_2(x) &= \sin(x). \end{aligned} \tag{106}$$

For this problem, we use $\omega = \frac{1}{\left(s_1(x)^2+s_2(x)^2\right)^{\frac{3}{4}}}$.

Using the techniques outlined in Section 16.1, the numerical solution to the system of Equation (105) has been obtained for $0 \leq x \leq 100{,}000$.

The following may be seen in Figure 17:

- Numer. Algor. I has approximately the same efficiency as Numer. Algor. VII;
- Numer. Algor. VI is more efficient than Numer. Algor. I;
- Numer. Algor. VIII is more efficient than Numer. Algor. VI;
- Numer. Algor. II has approximately the same efficiency as Numer. Algor. VIII, Numer. Algor. III, and Numer. Algor. V;
- Numer. Algor. IV is more efficient than Numer. Algor. II;
- Numer. Algor. IX is more efficient than Numer. Algor. IV;
- Numer. Algor. X is more efficient than Numer. Algor. IX;
- Numer. Algor. XI gives the most efficient results.

16.7. Perturbed Two-Body Gravitational Problem

16.7.1. Case $\mu = 0.1$

Here, we take into account the perturbed two-body Kepler's plane problem.

$$
\begin{aligned}
s_1''(x) &= -\frac{s_1(x)}{\left(s_1(x)^2+s_2(x)^2\right)^{\frac{3}{2}}} - \mu\left(\mu+2\right)\frac{s_1(x)}{\left(s_1(x)^2+s_2(x)^2\right)^{\frac{5}{2}}}, \\
&\qquad s_1(0)=1, \quad s_1'(0)=0, \\
s_2''(x) &= -\frac{s_2(x)}{\left(s_1(x)^2+s_2(x)^2\right)^{\frac{3}{2}}} - \mu\left(\mu+2\right)\frac{s_2(x)}{\left(s_1(x)^2+s_2(x)^2\right)^{\frac{5}{2}}}, \\
&\qquad s_2(0)=0, \quad s_2'(0)=1+\mu.
\end{aligned} \tag{107}
$$

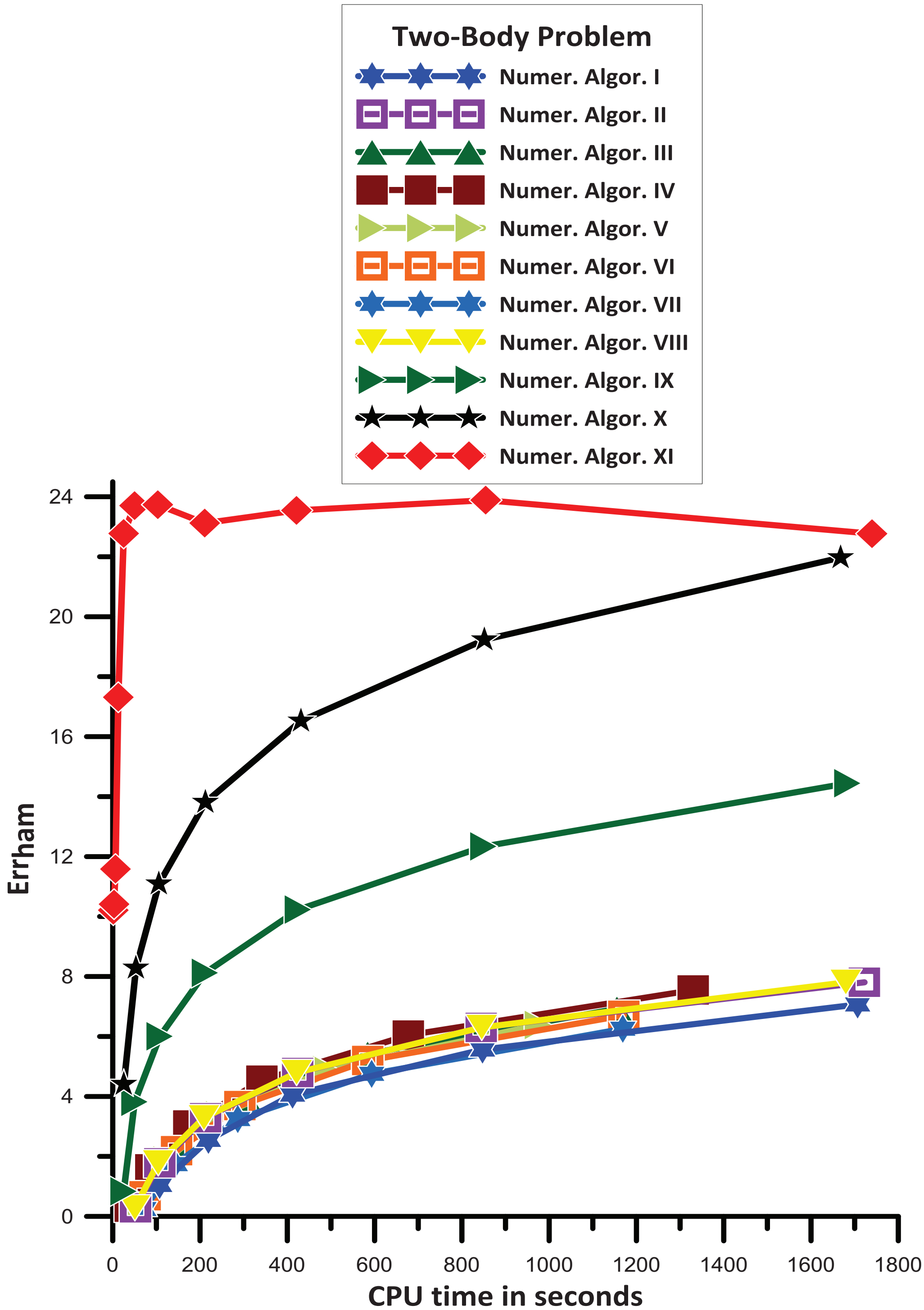


Figure 17. Numerical results for two-body gravitational problem (Kepler's plane problem).

The exact solution is

$$s_1(x) = \cos(x + \mu\, x),$$
$$s_2(x) = \sin(x + \mu\, x). \tag{108}$$

For this problem, we use $\omega = \frac{\sqrt{1+\mu\left(\mu+2\right)}}{\left(s_1(x)^2+s_2(x)^2\right)^{\frac{3}{4}}}$.

Numerical solutions have been found for $0 \le x \le 100{,}000$ using $\mu = 0.1$ and the techniques described in Section 16.1 for the system of Equation (107).

The following may be seen in Figure 18:

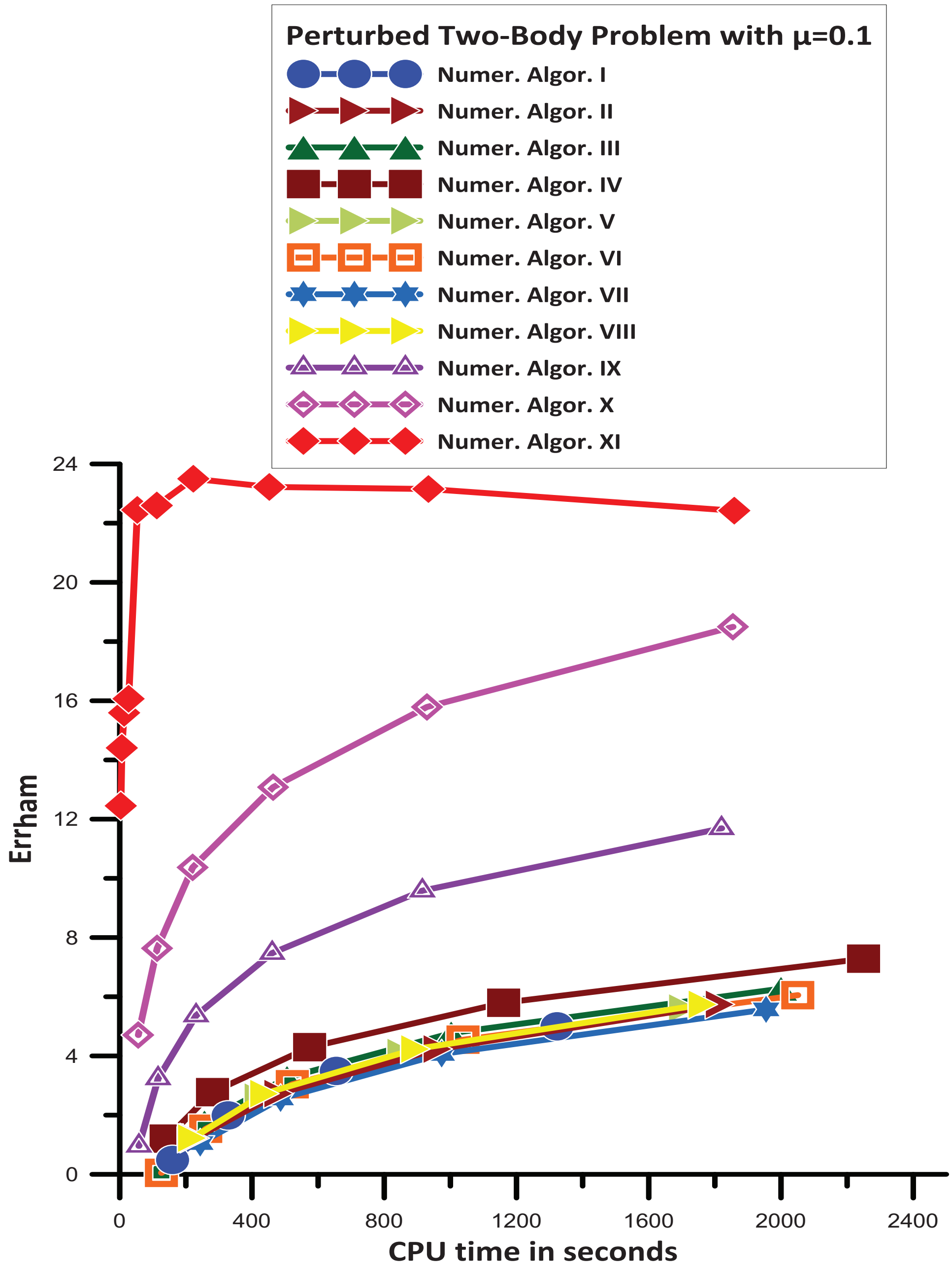


Figure 18. Numerical results for perturbed two-body gravitational problem (perturbed Kepler's problem) with $\mu = 0.1$.

- Numer. Algor. I is more efficient than Numer. Algor. VII;
- Numer. Algor. I, Numer. Algor. II, Numer. Algor. V, Numer. Algor. VI, and Numer. Algor. VIII, have approximately the same efficiency;
- Numer. Algor. III is more efficient than Numer. Algor. I;
- Numer. Algor. IV is more efficient than Numer. Algor. III;
- Numer. Algor. IX is more efficient than Numer. Algor. IV;
- Numer. Algor. X is more efficient than Numer. Algor. IX;
- Numer. Algor. XI gives the most efficient results.

16.7.2. Case $\mu = 0.4$

Numerical solutions have been found for $0 \leq x \leq 100{,}000$ using $\mu = 0.1$ and the techniques described in Section 16.1 for the system of Equation (107).

The following may be seen in Figure 19:

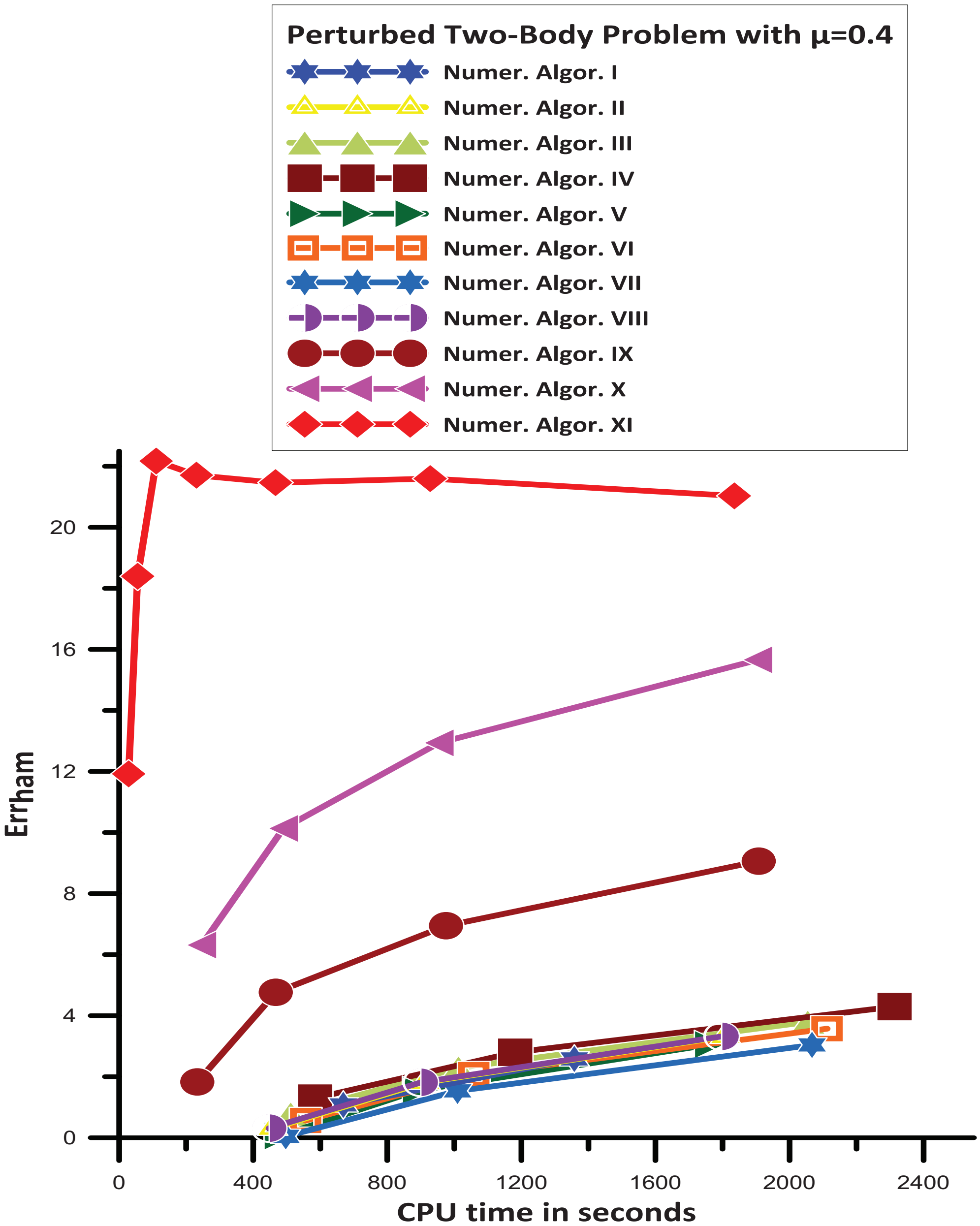


Figure 19. Numerical results for perturbed two-body gravitational problem (perturbed Kepler's problem) with $\mu = 0.4$.

- Numer. Algor. I is more efficient than Numer. Algor. VII;
- Numer. Algor. I, Numer. Algor. II, Numer. Algor. V, Numer. Algor. VI, and Numer. Algor. VIII, have approximately the same efficiency;
- Numer. Algor. III is more efficient than Numer. Algor. I;
- Numer. Algor. IV is more efficient than Numer. Algor. III;
- Numer. Algor. IX is more efficient than Numer. Algor. IV;
- Numer. Algor. X is more efficient than Numer. Algor. IX;
- Numer. Algor. XI gives the most efficient results.

Based on the numerical examples provided above, we may deduce:

- Results for all problems are most efficiently produced by the phase-fitted and amplification-fitted approach (Numer. Algor. XI);
- Results for the majority of problems are second-best when using the amplification-fitted Adams–Bashforth–Moulton Algorithm of second algebraic order with a phase-lag of order eight (Numer. Algor. X);
- Results for the majority of problems are third-best when using the amplification-fitted Adams–Bashforth–Moulton Algorithm of second algebraic order with a phase-lag of order six (Numer. Algor. IX).

In light of the above, it is clear that the strategies offered in this work that provide the best results are:

- The strategy that disregards the algebraic order of the procedure in favor of minimizing the phase-lag;
- Strategies that concentrate on eliminating phase-lag and the amplification factor

The effectiveness of frequency-dependent approaches, such as the recently presented ones, is clearly influenced by the parameter v that is chosen. This option may often be defined directly from the problem's model in many cases. The literature (e.g., [123,124]) has proposed approaches for determining the parameter v in circumstances when this is not simple.

Remark 2. *One thing to keep in mind when solving systems of high-order ordinary differential equations using the recently introduced techniques is that there are already established ways to simplify such systems into first-order differential equations. For examples of such methods, see [125].*

In order to solve systems of partial differential equations using the recently introduced techniques mentioned earlier, it is important to note that there are already established methods, see [126], that can reduce such a system to a system of first-order differential equations.

17. Conclusions

For implicit multistep approaches to first-order initial-value problems, we presented here the theory of phase-lag and amplification-error analysis. Several strategies for the construction of efficient predictor–corrector methods were offered, based on the theory described above and that developed in [115] for explicit methods. Our development efforts focused on the following strategies:

- Strategies for reducing the phase-lag;
- A strategy for the construction of an amplification-fitted method;
- A strategy for the construction of a phase–fitted method.

We created a number of multistep predictor–corrector approaches by using the aforementioned strategies. We based on the fourth algebraic order the Adams–Bashforth explicit method and on the fifth algebraic order Adams–Moulton implicit method.

The effectiveness of the aforementioned strategies was evaluated by applying them to many problems involving oscillating solutions.

It is worth mentioning that the idea put forward in this work and [115] is novel in the literature in relation to the development of:

- multistep methods for first-order initial-value problems with minimal phase-lag;
- phase-fitted and amplification-fitted multistep methods for first-order initial-value problems.

The same theory can be applied to all categories of multistep methods for first-order initial-value problem with oscillating solutions.

We also note that the methods presented in this paper can be applied to any problem with oscillating solution.

The computations were carried out using a 64-bit quadruple-precision arithmetic data type-compatible personal computer that conformed to the IEEE Standard 754, using a *FORTRAN* package.

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Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. Development of Algorithm III

Eliminating the Amplification Factor

We obtain the following result when we use the straightforward approach for computing the amplification factor (28):

$$AF = \frac{\sin(4\,v) - \sin(3\,v) - K_0(vs.)\,v\,\cos(3\,v) - K_1(vs.)\,vs.\,\cos(2\,v) - K_2(vs.)\,v\,\cos(vs.) - v\,K_3(vs.)}{-9\,v^2\,K_0(vs.) - 4\,v^2\,K_1(vs.) - v^2\,K_2(vs.)}, \tag{A1}$$

where AF denotes the amplification factor.

Assuming that the amplification factor must be eliminated, or that $AF = 0$, we derive:

$$K_0(vs.)(vs.) = \frac{-K_1(vs.)\,v\,\cos(2\,v) - K_2(vs.)\,v\,\cos(vs.) - vs.\,K_3(vs.) + \sin(4\,v) - \sin(3\,v)}{v\,\cos(3\,v)}. \tag{A2}$$

Procedure for Minimizing the Phase-Lag

We can obtain the phase-lag by plugging the values of $K_0(vs.)(vs.)$ that is previously provided into the direct formula for calculating it (19):

$$PhErr = \frac{v\,NUMRT_1(vs.)}{\Xi_3(vs.)}, \tag{A3}$$

where

$$\begin{aligned}
NUMRT_1(vs.) &= 4\,\sin(vs.)(\cos(vs.))^2\,v\,K_3(vs.) + 2\,\sin(vs.)\cos(vs.)v\,K_2(vs.) \\
&+ K_1(vs.)v\sin(vs.) - \sin(vs.)\,v\,K_3(vs.) - \cos(vs.) + 1, \\
\Xi_3(vs.) &= 8\,(\cos(vs.))^3\,v\,K_1(vs.) + 4\,(\cos(vs.))^3\,v\,K_2(vs.) \\
&+ 24\,\sin(vs.)\,(\cos(vs.))^3 - 28\,(\cos(vs.))^3\,vs. \\
&- 6\,v\,K_1(vs.)\,(\cos(vs.))^2 - 12\,\sin(vs.)\,(\cos(vs.))^2 \\
&- 6\,\cos(vs.)\,v\,K_1(vs.) - 6\,K_2(vs.)\,v\,\cos(vs.) \\
&- 12\,\sin(vs.)\,\cos(vs.) + 21\,\cos(vs.)\,v \\
&+ 3\,K_1(vs.)\,vs. - 3\,v\,K_3(vs.) + 3\,\sin(vs.),
\end{aligned} \tag{A4}$$

and $PhErr$ denotes the phase-lag.

By applying the Taylor series expansion to Formula (A4), we are able to retrieve:

$$\begin{aligned}
PhErr &= \frac{\left(3\,K_3(vs.) + 2\,K_2(vs.) + K_1(vs.) + \frac{1}{2}\right)}{-K_1(vs.) - 2\,K_2(vs.) - 4 - 3\,K_3(vs.)}\,v^2 \\
&+ \frac{\Xi_4(vs.)}{-K_1(vs.) - 2\,K_2(vs.) - 4 - 3\,K_3(vs.)}\,v^4 \\
&+ \frac{\Xi_5(vs.)}{-K_1(vs.) - 2\,K_2(vs.) - 4 - 3\,K_3(vs.)}\,v^6 \\
&+ \frac{\Xi_6(vs.)}{-K_1(vs.) - 2\,K_2(vs.) - 4 - 3\,K_3(vs.)}\,v^8 \\
&+ \frac{\Xi_7(vs.)}{-K_1(vs.) - 2\,K_2(vs.) - 4 - 3\,K_3(vs.)}\,v^{10} + \ldots,
\end{aligned} \tag{A5}$$

where

$$\begin{aligned}
\Xi_4(vs.) &= -\frac{9}{2}\,K_3(vs.) - \frac{4}{3}\,K_2(vs.) - \frac{1}{6}\,K_1(vs.) - \frac{1}{24} \\
&+ \frac{1}{2}\,\frac{(6\,K_3(vs.) + 4\,K_2(vs.) + 2\,K_1(vs.) + 1)(-3\,K_1(vs.) - 3\,K_2(vs.) + 13)}{K_1(vs.) + 2\,K_2(vs.) + 4 + 3\,K_3(vs.)}, \\
\Xi_5(vs.) &= \frac{81\,K_3(vs.)}{40} + \frac{4\,K_2(vs.)}{15} + \frac{K_1(vs.)}{120} + \frac{1}{720} \\
&+ \frac{1}{2}\,\frac{6\,K_3(vs.) + 4\,K_2(vs.) + 2\,K_1(vs.) + 1}{K_1(vs.) + 2\,K_2(vs.) + 4 + 3\,K_3(vs.)}\left(\frac{19\,K_1(vs.)}{4} + \frac{13\,K_2(vs.)}{4} - \frac{41}{10}\right) \\
&- \frac{1}{24}\,\frac{\Psi_1(vs.)\,(-3\,K_1(vs.) - 3\,K_2(vs.) + 13)}{(K_1(vs.) + 2\,K_2(vs.) + 4 + 3\,K_3(vs.))^2}, \\
\Psi_1(vs.) &= 76\,K_1(vs.)^2 + 256\,K_1(vs.)\,K_2(vs.) + 336\,K_1(vs.)\,K_3(vs.) \\
&+ 208\,K_2(vs.)^2 + 528\,K_2(vs.)\,K_3 - 259\,K_1(vs.)(vs.) \\
&+ 324\,K_3(vs.)^2 - 458\,K_2(vs.) - 501\,K_3(vs.) - 152, \\
\Xi_6(vs.) &= -\frac{243\,K_3(vs.)}{560} - \frac{8\,K_2(vs.)}{315} - \frac{K_1(vs.)}{5040} - \frac{1}{40320} \\
&+ \frac{1}{2}\,\frac{6\,K_3(vs.) + 4\,K_2(vs.) + 2\,K_1(vs.) + 1}{K_1(vs.) + 2\,K_2(vs.) + 4 + 3\,K_3(vs.)}\,\Psi_2(vs.) \\
&- \frac{1}{24}\,\frac{\Psi_3(vs.)\,\Psi_4(vs.)}{(K_1(vs.) + 2\,K_2(vs.) + 4 + 3\,K_3(vs.))^2} \\
&+ \frac{\Psi_5(vs.)\,(-3\,K_1(vs.) - 3\,K_2(vs.) + 13)}{720\,(K_1(vs.) + 2\,K_2(vs.) + 4 + 3\,K_3(vs.))^3}, \\
\Psi_2(vs.) &= -\frac{211\,K_1(vs.)}{120} - \frac{121\,K_2(vs.)}{120} - \frac{229}{168}, \\
\Psi_3(vs.) &= 76\,K_1(vs.)^2 + 256\,K_1(vs.)K_2(vs.) + 336\,K_1(vs.)K_3(vs.) \\
&+ 208\,K_2(vs.)^2 + 528\,K_2(vs.)K_3(vs.) + 324\,K_3(vs.)^2 \\
&- 259\,K_1(vs.) - 458\,K_2(vs.) - 501\,K_3(vs.) - 152, \\
\Psi_4(vs.) &= \frac{19\,K_1(vs.)}{4} + \frac{13\,K_2(vs.)}{4} - \frac{41}{10},
\end{aligned}$$

$$
\begin{aligned}
\Psi_5(vs.) &= 10266\,K_1(vs.)^3 + 46116\,K_1(vs.)^2\,K_2(vs.) + 52254\,K_1(vs.)^2\,K_3(vs.) \\
&+ 65592\,K_1(vs.)\,K_2(vs.)^2 + 139896\,K_1(vs.)\,K_2(vs.)\,K_3(vs.) + 68742\,K_1(vs.)\,K_3(vs.)^2 \\
&+ 28848\,K_2(vs.)^3 + 83736\,K_2(vs.)^2\,K_3(vs.) + 69444\,K_2(vs.)\,K_3(vs.)^2 \\
&+ 13122\,K_3(vs.)^3 - 40463\,K_1(vs.)^2 - 133232\,K_1(vs.)\,K_2(vs.) \\
&- 135858\,K_1(vs.)\,K_3(vs.) - 110012\,K_2(vs.)^2 - 226896\,K_2(vs.)\,K_3(vs.) \\
&- 117927\,K_3(vs.)^2 + 80990\,K_1(vs.) + 146140\,K_2(vs.) + 178890\,K_3(vs.) + 53392, \\
\Xi_7(vs.) &= \frac{243\,K_3(vs.)}{4480} + \frac{4\,K_2(vs.)}{2835} + \frac{K_1(vs.)}{362880} + \frac{1}{3628800} \\
&+ \frac{1}{2}\,\frac{6\,K_3(vs.) + 4\,K_2(vs.) + 2\,K_1(vs.) + 1}{K_1(vs.) + 2\,K_2(vs.) + 4 + 3\,K_3(vs.)}\,\Psi_6(vs.) \\
&- \frac{1}{24}\,\frac{\Psi_7(vs.)}{(K_1(vs.) + 2\,K_2(vs.) + 4 + 3\,K_3(vs.))^2}\,\Psi_8(vs.) \\
&+ \frac{1}{720}\,\frac{\Psi_9(vs.)}{(K_1(vs.) + 2\,K_2(vs.) + 4 + 3\,K_3(vs.))^3}\,\Psi_{10}(vs.) \\
&- \frac{1}{40320}\,\frac{1}{(K_1(vs.) + 2\,K_2(vs.) + 4 + 3\,K_3(vs.))^4}\,\Psi_{11}(vs.)\Big(-3\,K_1(vs.) - 3\,K_2(vs.) + 13\Big), \\
\Psi_6(vs.) &= \frac{2059\,K_1(vs.)}{6720} + \frac{1093\,K_2(vs.)}{6720} + \frac{2617}{3024}, \\
\Psi_7(vs.) &= 76\,K_1(vs.)^2 + 256\,K_1(vs.)\,K_2(vs.) + 336\,K_1(vs.)\,K_3(vs.) \\
&+ 208\,K_2(vs.)^2 + 528\,K_2(vs.)\,K_3(vs.) + 324\,K_3(vs.)^2 \\
&- 259\,K_1(vs.) - 458\,K_2(vs.) - 501\,K_3(vs.) - 152, \\
\Psi_8(vs.) &= -\frac{211\,K_1(vs.)}{120} - \frac{121\,K_2(vs.)}{120} - \frac{229}{168}, \\
\Psi_9(vs.) &= 10266\,K_1(vs.)^3 + 46116\,K_1(vs.)^2\,K_2(vs.) + 52254\,K_1(vs.)^2\,K_3(vs.) \\
&+ 65592\,K_1(vs.)\,K_2(vs.)^2 + 139896\,K_1(vs.)\,K_2(vs.)\,K_3(vs.) + 68742\,K_1(vs.)\,K_3(vs.)^2 \\
&+ 28848\,K_2(vs.)^3 + 83736\,K_2(vs.)^2\,K_3(vs.) + 69444\,K_2(vs.)\,K_3(vs.)^2 \\
&+ 13122\,K_3(vs.)^3 - 40463\,K_1(vs.)^2 - 133232\,K_1(vs.)\,K_2(vs.) \\
&- 135858\,K_1(vs.)\,K_3(vs.) - 110012\,K_2(vs.)^2 - 226896\,K_2(vs.)\,K_3(vs.) \\
&- 117927\,K_3(vs.)^2 + 80990\,K_1(vs.) + 146140\,K_2(vs.) \\
&+ 178890\,K_3(vs.) + 53392, \\
\Psi_{10}(vs.) &= \frac{19\,K_1(vs.)}{4} + \frac{13\,K_2(vs.)}{4} - \frac{41}{10},
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{11}(vs.) &= 2402072\,K_1(vs.)^4 + 13610080\,K_1(vs.)^3\,K_2(vs.) \\
&+ 27841152\,K_1(vs.)^2\,K_2(vs.)^2 + 54097920\,K_1(vs.)^2\,K_2(vs.)\,K_3(vs.) \\
&+ 24249888\,K_1(vs.)^2\,K_3(vs.)^2 + 13935024\,K_1(vs.)^3\,K_3(vs.) \\
&+ 24184192\,K_1(vs.)\,K_2(vs.)^3 + 65984832\,K_1(vs.)\,K_2(vs.)^2\,K_3(vs.) \\
&+ 53883360\,K_1(vs.)\,K_2(vs.)\,K_3(vs.)^2 - 72514521\,K_1(vs.)\,K_3(vs.)^2 \\
&+ 12347856\,K_1(vs.)\,K_3(vs.)^3 + 7451264\,K_2(vs.)^4 + 24880896\,K_2(vs.)^3\,K_3(vs.) \\
&+ 26733888\,K_2(vs.)^2\,K_3(vs.)^2 + 9581760\,K_2(vs.)\,K_3(vs.)^3 + 472392\,K_3(vs.)^4 \\
&- 13778147\,K_1(vs.)^3 - 61836294\,K_1(vs.)^2\,K_2(vs.) - 59902335\,K_1(vs.)^2\,K_3(vs.) \\
&- 89481732\,K_1(vs.)\,K_2(vs.)^2 - 165890772\,K_1(vs.)\,K_2(vs.)\,K_3(vs.) \\
&- 40936264\,K_2(vs.)^3 - 104978844\,K_2(vs.)^2K_3(vs.) - 78163110\,K_2(vs.)K_3(vs.)^2 \\
&- 12874437\,K_3(vs.)^3 + 35158660\,K_1(vs.)^2 + 116855920\,K_1(vs.)\,K_2(vs.) \qquad \text{(A6)} \\
&+ 116770488\,K_1(vs.)\,K_3(vs.) - 44992832\,K_1(vs.) \\
&+ 97149520\,K_2(vs.)^2 + 193493136\,K_2(vs.)\,K_3(vs.) + 94000644\,K_3(vs.)^2 \\
&- 83437696\,K_2(vs.) - 108869952\,K_3(vs.) - 34241728 + \ldots. \qquad \text{(A7)}
\end{aligned}
$$

Requiring the phase-lag to be minimized, we obtain the system of equations mentioned below:

$$\begin{aligned}
\frac{\left(3\,K_3(vs.)+2\,K_2(vs.)+K_1(vs.)+\frac{1}{2}\right)}{-K_1(vs.)-2\,K_2(vs.)-4-3\,K_3(vs.)} &= 0, \\
\frac{\Xi_4(vs.)}{-K_1(vs.)-2\,K_2(vs.)-4-3\,K_3(vs.)} &= 0, \\
\frac{\Xi_5(vs.)}{-K_1(vs.)-2\,K_2(vs.)-4-3\,K_3(vs.)} &= 0.
\end{aligned} \tag{A8}$$

We obtain the following result after solving the system of Equations (A8):

$$\begin{aligned}
K_1(vs.) &= -\frac{179}{288}, \\
K_2(vs.) &= \frac{13}{180}, \\
K_3(vs.) &= -\frac{11}{1440}.
\end{aligned} \tag{A9}$$

Appendix B. Development of Algorithm VII

Eliminating the Amplification Factor

We obtain the following result when we use the straightforward approach for computing the amplification factor (21):

$$AF = \frac{\Psi_{14}\,(vs.)}{-v^2\,Q_3(vs.)-16\,v^2\,Q_0(vs.)-\frac{793}{120}\,v^2-1}. \tag{A10}$$

where AF denotes the amplification factor, and

$$\begin{aligned}
\Psi_{14}(vs.) &= \sin(4\,v)-\sin(3\,v)-Q_0(vs.)\,v\,\cos(4\,v)-\frac{323}{360}\,v\,\cos(3\,v) \\
&+ \frac{11}{30}\,v\,\cos(2\,v)-Q_3(vs.)\,v\,\cos(vs.)-v\,Q_4(vs.).
\end{aligned} \tag{A11}$$

Assuming that the amplification factor must be eliminated, or that $AF=0$, we derive:

$$Q_0(vs.) = \frac{\Psi_{15}(vs.)}{360\,v\,\cos(4\,v)}, \tag{A12}$$

where

$$\begin{aligned}
\Psi_{15}(vs.) &= -360\,Q_3(vs.)\,v\,\cos(vs.)-323\,v\,\cos(3\,v)+132\,v\,\cos(2\,v) \\
&- 360\,v\,Q_4(vs.)+360\,\sin(4\,v)-360\,\sin(3\,v).
\end{aligned} \tag{A13}$$

Procedure for Minimizing the Phase-Lag

We can obtain the phase-lag by plugging the value of $Q_0(vs.)$ that was previously provided into the direct formula for calculating it (19):

$$PhErr = \frac{v\,\Psi_{16}(vs.)}{\Psi_{17}(vs.)}, \tag{A14}$$

where

$$
\begin{aligned}
\Psi_{16}(vs.) &= 2880\,(\cos(vs.))^3 \sin(vs.)\,v\,Q_4(vs.) + 1440\,(\cos(vs.))^2 \sin(vs.)\,v\,Q_3(vs.) \\
&- 1440\,\cos(vs.)\,\sin(vs.)\,v\,Q_4(vs.) - 264\,\cos(vs.)\,\sin(vs.)\,vs. + 323\,v\,\sin(vs.) \\
&- 360\,Q_3(vs.)\,v\,\sin(vs.) + 360\,\cos(vs.) - 360, \\
\Psi_{17}(vs.) &= 2880\,(\cos(vs.))^4\,v\,Q_3(vs.) - 14520\,(\cos(vs.))^4\,v + 11520\,(\cos(vs.))^3 \sin(vs.) \\
&- 5168\,v\,(\cos(vs.))^3 - 2880\,(\cos(vs.))^2\,v\,Q_3(vs.) - 5760\,(\cos(vs.))^2 \sin(vs.) \\
&+ 15576\,(\cos(vs.))^2\,v - 1440\,Q_3(vs.)\,v\,\cos(vs.) - 5760\,\cos(vs.)\,\sin(vs.) \\
&+ 3876\,v\cos(vs.) + 360\,v\,Q_3(vs.) - 1440\,v\,Q_4(vs.) + 1440\,\sin(vs.) - 2343\,v,
\end{aligned} \tag{A15}
$$

and $PhErr$ denotes the phase-lag.

By applying the Taylor series expansion to Formula (A15), we are able to retrieve:

$$
\begin{aligned}
PhErr &= \frac{(1440\,Q_4(vs.) + 1080\,Q_3(vs.) - 121)}{-1080\,Q_3(vs.) - 1139 - 1440\,Q_4(vs.)}\,v^2 \\
&+ \frac{1}{-1080\,Q_3(vs.) - 1139 - 1440\,Q_4(vs.)}\,v^4\,\Psi_{18}(vs.) \\
&+ \frac{1}{-1080\,Q_3(vs.) - 1139 - 1440\,Q_4(vs.)}\,v^6\,\Psi_{19}(vs.) + \ldots,
\end{aligned} \tag{A16}
$$

where

$$
\begin{aligned}
\Psi_{18}(vs.) &= -3840\,Q_4(vs.) - 1620\,Q_3(vs.) + \frac{823}{6} \\
&+ \frac{(1440\,Q_4(vs.) + 1080\,Q_3(vs.) - 121)(-2160\,Q_3(vs.) + 10398)}{1080\,Q_3(vs.) + 1139 + 1440\,Q_4(vs.)}, \\
\Psi_{19}(vs.) &= 3072\,Q_4(vs.) + 729\,Q_3(vs.) - \frac{3961}{120} \\
&+ \frac{\left(1440\,Q_4(vs.) + 1080\,Q_3(vs.) - 121\right)\left(3780\,Q_3(vs.) - \frac{27993}{2}\right)}{1080\,Q_3(vs.) + 1139 + 1440\,Q_4(vs.)} \\
&- \frac{1}{6}\,\frac{\Psi_{20}(vs.)\left(-2160\,Q_3(vs.) + 10398\right)}{\left(1080\,Q_3(vs.) + 1139 + 1440\,Q_4(vs.)\right)^2}, \\
\Psi_{20}(vs.) &= 24494400\,Q_3(vs.)^2 + 57542400\,Q_{1(vs.)}\,Q_4(vs.) \\
&+ 33177600\,Q_4(vs.)^2 - 58764960\,Q_3(vs.) \\
&- 64781280\,Q_4(vs.) + 6611551.
\end{aligned} \tag{A17}
$$

Requiring the phase-lag to be minimized, we obtain the system of equations mentioned below:

$$
\begin{aligned}
\frac{(1440\,Q_4(vs.) + 1080\,Q_3(vs.) - 121)}{-1080\,Q_3(vs.) - 1139 - 1440\,Q_4(vs.)} &= 0, \\
\frac{1}{-1080\,Q_3(vs.) - 1139 - 1440\,Q_4(vs.)}\,v^4\,\Psi_{18}(vs.) &= 0.
\end{aligned} \tag{A18}
$$

The solution of the above system of equations is given by:

$$
Q_3(vs.) = \frac{53}{360}, \;\; Q_4(vs.) = -\frac{19}{720}. \tag{A19}
$$

Appendix C. Development of Algorithm VIII

Eliminating the Amplification Factor

We obtain the following result when we use the straightforward approach for computing the amplification factor (21):

$$AF = \frac{\Psi_{21}(vs.)}{-v^2\, Q_3(vs.) - 4\, v^2\, Q_2(vs.) - 16\, v^2\, Q_0(vs.) - \frac{323}{40}\, v^2 - 1}, \tag{A20}$$

where AF denotes the amplification factor, and

$$\begin{aligned} \Psi_{21}(vs.) &= \sin(4\,v) - \sin(3\,v) - Q_0(vs.)\, v\, \cos(4\,v) - \frac{323}{360}\, v\, \cos(3\,v) \\ &- Q_2(vs.)\, v\, \cos(2\,v) - Q_3(vs.)\, v\, \cos(vs.) - v\, Q_4(vs.). \end{aligned} \tag{A21}$$

Assuming that the amplification factor must be eliminated, or that $AF = 0$, we derive:

$$Q_0(vs.) = \frac{\Psi_{22}(vs.)}{360\, v\, \cos(4\,v)}, \tag{A22}$$

where

$$\begin{aligned} \Psi_{22}(vs.) &= -360\, Q_2(vs.)\, v\, \cos(2\,v) - 360\, Q_3(vs.)\, v\, \cos(vs.) - 323\, v\, \cos(3\,v) \\ &- 360\, v\, Q_4(vs.) + 360\, \sin(4\,v) - 360\, \sin(3\,v). \end{aligned} \tag{A23}$$

Procedure for Minimizing the Phase-Lag

We can obtain the phase-lag by plugging the value of $Q_0(vs.)$ that was previously provided into the direct formula for calculating it (19):

$$PhErr = \frac{v\, \Psi_{23}(vs.)}{\Psi_{24}(vs.)}, \tag{A24}$$

where

$$\begin{aligned} \Psi_{23}(vs.) &= 2880\, \sin(vs.)\, (\cos(vs.))^3\, v\, Q_4(vs.) + 1440\, \sin(vs.)\, (\cos(vs.))^2\, v\, Q_3(vs.) \\ &+ 720\, \sin(vs.)\, \cos(vs.)\, v\, Q_2(vs.) - 1440\, \sin(vs.)\, \cos(vs.)\, v\, Q_4(vs.) \\ &- 360\, Q_3(vs.)\, v\, \sin(vs.) + 323\, v \sin(vs.) + 360\, \cos(vs.) - 360, \\ \Psi_{24}(vs.) &= 5760\, (\cos(vs.))^4\, v\, Q_2(vs.) + 2880\, (\cos(vs.))^4\, v\, Q_3(vs.) \\ &- 12408\, (\cos(vs.))^4\, v + 11520\, \sin(vs.)\, (\cos(vs.))^3 - 5168\, v\, (\cos(vs.))^3 \\ &- 8640\, (\cos(vs.))^2\, v\, Q_2(vs.) - 2880\, (\cos(vs.))^2\, v\, Q_3(vs.) - 5760\, \sin(vs.)\, (\cos(vs.))^2 \\ &+ 12408\, (\cos(vs.))^2\, v - 1440\, Q_3(vs.)\, v\, \cos(vs.) - 5760\, \sin(vs.)\, \cos(vs.) \\ &+ 3876\, v\, \cos(vs.) + 2160\, v\, Q_2(vs.) + 360\, v\, Q_3(vs.) \\ &- 1440\, v\, Q_4(vs.) + 1440\, \sin(vs.) - 1551\, v, \end{aligned} \tag{A25}$$

and $PhErr$ denotes the phase-lag.

By applying the Taylor series expansion to Formula (A25), we are able to retrieve:

$$
\begin{aligned}
PhErr &= \frac{(1440\,Q_4(vs.)+1080\,Q_3(vs.)+720\,Q_2(vs.)+143)}{-720\,Q_2(vs.)-1080\,Q_3(vs.)-1403-1440\,Q_4(vs.)}\,v^2\\
&+ \frac{1}{-720\,Q_2(vs.)-1080\,Q_3(vs.)-1403-1440\,Q_4(vs.)}\,\Psi_{25}(vs.)\,v^4\\
&+ \frac{1}{-720\,Q_2(vs.)-1080\,Q_3(vs.)-1403-1440\,Q_4(vs.)}\,\Psi_{27}(vs.)\,v^6\\
&+ \frac{1}{-720\,Q_2(vs.)-1080\,Q_3(vs.)-1403-1440\,Q_4(vs.)}\,\Psi_{29}(vs.)\,v^8+\ldots,
\end{aligned} \tag{A26}
$$

where

$$
\begin{aligned}
\Psi_{25}(vs.) &= -3840\,Q_4(vs.)-1620\,Q_3(vs.)-480\,Q_2(vs.)\\
&- \frac{233}{6}+\frac{\Psi_{26}(vs.)\left(-2880\,Q_2(vs.)-2160\,Q_3(vs.)+9342\right)}{720\,Q_2(vs.)+1080\,Q_3(vs.)+1403+1440\,Q_4(vs.)},\\
\Psi_{26}(vs.) &= \left(1440\,Q_4(vs.)+1080\,Q_3(vs.)+720\,Q_2(vs.)+143\right),\\
\Psi_{27}(vs.) &= 3072\,Q_4(vs.)+729\,Q_3(vs.)+96\,Q_2(vs.)\\
&+ \frac{263}{120}+\frac{1440\,Q_4(vs.)+1080\,Q_3(vs.)+720\,Q_2(vs.)+143}{720\,Q_2(vs.)+1080\,Q_3(vs.)+1403+1440\,Q_4(vs.)}\Big(6720\,Q_2(vs.)\\
&+ 3780\,Q_3(vs.)-\frac{23065}{2}\Big)-\frac{1}{6}\,\frac{\Psi_{28}(vs.)\left(-2880\,Q_2(vs.)-2160\,Q_3(vs.)+9342\right)}{(720\,Q_2(vs.)+1080\,Q_3(vs.)+1403+1440\,Q_4(vs.))^2},\\
\Psi_{28}(vs.) &= 14515200\,Q_2(vs.)^2+38102400\,Q_2(vs.)Q_3(vs.)+45619200\,Q_2(vs.)Q_4(vs.)\\
&+ 24494400\,Q_3(vs.)^2+57542400\,Q_3(vs.)Q_4(vs.)+33177600\,Q_4(vs.)^2\\
&- 33678000\,Q_2(vs.)-44794080\,Q_3(vs.)-48054240\,Q_4(vs.)-7688537,\\
\Psi_{29}(vs.) &= -\frac{8192\,Q_4(vs.)}{7}-\frac{2187\,Q_3(vs.)}{14}-\frac{64\,Q_2(vs.)}{7}\\
&- \frac{139}{2520}+\frac{1440\,Q_4(vs.)+1080\,Q_3(vs.)+720\,Q_2(vs.)+143}{720\,Q_2(vs.)+1080\,Q_3(vs.)+1403+1440\,Q_4(vs.)}\,\Psi_{30}(vs.)\\
&- \frac{1}{6}\,\frac{\Psi_{31}(vs.)}{\left(720\,Q_2(vs.)+1080\,Q_3(vs.)+1403+1440\,Q_4(vs.)\right)^2}\,\Psi_{32}(vs.)\\
&+ \frac{\Psi_{33}(vs.)\left(-2880\,Q_2(vs.)-2160\,Q_3(vs.)+9342\right)}{120\,(720\,Q_2(vs.)+1080\,Q_3(vs.)+1403+1440\,Q_4(vs.))^3},\\
\Psi_{30}(vs.) &= -3968\,Q_2(vs.)-2046\,Q_3(vs.)+\frac{2551639}{420},\\
\Psi_{31}(vs.) &= 14515200\,Q_2(vs.)^2+38102400\,Q_2(vs.)Q_3(vs.)+45619200\,Q_2(vs.)Q_4(vs.)\\
&+ 24494400\,Q_3(vs.)^2+57542400\,Q_3(vs.)Q_4(vs.)+33177600\,Q_4(vs.)^2\\
&- 33678000\,Q_2(vs.)-44794080\,Q_3(vs.)-48054240\,Q_4(vs.)-7688537,\\
\Psi_{32}(vs.) &= 6720\,Q_2(vs.)+3780\,Q_3(vs.)-\frac{23065}{2},
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{33}(vs.) &= 1260085248000\, Q_2(vs.)^3 + 4374279936000\, Q_2(vs.)^2\, Q_3(vs.) \\
&+ 4852410624000\, Q_2(vs.)\, Q_3(vs.)^2 + 9524542464000\, Q_2(vs.)\, Q_3(vs.) Q_4(vs.) \\
&+ 4371480576000\, Q_2(vs.)\, Q_4(vs.)^2 + 1689273792000\, Q_3(vs.)^3 \\
&+ 4598788608000\, Q_3(vs.)^2\, Q_4(vs.) + 4514807808000\, Q_2(vs.)^2\, Q_4(vs.) \\
&+ 3701873664000\, Q_3(vs.)\, Q_4(vs.)^2 + 764411904000\, Q_4(vs.)^3 - 4448256134400\, Q_2(vs.)^2 \\
&- 11242923379200\, Q_2(vs.)\, Q_3(vs.) - 11573966361600\, Q_2(vs.)\, Q_4(vs.) \qquad \text{(A27)} \\
&- 7103038910400\, Q_3(vs.)^2 - 14650375910400\, Q_3(vs.)\, Q_4(vs.) \\
&- 7578470937600\, Q_4(vs.)^2 + 4494085662240\, Q_2(vs.) + 1159393471547 \\
&+ 5990509706760\, Q_3(vs.) + 6624257647680\, Q_4(vs.). \qquad \text{(A28)}
\end{aligned}
$$

Requiring the phase-lag to be minimized, we obtain the system of equations mentioned below:

$$
\begin{aligned}
\frac{(1440\, Q_4(vs.) + 1080\, Q_3(vs.) + 720\, Q_2(vs.) + 143)}{-720\, Q_2(vs.) - 1080\, Q_3(vs.) - 1403 - 1440\, Q_4(vs.)} &= 0, \\
\frac{1}{-720\, Q_2(vs.) - 1080\, Q_3(vs.) - 1403 - 1440\, Q_4(vs.)}\, \Psi_{25}(vs.) &= 0, \\
\frac{1}{-720\, Q_2(vs.) - 1080\, Q_3(vs.) - 1403 - 1440\, Q_4(vs.)}\, \Psi_{27}(vs.) &= 0. \qquad \text{(A29)}
\end{aligned}
$$

The solution of the above system of equations is given by:

$$
Q_2(vs.) = -\frac{167}{480}, \;\; Q_3(vs.) = \frac{317}{2520}, \;\; Q_4(vs.) = -\frac{397}{20160}. \qquad \text{(A30)}
$$

Appendix D. Development of Algorithm IX

Eliminating the Amplification Factor

We obtain the following result when we use the straightforward approach for computing the amplification factor (21):

$$
AF = \frac{\Psi_{35}(vs.)}{-16\, v^2\, Q_0(vs.) - 9\, v^2\, Q_1(vs.) - 4\, v^2\, Q_2(vs.) - v^2\, Q_3(vs.) - 1}, \qquad \text{(A31)}
$$

where AF denotes the amplification factor, and

$$
\begin{aligned}
\Psi_{35}(vs.) &= \sin(4\,v) - \sin(3\,v) - Q_0(vs.)\, v\, \cos(4\,v) - Q_1(vs.)\, v\, \cos(3\,v) \\
&- Q_2(vs.)\, v\, \cos(2\,v) - Q_3(vs.)\, v\, \cos(vs.) - v\, Q_4(vs.). \qquad \text{(A32)}
\end{aligned}
$$

Assuming that the amplification factor must be eliminated, or that $AF = 0$, we derive:

$$
Q_0(vs.) = \frac{\Psi_{36}(vs.)}{v\, \cos(4\,v)}, \qquad \text{(A33)}
$$

where

$$
\begin{aligned}
\Psi_{36}(vs.) &= -Q_1(vs.)\, v\, \cos(3\,v) - Q_2(vs.)\, v\, \cos(2\,v) - Q_3(vs.)\, v\, \cos(vs.) \\
&- v\, Q_4(vs.) + \sin(4\,v) - \sin(3\,v). \qquad \text{(A34)}
\end{aligned}
$$

Procedure for Minimizing the Phase-Lag

We can obtain the phase-lag by plugging the value of $Q_0(vs.)$ that was previously provided into the direct formula for calculating it (19):

$$PhErr = \frac{\Psi_{37}(vs.)}{\Psi_{38}(vs.)}, \tag{A35}$$

where

$$\begin{aligned} \Psi_{37}(vs.) &= vs. \Big(8 \sin(vs.) (\cos(vs.))^3 v\, Q_4(vs.) + 4 \sin(vs.)(\cos(vs.))^2 v\, Q_3(vs.) \\ &+ 2 \sin(vs.) \cos(vs.)\, v\, Q_2(vs.) - 4 \sin(vs.) \cos(vs.)\, v\, Q_4(vs.) \\ &+ v\, Q_1(vs.) \sin(vs.) - Q_3(vs.)\, v \sin(vs.) + \cos(vs.) - 1\Big), \end{aligned}$$

$$\begin{aligned} \Psi_{38}(vs.) &= 24 (\cos(vs.))^4 v\, Q_1(vs.) + 16 (\cos(vs.))^4 v\, Q_2(vs.) \\ &+ 8 (\cos(vs.))^4 v\, Q_3(vs.) - 56 (\cos(vs.))^4 v \\ &- 16\, v\, Q_1(vs.) (\cos(vs.))^3 + 32 \sin(vs.) (\cos(vs.))^3 \\ &- 24 (\cos(vs.))^2 v\, Q_1(vs.) - 24 (\cos(vs.))^2 v\, Q_2(vs.) \\ &- 8 (\cos(vs.))^2 v\, Q_3(vs.) - 16 \sin(vs.) (\cos(vs.))^2 \\ &+ 56 (\cos(vs.))^2 v + 12 \cos(vs.)\, v\, Q_1(vs.) \\ &- 4\, Q_3(vs.) v \cos(vs.) - 16 \sin(vs.) \cos(vs.) + 3\, v\, Q_1(vs.) \\ &+ 6\, Q_2(vs.)\, v + Q_3(vs.)\, v - 4\, v\, Q_4(vs.) + 4 \sin(vs.) - 7\, v, \end{aligned} \tag{A36}$$

and $PhErr$ denotes the phase-lag.

By applying the Taylor series expansion to Formula (A36), we are able to retrieve:

$$\begin{aligned} PhErr &= \frac{4\, Q_4(vs.) + 3\, Q_3(vs.) + 2\, Q_2(vs.) + Q_1(vs.) - \frac{1}{2}}{-Q_1(vs.) - 2\, Q_2(vs.) - 3\, Q_3(vs.) - 3 - 4\, Q_4(vs.)}\, v^2 \\ &+ \frac{\Psi_{39}(vs.)}{-Q_1(vs.) - 2\, Q_2(vs.) - 3\, Q_3(vs.) - 3 - 4\, Q_4(vs.)}\, v^4 \\ &+ \frac{\Psi_{41}(vs.)}{-Q_1(vs.) - 2\, Q_2(vs.) - 3\, Q_3(vs.) - 3 - 4\, Q_4(vs.)}\, v^6 \\ &+ \frac{\Psi_{46}(vs.)}{-Q_1(vs.) - 2\, Q_2(vs.) - 3\, Q_3(vs.) - 3 - 4\, Q_4(vs.)}\, v^8 \\ &+ \frac{\Psi_{52}(vs.)}{-Q_1(vs.) - 2\, Q_2(vs.) - 3\, Q_3(vs.) - 3 - 4\, Q_4(vs.)}\, v^{10} + \ldots, \end{aligned} \tag{A37}$$

where

$$
\begin{aligned}
\Psi_{39}(vs.) &= -\frac{32\,Q_4(vs.)}{3} - \frac{9}{2}\,Q_3(vs.) - \frac{4}{3}\,Q_2(vs.) - \frac{1}{6}\,Q_1(vs.) + \frac{1}{24} + \Psi_{40}(vs.),\\
\Psi_{40}(vs.) &= \frac{1}{2}\,\frac{8\,Q_4(vs.) + 6\,Q_3(vs.) + 4\,Q_2(vs.) + 2\,Q_1(vs.) - 1}{Q_1(vs.) + 2\,Q_2(vs.) + 3\,Q_3(vs.) + 3 + 4\,Q_4(vs.)}\\
&\quad \left(-6\,Q_1(vs.) - 8\,Q_2(vs.) - 6\,Q_3(vs.) + \frac{94}{3}\right),\\
\Psi_{41}(vs.) &= \frac{128}{15}\,Q_4(vs.) + \frac{81}{40}\,Q_3(vs.) + \frac{4}{15}\,Q_2(vs.)\\
&+ \frac{Q_1(vs.)}{120} - \frac{1}{720} + \frac{1}{2}\,\Psi_{42}(vs.)\,\Psi_{43}(vs.)\\
&- \frac{1}{24}\,\frac{\Psi_{44}(vs.)}{\left(Q_1(vs.) + 2\,Q_2(vs.) + 3\,Q_3(vs.) + 3 + 4\,Q_4(vs.)\right)^2}\,\Psi_{45}(vs.),\\
\Psi_{42}(vs.) &= \frac{8\,Q_4(vs.) + 6\,Q_3(vs.) + 4\,Q_2(vs.) + 2\,Q_1(vs.) - 1}{Q_1(vs.) + 2\,Q_2(vs.) + 3\,Q_3(vs.) + 3 + 4\,Q_4(vs.)},\\
\Psi_{43}(vs.) &= \frac{37}{2}\,Q_1(vs.) + \frac{56}{3}\,Q_2(vs.) + \frac{21}{2}\,Q_3(vs.) - \frac{1459}{30},\\
\Psi_{44}(vs.) &= 148\,Q_1(vs.)^2 + 520\,Q_1(vs.)\,Q_2(vs.) + 696\,Q_1(vs.)\,Q_3(vs.)\\
&+ 848\,Q_1(vs.)\,Q_4(vs.) + 448\,Q_2(vs.)^2 + 1176\,Q_2(vs.)\,Q_3(vs.)\\
&+ 1408\,Q_2(vs.)\,Q_4(vs.) + 756\,Q_3(vs.)^2 + 1776\,Q_3(vs.)\,Q_4(vs.)\\
&+ 1024\,Q_4(vs.)^2 - 813\,Q_1(vs.) - 1506\,Q_2(vs.)\\
&- 2007\,Q_3(vs.) - 2244\,Q_4(vs.) + 373,\\
\Psi_{45}(vs.) &= -6\,Q_1(vs.) - 8\,Q_2(vs.) - 6\,Q_3(vs.) + \frac{94}{3},
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{46}(vs.) &= -6\,Q_1(vs.) - 8\,Q_2(vs.) - 6\,Q_3(vs.) + \frac{94}{3}\\
&- \frac{1024}{315}\,Q_4(vs.) - \frac{243}{560}\,Q_3(vs.) - \frac{8}{315}\,Q_2(vs.)\\
&- \frac{Q_1(vs.)}{5040} + \frac{1}{40320}\\
&+ \frac{1}{2}\,\frac{8\,Q_4(vs.) + 6\,Q_3(vs.) + 4\,Q_2(vs.) + 2\,Q_1(vs.) - 1}{Q_1(vs.) + 2\,Q_2(vs.) + 3\,Q_3(vs.) + 3 + 4\,Q_4(vs.)}\,\Psi_{47}(vs.)\\
&- \frac{1}{24}\,\frac{\Psi_{48}(vs.)}{(Q_1(vs.) + 2\,Q_2(vs.) + 3\,Q_3(vs.) + 3 + 4\,Q_4(vs.))^2}\,\Psi_{49}(vs.)\\
&+ \frac{\Psi_{50}(vs.)}{720\,(Q_1(vs.) + 2\,Q_2(vs.) + 3\,Q_3(vs.) + 3 + 4\,Q_4(vs.))^3}\,\Psi_{51}(vs.),
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{47}(vs.) &= -\frac{781}{60}\,Q_1(vs.) - \frac{496}{45}\,Q_2(vs.) - \frac{341}{60}\,Q_3(vs.) + \frac{11993}{420},\\
\Psi_{48}(vs.) &= 148\,Q_1(vs.)^2 + 520\,Q_1(vs.)\,Q_2(vs.) + 696\,Q_1(vs.)\,Q_3(vs.)\\
&+ 848\,Q_1(vs.)\,Q_4(vs.) + 448\,Q_2(vs.)^2 + 1176\,Q_2(vs.)\,Q_3(vs.)\\
&+ 1408\,Q_2(vs.)\,Q_4(vs.) + 756\,Q_3(vs.)^2 + 1776\,Q_3(vs.)\,Q_4(vs.)\\
&+ 1024\,Q_4(vs.)^2 - 813\,Q_1(vs.) - 1506\,Q_2(vs.)\\
&- 2007\,Q_3(vs.) - 2244\,Q_4(vs.) + 373,\\
\Psi_{49}(vs.) &= \frac{37}{2}\,Q_1(vs.) + \frac{56}{3}\,Q_2(vs.) + \frac{21}{2}\,Q_3(vs.) - \frac{1459}{30},
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{50}(vs.) &= 39966\, Q_1(vs.)^3 + 196056\, Q_1(vs.)^2\, Q_2(vs.) + 240894\, Q_1(vs.)^2\, Q_3(vs.) \\
&+ 265392\, Q_1(vs.)^2\, Q_4(vs.) + 313272\, Q_1(vs.)\, Q_2(vs.)^2 + 750096\, Q_1(vs.)\, Q_2(vs.)\, Q_3(vs.) \\
&+ 803808\, Q_1(vs.)\, Q_2(vs.)\, Q_4(vs.) + 435402\, Q_1(vs.)\, Q_3(vs.)^2 + 901152\, Q_1(vs.)\, Q_3(vs.)\, Q_4(vs.) \\
&+ 446688\, Q_1(vs.)\, Q_4(vs.)^2 + 162048\, Q_2(vs.)^3 + 562536\, Q_2(vs.)^2\, Q_3(vs.) \\
&+ 580608\, Q_2(vs.)^2 Q_4(vs.) + 624024\, Q_2(vs.)\, Q_3(vs.)^2 + 1224864\, Q_2(vs.)\, Q_3(vs.)\, Q_4(vs.) \\
&+ 562176\, Q_2(vs.) Q_4(vs.)^2 + 217242\, Q_3(vs.)^3 + 591408\, Q_3(vs.)^2\, Q_4(vs.) \\
&+ 476064\, Q_3(vs.)\, Q_4(vs.)^2 + 98304\, Q_4(vs.)^3 - 287141\, Q_1(vs.)^2 \\
&- 993644\, Q_1(vs.)\, Q_2(vs.) - 1244286\, Q_1(vs.)\, Q_3(vs.) - 1310968\, Q_1(vs.)\, Q_4(vs.) \\
&- 853124\, Q_2(vs.)^2 - 2118852\, Q_2(vs.)\, Q_3(vs.) - 2209616\, Q_2(vs.)\, Q_4(vs.) \\
&- 1304109\, Q_3(vs.)^2 - 2692584\, Q_3(vs.)\, Q_4(vs.) - 1375376\, Q_4(vs.)^2 \\
&+ 723888\, Q_1(vs.) + 1311636\, Q_2(vs.) + 1692864\, Q_3(vs.) \\
&+ 1814472\, Q_4(vs.) - 298105, \\
\Psi_{51}(vs.) &= -6\, Q_1(vs.) - 8\, Q_2(vs.) - 6\, Q_3(vs.) + \frac{94}{3}, \\
\Psi_{52}(vs.) &= \frac{2048\, Q_4(vs.)}{2835} + \frac{243\, Q_3(vs.)}{4480} + \frac{4\, Q_2(vs.)}{2835} \\
&+ \frac{Q_1(vs.)}{362880} - \frac{1}{3628800} + \frac{1}{2}\, \frac{8\, Q_4(vs.) + 6\, Q_3(vs.) + 4\, Q_2(vs.) + 2\, Q_1(vs.) - 1}{Q_1(vs.) + 2\, Q_2(vs.) + 3\, Q_3(vs.) + 3 + 4\, Q_4(vs.)}\, \Psi_{53}(vs.) \\
&- \frac{1}{24}\, \frac{\Psi_{54}(vs.)}{\Big(Q_1(vs.) + 2\, Q_2(vs.) + 3\, Q_3(vs.) + 3 + 4\, Q_4(vs.)\Big)^2}\, \Psi_{55}(vs.) \\
&+ \frac{1}{720}\, \frac{\Psi_{56}(vs.)}{\Big(Q_1(vs.) + 2\, Q_2(vs.) + 3\, Q_3(vs.) + 3 + 4\, Q_4(vs.)\Big)^3}\, \Psi_{57}(vs.) \\
&- \frac{1}{120960}\, \frac{\Psi_{58}(vs.)}{\Big(Q_1(vs.) + 2\, Q_2(vs.) + 3\, Q_3(vs.) + 3 + 4\, Q_4(vs.)\Big)^4}\, \Psi_{59}(vs.), \\
\Psi_{53}(vs.) &= \frac{14197}{3360}\, Q_1(vs.) + \frac{1016}{315}\, Q_2(vs.) \\
&+ \frac{5461}{3360}\, Q_3(vs.) - \frac{789731}{90720}, \\
\Psi_{54}(vs.) &= 148\, Q_1(vs.)^2 + 520\, Q_1(vs.)\, Q_2(vs.) + 696\, Q_1(vs.)\, Q_3(vs.) \\
&+ 848\, Q_1(vs.)\, Q_4(vs.) + 448\, Q_2(vs.)^2 + 1176\, Q_2(vs.)\, Q_3(vs.) \\
&+ 1408\, Q_2(vs.)\, Q_4(vs.) + 756\, Q_3(vs.)^2 + 1776\, Q_3(vs.)\, Q_4(vs.) \\
&+ 1024\, Q_4(vs.)^2 - 813\, Q_1(vs.) - 1506\, Q_2(vs.) \\
&- 2007\, Q_3(vs.) - 2244\, Q_4(vs.) + 373, \\
\Psi_{55}(vs.) &= -\frac{781}{60}\, Q_1(vs.) - \frac{496}{45}\, Q_2(vs.) \\
&- \frac{341}{60}\, Q_3(vs.) + \frac{11993}{420},
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{56}(vs.) &= 39966\, Q_1(vs.)^3 + 196056\, Q_1(vs.)^2\, Q_2(vs.) + 240894\, Q_1(vs.)^2\, Q_3(vs.) \\
&+ 265392\, Q_1(vs.)^2\, Q_4(vs.) + 313272\, Q_1(vs.)\, Q_2(vs.)^2 + 750096\, Q_1(vs.)\, Q_2(vs.)\, Q_3(vs.) \\
&+ 803808\, Q_1(vs.)\, Q_2(vs.)\, Q_4(vs.) + 435402\, Q_1(vs.)\, Q_3(vs.)^2 + 901152\, Q_1(vs.)\, Q_3(vs.)\, Q_4(vs.) \\
&+ 446688\, Q_1(vs.)\, Q_4(vs.)^2 + 162048\, Q_2(vs.)^3 + 562536\, Q_2(vs.)^2\, Q_3(vs.) \\
&+ 580608\, Q_2(vs.)^2\, Q_4(vs.) + 624024\, Q_2(vs.)\, Q_3(vs.)^2 + 1224864\, Q_2(vs.)\, Q_3(vs.)\, Q_4(vs.) \\
&+ 562176\, Q_2(vs.)\, Q_4(vs.)^2 + 217242\, Q_3(vs.)^3 + 591408\, Q_3(vs.)^2\, Q_4(vs.) \\
&+ 476064\, Q_3(vs.)\, Q_4(vs.)^2 + 98304\, Q_4(vs.)^3 - 287141\, Q_1(vs.)^2 \\
&- 993644\, Q_1(vs.)\, Q_2(vs.) - 1244286\, Q_1(vs.)\, Q_3(vs.) - 1310968\, Q_1(vs.)\, Q_4(vs.) \\
&- 853124\, Q_2(vs.)^2 - 2118852\, Q_2(vs.)\, Q_3(vs.) - 2209616\, Q_2(vs.)\, Q_4(vs.) \\
&- 1304109\, Q_3(vs.)^2 - 2692584\, Q_3(vs.)\, Q_4(vs.) - 1375376\, Q_4(vs.)^2 \\
&+ 723888\, Q_1(vs.) + 1311636\, Q_2(vs.) + 1692864\, Q_3(vs.) \\
&+ 1814472\, Q_4(vs.) - 298105, \\
\Psi_{57}(vs.) &= \frac{37}{2}\, Q_1(vs.) + \frac{56}{3}\, Q_2(vs.) + \frac{21}{2}\, Q_3(vs.) - \frac{1459}{30},
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{58}(vs.) &= 55659768\,Q_1(vs.)^4 + 352129872\,Q_1(vs.)^3\,Q_2(vs.) \\
&+ 412143264\,Q_1(vs.)^3\,Q_3(vs.) + 1885960656\,Q_1(vs.)^2\,Q_2(vs.)\,Q_3(vs.) \\
&+ 421068192\,Q_1(vs.)^3\,Q_4(vs.) + 821700000\,Q_1(vs.)^2\,Q_2(vs.)^2 \\
&+ 1879728768\,Q_1(vs.)^2\,Q_2(vs.)\,Q_4(vs.) + 1056381264\,Q_1(vs.)^2\,Q_3(vs.)^2 \\
&+ 2040241824\,Q_1(vs.)^2\,Q_3(vs.)\,Q_4(vs.) + 2816335296\,Q_1(vs.)\,Q_2(vs.)^2\,Q_3(vs.) \\
&+ 942304896\,Q_1(vs.)^2\,Q_4(vs.)^2 + 836548032\,Q_1(vs.)\,Q_2(vs.)^3 \\
&+ 2727370368\,Q_1(vs.)\,Q_2(vs.)^2\,Q_4(vs.) + 3073982832\,Q_1(vs.)\,Q_2(vs.)\,Q_3(vs.)^2 \\
&+ 5734985472\,Q_1(vs.)\,Q_2(vs.)\,Q_3(vs.)\,Q_4(vs.) + 2893125600\,Q_1(vs.)\,Q_3(vs.)^2 Q_4(vs.) \\
&+ 2532734208\,Q_1(vs.)\,Q_2(vs.)\,Q_4(vs.)^2 + 1082328480\,Q_1(vs.)\,Q_3(vs.)^3 \\
&+ 2403198720\,Q_1(vs.)\,Q_3(vs.)\,Q_4(vs.)^2 + 600645120\,Q_1(vs.)\,Q_4(vs.)^3 \\
&+ 312778752\,Q_2(vs.)^4 + 2174279328\,Q_2(vs.)^2\,Q_3(vs.)^2 \\
&+ 1368555840\,Q_2(vs.)^3\,Q_3(vs.) + 1281146880\,Q_2(vs.)^3\,Q_4(vs.) \\
&+ 3891547008\,Q_2(vs.)^2\,Q_3(vs.)\,Q_4(vs.) + 1625260032\,Q_2(vs.)^2\,Q_4(vs.)^2 \\
&+ 1477117296\,Q_2(vs.)\,Q_3(vs.)^3 + 640745472\,Q_2(vs.)\,Q_4(vs.)^3 \\
&+ 3747821184\,Q_2(vs.)\,Q_3(vs.)^2\,Q_4(vs.) + 2890374912\,Q_2(vs.)\,Q_3(vs.)\,Q_4(vs.)^2 \\
&+ 358980984\,Q_3(vs.)^4 + 1128657888\,Q_3(vs.)^3\,Q_4(vs.) \\
&+ 1167405696\,Q_3(vs.)^2\,Q_4(vs.)^2 - 565295331\,Q_1(vs.)^3 \\
&+ 419830272\,Q_3(vs.)\,Q_4(vs.)^3 + 25165824\,Q_4(vs.)^4 \\
&- 2779573266\,Q_1(vs.)^2 Q_2(vs.) - 3305432331\,Q_1(vs.)^2\,Q_3(vs.) \\
&- 3315314340\,Q_1(vs.)^2\,Q_4(vs.) - 10309349520\,Q_1(vs.)\,Q_2(vs.)\,Q_4(vs.) \\
&- 4489981092\,Q_1(vs.) Q_2(vs.)^2 - 10500895404\,Q_1(vs.)\,Q_2(vs.) Q_3(vs.) \\
&- 6022349649\,Q_1(vs.)\,Q_3(vs.)^2 - 11541767256\,Q_1(vs.)\,Q_3(vs.)\,Q_4(vs.) \\
&- 5369568912\,Q_1(vs.)\,Q_4(vs.)^2 - 7780962960\,Q_2(vs.)^2\,Q_4(vs.) \\
&- 2374354968\,Q_2(vs.)^3 - 8152860204\,Q_2(vs.)^2\,Q_3(vs.) \\
&- 9095875938\,Q_2(vs.)\,Q_3(vs.)^2 - 16783131312\,Q_2(vs.)\,Q_3(vs.)\,Q_4(vs.) \\
&- 7402899744\,Q_2(vs.)\,Q_4(vs.)^2 - 7043464368\,Q_3(vs.)\,Q_4(vs.)^2 \\
&- 3267603585\,Q_3(vs.)^3 - 8612711748\,Q_3(vs.)^2\,Q_4(vs.) \\
&- 1685883072\,Q_4(vs.)^3 + 2129452501\,Q_1(vs.)^2 \\
&+ 7315812004\,Q_1(vs.)\,Q_2(vs.) + 6247837684\,Q_2(vs.)^2 \\
&+ 8917489566\,Q_1(vs.)\,Q_3(vs.) + 8861348168\,Q_1(vs.)\,Q_4(vs.) \\
&+ 15124089132\,Q_2(vs.)\,Q_3(vs.) + 14867966416\,Q_2(vs.)\,Q_4(vs.) \\
&+ 9077381469\,Q_3(vs.)^2 - 3528107889\,Q_1(vs.) \\
&+ 17635437144\,Q_3(vs.)\,Q_4(vs.) + 8425636816\,Q_4(vs.)^2 \\
&- 6322631682\,Q_2(vs.) - 8014817763\,Q_3(vs.) \\
&- 8339272836\,Q_4(vs.) + 1310487239, \\
\Psi_{59}(vs.) &= -6\,Q_1(vs.) - 8\,Q_2(vs.) - 6\,Q_3(vs.) + \frac{94}{3}.
\end{aligned}
\tag{A38}
$$

Requiring the phase-lag to be minimized, we obtain the system of equations mentioned below:

$$
\begin{aligned}
\frac{4\,Q_4(vs.) + 3\,Q_3(vs.) + 2\,Q_2(vs.) + Q_1(vs.) - \frac{1}{2}}{-Q_1(vs.) - 2\,Q_2(vs.) - 3\,Q_3(vs.) - 3 - 4\,Q_4(vs.)} &= 0, \\
\frac{\Psi_{39}(vs.)}{-Q_1(vs.) - 2\,Q_2(vs.) - 3\,Q_3(vs.) - 3 - 4\,Q_4(vs.)} &= 0, \\
\frac{\Psi_{41}(vs.)}{-Q_1(vs.) - 2\,Q_2(vs.) - 3\,Q_3(vs.) - 3 - 4\,Q_4(vs.)} &= 0, \\
\frac{\Psi_{46}(vs.)}{-Q_1(vs.) - 2\,Q_2(vs.) - 3\,Q_3(vs.) - 3 - 4\,Q_4(vs.)} &= 0.
\end{aligned}
\tag{A39}
$$

The solution of the above system of equations is given by:

$$\begin{aligned} Q_1(vs.) &= \frac{5561}{8640}, \ Q_2(vs.) = -\frac{163}{1728}, \\ Q_3(vs.) &= \frac{23}{1344}, \ Q_4(vs.) = -\frac{191}{120960}. \end{aligned} \quad \text{(A40)}$$

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