symmetry

MDPI

Article

On Maximum Guaranteed Payoff in a Fuzzy Matrix Decision-Making Problem with a Fuzzy Set of States

Svajone Bekesiene [1,*] and Serhii Mashchenko [2]

[1] Logistics and Defense Technology Management Science Group, General Jonas Zemaitis Military Academy of Lithuania, Silo 5a, LT-10322 Vilnius, Lithuania
[2] Department of System Analysis and Decision-Making Theory, Faculty of Computer Science and Cybernetics, Taras Shevchenko National University of Kyiv, 64/13, Volodymyrska Street, 01601 Kyiv, Ukraine; mashchenko@knu.ua
* Correspondence: svajone.bekesiene@lka.lt

Abstract: The current study delves into a fuzzy matrix decision-making problem involving fuzzy sets of states. It establishes that a maximum guaranteed payoff constitutes a type-2 fuzzy set defined on the real line. Additionally, it provides the associated type-2 membership function. Moreover, the paper illustrates that the maximum guaranteed payoff type-2 fuzzy set of the decision-making problem can be broken down, based on the secondary membership grades, into a finite collection of fuzzy numbers. Each of these fuzzy numbers represents the maximum guaranteed payoff of the corresponding decision-making problem with a crisp set of states. This set corresponds to a specific cut of the original fuzzy set of states. Some properties of the maximum guaranteed payoff type-2 fuzzy set are investigated, and illustrative examples are provided. Since the problem formulation is symmetrical with respect to alternatives and states of nature, the results obtained can be used in the case of a fuzzy set of alternatives.

Keywords: decision-making problem; maximum guaranteed payoff; type-2 fuzzy set

Citation: Bekesiene, S.; Mashchenko, S. On Maximum Guaranteed Payoff in a Fuzzy Matrix Decision-Making Problem with a Fuzzy Set of States. *Symmetry* **2024**, *16*, 510. https://doi.org/10.3390/sym16050510

Academic Editors: Aleksandar Djordjevic and Miladin Stefanović

Received: 28 March 2024
Revised: 11 April 2024
Accepted: 15 April 2024
Published: 23 April 2024

1. Introduction

A decision problem modeling in the formal theory requires [1]:

- alternatives (choices of a DM);
- states (events);
- the payoff associated with each alternative and each state.

This is often presented as a payoff matrix (decision matrix). The rows of this matrix are associated with alternatives, and the columns are associated with states [1].

We note that, at its core, the matrix decision-making problem is symmetrical with respect to alternatives and states of nature. These elements of the problem differ only in that the alternatives characterize the actions of the DM, and the states of nature are characterized by some objective reality, which should not be taken literally. There may well be situations in which actions are dictated by nature (for example, circumstances related to weather conditions or natural forces). Therefore, the division into alternatives and states of nature is conditional and is used only as an element of system analysis that uses symmetry.

The principle of utility maximization was axiomatically substantiated in Von Neumann and Morgenstern [2] and Savage [3]. This principle consists of choosing an alternative that maximizes the so-called decision criterion (the utility function of alternatives). The type of a decision criterion depends on the availability of some additional information about:

- the type of uncertainty;
- the set of states;
- the risk features of a DM.

In conventional decision-making scenarios under uncertainty, it is typically assumed that there exists a probability distribution across the space of states, leading to what is known as decision-making under risk. In this context, DMs assess and choose among alternatives while considering the probabilities associated with various states of nature. To solve this problem, statistical criteria are used—for instance, Bayesian criteria, variance minimization, etc. In situations where information regarding the probability distribution is unavailable, this model gives rise to what is termed as a decision-making problem under ignorance. In this case, one uses various decision-making criteria like the maxmin (the Wald criterion), the maxmax (the optimistic criterion), the minimax regret (the Savage criterion), and the Hurwicz criterion. Unfortunately, the assumptions of the classical decision-making theory do not necessarily hold, for probabilities of real-world events are sometimes imprecise or non-measurable. In addition, a set of alternatives, a set of states, and payoffs may not be known precisely both in conditions of ignorance and in conditions of risk. The desire to take these factors into account led to the application of fuzzy set (FS) theory [4–12]. The corresponding works include approaches to solving decision problems with:

- a fuzzy set of alternatives [4,5];
- fuzzy states [4–6];
- a fuzzy set of states [1,7];
- fuzzy probabilities of states [8–12];
- fuzzy payoffs [1,8];
- fuzzy information [4–6].

The following contributions transform classical decision theory into fuzzy decision theory. Ignorance represents a specific instance of uncertainty within decision-making problems; thus, certain approaches rooted in fuzzy set (FS) theory designed for handling uncertainty can be utilized in cases of ignorance as well. In [6], Tanaka et al. explored a decision-making problem wherein the uncertainty surrounding the interpretation of events was captured using fuzzy sets (FSs), while the uncertainty regarding the likelihood of events occurring was quantified using probabilities. The authors introduced definitions related to worth, entropy, and quantity specifically tailored to fuzzy information within their paper. These concepts play crucial roles in analyzing decision-making processes under the combined influence of fuzzy logic and statistic uncertainty. Based on these definitions, an investment problem was analyzed. In [4], Tanaka et al. investigated properties of the fuzzy Bayes formula and the fuzzy observation system derived from FS theory. The analysis made it possible to consider the main elements of a fuzzy decision-making problem, which are states, alternatives, and other available information as fuzzy events. In [5], Tanaka et al. merged fuzzy set (FS) theory with statistical decision theory to address decision problems involving fuzzy events. They formulated fuzzy decision-making problems based on the concept of fuzzy events and introduced definitions related to entropy, the value of information, and the amount of information within this context. On the other hand, in [7], Mashchenko explored a method for resolving decision-making problems under uncertainty using a fuzzy set of states and crisp payoffs. By combining fuzzy set theory with crisp payoffs, Mashchenko proposed a method to handle decision-making scenarios where states are described imprecisely but payoffs are known crisply. In this method, a DM maximizes utilities simultaneously for all states. A type-2 fuzzy relation was constructed, which characterizes the utility of alternatives. The concept of a maximizing weak solution was introduced and its properties were explored in the referenced paper. Subsequently, this method was further developed in [13,14] for decision-making problems with objectives represented as fuzzy sets of preference relations. Additionally, in [1], Jain investigated a decision-making approach tailored for systems where either the state of the system or the utilities of alternatives, or both, were described using fuzzy sets (FSs). Through this research, Jain aimed to provide a methodology to address decision-making scenarios characterized by imprecise or uncertain information. The problem reduces to choosing, in a certain sense, the best FS from a collection of fuzzy estimates of alternatives. Each fuzzy estimate is an image of the FS of states under a mapping, which is specified by the payoff matrix. The method uses the Zadeh concept of a maximizing set to determine the FS of a solution to the problem. Next, an alternative is selected. This alternative is a compromise between the value of the utility and the degree of the membership to the solution FS. The idea of using a maximizing set to determine the solution FS does not correspond to any of the known optimality criteria of expected utility theory, which is a drawback of the method and limits its application. In [8], Whalen conducted a comparative analysis of different types of methodologies for choosing alternatives under uncertainty. These are:

- the classical statistical analysis;
- the maxmin approach;
- the analysis of fuzzy statistical solutions [9,10];
- the possibilistic decision-making algorithm [11] and its development;
- the L-fuzzy risk minimization algorithm [12], which uses Z-fuzzy sets [15].

These approaches vary based on their underlying assumptions regarding the quality and quantity of information available concerning the likelihood or probability of different states and the utility associated with the outcomes (alternative-state pairs). Each approach may make different assumptions about the precision of the information available, the form in which it is represented (e.g., probabilistic or fuzzy), and the degree of uncertainty accounted for in the decision-making process. These differences in assumptions can significantly influence the strategies and methodologies employed in addressing decision-making problems under uncertainty.

Advances in the development of the theory of fuzzy numbers (FNs) have made it possible to generalize the standard criteria of crisp decision-making to cases of payoffs in the form of FNs and use them directly. The following decision-making criteria under ignorance were generalized [16]:

- the maxmin (the Wald criterion);
- the maxmax (an optimistic criterion);
- the minimax regret (the Savage criterion);
- the Hurwicz criterion.

In [16], when studying a multicriteria problem with fuzzy parameters in the form of triangular FNs, Larbani examined a multi-objective two-person zero-sum game scenario, where the first player acted as the decision maker (DM), and the second player represented nature. In addressing this problem, Larbani employed the maxmin criterion, a decision-making approach under uncertainty. While our review does not encompass all the relevant literature, several conclusions can still be drawn from the existing research:

- Utilizing fuzzy sets (FSs) to represent states offers a more intrinsic and nuanced depiction of decision-making models under conditions of uncertainty compared to crisp sets.
- A FS of alternatives, a FS of states, and fuzzy payoffs describe different elements of a decision-making model under uncertainty and are of interest for research.
- Using FNs allows one to quite effectively solve decision-making problems under uncertainty by means of the fuzzy arithmetic.
- In the context of decision-making, there has been a lack of investigation into models utilizing payoffs represented as FNs alongside decision-making criteria under ignorance, specifically concerning fuzzy sets of states. This gap in research suggests an opportunity for further exploration and development in this area.

This article focuses on the exploration of decision-making processes under conditions of ignorance, specifically within the framework of fuzzy sets of states. The central place among criteria for decision-making under ignorance is occupied by the maxmin criterion (the Wald criterion). This stems from the fact that the maxmin criterion is widely used in practice in cases where it is necessary to completely eliminate a risk. In addition, the maxmin criterion is the mathematical basis of other criteria in conditions of ignorance. For instance, the minimax regret criterion is represented as the maxmin criterion for a negative relative loss function, the optimistic criterion is represented as the maxmin criterion for negative payoffs, and the Hurwicz criterion is represented as the linear convolution of the maxmin and optimistic criteria. In this article, we are going to investigate an application of the maxmin criterion for a decision-making model under ignorance with a FS of states. For the sake of simplicity, we investigate the maximum guaranteed payoff without focusing on an 'optimal' alternative. Furthermore, the significance of such a study can be elucidated by the understanding that decision-making problems under uncertainty hold interest beyond the pursuit of identifying an 'optimal' alternative. At times, the maximum guaranteed payoff holds independent interest, as decision-making theory's scope extends to encompass forecasting and analyzing uncertainties from the perspective of a decision-maker, as well as external agents. The aim of such an analysis is often directed towards informing decisions concerning alterations in the prerequisites, conditions, and outcomes of situations characterized by uncertainty. These decisions may involve adjustments in factors such as awareness levels, model formulations, and other relevant parameters. The formalization of a set of states through the utilization of a fuzzy set framework enables the incorporation of not only uncertainty stemming from the decision-maker's assessment of state admissibility but also the inherent fuzziness associated with delineating which states are feasible within the decision-making model. An example of such FSs of states could be 'possible states', 'probable states', 'expected states', etc.

The principal aims of this article are delineated as follows:

- To provide a rationale supporting the claim that a fuzzy set (FS) of states within a decision-making scenario under ignorance leads to a type-2 fuzzy set (T2FS) of maximum guaranteed payoff. This T2FS is characterized by a simplified and practical form tailored for real-world applications, in contrast to the more generalized form of T2FS.
- To thoroughly investigate the properties of this specific T2FS.
- To develop a decomposition method aimed at constructing a T2FS representing the maximum guaranteed payoff.
- To demonstrate, with the help of an example, that the approach that we develop here enables us to solve a fuzzy decision-making problem under ignorance in the situation where standard decision-making criteria fail.

The practical significance of our findings lies in their ability to effectively model the inherent vagueness in human judgment and the ambiguity surrounding the acceptability of states during the decision-making process under conditions of ignorance. By acknowledging and accounting for these factors, our research offers valuable insights and methodologies for addressing uncertainty and making informed decisions in real-world contexts.

This article is organized as follows: After this introduction, in Section 2, we recall some definitions and some basic results about MAX and MIN operations on FNs, a maximum guaranteed payoff in a fuzzy matrix decision-making problem, and T2FSs to be used later. In Section 3, we formulate the decision-making problem with a fuzzy set of states. Section 4 proposes an idea for solving this problem. At the end of Section 4, we conclude that the resulting maximum guaranteed payoff is a T2FS on the real line. Section 5.1 is devoted to the definition of a maximum guaranteed payoff T2FS and results that simplify its construction method. Section 5.2 is concerned with the calculation algorithm. Some useful properties of a maximum guaranteed payoff T2FS to a decision-making problem for a FS of states are studied in Section 5.3. In Section 5.4, we consider the example of constructing the resulting T2FS and give a simple interpretation of it. In addition, we study the ability of the developed approach to find a solution to a decision-making problem under ignorance in the case where the standard decision-making criteria fail. Section 6 is devoted to a discussion of the results. In the last section, some conclusions are pointed out.

2. Materials and Methods

2.1. MAX and MIN Operations on Fuzzy Numbers

In [17], Dubois and Prade introduced the concept of a fuzzy number, defining it as a normal and convex fuzzy set (FS) defined on the real line $\mathbb{R}$. In broader terms, a fuzzy number (FN) is conceptualized as a normal fuzzy set (FS) with bounded support, characterized by a membership function (MF) that is upper semicontinuous and quasi-concave, as described in reference [18]. This definition encapsulates fuzzy numbers as having continuous membership values across their domains, with a degree of concavity in their MF, contributing to their smooth and well-defined properties. One pivotal challenge in the theory of fuzzy numbers revolves around ordering them, a task facilitated by employing mathematical operations such as maximum (MAX) and minimum (MIN). This process is crucial for comparing and arranging fuzzy numbers, which is vital for various decision-making and analytical purposes within fuzzy systems and applications. The idea of using the MAX and MIN operations was first proposed by Ambrosio and Martini [19] for fuzzy symbols. A fuzzy symbol is an analog of a FS, according to Fung and Fu [20]. The MF of a fuzzy symbol is a mapping from one complete and linearly ordered space to another. This MF and the MF of a FN enjoy similar properties. In [21], Klir and Yuan defined the MAX and MIN operations for classical FNs and investigated some basic properties of these operations. They showed that the threesome (Φ, MIN, and MAX) is a distributive lattice, where Φ is the set of FNs on $\mathbb{R}$. In [22], Zhang and Hirota developed an algebraic theory of lattices for FNs. In [23], Tahayoria et al. extended the employment of the MAX and MIN operations to the case of convex FSs on $\mathbb{R}$ with MFs, which do not necessarily have to be continuous. In [24], Chiu and Wang suggested an approach towards simplifying the application of the MAX and MIN operations for two FNs with continuous MFs. In [25], Shirin and Saha worked out an algorithm for computing the MAX and MIN of any two 'triangular' FNs and visualizing the resulting MF. In [26], Mashchenko investigated the MIN operation in the case of a FS of operands.

According to [21], for FNs A and B with the MFs $\mu_A(r)$ and $\mu_B(r)$, $r \in \mathbb{R}$, respectively, the minimum $MIN = \min\{A, B\}$ is defined by the FN $MIN = \{(r, \mu_{MIN}(r)) : r \in \mathbb{R}\}$ with the MF

$$\mu_{MIN}(r) = \max\{\min\{\mu_A(r_A), \mu_B(r_B)\} : r = \min\{r_A, r_B\},\ r_A, r_B \in \mathbb{R}\},\ r \in \mathbb{R}. \tag{1}$$

To calculate $\mu_{MIN}(r)$, the idea of demonstrating a FN by its cuts is employed in [17]. To be more specific, one represents MFs of the FNs A, B, and MIN in the form

$$\mu_A(r) = \max_{u\in[0,1]} u1_{[A]_u}(r),\ \mu_B(r) = \max_{u\in[0,1]} u1_{[B]_u}(r),\ \mu_{MIN}(r) = \max_{u\in[0,1]} u1_{[MIN]_u}(r), \tag{2}$$

where the closed intervals $[A]_u = [(A)_u^L, (A)_u^H]$, $[B]_u = [(B)_u^L, (B)_u^H]$, and $[MIN]_u = [(MIN)_u^L, (MIN)_u^H]$ are u-cuts, $u \in [0,1]$ of the FNs A, B, and MIN, respectively. Here, we denote by $(\cdot)_u^L$ and $(\cdot)_u^H$ the lower and upper end points of the closed interval $[\cdot]_u = [(\cdot)_u^L, (\cdot)_u^H]$ of the u-cut, $u \in [0,1]$. These u-cuts are crisp sets with the MFs

$$1_{[A]_u}(r) = \begin{cases} 1, & r \in [A]_u; \\ 0, & r \notin [A]_u; \end{cases}\ 1_{[B]_u}(r) = \begin{cases} 1, & r \in [B]_u; \\ 0, & r \notin [B]_u; \end{cases}\ 1_{[MIN]_u}(r) = \begin{cases} 1, & r \in [MIN]_u; \\ 0, & r \notin [MIN]_u; \end{cases}$$

$r \in \mathbb{R}$, respectively. According to [17], Formula (1) implies that the u-cut $[MIN]_u$ of the FN $MIN = \min\{A, B\}$ is provided by

$$[MIN]_u = [(MIN)_u^L, (MIN)_u^H] = [\min\{(A)_u^L, (B)_u^L\}, \min\{(A)_u^H, (B)_u^H\}]. \tag{3}$$

Thus, Formula (2) entails the representations

$$\mu_{MIN}(r) = \max\Big\{u \in [0,1] :\ (MIN)_u^L \le r \le (MIN)_u^H\Big\},\ r \in \mathbb{R} \tag{4}$$

and

$$MIN = \{(r,u) : r \in [(MIN)_u^L, (MIN)_u^H],\ u \in [0,1]\} = \{([(MIN)_u^L, (MIN)_u^H], u) : u \in [0,1]\} \quad (5)$$

Similarly, we obtain formulae for calculating the maximum $MAX = \max\{A,B\}$ as the FN $MAX = \{(r, \mu_{MAX}(r)) : r \in \mathbb{R}\}$ with the MF

$$\mu_{MAX}(r) = \max\{\min\{\mu_A(r_A), \mu_B(r_B)\} : r = \max\{r_A, r_B\},\ r_A, r_B \in \mathbb{R}\},\ r \in \mathbb{R}. \quad (6)$$

The u-cut $[MAX]_u$ of the FN $MAX = \max\{A,B\}$ is provided by

$$[MAX]_u = [(MAX)_u^L, (MAX)_u^H] = [\max\{(A)_u^L, (B)_u^L\}, \max\{(A)_u^H, (B)_u^H\}]. \quad (7)$$

We represent the MF μ_{MAX} and the FN MAX as follows:

$$\mu_{MAX}(r) = \max\{u \in [0,1] :\ (MAX)_u^L \le r \le (MAX)_u^H\},\ r \in \mathbb{R} \quad (8)$$

and

$$MAX = \{(r,u) : r \in [(MAX)_u^L, (MAX)_u^H],\ u \in [0,1]\} = \{([(MAX)_u^L, (MAX)_u^H], u) : u \in [0,1]\}, \quad (9)$$

respectively.

The lattice (Φ, MIN, MAX) can also be expressed as the pair $(\Phi, \precsim)$, where $\precsim$ is a fuzzy partial order, which is defined by $A \precsim B \Leftrightarrow MIN\{A,B\} = A$ or otherwise $A \precsim B \Leftrightarrow MAX\{A,B\} = B$. This method of FNs ranking leads to the conclusion that, in some cases, the FNs A and B are incomparable. Therefore, the set Φ of FNs is not linearly ordered, unlike the set of real numbers. This property of fuzzy partial order limits its use for FNs ranking, since it does not guarantee the uniqueness of choice. To resolve this problem, a lot of special methods for FNs ranking have been developed, which ensure a unique choice. A review of ranking methods can be found in [27]. However, applying these FNs ranking methods for calculating the minimum (maximum) leads to a subset of the MIN (MAX). Thus, as opposed to crisp numbers, as far as FNs are concerned, it is important to understand the main problem that a DM faces. If it is more important to choose a unique minimum (maximum) FN, then it is necessary to use one of the methods for FNs ranking that is suitable under the given conditions and then take the minimum (maximum) that is equal to this FN. This FN is a subset of the MIN (MAX) only. If it is more important to obtain the correct value of the minimum (maximum), then Formula (1) (Formula (6)) should be used. In this case, the uniqueness of the choice is not guaranteed. Since, in this article, our concern is the correct values of the minimum and maximum, we shall use Formulae (1) and (6).

2.2. Maximum Guaranteed Payoff in a Fuzzy Matrix Decision-Making Problem

The problem of decision-making under ignorance is conveniently formalized using a matrix (we denote this by $\tilde{F} = (F_{ij})_{i \in M, j \in J}$) of the payoffs presented in Table 1. In the matrix $\tilde{F} = (F_{ij})_{i \in M, j \in J}$, to the rows $i \in M = \{1, \ldots, m\}$, $|M| = m$, we associate the alternatives feasible for the DM. The DM needs to choose one of these alternatives. To the columns $j \in J = \{1, \ldots, l\}$, $|J| = l$, we associate the states (events), one of which can occur.

Table 1. The matrix representing the payoffs.

Alternatives	**States**			
	1	**2**	...	***l***
1	F_{11}	F_{12}	...	F_{1l}
2	F_{21}	F_{22}	...	F_{2l}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
m	F_{m1}	F_{m2}	...	F_{ml}

In the matrix $\tilde{F} = (F_{ij})_{i \in M, j \in J}$, to the rows $i \in M = \{1, \ldots, m\}$, $|M| = m$, we associate the alternatives feasible for the DM. The DM needs to choose one of these alternatives. To the columns $j \in J = \{1, \ldots, l\}$, $|J| = l$, we associate the states (events), one of which can occur. We denote by $|\cdot|$ the cardinality of a set. Each element F_{ij} of the matrix, $\tilde{F}$ represents a payoff (a utility, a win) in the form of the FN with the MF $\mu_{F_{ij}}(r)$, $r \in \mathbb{R}$, that the DM obtains if the alternative $i \in M$ has been chosen and the state $j \in J$ has occurred. For the DM, the objective is to select an alternative with a maximum payoff when the state is unknown. We represent this problem in the form $D(J) = \langle M, J, \tilde{F} \rangle$. Assume that information about the probabilities of states is not available or that a decision is made so seldom that we cannot use this information. According to the maxmin criterion (the Wald criterion), a maximum guaranteed payoff is calculated by the formula

$$G(J) = \max_{i \in M} \min_{j \in J} F_{ij}. \tag{10}$$

For the convenience of subsequent presentation, the set J of states is indicated hereinafter as the parameter in the maximum guaranteed payoff $G(J)$. According to Section 2.1, the maximum guaranteed payoff is the FN $G(J) = \left\{ (r, \mu_{G(J)}(r)) : r \in \mathbb{R} \right\}$ on $\mathbb{R}$. Formulae (1) and (6) imply that the MF of the FN $G(J) = \max_{i \in M} \min_{j \in J} F_{ij}$ is given by

$$\mu_{G(J)}(r) = \max\left\{ \min_{j \in J} \mu_{F_{ij}}(r_{ij}) : \ r = \max_{i \in M} \min_{j \in J} r_{ij},\ r_{ij} \in \mathbb{R},\ i \in M,\ j \in J \right\}. \tag{11}$$

To calculate the MF $\mu_{G(J)}(r)$, we represent the FN $G(J)$ by its cuts. To this end, we write the MFs of the FNs F_{ij}, $i \in M$, $j \in J$, and $G(J)$ as follows:

$$\mu_{F_{ij}}(r) = \max_{u \in [0,1]} u 1_{[F_{ij}]_u}(r),\ i \in M,\ j \in J \text{ and } \mu_{G(J)}(r) = \max_{u \in [0,1]} u 1_{[G(J)]_u}(r), \tag{12}$$

where the closed intervals $[F_{ij}]_u = [(F_{ij})_u^L, (F_{ij})_u^H]$ and $[G(J)]_u = [(G(J))_u^L, (G(J))_u^H]$ are u-cuts, $u \in [0,1]$, of the FNs F_{ij}, $i \in M$, $j \in J$, and $G(J)$, respectively. These u-cuts are crisp sets with the MFs

$$1_{[F_{ij}]_u}(r) = \begin{cases} 1, & r \in [F_{ij}]_u; \\ 0, & r \notin [F_{ij}]_u; \end{cases} \quad 1_{[G(J)]_u}(r) = \begin{cases} 1, & r \in [G(J)]_u; \\ 0, & r \notin [G(J)]_u; \end{cases}$$

$r \in \mathbb{R}$, respectively. Formulae (3) and (7) imply that the u-cut $[G(J)]_u$ of the FN $G(J) = \max_{i \in M} \min_{j \in J} F_{ij}$ is given by

$$[G(J)]_u = [G(J)_u^L, G(J)_u^H] = [\max_{i \in M} \min_{j \in J} (F_{ij})_u^L, \max_{i \in M} \min_{j \in J} (F_{ij})_u^H]. \tag{13}$$

By (12), this entails the following representations of the MF $\mu_{G(J)}$ and the FN $G(J)$

$$\mu_{G(J)}(r) = \max\{u \in [0,1] : \ (G(J))_u^L \le r \le (G(J))_u^H\},\ r \in \mathbb{R} \tag{14}$$

and

$$G(J) = \{(r,u) : r \in [(G(J))_u^L, (G(J))_u^H],\ u \in [0,1]\} = \{([(G(J))_u^L, (G(J))_u^H], u) : u \in [0,1]\}, \quad (15)$$

respectively.

2.3. Type-2 Fuzzy Sets

The concept of type-2 fuzzy sets (T2FSs) was introduced by Zadeh in reference [28] as a broader framework encompassing type-1 fuzzy sets (T1FSs). As outlined by Mizumoto and Tanaka in reference [29], a T2FS, symbolized as $\tilde{C}$, defined on a crisp set X is distinguished by its fuzzy MF $M_{\tilde{C}} : X \to [0,1]^{[0,1]}$. For fixed $x' \in X$, the value of $M_{\tilde{C}}(x')$ is the T1FS $M_{\tilde{C}}(x') = \{(u, \mu_{\tilde{M}_{\tilde{C}}(x')}(u)) : u \in U_{x'}\}$ on the set $U_{x'} \subseteq [0,1]$ of the primary membership degrees u of x' to the T2FS $\tilde{C}$ with the corresponding MF $\mu_{\tilde{M}_{\tilde{C}}(x')}(u)$, $u \in U_{x'}$, where the value $\mu_{\tilde{M}_{\tilde{C}}(x')}(u)$ is the secondary grade of the pair (x,u). The following depiction of the T2FS

$$\tilde{C} = \{(x, \tilde{M}_{\tilde{C}}(x')) : x \in X\} = \{(x, \{(u, \mu_{\tilde{M}_{\tilde{C}}(x)}(u)) : u \in U_x\}) : x \in X\}$$

is referred to as the vertical slice approach.

Building upon the concepts introduced by Karnik and Mendel [30], Mendel and John proposed an alternative definition in their work [31]. Additionally, Harding et al. [32] made several amendments to these definitions: the T2FS $\tilde{C}$ on X is given by $\tilde{C} = \{((x,u), \eta_{\tilde{C}}(x,u)) : x \in X,\ u \in [0,1]\}$, where

$$\eta_{\tilde{C}}(x,u) = \begin{cases} \mu_{M_{\tilde{C}}(x)}(u), & u \in U_x; \\ 0, & \text{otherwise} \end{cases}$$

is the type-2 membership function (T2MF).

Remark 1. *The primary degree u is related to the degree of presence of some property (defining a given fuzzy set) for $x \in X$. By secondary degree, we mean, by [33], the degree of truth of the corresponding primary u degree of this property for x.*

According to [31], we use notions of the embedded T2FSs and T1FSs for a T2FS $\tilde{C} = \{((x,u), \eta_{\tilde{C}}(x,u)) : x \in X,\ u \in [0,1]\}$. Letting $u_x = \mu_{C^{e1}}(x) \in [0,1]$ is a unique primary degree of membership for each $x \in X$, where $\mu_{C^{e1}}(x)$, $x \in X$ is the MF of the T1FS $C^{e1} = \{(x, \mu_{C^{e1}}(x)) : x \in X\}$. The T1FS C^{e1} and the T2FS $\tilde{C}^{e2} = \{((x,u_x), \eta_{\tilde{C}^{e2}}(x,u_x)) : x \in X\}$ with $\eta_{\tilde{C}^{e2}}(x,u_x)) = \eta_{\tilde{C}}(x, \mu_{C^{e1}}(x)))$, $x \in X$ are called embedded in the T2FS $\tilde{C}$.

Remark 2. *According to [31], the collection $\tilde{C} = \{((x,u), \eta_{\tilde{C}}(x,u)) : x \in X,\ u \in [0,1]\}$ is the classical union of its elements in the sense of T1FSs. In addition, each T2FS can be represented as a collection of all possible embedded T2FSs.*

We use two special cases of T2FSs in view of [13,26]. Letting $\Omega = \{\eta_{\tilde{C}}(x,u) : \eta_{\tilde{C}}(x,u) > 0, x \in X, u \in [0,1]\}$ is the finite set of all possible positive values of secondary grades for the T2FS $\tilde{C} = \{((x,u), \eta_{\tilde{C}}(x,u)) : x \in X,\ u \in [0,1]\}$.

Definition 1 [13]. *We say that an embedded T2FS $\tilde{C}^{e2}_\alpha = \{((x,u_x), \eta_{\tilde{C}^{e2}_\alpha}(x,u_x)) : x \in X\}$ in the T2FS $\tilde{C}$ has a constant secondary grade $\alpha \in \Omega$ if, for each $x \in X$, the unique primary degree $u_x = \mu_{C^{e1}_\alpha}(x) \in [0,1]$ exists for which $\eta_{\tilde{C}^{e2}_\alpha}(x, \mu_{C^{e1}(\alpha)}(x)) \equiv \alpha$.*

In this definition, $\mu_{C^{e1}_\alpha}(x)$, $x \in X$ is the MF of the embedded T1FS $C^{e1}_\alpha = \{(x, \mu_{C^{e1}_\alpha}(x)) : x \in X\}$ in the T2FS $\tilde{C}$.

Remark 3 [26]. *For the T2FS $\tilde{C}$ and each $\alpha \in \Omega$, there is the unique embedded T1FS $C^{e1}_\alpha = \{(x, \mu_{C^{e1}_\alpha}(x)) : x \in X\}$ that corresponds to the embedded T2FS $\tilde{C}^{e2}_\alpha$ with a con-*

stant secondary grade α. *Hence,* $\tilde{C}^{e2}_{\alpha} = \{(C^{e1}_{\alpha}, \alpha)\} = \{(\{(x, \mu_{C^{e1}_{\alpha}}(x)) : x \in X\}, \alpha)\} = \{((x, \mu_{C^{e1}_{\alpha}}(x)), \alpha) : x \in X\}$.

Moreover, we examine an additional specific instance of a T2FS.

Definition 2 [26]. *We say that the T2FS* $\tilde{C}$ *is decomposable by secondary grades into a collection of embedded T2FSs with constant secondary grades if there are the T2FSs* $\tilde{C}^{e2}_{\alpha} = \{((x, \mu_{C^{e1}_{\alpha}}(x)), \alpha) : x \in X\} = \{(C^{e1}_{\alpha}, \alpha)\}$ *with constant secondary grades* $\alpha \in \Omega$*, respectively, which are embedded in the T2FS* $\tilde{C}$*, satisfying* $\tilde{C} = \{\tilde{C}^{e2}_{\alpha} : \alpha \in \Omega\}$.

Remark 4. *If the T2FS* $\tilde{C}$ *is decomposable by secondary grades* $\alpha \in \Omega$ *into the collection* $\tilde{C} = \{\tilde{C}^{e2}_{\alpha} : \alpha \in \Omega\}$ *of embedded T2FSs with constant secondary grades, then the T2FS* $\tilde{C}$ *is represented in the form of a collection* $\tilde{C} = \{(C^{e1}_{\alpha}, \alpha) : \alpha \in \Omega\}$ *of embedded T1FSs* C^{e1}_{α}, $\alpha \in \Omega$*, each of which is assigned the constant secondary grade* $\alpha \in \Omega$*, respectively.*

3. Formulation of the Problem

Consider a fuzzy decision-making problem $D(N) = \langle M, N, \tilde{F} \rangle$ with a matrix of fuzzy payoffs F_{ij}, $i \in M$, $j \in N$ specified in the form of FNs with the corresponding membership functions $\mu_{F_{ij}}(r)$, $r \in \mathbb{R}$. Here, $M = \{1, 2, \ldots, m\}$ is the set of alternatives, $|M| = m$; $N = \{1, 2, \ldots, n\}$ is the set of states, $|N| = n$. According to (10) and (11), with $J = N$, the maximum guaranteed payoff is given by

$$G(N) = \max_{i \in M} \min_{j \in N} F_{ij} \tag{16}$$

with the MF

$$\mu_{G(N)}(r) = \max\{\min_{j \in N} \mu_{F_{ij}}(r_{ij}) : r = \max_{i \in M} \min_{j \in N} r_{ij},\ r_{ij} \in \mathbb{R},\ i \in M,\ j \in N\}. \tag{17}$$

Let $\tilde{N} = \{(j, \mu_{\tilde{N}}(j)) : j \in N\}$ be some FS with the MF $\mu_{\tilde{N}}(j)$, $j \in N$ on the set N of states. We shall call $\tilde{N}$ the FS of states. The following question arises: 'What is the maximum guaranteed payoff to a fuzzy decision-making problem in the case when the set of states is fuzzy?'. The corresponding value will be denoted by $\max\limits_{i \in M} \min\limits_{(j, \mu_{\tilde{N}}(j)) \in \tilde{N}} F_{ij}$. In addition, we represent a fuzzy decision-making problem for the FS $\tilde{N}$ of states in the form $D(\tilde{N}) = \langle M, \tilde{N}, \tilde{F} \rangle$. Another natural question is 'When is there a need for such problem formulation?' To answer this question, we consider the following examples.

Assume we know expert estimates of the state probabilities but we may make a decision only once. In this case, it is not reasonable to use statistical decision-making criteria, for example, the expectation value criterion. Completely ignoring information about probabilities is also not rational. Why do not we interpret expert estimates of probabilities as degrees of membership of a FS of states? In this example, the maximum guaranteed payoff is the result of calculating the value $\max\limits_{i \in M} \min\limits_{(j, \mu_{\tilde{N}}(j)) \in \tilde{N}} F_{ij}$. The idea of interpreting the probabilities of states as degrees of membership in some FS of states is only given as a hypothetical example. We shall not consider the validity of such an interpretation in this article. Also, this should not be considered as the only case in which the problem formulation under consideration may occur.

4. Main Idea

For each fixed $r \in \mathbb{R}$, we consider the mapping $G^r : 2^N \to [0, 1]$ given by

$$G^r(J) = \max\{\min_{j \in N} \mu_{F_{ij}}(r_{ij}) : r = \max_{i \in M} \min_{j \in J} r_{ij},\ r_{ij} \in \mathbb{R},\ i \in M,\ j \in J\}, J \subseteq N. \tag{18}$$

In accordance with (11), the mapping G^r links each subset $J \subseteq N$ of states with the value of the MF $\mu_{G(J)}(r)$ of the FN

$$G(J) = \{(r, \mu_{G(J)}(r)) : r \in \mathbb{R}\} \tag{19}$$

of the maximum guaranteed payoff $G(J) = \max_{i \in M} \min_{j \in J} F_{ij}$ to the decision-making problem $D(J) = \langle M, J, \tilde{F} \rangle$, which is

$$G^r(J) = \mu_{G(J)}(r),\ r \in \mathbb{R}. \tag{20}$$

With Zadeh's extension principle [34] at hand, we extend the domain 2^N of the mapping G^r to the collection of FSs $\tilde{N}$ that are defined on the set N of states and generalize Formulae (18) and (19) to this case. We denote by $\tilde{G} = \max_{i \in M} \min_{(j, \mu_{\tilde{N}}(j)) \in \tilde{N}} F_{ij}$ the maximum guaranteed payoff to the decision-making problem $D(\tilde{N}) = \langle M, \tilde{N}, \tilde{F} \rangle$ for the FS $\tilde{N}$ of states, and we denote by $M_{\tilde{G}}(r)$ the relevant MF. In this case, for each fixed $r = r^*$, the value of the MF $M_{\tilde{G}}(r)$ coincides with the image $G^{r^*}(\tilde{N})$ of the FS $\tilde{N}$ of states under the mapping G^{r^*}, which is

$$M_{\tilde{G}}(r^*) = G^{r^*}(\tilde{N}). \tag{21}$$

Following Zadeh's extension principle [34], it can be shown that the image of the FS $\tilde{N}$ of states under the mapping G^{r^*} is the FS:

$$G^{r^*}(\tilde{N}) = \{(u, \mu_{G^{r^*}(\tilde{N})}(u)) : u \in [0,1]\} \tag{22}$$

with the MF

$$\mu_{G^{r^*}(\tilde{N})}(u) = \max\{\alpha :\ \alpha \in [0,1],\ u = G^{r^*}(N_\alpha)\}, \tag{23}$$

$u \in \mathrm{supp}(G^{r^*}(\tilde{N}))$, where the set

$$\mathrm{supp}(G^{r^*}(\tilde{N})) = \{u \in [0,1] :\ u = G^{r^*}(N_\alpha);\ \alpha \in [0,1]\} \tag{24}$$

is the support of the FS $G^{r^*}(\tilde{N})$;

$N_\alpha = \{j \in N : \mu_{\tilde{N}}(j) \geq \alpha\}$ is the α-cut, $\alpha \in [0,1]$ of the FS $\tilde{N} = \{(j, \mu_{\tilde{N}}(j)) : j \in N\}$ of states; $G^{r^*}(N_\alpha)$ is the image of the α-cut N_α, $\alpha \in [0,1]$ of the FS $\tilde{N}$ of states under the mapping G^{r^*} (see (18)); the equality

$$G^{r^*}(N_\alpha) = \mu_{G(N_\alpha)}(r^*) \tag{25}$$

holds, where $\mu_{G(N_\alpha)}(r^*)$ is the MF value of the FN $G(N_\alpha)$ by (20) with $J = N_\alpha$.

Remark 5. *Let $\Omega = \{\mu_{\tilde{N}}(j) : j \in N\}$ be the set of membership degrees values $\mu_{\tilde{N}}(j)$, $j \in N$ of the FS $\tilde{N} = \{(j, \mu_{\tilde{N}}(j)) : j \in N\}$ of states. Note that the cardinality of the set Ω is $|\Omega| < n$. It is clear that, when obtaining the α-cut $N_\alpha = \{j \in N : \mu_{\tilde{N}}(j) \geq \alpha\} \neq \varnothing$ of the FS $\tilde{N}$, we can assume that $\alpha \in \Omega$ rather than $\alpha \in (0,1]$.*

Proposition 1 highlights a valuable characteristic of the FS $\tilde{G}$.

Proposition 1. *For each fixed $r = r^* \in \mathbb{R}$, the values of the MF $M_{\tilde{G}}(r^*)$ form the FS $M_{\tilde{G}}(r^*) = \{(u, \mu_{M_{\tilde{G}}(r^*)}(u)) : u \in [0,1]\}$ on $[0,1]$ with the MF*

$$\mu_{M_{\tilde{G}}(r^*)}(u) = \max\{\alpha :\ \alpha \in \Omega,\ u = \mu_{G(N_\alpha)}(r^*)\} \tag{26}$$

for $u \in \mathrm{supp}(M_{\tilde{G}}(r^*))$, where the support of the FS $M_{\tilde{G}}(r^*)$ is given by

$$\mathrm{supp}(M_{\tilde{G}}(r^*)) = \{u \in [0,1] :\ u = \mu_{G(N_\alpha)}(r^*),\ \alpha \in \Omega\}. \tag{27}$$

Proof of Proposition 1. According to (21) and (25), Formula (23) implies that $\mu_{M_{\tilde{G}}(r^*)}(u) = \max\{\alpha :\ \alpha \in [0,1],\ u = \mu_{G(N_\alpha)}(r^*)\}$ for $u \in \text{supp}(M_{\tilde{G}}(r^*))$. Then, Remark 5 entails (26). Next, we prove (27). Assume that

$$u^* \in \text{supp}(M_{\tilde{G}}(r^*)). \tag{28}$$

We aim to verify that

$$u^* \in \{u \in [0,1] :\ u = \mu_{G(N_\alpha)}(r^*),\ \alpha \in \Omega\}. \tag{29}$$

According to Remark 5, it suffices to show that there exists $\alpha^* \in (0,1]$, such that $u^* = \mu_{G(N_{\alpha^*})}(r^*)$. Assume, on the contrary, that the inequality $u^* \neq \mu_{G(N_\alpha)}(r^*)$ holds for any $\alpha \in (0,1]$, whence $u^* \neq G^{r^*}(N_\alpha)$ by (25). Therefore, $u^* \notin \text{supp}(G^{r^*}(\tilde{N}))$ by (24). An appeal to (21) yields $u \notin \text{supp}(M_{\tilde{G}}(r^*))$, a contradiction to (28). Therefore, (29) holds true.

On the other hand, let us assume that the inclusion of (29) is valid. Then, there exists $\alpha^* \in \Omega$, for which

$$u^* = \mu_{G(N_{\alpha^*})}(r^*). \tag{30}$$

We intend to prove Formula (28). Assume, on the contrary, that the inequality $u^* \notin \text{supp}(M_{\tilde{G}}(r^*))$ holds. Then, Formula (21) implies that $u^* \notin \text{supp}(G^{r^*}(\tilde{N}))$. Therefore, $u^* \neq G^{r^*}(N_\alpha)$ for all $\alpha \in (0,1]$ by (24). Hence, we infer $u^* \neq \mu_{G(N_\alpha)}(r^*)$ by (25), a contradiction to (30). Therefore, (27) holds true. □

Based on Proposition 1, we state that $\tilde{G}$ is a FS on $\mathbb{R}$ with the MF which values form a FS on [0, 1]. Thus, $\tilde{G}$ is the T2FS on $\mathbb{R}$ by [28]. In the manner of vertical slices (see Section 2.3), the T2FS $\tilde{G}$ is given by $\tilde{G} = \{(r, M_{\tilde{G}}(r)) : r \in \mathbb{R}\} = \{(r, \{(u, \mu_{M_{\tilde{G}}(r)}(u)) :\ u \in U_r\}) :\ r \in \mathbb{R}\}$, where $\mu_{M_{\tilde{G}}(r)}(u)$, $u \in [0,1]$ is the MF of the FS $M_{\tilde{G}}(r) = \{(u, \mu_{M_{\tilde{G}}(r)}(u)) :\ u \in [0,1]\}$ of values of fuzzy degree of membership of the number $r \in \mathbb{R}$ to the T2FS $\tilde{G}$, and $U_r = \text{supp}(M_{\tilde{G}}(r))$ is the set of primary membership degrees. We can also characterize the T2FS $\tilde{G}$ by the T2MF:

$$\eta_{\tilde{G}}(r,u) = \begin{cases} \mu_{M_{\tilde{G}}(r)}(u),\ u \in U_r; \\ 0, \qquad \text{otherwise} \end{cases}$$

(see Section 2.3). Then, the T2FS $\tilde{G}$ of the maximum guaranteed payoff is $\tilde{G} = \{((r,u), \eta_{\tilde{G}}(r,u)) :\ u \in [0,1],\ r \in \mathbb{R}\}$.

5. Maximum Guaranteed Payoff for a FS of States

5.1. Maximum Guaranteed Payoff T2FS

The conclusion drawn in Section 4 leads us to introduce the following concept.

Definition 3. *By the maximum guaranteed payoff to the decision-making problem $D(\tilde{N}) = \langle M, \tilde{N}, \tilde{F} \rangle$ for the FS $\tilde{N} = \{(j, \mu_{\tilde{N}}(j)) : j \in N\}$ of states is meant the T2FS.*

$$\tilde{G} = \{((r,u), \eta_{\tilde{G}}(r,u)) :\ u \in [0,1],\ r \in \mathbb{R}\} \tag{31}$$

with the T2MF

$$\eta_{\tilde{G}}(r,u) = \begin{cases} \max\{\alpha \in \Omega :\ u = \mu_{G(N_\alpha)}(r)\}, & u \in U_r; \\ 0, & u \notin U_r; \end{cases} \tag{32}$$

$r \in \mathbb{R}$, $u \in [0,1]$.

Here,

$$U_r = \{u \in [0,1] :\ u = \mu_{G(N_\alpha)}(r),\ \alpha \in \Omega\} \tag{33}$$

is the set of primary membership degrees that coincides with the support $\mathrm{supp}(M_{\tilde{G}}(r))$ (refer to Equality (27)) of the FS $M_{\tilde{G}}(r)$ of fuzzy membership degrees of the number $r \in \mathbb{R}$;

$$\mu_{G(N_\alpha)}(r) = \max\{\min_{j \in N_\alpha} \mu_{F_{ij}}(r_{ij}) : \ r = \max_{i \in M} \min_{j \in N_\alpha} r_{ij}, \ r_{ij} \in \mathbb{R}, \ i \in M, \ j \in N_\alpha\} \tag{34}$$

is the MF of the FN

$$G(N_\alpha) = \{(r, \mu_{G(N_\alpha)}(r)) : r \in \mathbb{R}\} \tag{35}$$

which is the maximum guaranteed payoff $G(N_\alpha) = \max_{i \in M} \min_{j \in N_\alpha} F_{ij}$ to the decision-making problem $D(N_\alpha) = \langle M, N_\alpha, \tilde{F} \rangle$ for the set N_α of states (see (10) and (11) with $J = N_\alpha$);

$$N_\alpha = \{j \in N : \mu_{\tilde{N}}(j) \geq \alpha\} \tag{36}$$

is the α-cut of the FS $\tilde{N} = \{(j, \mu_{\tilde{N}}(j)) : j \in N\}$ of states, $\alpha \in \Omega$;

$$\Omega = \{\mu_{\tilde{N}}(j) : j \in N\} \tag{37}$$

is the set of the positive membership degree values of the FS $\tilde{N} = \{(j, \mu_{\tilde{N}}(j)) : j \in N\}$ of states (see Remark 5).

Remark 6. *In the case of a decision-making problem with a crisp payoff matrix and a fuzzy set of states, the maximum guaranteed payoff is $\tilde{G} = \{((r, u), \eta_{\tilde{G}}(r, u)) : u \in \{0, 1\}, r \in \mathbb{R}\}$, which is a special case of a T2FS. Since the primary membership degrees $u \in \{0, 1\}$ of the T2FS $\tilde{G}$ take only two values, 0 or 1, this yields an interesting interpretation of the T2FS $\tilde{G}$ by Remark 1. Similar to a crisp set, there are only two options for each number $r \in \mathbb{R}$: either r completely belongs to the T2FS $\tilde{G}$ (the primary membership degree is u = 1) or it completely does not belong (u = 0). Unlike a crisp set, the degrees $\eta_{\tilde{G}}(r, 0)$ and $\eta_{\tilde{G}}(r, 1)$ of truth of the identification of these two facts can differ from 1 and take values in the closed interval* [0, 1].

We note that this type of a T2FS is already known (see, for instance, [35]).

Remark 7. *In the case of a decision-making problem with payoffs in the form of crisp numbers, we also use MFs to represent these numbers. For example, we represent a crisp number $a \in \mathbb{R}$ by $\{(a, \mu_a(r)) : r \in \mathbb{R}\} = \{(a, 1)\} \cup \{(r, 0) : r \neq a, r \in \mathbb{R}\}$, where*

$$\mu_a(r) = \begin{cases} 1, & r = a; \\ 0, & otherwise \end{cases}$$

is the MF (the characteristic function) of the crisp number a.

Proposition 2 validates the decomposability, as defined in Definition 2, of a maximum guaranteed payoff T2FS for a decision-making problem involving a fuzzy set of states. This decomposition involves breaking down the T2FS into a collection of embedded T2FSs characterized by constant secondary grades.

Proposition 2. *The maximum guaranteed payoff T2FS $\tilde{G}$ to the decision-making problem $D(\tilde{N}) = \langle M, \tilde{N}, \tilde{F} \rangle$ for the FS $\tilde{N} = \{(j, \mu_{\tilde{N}}(j)) : j \in N\}$ of states is decomposable by secondary grades $\alpha \in \Omega$ into the collection*

$$\tilde{G} = \{\tilde{G}^{e2}_\alpha : \ \alpha \in \Omega\} \tag{38}$$

of the embedded T2FSs

$$\tilde{G}^{e2}_\alpha = \{(G(N_\alpha), \alpha)\} \tag{39}$$

and given by

$$\tilde{G} = \{(G(N_\alpha), \alpha) : \ \alpha \in \Omega\} \tag{40}$$

where $G(N_\alpha) = \{(r, \mu_{G(N_\alpha)}(r)) : r \in \mathbb{R}\}$ *is the embedded T1FS* $G^{e1}_\alpha = G(N_\alpha)$ *with the MF* $\mu_{G^{e1}_\alpha}(r) = \mu_{G(N_\alpha)}(r)$ *in Form (34). It is the FN that is the maximum guaranteed payoff* $G(N_\alpha) = \max_{i\in M}\min_{j\in N_\alpha} F_{ij}$ *to the decision-making problem* $D(N_\alpha) = \langle M, N_\alpha, \tilde{F}\rangle$ *for the crisp set* N_α, $\alpha \in \Omega$ *of states.*

Proof of Proposition 2. In accordance with (31), the maximum guaranteed payoff T2FS is $\tilde{G} = \{((r,u), \eta_{\tilde{G}}(r,u)) : u \in [0,1], r \in \mathbb{R}\}$. Then, invoking (32) yields

$$\tilde{G} = \{\{((r,u), \max\{\alpha : \alpha \in \Omega, u = \mu_{G(N_\alpha)}(r)\}) : u \in U_r\} \cup \{((r,u),0) : r \notin U_r\} : r \in \mathbb{R}\}.$$

Remark 2 allows us to ignore pairs (r,u) that have secondary grades that are equal to 0; therefore, $\tilde{G} = \{\{((r,u), \max\{\alpha : \alpha \in \Omega, u = \mu_{G(N_\alpha)}(r)\}) : u \in U_r, r \in \mathbb{R}\}$. Then, invoking (33) yields $\tilde{G} = \{(r, \{(\mu_{G(N_\alpha)}(r), \alpha) : \alpha \in \Omega\}), r \in \mathbb{R}\}$. Further, regrouping the elements, we obtain

$$\tilde{G} = \{(r, (\mu_{G(N_\alpha)}(r), \alpha)) : \alpha \in \Omega, r \in \mathbb{R}\} = \{\{((r, \mu_{G(N_\alpha)}(r)), \alpha) : r \in \mathbb{R}\} : \alpha \in \Omega\}$$

Then, (35) entails (40) and, thereupon, (38) by (39). □

Proposition 2 allows us to represent the T2FS $\tilde{G}$ in a form that is more suitable for calculations and interpretation. Following Remark 1, the T2FS $\tilde{G}$ can be interpreted as the collection of maximum guaranteed payoffs $G(N_\alpha)$ to the decision-making problem $D(N_\alpha)$ for the crisp set N_α of states with the degree of truth of the FN $G(N_\alpha)$ being equal to $\alpha \in \Omega$.

5.2. Calculation Algorithm of the Maximum Guaranteed Payoff T2FS

In this section, we consider an algorithm for calculating the maximum guaranteed payoff T2FS.

Step 0. We construct the finite set $\Omega = \{\mu_{\tilde{N}}(j) : j \in N\}$ of membership degrees values of the FS $\tilde{N} = \{(j, \mu_{\tilde{N}}(j)) : j \in N\}$ of states and represent Ω in the form $\Omega = \{\alpha_1, \ldots, \alpha_{|\Omega|}\}$.

Step $k \in \{1, \ldots, |\Omega|\}$. For $\alpha = \alpha_k$, following (36), we construct the α-cut $N_\alpha = \{j \in N : \mu_{\tilde{N}}(j) \geq \alpha\}$ of the FS $\tilde{N}$. We calculate a solution $G(N_\alpha) = \max_{i\in M}\min_{j\in N_\alpha} F_{ij}$ to the problem $D(N_\alpha) = \langle M, N_\alpha, \tilde{F}\rangle$ for the set N_α of states. This is the maximum guaranteed payoff FN $G(N_\alpha) = \{(r, \mu_{G(N_\alpha)}(r)) : r \in \mathbb{R}\}$ (see (35)), which is the embedded T1FS—that is, $G^{e1}_\alpha = G(N_\alpha)$ by Proposition 2. To calculate the FN $G(N_\alpha)$, we use the representation of this FN by its cuts. According to (15) with $J = N_\alpha$, we use the formula

$$G(N_\alpha) = \{([(G(N_\alpha))^L_u, (G(N_\alpha))^H_u], u) : u \in [0,1]\}. \quad (41)$$

The MF of the FN $G(N_\alpha)$ is given by

$$\mu_{G(N_\alpha)}(r) = \max\{u \in [0,1] : (G(N_\alpha))^L_u \leq r \leq (G(N_\alpha))^H_u\}, \ r \in \mathbb{R} \quad (42)$$

as a consequence of (14) with $J = N_\alpha$. For each $u \in [0,1]$, we calculate the boundaries

$$(G(N_\alpha))^L_u = \max_{i\in M}\min_{j\in N_\alpha} (F_{ij})^L_u \quad (43)$$

and

$$(G(N_\alpha))^H_u = \max_{i\in M}\min_{j\in N_\alpha} (F_{ij})^H_u \quad (44)$$

of the closed interval

$$[G(N_\alpha)]_u = [(G(N_\alpha))^L_u, (G(N_\alpha))^H_u] \quad (45)$$

according to (13) with $J = N_\alpha$. In these formulae, $(F_{ij})^L_u$ and $(F_{ij})^H_u$ are the lower and the upper bounds, respectively, of the closed interval $[F_{ij}]_u = [(F_{ij})^L_u, (F_{ij})^H_u]$. This interval is the u-cut of the payoff FN F_{ij} of the alternative $i \in M$ for the state $j \in N_\alpha$. For approximate calculations, we choose the values $u = u_s = s/S$, $s = 0, \ldots, S$, where $(S + 1)$ is the number of cut levels.

The final step. Once all FNs $G(N_\alpha)$, $\alpha \in \Omega$ have been obtained, the resulting T2FS $\tilde{G}$ is given by $\tilde{G} = \{(G(N_\alpha), \alpha) : \alpha \in \Omega\}$ according to (40). The T2MF $\eta_{\tilde{G}}(r, u)$ can be calculated with the help of Formulae (32) and (33). According to Remark 1, the T2FS $\tilde{G}$ can be interpreted as follows. The maximum guaranteed payoff T2FS $\tilde{G}$ is equal to the FN $G(N_{\alpha_1})$ with the degree of truth being equal to α_1; the FN $G(N_{\alpha_2})$ with the degree of truth being equal to α_2;. . .; and the FN $G(N_{\alpha_{|\Omega|}})$ with the degree of truth being equal to $\alpha_{|\Omega|}$.

5.3. Properties of the Maximum Guaranteed Payoff T2FS

Proposition 3 highlights several valuable properties of a maximum guaranteed payoff T2FS to a decision-making problem for a FS of states.

Proposition 3. *Let $\mu_{G(N_{\alpha^*})}(r)$ be the membership degree of a number $r \in \mathbb{R}$ to the FN $G(N_{\alpha^*})$, which is the maximum guaranteed payoff to the decision-making problem $D(N_{\alpha^*}) = \langle M, N_{\alpha^*}, \tilde{F} \rangle$ for α*-cut, $\alpha^* \in \Omega$ N_{α^*} of the FS $\tilde{N}$ of states. Then $u = \mu_{G(N_{\alpha^*})}(r)$ is the primary membership degree of the number r to the maximum guaranteed payoff T2FS $\tilde{G}$ with the secondary grade not smaller than α*—that is, $\eta_{\tilde{G}}(r, \mu_{G(N_{\alpha^*})}(r)) \geq \alpha^*$.*

Proof of Proposition 3. Assume that $\alpha^* \in \Omega$ and $u = \mu_{G(N_{\alpha^*})}(r)$. Then, $u \in U_r$ according to (33). Therefore, $\eta_{\tilde{G}}(r, u) = \max\{\alpha : \alpha \in \Omega, \mu_{G(N_{\alpha^*})}(r) = u = \mu_{G(N_\alpha)}(r)\} \geq \alpha^*$ by (32). □

According to Proposition 3, the maximum guaranteed degree of truth of the primary degree of membership of some number $r \in \mathbb{R}$ to the T2FS $\tilde{G}$ is determined by the smallest degree of membership to the FS $\tilde{N}$ of those states for which this number r is the maximum guaranteed payoff.

In Proposition 4, we employ the fuzzy number ordering proposed by Ramik and Rimanek [36].

Definition 4. *Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers and $[\tilde{a}]_u = [(\tilde{a})_u^L, (\tilde{a})_u^H]_u$, $[\tilde{b}]_u = [(\tilde{b})_u^L, (\tilde{b})_u^H]_u$ be their u-cuts, $u \in [0, 1]$. By the fuzzy partial order is meant the relation $\tilde{a} \succsim \tilde{b} \Leftrightarrow (\tilde{a})_u^L \geq (\tilde{b})_u^L$, $(\tilde{a})_u^H \geq (\tilde{b})_u^H$ for all $u \in [0, 1]$. Here $(\tilde{a})_u^L$ and $(\tilde{a})_u^H$ are the lower and upper bounds of $[\tilde{a}]_u$.*

Proposition 4. *Suppose that the maximum guaranteed payoff T2FS $\tilde{G}$ to the decision-making problem for the FS $\tilde{N}$ of states is represented as in (40). Then, for any α', $\alpha'' \in \Omega$ with $\alpha' \geq \alpha''$, the relations $N_{\alpha'} \subseteq N_{\alpha''}$ and $G(N_{\alpha''}) \succsim G(N_{\alpha'})$ hold true, where '$\succsim$' is the fuzzy partial order.*

Proof of Proposition 4. Formula (36) implies that

$$N_{\alpha'} \subseteq N_{\alpha''}. \tag{46}$$

According to Definition 4, to prove the relation $G(N_{\alpha''}) \succsim G(N_{\alpha'})$, it suffices to show that the inequalities $G(N_{\alpha''})_u^L \geq G(N_{\alpha'})_u^L$ and $G(N_{\alpha''})_u^H \geq G(N_{\alpha'})_u^H$ hold for each $u \in [0, 1]$. Assume, on the contrary, that there exists $u^* \in [0, 1]$, such that the inequalities

$$G(N_{\alpha'})_{u^*}^L < G(N_{\alpha''})_{u^*}^L \tag{47}$$

or (and)

$$G(N_{\alpha'})_{u^*}^H < G(N_{\alpha''})_{u^*}^H \tag{48}$$

hold. In the former case, we obtain

$$(G(N_{\alpha'}))^L_{u^*} = \max_{i\in M}\min_{j\in N_{\alpha'}}(F_{ij})^L_{u^*} < \max_{i\in M}\min_{j\in N_{\alpha''}}(F_{ij})^L_{u^*} = (G(N_{\alpha''}))^L_{u^*}$$

This entails $\min_{j\in N_{\alpha'}}(F_{ij})^L_{u^*} < \min_{j\in N_{\alpha''}}(F_{i^*j})^L_{u^*}$ for all $i \in M$ and particularly for $i^* = \operatorname{argmax}_{i\in M}\min_{j\in N_{\alpha''}}(F_{ij})^L_{u^*}$. With this at hand, we conclude that $\min_{j\in N_{\alpha'}}(F_{i^*j})^L_{u^*} < \min_{j\in N_{\alpha''}}(F_{i^*j})^L_{u^*}$ and thereupon $N_{\alpha'} \supset N_{\alpha''}$, a contradiction to (46). We also obtain a similar contradiction in the case where Inequality (48) holds. □

According to Proposition 4, FNs with more favorable maximum guaranteed payoffs, corresponding to smaller cut levels of the FS of states, exhibit larger secondary membership degrees (degrees of truth) to the maximum guaranteed payoff T2FS $\tilde{G}$. This is quite natural, for large degrees of truth correspond to large degrees of membership in a fuzzy set of states.

5.4. Numerical Examples

In this section, we examine examples that demonstrate the construction of a maximum guaranteed payoff T2FS for a fuzzy set of states. Example 1 serves to illustrate the algorithm for calculating a maximum guaranteed payoff T2FS. Example 2 demonstrates the ability of the developed approach to find a solution to a decision-making problem under ignorance in the case where the standard decision-making criteria fail.

Example 1. *We consider the problem of decision-making with the matrix of fuzzy payoffs given in Table 2 in the form of the 'triangular' FNs F_{ij}, $i \in M$, $j \in N$ with the MFs $\mu_{F_{ij}}(r)$, $r \in \mathbb{R}$, $i \in M$, $j \in N$, respectively, where M = {1, 2, 3} is the set of alternatives and N = {1, 2, 3, 4} is the set of states. The input data are given in the form of 'triangular' FNs only for the purpose of simplification and clarity of the description. Recall that the MF for the 'triangular' FN B = (a, b, c) is given by*

$$\mu_B(x) = \begin{cases} 1-(b-x)/(b-a), & a \le x \le b; \\ 1-(x-b)/(c-b), & b \le x \le c; \\ 0, & \textit{otherwise}. \end{cases} \quad (49)$$

The cuts $[B]_u = [(B)^L_u, (B)^H_u]$, $u \in [0,1]$ of the 'triangular' FN B with the MF in Form (49) are calculated using the formula $[B]_u = [a + (b-a)u, c-(c-b)u]$. In this example, we use approximate calculations with the number of u-cuts, $u = u_s = s/10$, $s = 0,\ldots,10$ being equal to 11. The graphs of the MFs $\mu_{F_{ij}}(r)$, $r \in \mathbb{R}$, $j \in \{1,\ 2,\ 3,\ 4\}$ are represented in Figure 1a for the alternative $i = 1$, in Figure 1b for $i = 2$, and Figure 1c for $i = 3$. In Figure 1a–c, for each alternative $i \in \{1,\ 2,\ 3\}$, the MFs $\mu_{F_{ij}}(r)$, $r \in \mathbb{R}$ are drawn blue for the state j = 1, yellow for the state j = 2, green for state j = 3, and red for the state j = 4.

Assume that a DM perceives the set N = {1, 2, 3, 4} of states in the form of the FS $\tilde{N} = \{(1, 0.7), (2, 0.9), (3, 1), (4, 1)\}$ with the MF values $\mu_{\tilde{N}}(1) = 0.7$, $\mu_{\tilde{N}}(2) = 0.9$ u $\mu_{\tilde{N}}(3) = \mu_{\tilde{N}}(4) = 1$. The DM intends to predict a maximum guaranteed payoff using the algorithm from Section 5.2.

Step 0. According to (37), the set of membership degrees of the FS Ñ is $\Omega = \{0.7, 0.9, 1\}$.

Step k = 1. For $\alpha = 0.7$, according to (36), we construct the 0.7-cut $N_{0.7}$ = {1, 2, 3, 4} of the FS Ñ. We intend to find a solution $G(N_{0.7}) = \max_{i\in M}\min_{j\in N_{0.7}} F_{ij}$ to the problem $D(N_{0.7}) = \langle M, N_{0.7}, \tilde{F}\rangle$ for the set $N_{0.7}$ of states. To calculate the FN $G(N_{0.7})$ in Form (41), we represent this FN by its cuts $[G(N_{0.7})]_u = [(G(N_{0.7}))^L_u, (G(N_{0.7}))^H_u]$, $u = u_s = s/10$, $s = 0,\ldots,10$ with the help of Formulae (43) and (44). In these formulae, we use u-cuts $[F_{ij}]_u = [(F_{ij})^L_u, (F_{ij})^H_u]$ of the payoffs F_{ij} of the alternatives $i \in M$ for the states $j \in N_{0.7}$. The MF $\mu_{G(N_{0.7})}$ of the FN $G(N_{0.7})$ is drawn red in Figure 1d. We call the FN $G(N_{0.7})$ 'approximately 350' and denote it by $\widetilde{350}$.

Step $k = 2$. For $\alpha = 0.9$, according to (36), we construct the 0.9-cut $N_{0.9} = \{2, 3, 4\}$ of the FS Ñ. We intend to find a solution $G(N_{0.9}) = \max_{i\in M}\min_{j\in N_{0.9}} F_{ij}$ to the problem $D(N_{0.9}) = \langle M, N_{0.9}, \tilde{F}\rangle$ for *the set $N_{0.9}$ of states. To calculate the FN $G(N_{0.9})$ in Form (41), we represent this FN by its cuts $[G(N_{0.9})]_u = [(G(N_{0.9}))_u^L, (G(N_{0.9}))_u^H]$, $u = u_s = s/10$, $s = 0, \ldots, 10$ with the help of Formulae (43) and (44). The MF $\mu_{G(N_{0.9})}$ of the FN $G(N_{0.9})$ is drawn green in Figure 1d. We call the FN $G(N_{0.9})$ 'approximately 400' and denote it by $\widetilde{400}$.*

Step $k = 3$. For $\alpha = 1$, according to (36), we construct the 1-cut $N_1 = \{3, 4\}$ of the FS $\tilde{N}$. We intend to find a solution $G(N_1) = \max\limits_{i \in M} \min\limits_{j \in N_1} F_{ij}$ to the problem $D(N_1) = \langle M, N_1, \tilde{F} \rangle$ for the set N_1 of states. To calculate the FN $G(N_1)$ in Form (41), we represent this FN by its cuts $[G(N_1)]_u = [(G(N_1))_u^L, (G(N_1))_u^H]$, $u = u_s = s/10$, $s = 0, \ldots, 10$ using Formulae (43) and (44). The MF $\mu_{G(N_1)}$ of the FN $G(N_1)$ is drawn blue in Figure 1d. We call the FN $G(N_1)$ 'approximately 470' and denote it by $\widetilde{470}$.

The final step. Once all FNs $G(N_{0.7}) = \widetilde{350}$, $G(N_{0.9}) = \widetilde{400}$, and $G(N_1) = \widetilde{470}$ have been obtained, the resulting T2FS $\tilde{G} = \{(G(N_{0.7}), 0.7), (G(N_{0.9}), 0.9), (G(N_1), 1)\}$ is given by $\tilde{G} = \{(\widetilde{350}, 0.7), (\widetilde{400}, 0.9), (\widetilde{470}, 1)\}$. The T2MF $\eta_{\tilde{G}}(r, u)$ can be calculated with the help of Formulae (32) and (33). The levels $\eta_{\tilde{G}}(r, u) \in \{0.7, 0.9, 1\}$ are given by blue (for $\alpha = 1$), green (for $\alpha = 0.9$), and red (for $\alpha = 0.7$) lines in Figure 1d. The obtained T2FS $\tilde{G} = \{(\widetilde{350}, 0.7), (\widetilde{400}, 0.9), (\widetilde{470}, 1)\}$ can be interpreted as follows. The maximum guaranteed payoff is equal to the FN $\widetilde{350}$ with the degree of truth being equal to 1, the FN $\widetilde{400}$ with the degree of truth being equal to 0.9, and the FN $\widetilde{470}$ with the degree of truth being equal to 1.

Table 2. Payoffs of Example 1.

Alternatives	States			
	1	2	3	4
1	(300, 400, 500)	(310, 350, 380)	(520, 550, 600)	(500, 600, 650)
2	(270, 300, 450)	(250, 350, 600)	(600, 700, 800)	(620, 650, 700)
3	(400, 600, 800)	(150, 250, 350)	(180, 200, 220)	(450, 500, 550)

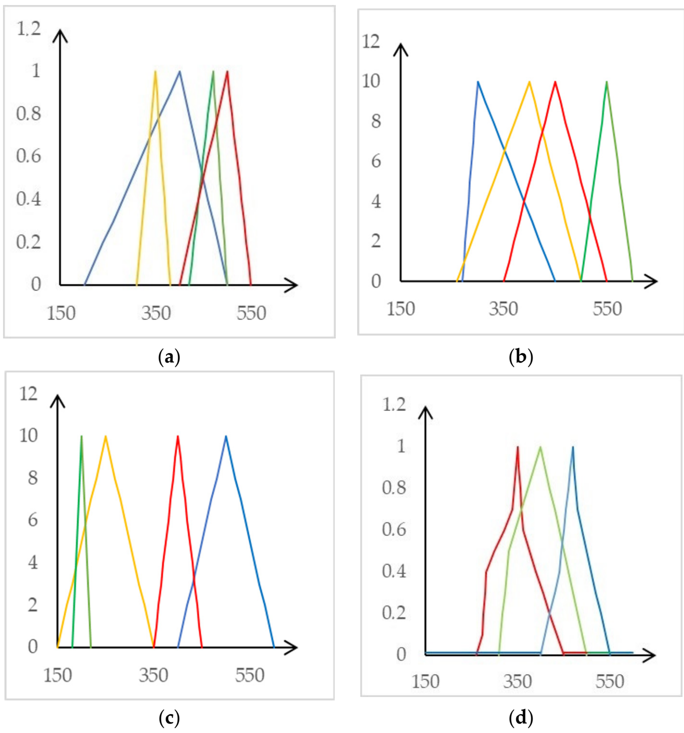

Figure 1. The graphs of the MFs $\mu_{F_{ij}}$ of the payoffs F_{ij}, $j \in \{1, 2, 3, 4\}$: (**a**) for the alternative $i = 1$; (**b**) $i = 2$; (**c**) $i = 3$; (**d**) the levels lines of the T2MF $\eta_{\tilde{G}}$. *For each alternative $i \in \{1, 2, 3\}$, the MFs $\mu_{F_{ij}}(r)$, $r \in$ R are drawn blue for the state $j = 1$, yellow for the state $j = 2$, green for state $j = 3$, and red for the state $j = 4$.*

Example 2. *In this example, we take crisp payoffs to transparently compare the results of solving a decision-making problem using different methods and under different assumptions about a set of states. The payoff matrix F_{ij}, $i \in M = \{1,2,3,4\}$, $j \in N = \{1,2,3,4,5\}$ is given in Table 3.*

We consider the following cases.

In the first case, assume that we make a decision when the probabilities $p_1 = 0.2$, $p_2 = 0.3$, $p_3 = 0.2$, $p_4 = 0.15$, and $p_5 = 0.15$ are available for the states $j \in \{1,2,3,4,5\}$, respectively, and a decision is made sufficiently often so that we can hope for a proper application of statistical criteria. If we use the Bayesian decision criterion, we obtain equal expected utilities $G_i^B = \sum\limits_{j \in N} F_{ij} p_j = 1$ for all alternatives $i \in N$. Therefore, a unique decision is not possible.

In the second case, assume that we make a decision under complete ignorance, when state probabilities are not available and/or a decision is made only once. We consider the following situations:

1. *Suppose that the set $\tilde{N}$ of states is crisp—that is $\mu_{\tilde{N}}(1) = \mu_{\tilde{N}}(2) = \mu_{\tilde{N}}(3) = \mu_{\tilde{N}}(4) = 1$. Then, in the case of using:*
    a. *the maxmin criterion (the Wald criterion), we obtain equal maximum guaranteed payoffs $G_i^W = \min\limits_{j \in N} F_{ij} = 0$ for all alternativesi $\in N$;*
    b. *the optimistic criterion, we obtain equal optimistic payoffs $G_i^O = \max\limits_{j \in N} F_{ij} = 2$ for all alternatives $i \in N$;*
    c. *the compromise criterion (the Hurwicz criterion), we obtain equal compromise payoffs $G_i^H = \lambda \max\limits_{j \in N} F_{ij} + (1-\lambda) \min\limits_{j \in N} F_{ij} = 2$ for all alternatives $i \in N$ for any value of the parameter $\lambda \in [0,1]$ characterizing the DM's propensity to take risks;*
    d. *the minmax regret criterion (the Savage criterion), we obtain equal relative utility losses $G_i^S = \max\limits_{j \in N}(\max\limits_{k \in M} F_{kj} - F_{ij}) = 2$ for all alternatives $i \in N$.*

    Thus, in all these cases, applying basic standard decision criteria does not help us to make a unique decision.
2. *Assume that the set $\tilde{N}$ of states is fuzzy. It is quite possible that we can consider the FS $\tilde{N}$ as 'probable states' and choose degrees of membership to this FS being equal to the corresponding probabilities—that is, $\mu_{\tilde{N}}(1) = p_1 = 0.2$, $\mu_{\tilde{N}}(2) = p_2 = 0.3$, $\mu_{\tilde{N}}(3) = p_3 = 0.2$, $\mu_{\tilde{N}}(4) = p_4 = 0.15$, and $\mu_{\tilde{N}}(5) = p_5 = 0.15$. Using the algorithm from Section 5.2, we infer:*
    $\Omega = \{0.15, 0.2, 0.3\}$ is the set of degrees of membership to the FS $\tilde{N}$;
    $N_{0.3} = \{2\}$, $N_{0.2} = \{1,2,3\}$, $N_{0.15} = \{1,\ldots,5\}$ are the α-cuts, $\alpha \in \Omega$ of the FS $\tilde{N}$ of states;
    $G(N_{0.3}) = \{(2,1)\} \cup \{(r,0) : r \neq 2\}$ (denote it by '2');
    $G(N_{0.2}) = \{(1,1)\} \cup \{(r,0) : r \neq 1\}$ (denote it by '1');
    $G(N_{0.15}) = G(N_{0.1}) = \{(0,1)\} \cup \{(r,0) : r \neq 1\}$ (denote it by '0')

Are the maximum guaranteed payoffs for the α-cuts, $\alpha \in \Omega$ of the FS $\tilde{N}$ of states, respectively; hereinafter, we use Remark 7 to represent crisp numbers using MFs, for example, a crisp number $G(N_{0.3})$ is given by

$$G(N_{0.3}) = \text{'2'} = \left\{(2, \mu_{G(N_{0.3})}(r)) : r \in \mathbb{R}\right\} = \{(2,1)\} \cup \{(r,0) : r \neq 2\}$$

with the MF

$$\mu_{G(N_{0.3})}(r) = \begin{cases} 1, & r = 2; \\ 0, & \text{otherwise.} \end{cases}$$

Once all maximum guaranteed payoffs $G(N_{0.3})$ = '2', $G(N_{0.2})$ = '1', and $G(N_{0.15})$ = '0' have been obtained, the resulting T2FS $\tilde{G} = \{(G(N_{0.3}), 0.3), (G(N_2), 0.2), (G(N_{0.15}), 0.15)\}$ is given by $\tilde{G}$ = {('2',0.3),('1',0.2),('0',0.15)} . The obtained T2FS $\tilde{G}$ can be interpreted as follows. The maximum guaranteed payoff is equal to '2' with the degree of truth being equal to 0.3, '1' with the degree of truth being equal to 0.2, and '0' with the degree of truth being equal to 0.15.

Table 3. Payoffs of Example 2.

Alternatives	**States**				
	1	**2**	**3**	**4**	**5**
1	0	1	1.25	2	1
2	1.5	1	2	0	0
3	1	2	0	0	4/3
4	2	0	1	2/3	2

6. Discussion and Results

Since the purpose of this article was to demonstrate that, in a fuzzy decision-making problem under ignorance, a FS of states generates a T2FS of the maximum guaranteed payoffs, we directed our attention towards outcomes that facilitated this objective. In Section 4, we provided the rationale supporting the assertion that the maximum guaranteed payoffs for a fuzzy set of states form a T2FS. This enabled us to outline the corresponding definition in Section 5.1. Using a decomposition approach, we represented a maximum guaranteed payoffs T2FS for a FS of states by a collection of embedded T2FSs with constant secondary grades. In Section 5.2, we worked out the algorithm for constructing a maximum guaranteed payoffs T2FS. This study clarified that, while a type-2 fuzzy set (T2FS) is typically a complex mathematical construct, T2FSs with constant secondary grades exhibit simplicity suitable for practical applications. This was illustrated through the examples provided in Section 5.4. In addition, in Example 2 of Section 5.4, we showed how to use our approach for solving a fuzzy decision-making problem under ignorance in the case where the standard decision-making criteria fail. Our investigation of some properties of the maximum guaranteed payoffs T2FS in Section 5.3 showed that:

- The maximum guaranteed degree of truth of the primary degree of membership of some number $r \in \mathbb{R}$ to the resulting T2FS was determined by the smallest degree of membership to the FS $\tilde{N}$ of those states for which this number r was the maximum guaranteed payoff.
- More preferable FNs of maximum guaranteed payoffs (which corresponded to smaller cut levels of the FS of states) had larger secondary membership degrees (degrees of truth) to the maximum guaranteed payoff T2FS. This was quite natural, for large degrees of truth corresponded to large degrees of membership in a fuzzy set of states.

Upon comparing the maxmin criterion with our approach, we draw the following conclusion: Our approach exhibits a notable limitation, namely an escalation in computational complexity when juxtaposed with the maxmin criterion. This is caused by the need of calculating a maximum guaranteed payoff for each α-cut $N_\alpha = \{j \in N : \mu_{\tilde{N}}(j) \geq \alpha\}$ of the FS $\tilde{N}$ of states. This drawback constrains the applicability of our approach to scenarios involving a substantial number of alternatives and states, which may pose challenges for solving optimization Problems (43) and (44). However, this limitation can be mitigated by refraining from undertaking the full computation of the maximum guaranteed payoff T2FS and instead focusing solely on obtaining the T2FS $\tilde{G}^{e2}(N_\alpha) = \{(G(N_\alpha), \alpha)\}$ with the constant secondary grade α corresponding to an acceptable fixed value $\alpha \in \Omega$ of the degree of truth.

7. Conclusions

The present research shows that, in addition to classical decision criteria, a DM can use our approach, which is based on representing a set of states by a FS. This FS can describe some property of the set of states, for example, an expected ability of the states. An application of the FS theory for solving a decision-making problem in such a formulation appears quite rational. Since we use a maximum guaranteed payoff, our methodology accedes to the benefits, drawbacks, and possibility of employment of a guaranteed outcome in practice.

Accordingly, our approach guarantees a risk-free decision and an opportunity to make a decision only once. In addition, using the approach, one constructs a maximum guaranteed payoff depending on the set of states (fuzzy, in general) that are considered during the decision-making process. These techniques enable us to optimize decision-making processes by considering and leveraging the inherent symmetries and patterns present in the decision-making problem. Since the problem formulation is symmetrical with respect to alternatives and states of nature, the results obtained can be used in the case of a fuzzy set of alternatives. Looking ahead, one potential avenue for future research could involve further exploring the integration of symmetry-aware optimization techniques with other decision criteria and to develop a similar approach for decision problems with FSs of alternatives and states. By expanding and refining our approach in this manner, we can continue to advance decision-making theory across various disciplines, including social sciences and artificial intelligence. Ultimately, we anticipate that our approach, alongside contributions from other researchers, will broaden the scope and applicability of decision-making theory in diverse fields of study.

Author Contributions: Conceptualization, S.M.; Methodology, S.M.; Resources, S.B.; Writing—original draft, S.B. and S.M.; Writing—review and editing, S.B. and S.M.; Project administration, S.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Ministry of National Defense of Lithuania as part of the study project Study Support Projects No VI-18, dated 2 December 2021 (2021–2024), General Jonas Žemaitis Military Academy of Lithuania, Vilnius, Lithuania, and was partially supported by the Ministry of Education and Science of Ukraine (grant BF/30-2021 (2021–2025) for the perspective development of the scientific direction "Mathematical Sciences and Natural Sciences" at the Taras Shevchenko National University of Kyiv.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest. Any opinions, findings, conclusions, or recommendations expressed in this material are those of the authors and do not necessarily reflect the view of the funding agency.

References

1. Jain, R. Decision-making in the presence of fuzzy variables. *IEEE Trans. Syst. Man Cybern.* **1976**, *6*, 698–703. [CrossRef]
2. Von Neumann, J.; Morgenstern, O. *Theory of Games and Economic Behavior*; Princeton University Press: Princeton, NY, USA, 1944.
3. Savage, L.J. *The Foundations of Statistics*; John Wiley and Son: New York, NY, USA, 1954.
4. Tanaka, H.; Okuda, T.; Asai, K. On decision-making in fuzzy environment, fuzzy information and decision-making. *Int. J. Prod. Res.* **2007**, *15*, 623–635. [CrossRef]
5. Guo, L.; Zhan, J.; Xu, Z.; José Carlos, R.; Alcantud, J.C.R. A consensus measure-based three-way clustering method for fuzzy large group decision making. *Inf. Sci.* **2023**, *632*, 144–163. [CrossRef]
6. Tanaka, H.; Okuda, T.; Asai, K. A formulation of fuzzy decision problems and its application to an investment problem. *Kybernetes* **1976**, *5*, 25–30. [CrossRef]
7. Mashchenko, S.O. Generalization of Germeyer's Criterion in the Problem of Decision-making under the Uncertainty Conditions with the Fuzzy Set of the States of Nature. *J. Autom. Inf. Sci.* **2012**, *44*, 26–34. [CrossRef]
8. Whalen, T. Decision-making under uncertainty with various assumptions about available information. *IEEE Trans. Syst. Man Cybern.* **1984**, *14*, 888–900. [CrossRef]
9. Watson, S.R.; Weiss, J.J.; Donnell, M.L. Fuzzy decision analysis. *IEEE Trans. Syst. Man Cybern.* **1979**, *9*, 1–9. [CrossRef]
10. Freeling, A.N. Fuzzy sets and decision analysis. *IEEE Trans. Syst. Man Cybern.* **1980**, *10*, 341–354. [CrossRef]
11. Yager, R.R. Possibilistic decision making. *IEEE Trans. Syst. Man Cybern.* **1979**, *9*, 388–392.
12. Whalen, T.H. Risk minimization using L-fuzzy sets. In Proceedings of the International Conference on Cybernetics and Society, Hyatt Regency, Cambridge, MA, USA, 8–10 October 1980; pp. 821–825.
13. Mashchenko, S.O.; Bovsunivskyi, O.M. Effective alternatives of decision-making problems with the fuzzy set of preference relations. *J. Autom. Inf. Sci.* **2013**, *45*, 32–42. [CrossRef]
14. Mashchenko, S.O. Non-dominated alternatives of fuzzy set of experts. *Cybern. Syst. Anal.* **2023**, *59*, 407–416. [CrossRef]
15. Zadeh, L.A. A note on Z-numbers. *Inf. Sci.* **2011**, *181*, 2923–2932. [CrossRef]
16. Larbani, M. Multiobjective problems with fuzzy parameters and games against nature. *Fuzzy Sets Syst.* **2010**, *161*, 2642–2660. [CrossRef]
17. Dubois, D.; Prade, H. Operations on fuzzy numbers. *Int. J. Syst. Sci.* **1978**, *9*, 613–626. [CrossRef]

18. Heilpern, S. Representation and application of fuzzy numbers. *Fuzzy Sets Syst.* **1997**, *9*, 259–268. [CrossRef]
19. Ambrosio, R.; Martini, G.B. Maximum and minimum between fuzzy symbols in noninteractive and weakly non-interactive situations. *Fuzzy Sets Syst.* **1984**, *12*, 27–35. [CrossRef]
20. Fung, L.W.; Fu, K.S. An axiomatic approach to rational decision-making in a fuzzy environment. In *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*; Academic Press: New York, NY, USA, 1975; pp. 227–256.
21. Klir, G.J.; Yuan, B. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*; Prentice-Hall PTR: Englewood Cliffs, NJ, USA, 1995.
22. Zhang, K.L.; Hirota, K. On fuzzy number lattice (R, <). *Fuzzy Sets Syst.* **1997**, *92*, 113–122. [CrossRef]
23. Tahayori, H.; Tettamanzi, A.G.; Degli Antoni, G.; Visconti, A. On the calculation of extended max and min operations between convex fuzzy sets of the real line. *Fuzzy Sets Syst.* **2009**, *160*, 3103–3114. [CrossRef]
24. Chiu, C.H.; Wang, W.J. A simple computation of MIN and MAX operations for fuzzy numbers. *Fuzzy Sets Syst.* **2002**, *126*, 273–276. [CrossRef]
25. Shirin, S.; Saha, G. Graphical representations of membership functions of maximum and minimum of two fuzzy numbers using computer program. *GANIT J. Bangladesh Math. Soc.* **2012**, *31*, 105–115. [CrossRef]
26. Mashchenko, S.O. Minimum of fuzzy numbers with a fuzzy set of operands. *Cybern. Syst. Anal.* **2022**, *58*, 210–219. [CrossRef]
27. Skalna, I.; Rebiasz, B.; Gawel, B.; Basiura, B.; Duda, J.; Opila, J.; Pelech-Pilichowski, T. *Advances in Fuzzy Decision-Making. Theory and Practice*; Springer: Cham, Switzerland, 2015. [CrossRef]
28. Zadeh, L.A. Quantitative fuzzy semantics. *Inf. Sci.* **1971**, *3*, 159–176. [CrossRef]
29. Mizumoto, M.; Tanaka, K. Some properties of fuzzy sets of type-2. *Inf. Control* **1976**, *31*, 312–340. [CrossRef]
30. Karnik, N.N.; Mendel, J.M. Introduction to type-2 fuzzy logic systems. *IEEE Int. Conf. Fuzzy Syst.* **1998**, *2*, 915–920. [CrossRef]
31. Mendel, J.M.; John, R.I. Type-2 fuzzy sets made simple. *IEEE Trans. Fuzzy Syst.* **2002**, *10*, 117–127. [CrossRef]
32. Harding, J.; Walker, C.; Walker, E. The variety generated by the truth value algebra of T2FSs. *Fuzzy Sets Syst.* **2010**, *161*, 735–749. [CrossRef]
33. Mendel, J.M. Type-2 fuzzy sets: Some questions and answers. *IEEE Connect. Newsl. IEEE Neural Netw. Soc.* **2003**, *1*, 10–13.
34. Zadeh, L. The concept of a linguistic variable and its application to approximate reasoning—I. *Inf. Sci.* **1975**, *8*, 199–249. [CrossRef]
35. Bekesiene, S.; Mashchenko, S. On Nash Equilibria in a Finite Game for Fuzzy Sets of Strategies. *Mathematics* **2023**, *11*, 4619. [CrossRef]
36. Ramik, J.; Rimanek, J. Inequality relation between fuzzy numbers and its use in fuzzy optimization. *Fuzzy Sets Syst.* **1985**, *16*, 123–150. [CrossRef]