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(Non-Symmetric) Yetter–Drinfel'd Module Category and Invariant Coinvariant Jacobians

Zhongwei Wang [1,2,*] and Yong Wang [3]

1 College of Science, Jinling Institute of Technology, Nanjing 211169, China
2 Department of Mathematics, Southeast University, Nanjing 210096, China
3 College of Information Engineering, Nanjing Xiaozhuang University, Nanjing 211171, China; wangyongmath@njxzc.edu.cn
* Correspondence: wangzhongwei@jit.edu.cn

Abstract: In this paper, we generalize the homomorphisms of modules over groups and Lie algebras as being morphisms in the category of (non-symmetric) Yetter–Drinfel'd modules. These module homomorphisms play a key role in the conjecture of Yau.

Keywords: Hopf algebra; Yetter–Drinfel'd module; invariant and coinvariant Jacobian

MSC: 16T05; 16S40; 18D10

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1. Introduction

The singularity theory is an important research field in differential topology. It has important applications in judging and determining the number of solutions of differential equations, giving the classification and counterexamples of the differential structure of differential manifolds, and describing the geometric properties of specific positions on differential manifolds.

It is well known that, the Laplace-Beltrami operator is an extremely important operator that acts on $\mathbb{C}^\infty$ functions on a Riemannian manifold. Over past several decades, research on the spectrum of the Laplace-Beltrami operator has always become a core issue in the study of geometry. For example, the geometry of closed minimal submanifolds in the unit sphere is closely related to the eigenvalue problem. In order to classify the isolated hypersurface singularities in complex geometry, Yau conjectured in [1] that the Laplace eigenfunctions ϕ_λ:

$$\Delta\phi_\lambda + \lambda\phi_\lambda = 0,$$

satisfy

$$c\sqrt{\lambda} \le \mathcal{H}^{n-1}(\{\phi_\lambda = 0\}) \le C\sqrt{\lambda},$$

where M is an arbitrary n-dimensional $\mathbb{C}^\infty$-smooth closed Riemannian manifold (compact and without boundary), and the symbol $\mathcal{H}^k$ denotes the k dimensional Hausdorff measure. Here c, C depend only on the Riemannian metric on M and are independent of the eigenvalue λ.

Aiming at Yau's conjecture, in [2], it was pointed out that the moduli algebra

$$A(f) = \mathbb{C}[[x_1, \dots, x_n]]/(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}).$$

of an analytic function f completely determines theisolated hypersurface singularities' complex structure. Therefore, this classification problem can be translated into classifying the moduli algebras, up to isomorphism. For further study of Yau's conjecture, one may refer [3–6].

Let k be an algebraically closed field, and G is a simply connected algebraic group over it. Suppose that M is a finite dimensional rational G-module. Take a basis $\{e_i\}_{i=1}^n$ of M, and let $\{x_i\}_{i=1}^n \subseteq M^*$ be its dual basis. Denote by $A = S^\bullet M^* \cong k[x_1, \ldots, x_n]$ the polynomial function ring on M (with the usual G-action), and let $\partial_i := \partial/\partial x_i$ be the usual differential operators on A. If we let A_d be the set of homogeneous polynomials of degree d, then for each $v \in A_d$, it is known that the Jacobian $J(v)$ of f is a subspace of A_{d-1} spanned by $\{\partial_i(f)\}_{i=1}^n$. Under the notions as above, Yau conjectured that if $G = SL_2(\mathbb{C})$ and $J(v)$ is G-invariant, then the highest weights of $J(v)$ is a subset of the highest weights of A_1 ($\cong M^*$). In [6], Xi constructed the following homomorphism of G-modules

$$\phi : M \otimes A \to A, \;\; \phi(e_i \otimes v) = \partial_{\lambda,i}(v),$$

where G is an arbitrary simply connected algebraic group. Combining this with Theorem 13 [4], each invariant Jacobian $J(v)$ is a quotient of $ku \otimes M$ ($\cong M$) if the characteristic of k equals 0. As a consequence, Yau's conjecture (see [7]) is particularly true for simply connected complex algebraic groups.

Suppose that $m(x)$ is the minimal polynomial of $\Phi_{M^*,M^*} \in End(M^* \otimes M^*)$. Fixing an eigenvalue λ of Φ_{M^*,M^*}, let $m_\lambda(x) := m(x)/(x - \lambda)$ and $A(M,\lambda) = T(M^*)/I(m_\lambda(x), M^*)$. In [8], using braided derivations, Chen generalized the setting in [6] as a representation of a quasi-triangular Hopf algebra H, and showed that

$$\phi : M \otimes A(M,\lambda) \to A(M,\lambda), \;\; \phi(e_i \otimes v) = \partial_{\lambda,i}(v),$$

is a homomorphism of left H modules (for details, see ([8], Theorem 3.5)).

It is known that the category of Yetter–Drinfel'd modules is an important category in the theory of Hopf algebra. Under some favourable conditions (e.g., H is a Hopf algebra with a bijective antipode), the category of Yetter–Drinfel'd modules is indeed braided monoidal through Drinfel'd double construction (see [9]). In [10], it is pointed out that symmetric Yetter–Drinfel'd categories are trivial, i.e., if H is a Hopf algebra, such that the canonical braiding of the category of Yetter–Drinfel'd modules is a symmetry, then $H = k$ in the field. Via braiding structures, the notion of the Yetter–Drinfel'd module plays an important role in the relations between knot theory and quantum groups.

It is known that the category of H modules is a spacial case in Yetter–Drinfel'd modules. So, it is natural but meaningful to ask whether the map ϕ, defined above, is a morphism of Yetter–Drinfel'd modules or not when M is a Yetter–Drinfel'd module. This is where the motivation for our paper comes. In this case, the results in [8] also hold in the coquasi-triangular Hopf algebra.

The paper is organized as follows. In Section 2, we mainly present some useful definitions about Yetter–Drinfel'd modules. In Section 3, we generalize the homomorphisms of the module over the groups and Lie algebras established in [6] as being morphisms in the category of (non-symmetric) Yetter–Drinfel'd modules. In Section 4, we provide a brief conclusion in this paper.

2. Preliminaries and Useful Materials

Let k be a ground field. All algebra, linear spaces, etc., will be over k; unadorned $\otimes$ means $\otimes_k$. Unless otherwise stated, H will denote a Hopf algebra with comultiplication Δ, counit ε, and bijective antipode S. Then, the opposite H^{op} is again a Hopf algebra with antipode S^{-1}. We will use the version of Sweedler's sigma notation: $\Delta(h) = h_1 \otimes h_2$ for all $h \in H$. For unexplained concepts and notations about Hopf algebras, we refer to [11,12]. If M is a vector space, a left H-module (right H comodule) structure on M will be usually denoted by $\psi_M : H \otimes M \to M, h \otimes m \mapsto h \cdot m$ ($\rho_M : M \to M \otimes H, m \mapsto m_{(0)} \otimes m_{(1)}$, respectively).

Definition 1. *A Yetter–Drinfel'd module (cf. [13]), sometimes also called a quantum Yang–Baxter module (cf. [14]), is a vector space M, such that M is a left H-module and a right H-comodule satisfying the following equivalent compatibility conditions:*

$$h_1 \cdot m_{(0)} \otimes h_2 m_{(1)} = (h_2 \cdot m)_{(0)} \otimes (h_2 \cdot m)_{(1)} h_1, \tag{1}$$

$$\rho_M(h \cdot m) = h_2 \cdot m_{(0)} \otimes h_3 m_{(1)} S^{-1}(h_1), \tag{2}$$

for all $h \in H$ and $m \in M$.

We denote the category of Yetter–Drinfel'd modules by ${}_H\mathcal{YD}^H$, with the morphisms being H linear and H colinear maps. It is known that $({}_H\mathcal{YD}^H, \tilde{\otimes}, k)$ forms a braided tensor category as follows:

(i) For any $M, N \in {}_H\mathcal{YD}^H$, we have $M\tilde{\otimes}N \in {}_H\mathcal{YD}^H$, where $M\tilde{\otimes}N = M \otimes N$ are the spaces, and the Yetter–Drinfel'd module structure is provided by

$$h \cdot (m\tilde{\otimes}n) = h_1 \cdot m\tilde{\otimes}h_2 \cdot n,$$

$$\rho_{M\tilde{\otimes}N}(m\tilde{\otimes}n) = m_{(0)}\tilde{\otimes}n_{(0)} \otimes n_{(1)}m_{(1)}.$$

(ii) The braiding is defined by

$$\Phi_{M,N} : M\tilde{\otimes}N \to N\tilde{\otimes}M, \;\; \Phi(m\tilde{\otimes}n) = n_{(0)}\tilde{\otimes}n_{(1)} \cdot m,$$

with inverse $\Phi^{-1}_{M,N}(n\tilde{\otimes}m) = S(n_{(1)}) \cdot m\tilde{\otimes}n_{(0)}$.

Remark 1. *With the notation as above, if the braiding $\Phi_{M,N} : M\tilde{\otimes}N \to N\tilde{\otimes}M$ satisfies $\Phi^2_{M,N} = id$, then the category of Yetter–Drinfel'd modules is* symmetric. *As symmetric Yetter–Drinfel'd categories are trivial, we do not consider the case that $\Phi_{M,N} = \Phi^{-1}_{M,N}$.*

Let $M \in {}_H\mathcal{YD}^H$ be of a finite dimension. We denote the dual vector space M^* by $M^\diamond$ as being endowed with the following Yetter–Drinfel'd module structure:

$$(h \cdot f)(m) = f(S(h) \cdot m),$$

$$f_{(0)}(m) \otimes f_{(1)} = f(m_{(0)}) \otimes S^{-1}(m_{(1)}),$$

for all $h \in H, f \in M^\diamond, m \in M$.

Definition 2. *An algebra A is called a* left H module algebra *if A is a left Hcmodule such that its structure maps are morphisms of H modules. Explicitly, for all $h \in H$ and $a, b \in A$,*

$$h \cdot 1_A = \varepsilon(h) 1_A, \tag{3}$$

$$h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b). \tag{4}$$

Similarly, an algebra A is called a right H-comodule algebra if A is a right H comodule with a comodule structure ρ_A as an algebra map. Explicitly, for all $a, b \in A$,

$$\rho_A(1_A) = 1_A \otimes 1_H, \tag{5}$$

$$\rho_A(ab) = a_{(0)}b_{(0)} \otimes a_{(1)}b_{(1)}. \tag{6}$$

Definition 3. *An algebra A, which is a Yetter–Drinfel'd module is said to be a Yetter–Drinfel'd module algebra (cf. [14,15]) if A is both a left H-module algebra and a right H^{op}-comodule algebra.*

3. Methods

In this paper, we mainly use the theory of the braided monoidal category to generalize the module homomorphisms for groups and Lie algebras for morphisms in the category of (non-symmetric) Yetter–Drinfel'd modules.

4. Discussion

We then generalize the morphisms in an arbitrary category of the (non-symmetric) braided monoidal category.

5. The Invariant Jacobians

Let M and N be right H comodules. We take $f \in Hom(M,N)$ and consider $\rho(f) \in Hom(M, N \otimes H)$ provided by

$$\rho(f)(m) = f(m_{(0)})_{(0)} \otimes S^{-1}(m_{(1)}) f(m_{(0)})_{(1)}.$$

As k is a field, $Hom(M,N) \otimes H \subseteq Hom(M, N \otimes H)$. Define

$$HOM(M,N) = \{f \in Hom(M,N) | \, \rho(f) \in Hom(M,N) \otimes H\}.$$

If a morphism f belongs in $HOM(M,N)$, then f is said to be *rational*. If the Hopf algebra H is finite dimensional, then we know that all morphisms are rational. Meanwhile, it is known from [16] that $HOM(M,N)$ is a right H comodule, and that it is actually the largest H comodule contained in the pace $Hom(M,N)$. Also, recall that $\rho(f) = f_{(0)} \otimes f_{(1)}$ if and only if

$$f_{(0)}(m) \otimes f_{(1)} = f(m_{(0)})_{(0)} \otimes S^{-1}(m_{(1)}) f(m_{(0)})_{(1)}, \tag{7}$$

for all $m \in M$.

Proposition 1. *Let $M \in {}_H\mathcal{YD}^H$. Then, $END(M)$ is a Yetter–Drinfel'd module algebra.*

Proof. The coaction defined by (7) makes $END(M)$ into an H comodule. We now define a left H action on $END(M)$ by

$$(h \cdot f)(m) = h_1 \cdot f(S(h_2) \cdot m), \tag{8}$$

for all $h \in H, f \in END(M)$ and $m \in M$. Then, the rest proof is similar to that of Proposition 4.1 [14]. □

Lemma 1. *Let $M \in {}_H\mathcal{YD}^H$ be of finite dimension and $f(x) \in k[x]$ is the ring of polynomials in indeterminate x. Set $I(f(x), M)$ as the two-sided ideal of $T(M)$ generated by the image $f(\Phi_{M,M})$ in $M \otimes M$. Then, $T(M)/I(f(x), M)$ is a Yetter–Drinfel'd module algebra, which inherits the Yetter–Drinfel'd module structure from $T(M)$.*

Proof. It is known from [14] that the tensor algebra $T(M)$ is a Yetter–Drinfel'd module algebra. Hence, it only needs to be shown that $I(f(x), M)$ is stable under the left H module action and the right H^{op} comodule coaction.

Indeed, as Φ is a Yetter–Drinfel'd module morphism, so is $f(\Phi_{M,M})$. Thus, $\mathrm{Im} f(\Phi_{M,M})$ is a Yetter–Drinfel'd submodule of $M \otimes M$, so that $I(f(x), M)$ is a Yetter–Drinfel'd submodule of $T(M)$ and hence $T(M)/I(f(x), M)$ is a Yetter–Drinfel'd module algebra. □

Let $M \in {}_H\mathcal{YD}^H$ be of finite dimension with a basis $\{e_i\}_{i=1}^n$ and let $\{x_i\}_{i=1}^n \subseteq M^*$ be the dual basis of M, such that $x_i(e_i) = \delta_{i,j}$. Let $m(x)$ be the minimal polynomial of $\Phi_{M^\diamond, M^\diamond} \in End(M^\diamond \tilde{\otimes} M^\diamond)$. For each eigenvalue λ of $\Phi_{M^\diamond, M^\diamond}$, set $m_\lambda(x) := m(x)/(x - \lambda)$ and $A(M, \lambda) = T(M^\diamond)/I(m_\lambda(x), M^\diamond)$. From Lemma 1 it is known that $A(M, \lambda)$ is a Yetter–Drinfel'd module algebra. We define partial differential operators of $\partial_{\lambda,i}$ $(1 \le i \le n)$ on

$A(M,\lambda)$, and it can be seen that the algebra $A(M,\lambda)$, together with $\partial_{\lambda,i}$, plays the role of the usual polynomial ring A with G-action and the usual ∂_i's, comparable to what we can see from [8].

Set $E := END(T(M^\diamond))$. Then, it follows from Proposition 1 that E is a Yetter–Drinfel'd module algebra. Let $\psi : E\tilde{\otimes}T(M^\diamond) \to T(M^\diamond), f\tilde{\otimes}\mathbf{u} \mapsto f(\mathbf{u})$. We first define the partial differential operators $\partial_{\lambda,i}$ $(1 \le i \le n)$ on $T(M^\diamond)$ for each eigenvalue λ of $\Phi_{M^\diamond,M^\diamond}$. Their action on $T^1(M^\diamond) = M^\diamond$ is provided by $\partial_{\lambda,i}(x_j) = \delta_{i,j}$, while the action on $T^d(M^\diamond)$ for $d > 1$ is defined as follows: for all $x_{i_1}\tilde{\otimes}x_{i_2}\cdots\tilde{\otimes}x_{i_d} \in T^d(M^\diamond)$, we set

$$\partial_{\lambda,i}(x_{i_1}\tilde{\otimes}x_{i_2}\cdots\tilde{\otimes}x_{i_d}) = \sum_{k=1}^{d}(-\lambda)^{k-1}\psi_{k,k+1}\circ(\Phi_{E,(M^\diamond)^{\tilde{\otimes}(k-1)}}\tilde{\otimes}id_{(M^\diamond)^{\tilde{\otimes}(d-k+1)}})(\partial_{\lambda,i}\tilde{\otimes}x_{i_1}\tilde{\otimes}x_{i_2}\cdots\tilde{\otimes}x_{i_d}),$$

where $\psi_{k,k+1} := id_{(M^\diamond)^{\tilde{\otimes}(k-1)}}\tilde{\otimes}\Phi\tilde{\otimes}id_{(M^\diamond)^{\tilde{\otimes}(d-k)}}$.

Lemma 2. *The $\partial_{\lambda,i}$ $(1 \le i \le n)$ obeys the following Leibniz rule:*

$$\partial_{\lambda,i}(\mathbf{u}\tilde{\otimes}\mathbf{v}) = \partial_{\lambda,i}(\mathbf{u})\tilde{\otimes}\mathbf{v} + (-\lambda)^s\mathbf{u}_{(0)}\tilde{\otimes}(\mathbf{u}_{(1)}\cdot\partial_{\lambda,i})(\mathbf{v}), \tag{9}$$

for $\mathbf{u} \in T^s(M^\diamond)$ and $\mathbf{v} \in T^t(M^\diamond)$.

Proof. As Φ is a braiding, for any triple (U,V,W) of Yetter–Drinfel'd modules, by Definition 10.4.1 [11], we have that

$$\Phi_{U,V\tilde{\otimes}W} = (id_V\tilde{\otimes}\Phi_{U,W})\circ(\Phi_{U,V}\tilde{\otimes}id_W). \tag{10}$$

Then, by the definitions of $\partial_{\lambda,i}$ and (10), we obtain

$$\begin{aligned}
\partial_{\lambda,i}(\mathbf{u}\tilde{\otimes}\mathbf{v}) &= \sum_{k=1}^{s+t}(-\lambda)^{k-1}\psi_{k,k+1}\circ(\Phi_{E,(M^\diamond)^{\tilde{\otimes}(k-1)}}\tilde{\otimes}id_{(M^\diamond)^{\tilde{\otimes}(s+t-k+1)}})(\partial_{\lambda,i}\tilde{\otimes}\mathbf{u}\tilde{\otimes}\mathbf{v})\\
&= (\sum_{k=1}^{s}+\sum_{k=s+1}^{s+t})(-\lambda)^{k-1}\psi_{k,k+1}\circ(\Phi_{E,(M^\diamond)^{\tilde{\otimes}(k-1)}}\tilde{\otimes}id_{(M^\diamond)^{\tilde{\otimes}(s+t-k+1)}})(\partial_{\lambda,i}\tilde{\otimes}\mathbf{u}\tilde{\otimes}\mathbf{v})\\
&= \partial_{\lambda,i}(\mathbf{u})\tilde{\otimes}\mathbf{v} + \sum_{k=s+1}^{s+t}(-\lambda)^{k-1}\psi_{k,k+1}\circ(id_{(M^\diamond)^{\tilde{\otimes}s}}\tilde{\otimes}\Phi_{E,(M^\diamond)^{\tilde{\otimes}(k-s-1)}}\tilde{\otimes}id_{(M^\diamond)^{\tilde{\otimes}(s+t-k+1)}})\\
&\quad\circ(\Phi_{E,(M^\diamond)^{\tilde{\otimes}s}}\tilde{\otimes}id_{(M^\diamond)^{\tilde{\otimes}t}})(\partial_{\lambda,i}\tilde{\otimes}\mathbf{u}\tilde{\otimes}\mathbf{v})\\
&= \partial_{\lambda,i}(\mathbf{u})\tilde{\otimes}\mathbf{v} + \sum_{k=s+1}^{s+t}(-\lambda)^{k-1}\psi_{k,k+1}\circ(id_{(M^\diamond)^{\tilde{\otimes}s}}\tilde{\otimes}\Phi_{E,(M^\diamond)^{\tilde{\otimes}(k-s-1)}}\tilde{\otimes}id_{(M^\diamond)^{\tilde{\otimes}(s+t-k+1)}})(\mathbf{u}_{(0)}\tilde{\otimes}\mathbf{u}_{(1)}\cdot\partial_{\lambda,i}\tilde{\otimes}\mathbf{v})\\
&= \partial_{\lambda,i}(\mathbf{u})\tilde{\otimes}\mathbf{v} + (-\lambda)^s\sum_{m=1}^{t}(-\lambda)^{m-1}(id_{(M^\diamond)^{\tilde{\otimes}s}}\tilde{\otimes}\psi_{m,m+1})(\mathbf{u}_{(0)}\tilde{\otimes}\mathbf{u}_{(1)}\cdot\partial_{\lambda,i}\tilde{\otimes}\mathbf{v})\\
&= \partial_{\lambda,i}(\mathbf{u})\tilde{\otimes}\mathbf{v} + (-\lambda)^s\mathbf{u}_{(0)}\tilde{\otimes}(\mathbf{u}_{(1)}\cdot\partial_{\lambda,i})(\mathbf{v}),
\end{aligned}$$

as required. □

For each $\mathscr{A} \in End(M^\diamond\tilde{\otimes}M^\diamond)$, denote $\mathscr{A}(x_i\tilde{\otimes}x_j) = \sum_{1\le s,t\le n}[\mathscr{A}]_{s,t}^{i,j}x_s\tilde{\otimes}x_t$.

Lemma 3. *We have $\partial_{\lambda,i}(x_j\tilde{\otimes}x_k) = \delta_{i,j}x_k - \lambda\sum_{s=1}^{n}[\Phi_{M^*,M^*}]_{i,s}^{j,k}x_s$, for all i,j,k.*

Proof. As a matter of fact,

$$
\begin{aligned}
\partial_{\lambda,i}(x_j \tilde{\otimes} x_k) &= \delta_{i,j}x_k - \lambda(id_{M^\diamond}\tilde{\otimes}\psi)\circ(\Phi_{E,M^\diamond}\tilde{\otimes}id_{M^\diamond})(\partial_{\lambda,i}\tilde{\otimes}x_j\tilde{\otimes}x_k) \\
&= \delta_{i,j}x_k - \lambda x_{j(0)}\tilde{\otimes}(x_{j(1)}\cdot\partial_{\lambda,i})(x_k) \\
&= \delta_{i,j}x_k - \lambda x_{j(0)}\tilde{\otimes}\partial_{\lambda,i}(S(x_{j(1)})x_k) \\
&= \delta_{i,j}x_k - \lambda(id_{M^\diamond}\tilde{\otimes}\partial_{\lambda,i})\circ\tau\circ\Phi^{-1}_{M^\diamond,M^\diamond}(x_j\tilde{\otimes}x_k) \\
&= \delta_{i,j}x_k - \lambda(id_{M^\diamond}\tilde{\otimes}\partial_{\lambda,i})(\sum_{1\le s,t\le n}[\Phi^{-1}_{M^\diamond,M^\diamond}]^{j,k}_{s,t}x_t\tilde{\otimes}x_s) \\
&= \delta_{i,j}x_k - \lambda\sum_{1\le s,t\le n}[\Phi^{-1}_{M^\diamond,M^\diamond}]^{j,k}_{s,t}x_t\tilde{\otimes}\partial_{\lambda,i}(x_s) \\
&= \delta_{i,j}x_k - \lambda\sum_{1\le t\le n}[\Phi^{-1}_{M^\diamond,M^\diamond}]^{j,k}_{i,t}x_t.
\end{aligned}
$$

Hence, the lemma is proven. □

Lemma 4. *For all i, the ideal $I(m_\lambda(x), M^\diamond)$ is stable under $\partial_{\lambda,i}$.*

Proof. First, we obtain that $\partial_{\lambda,i}(\mathrm{Im}(m_\lambda(\Phi_{M^\diamond,M^\diamond}))) = 0$. Indeed, for all i, j, k, by Lemma 3 we obtain that

$$
\begin{aligned}
\partial_{\lambda,i}(x_j\tilde{\otimes}x_k) &= \delta_{i,j}x_k - \lambda\sum_{s=1}^{n}[\Phi^{-1}_{M^\diamond,M^\diamond}]^{j,k}_{i,s}x_s \\
&= (\partial_{\lambda,i}\tilde{\otimes}id_{M^\diamond})(x_j\tilde{\otimes}x_k) - \lambda\sum_{1\le s,t\le n}[\Phi^{-1}_{M^\diamond,M^\diamond}]^{j,k}_{t,s}\partial_{\lambda,i}(x_t)\tilde{\otimes}x_k \\
&= (\partial_{\lambda,i}\tilde{\otimes}id_{M^\diamond})\circ(id_{M^\diamond\tilde{\otimes}M^\diamond} - \lambda\Phi^{-1}_{M^\diamond,M^\diamond})(x_j\tilde{\otimes}x_k).
\end{aligned}
$$

Hence, $\partial_{\lambda,i} = (\partial_{\lambda,i}\tilde{\otimes}id_{M^\diamond})\circ(id_{M^\diamond\tilde{\otimes}M^\diamond} - \lambda\Phi^{-1}_{M^\diamond,M^\diamond})$ in $Hom(M^\diamond\tilde{\otimes}M^\diamond, M^\diamond)$.

For each $\mathbf{x} \in \mathrm{Im}(m_\lambda(\Phi_{M^\diamond,M^\diamond}))$, there exists a $\mathbf{y} \in M^\diamond\tilde{\otimes}M^\diamond$ such that $\mathbf{x} = m_\lambda(\Phi_{M^\diamond,M^\diamond})(\mathbf{y})$. Then,

$$
\begin{aligned}
\partial_{\lambda,i}(\mathbf{x}) &= \partial_{\lambda,i}\circ m_\lambda(\Phi_{M^\diamond,M^\diamond})(\mathbf{y}) \\
&= (\partial_{\lambda,i}\tilde{\otimes}id_{M^\diamond})\circ(id_{M^\diamond\tilde{\otimes}M^\diamond} - \lambda\Phi^{-1}_{M^\diamond,M^\diamond})\circ m_\lambda(\Phi_{M^\diamond,M^\diamond})(\mathbf{y}) \\
&= (\partial_{\lambda,i}\tilde{\otimes}id_{M^\diamond})\circ\Phi^{-1}_{M^\diamond,M^\diamond}\circ(\Phi_{M^\diamond,M^\diamond} - \lambda id_{M^\diamond\tilde{\otimes}M^\diamond})\circ m_\lambda(\Phi_{M^\diamond,M^\diamond})(\mathbf{y}) \\
&= (\partial_{\lambda,i}\tilde{\otimes}id_{M^\diamond})\circ\Phi^{-1}_{M^\diamond,M^\diamond}\circ m(\Phi_{M^\diamond,M^\diamond})(\mathbf{y}) \\
&= 0,
\end{aligned}
$$

as $m(\Phi_{M^\diamond,M^\diamond}) = 0$.

Let $\mathbf{u} \in \mathrm{Im}(m_\lambda(\Phi_{M^\diamond,M^\diamond}))$. When $n \in \mathbb{N}$ and $\mathbf{v} \in T^n(M^\diamond)$ are proven, we obtain that $\partial_{\lambda,i}(\mathbf{u}\tilde{\otimes}\mathbf{v}), \partial_{\lambda,i}(\mathbf{v}\tilde{\otimes}\mathbf{u}) \in I(m_\lambda(x), M^\diamond)$.

Indeed, notice $\partial_{\lambda,i}(\mathbf{u}) = 0$ from the above discussion, then by (9), we have

$$\partial_{\lambda,i}(\mathbf{u}\tilde{\otimes}\mathbf{v}) = \partial_{\lambda,i}(\mathbf{u})\tilde{\otimes}\mathbf{v} + \lambda^2\mathbf{u}_{(0)}\tilde{\otimes}(\mathbf{u}_{(1)}\cdot\partial_{\lambda,i})(\mathbf{v}) = \lambda^2\mathbf{u}_{(0)}\tilde{\otimes}(\mathbf{u}_{(1)}\cdot\partial_{\lambda,i})(\mathbf{v}) \in I(m_\lambda(x), M^\diamond),$$

as the two-side ideal $I(m_\lambda(x), M^\diamond)$ of $T(M^\diamond)$ is stable under H action by Lemma 1. On the other hand,

$$
\begin{aligned}
\partial_{\lambda,i}(\mathbf{v}\tilde{\otimes}\mathbf{u}) &= \partial_{\lambda,i}(\mathbf{v})\tilde{\otimes}\mathbf{u} + (-\lambda)^n\mathbf{v}_{(0)}\tilde{\otimes}(\mathbf{v}_{(1)}\cdot\partial_{\lambda,i})(\mathbf{u}) \\
&= \partial_{\lambda,i}(\mathbf{v})\tilde{\otimes}\mathbf{u} + (-\lambda)^n\mathbf{v}_{(0)}\tilde{\otimes}\partial_{\lambda,i}(S(\mathbf{v}_{(1)})\cdot\mathbf{u}) \\
&= \partial_{\lambda,i}(\mathbf{v})\tilde{\otimes}\mathbf{u} \in I(m_\lambda(x), M^\diamond).
\end{aligned}
$$

We complete the proof of this lemma. □

The following proposition, which enables us to identify $\partial_{\lambda,i}$ with e_i, is key in the discussion below.

Proposition 2. *The subspace $M'(\lambda) := \oplus_{i=1}^{n} k\partial_{\lambda,i}$ is a Yetter–Drinfel'd submodule of $END(A(M,\lambda))$ which is isomorphic to $M^{\diamond}$ under the map*

$$\alpha : M'(\lambda) \to M^{\diamond},\ \ \alpha(\partial_{\lambda,i}) = e_i.$$

Proof. Obviously, α is isomorphic. Assume that $h \cdot e_i = \sum_j a_{j,i}(h) e_j$. For all $v \in A(M,\lambda)$, if we can show that $(h \cdot \partial_{\lambda,i})(v) = \sum_j a_{j,i}(h)\partial_{\lambda,j}(v)$, then α becomes a morphism of left H modules as a matter of course. It is clear that the above claim holds for $v \in M^{\diamond}$ (i.e., deg$v$ = 1). For arbitrary degree (≥ 2), the claim follows from the induction on deg(v) and the axiom: for any $v^1, v^2 \in A(M,\lambda)$ with $\deg(v^i) = d_i$ $(i = 1,2)$,

$$(h \cdot \partial_{\lambda,i})(v^1 \tilde{\otimes} v^2) = (h \cdot \partial_{\lambda,i})(v^1) \tilde{\otimes} v^2 + (-\lambda)^{d_1} v^1_{(0)} \tilde{\otimes} (v^1_{(1)} h \cdot \partial_{\lambda,i})(v^2).$$

To prove the axiom, we compute

$$\begin{aligned}
(h \cdot \partial_{\lambda,i})(v^1 \tilde{\otimes} v^2) &= h_1 \cdot \partial_{\lambda,i}\big(S(h_3) \cdot v^1 \tilde{\otimes} S(h_2) \cdot v^2\big) \\
&\overset{(9)}{=} \big(h_1 \cdot \partial_{\lambda,i}(S(h_4) \cdot v^1)\big) \tilde{\otimes} \big(h_2 S(h_3) \cdot v^2\big) \\
&\quad + (-\lambda)^{d_1} h_1 \cdot (S(h_4) \cdot v^1)_{(0)} \tilde{\otimes} h_2 \cdot \Big((S(h_4) \cdot v^1)_{(1)} \cdot \partial_{\lambda,i}\Big)(S(h_3) \cdot v^2) \\
&\overset{(2)}{=} \big(h_1 \cdot \partial_{\lambda,i}(S(h_2) \cdot v^1)\big) \tilde{\otimes} v^2 + (-\lambda)^{d_1} h_1 S(h_5) \cdot v^1_{(0)} \tilde{\otimes} h_2 \cdot \Big(S(h_4) v^1_{(1)} h_6 \cdot \partial_{\lambda,i}\Big)(S(h_3) \cdot v^2) \\
&\overset{(8)}{=} (h \cdot \partial_{\lambda,i})(v^1) \tilde{\otimes} v^2 + (-\lambda)^{d_1} h_1 S(h_6) \cdot v^1_{(0)} \tilde{\otimes} h_2 S(h_5) v^1_{(1)} h_7 \cdot \partial_{\lambda,i}\Big(S(h_3 S(h_4) v^1_{(2)} h_8) \cdot v^2\Big) \\
&= (h \cdot \partial_{\lambda,i})(v^1) \tilde{\otimes} v^2 + (-\lambda)^{d_1} h_1 S(h_4) \cdot v^1_{(0)} \tilde{\otimes} h_2 S(h_3) v^1_{(1)} h_5 \cdot \partial_{\lambda,i}\Big(S(v^1_{(2)} h_6) \cdot v^2\Big) \\
&= (h \cdot \partial_{\lambda,i})(v^1) \tilde{\otimes} v^2 + (-\lambda)^{d_1} h_1 S(h_2) \cdot v^1_{(0)} \tilde{\otimes} v^1_{(1)} h_3 \cdot \partial_{\lambda,i}\Big(S(v^1_{(2)} h_4) \cdot v^2\Big) \\
&= (h \cdot \partial_{\lambda,i})(v^1) \tilde{\otimes} v^2 + (-\lambda)^{d_1} v^1_{(0)} \tilde{\otimes} v^1_{(1)} h_1 \cdot \partial_{\lambda,i}\Big(S(v^1_{(2)} h_2) \cdot v^2\Big) \\
&= (h \cdot \partial_{\lambda,i})(v^1) \tilde{\otimes} v^2 + (-\lambda)^{d_1} v^1_{(0)} \tilde{\otimes} (v^1_{(1)} h \cdot \partial_{\lambda,i})(v^2).
\end{aligned}$$

The H colinearity of α can be proven dually. Indeed, writing $e_{i(0)} \otimes e_{i(1)} = \sum_j k_{j,i} e_j \otimes h_j$, we need to show that $\partial_{\lambda,i(0)}(v) \otimes \partial_{\lambda,i(1)} = \sum_j k_{j,i} \partial_{\lambda,j}(v) \otimes h_j$ for all $v \in A(M,\lambda)$. Clearly, this claim holds for $v \in M^{\diamond}$. It remains to be proven that

$$\partial_{\lambda,i(0)}(v^1 \tilde{\otimes} v^2) \otimes \partial_{\lambda,i(1)} = \partial_{\lambda,i(0)}(v^1) \tilde{\otimes} v^2 \otimes \partial_{\lambda,i(1)} + (-\lambda)^{d_1} v^1_{(0)} \tilde{\otimes} (v^1_{(1)} \cdot \partial_{\lambda,i(0)})(v^2) \otimes \partial_{\lambda,i(1)}.$$

from which and the induction on deg(v), the claim follows.

In fact,

$$\begin{aligned}
\partial_{\lambda,i(0)}(v^1 \tilde{\otimes} v^2) \otimes \partial_{\lambda,i(1)} &= \partial_{\lambda,i}(v^1_{(0)} \tilde{\otimes} v^2_{(0)})_{(0)} \otimes S^{-1}(v^2_{(1)} v^1_{(1)}) \partial_{\lambda,i}(v^1_{(0)} \tilde{\otimes} v^2_{(0)})_{(1)} \\
&= \Big(\partial_{\lambda,i}(v^1_{(0)}) \tilde{\otimes} v^2_{(0)}\Big)_{(0)} \otimes S^{-1}(v^2_{(1)} v^1_{(1)}) \Big(\partial_{\lambda,i}(v^1_{(0)}) \tilde{\otimes} v^2_{(0)}\Big)_{(1)} + (-\lambda)^{d_1} \Big(v^1_{(0)} \tilde{\otimes} (v^1_{(1)} \cdot \partial_{\lambda,i})(v^2_{(0)})\Big)_{(0)} \\
&\quad \otimes S^{-1}(v^2_{(1)} v^1_{(1)}) \Big(v^1_{(0)} \tilde{\otimes} (v^1_{(1)} \cdot \partial_{\lambda,i})(v^2_{(0)})\Big)_{(1)} \\
&= \partial_{\lambda,i}(v^1_{(0)})_{(0)} \tilde{\otimes} v^2_{(0)} \otimes S^{-1}(v^2_{(2)} v^1_{(1)}) v^2_{(1)} \partial_{\lambda,i}(v^1_{(0)})_{(1)} + (-\lambda)^{d_1} \sum v^1_{(0)} \tilde{\otimes} (v^1_{(2)} \cdot \partial_{\lambda,i})(v^2_{(0)})_{(0)} \\
&\quad \otimes S^{-1}(v^2_{(1)} v^1_{(3)}) (v^1_{(2)} \cdot \partial_{\lambda,i})(v^2_{(0)})_{(1)} v^1_{(1)} \\
&= \partial_{\lambda,i(0)}(v^1) \tilde{\otimes} v^2 \otimes \partial_{\lambda,i(1)} + (-\lambda)^{d_1} v^1_{(0)} \tilde{\otimes} v^1_{(3)} \cdot \partial_{\lambda,i}\Big(S(v^1_{(5)}) \cdot v^2_{(0)}\Big)_{(0)} \\
&\quad \otimes S^{-1}(v^2_{(1)} v^1_{(6)}) v^1_{(4)} \partial_{\lambda,i}\Big(S(v^1_{(5)}) \cdot v^2_{(0)}\Big)_{(1)} S^{-1}(v^1_{(2)}) v^1_{(1)} \\
&= \partial_{\lambda,i(0)}(v^1) \tilde{\otimes} v^2 \otimes \partial_{\lambda,i(1)} + (-\lambda)^{d_1} v^1_{(0)} \tilde{\otimes} v^1_{(1)} \cdot \partial_{\lambda,i}\Big(S(v^1_{(3)}) \cdot v^2_{(0)}\Big)_{(0)} \\
&\quad \otimes S^{-1}(v^2_{(1)} v^1_{(4)}) v^1_{(2)} \partial_{\lambda,i}\Big(S(v^1_{(3)}) \cdot v^2_{(0)}\Big)_{(1)},
\end{aligned}$$

however,

$$
\begin{aligned}
& v^1_{(0)}\tilde{\otimes}v^1_{(1)}\cdot\partial_{\lambda,i}\Big(S(v^1_{(3)})\cdot v^2_{(0)}\Big)_{(0)}\otimes S^{-1}(v^2_{(1)}v^1_{(4)})v^1_{(2)}\partial_{\lambda,i}\Big(S(v^1_{(3)})\cdot v^2_{(0)}\Big)_{(1)} \\
&= v^1_{(0)}\tilde{\otimes}v^1_{(1)}\cdot\partial_{\lambda,i}\Big(S(v^1_{(3)})\cdot v^2_{(0)}\Big)_{(0)}\otimes S^{-1}\Big(S(v^1_{(2)})v^2_{(1)}v^1_{(4)}\Big)\partial_{\lambda,i}\Big(S(v^1_{(3)})\cdot v^2_{(0)}\Big)_{(1)} \\
&= v^1_{(0)}\tilde{\otimes}v^1_{(1)}\cdot\partial_{\lambda,i}\Big((S(v^1_{(2)})\cdot v^2)_{(0)}\Big)_{(0)}\otimes S^{-1}\Big((S(v^1_{(2)})\cdot v^2)_{(1)}\Big)\partial_{\lambda,i}\Big((S(v^1_{(2)})\cdot v^2)_{(0)}\Big)_{(1)} \\
&= v^1_{(0)}\tilde{\otimes}v^1_{(1)}\cdot\partial_{\lambda,i(0)}\Big((S(v^1_{(2)})\cdot v^2)\Big)\otimes\partial_{\lambda,i(1)} \\
&= v^1_{(0)}\tilde{\otimes}(v^1_{(1)}\cdot\partial_{\lambda,i(0)})(v^2)\otimes\partial_{\lambda,i(1)}.
\end{aligned}
$$

So, the claim follows and the proof is completed. □

The following theorem generalizes Theorem 3.1(a) [6] and Theroem 3.5 [8].

Theorem 1. *Let H be a Hopf algebra over an algebraically closed field k, and M is a finite dimensional Yetter–Drinfel'd module. For each eigenvalue λ of $\Phi_{M^\diamond,M^\diamond}$ in k, the linear map*

$$\phi: M\tilde{\otimes}A(M,\lambda)\to A(M,\lambda),\ \ \phi(e_i\tilde{\otimes}v)=\partial_{\lambda,i}(v),$$

for each i and $v\in A(M,\lambda)$ is a morphism of Yetter–Drinfel'd modules.

Proof. For each i and $v\in A(M,\lambda), h\in H$, by Proposition 2 we have

$$
\begin{aligned}
\phi\big(h\cdot(e_i\tilde{\otimes}v)\big) &= \phi(h_1\cdot e_i\tilde{\otimes}h_2\cdot v)=(h_1\cdot\partial_{\lambda,i})(h_2\cdot v) \\
&= h_1\cdot\partial_{\lambda,i}(S(h_2)h_3\cdot v)=h\cdot\partial_{\lambda,i}(v)=h\cdot\phi(e_i\tilde{\otimes}v), \\
\phi\circ\rho(e_i\tilde{\otimes}v) &= \sum\phi(e_{i(0)}\tilde{\otimes}v_{(0)})\otimes v_{(1)}e_{i(1)}=\partial_{\lambda,i(0)}(v_{(0)})\otimes v_{(1)}\partial_{\lambda,i(1)} \\
&= \partial_{\lambda,i}(v_{(0)})_{(0)}\otimes v_{(2)}S^{-1}(v_{(1)})\partial_{\lambda,i}(v_{(0)})_{(1)} \\
&= \partial_{\lambda,i}(v)_{(0)}\otimes\partial_{\lambda,i}(v)_{(1)}=\rho\circ\phi(e_i\tilde{\otimes}v),
\end{aligned}
$$

which completes the proof. □

Definition 4. *Let $A=\oplus_{q\in\mathbb{N}}A_q=A_0\oplus A_1\oplus\cdots$ be a graded ring. Then, $a=a_0+a_1+\cdots$ is unique for all $a\in A$. Here, a_q is called the q-th homogeneous component of a.*

An ideal $I\subseteq A$ is said to be homogeneous if for all $x\in I$, its homogeneous component belongs to I.

As $I(m_\lambda(x),M^*)$ is a homogeneous ideal (the elements in $\mathrm{Im}(m_\lambda(\Phi_{M^\diamond,M^\diamond}))$ are homogeneous of degree 2), $A(M,\lambda)$ is a graded algebra. For each $q\in\mathbb{N}$, denote the q-th homogeneous component by $A(M,\lambda)_q$. As in the classical case, we define the λ-Jacobian $J_\lambda(v)$ of $v\in A(M,\lambda)_q$ to the subspace of $A(M,\lambda)_{q-1}$ spanned by $\{\partial_{\lambda,i}(v)^n_{i=1}\}$. Then, as a consequence of Theorem 1, we have

Corollary 1. *Let $v\in A(M,\lambda)_q$. If $J_\lambda(v)$ is H invariant and H coinvariant, then $J_\lambda(v)$ is a quotient Yetter–Drinfel'd module of $M\tilde{\otimes}A(M,\lambda)_q$. If a $u\in A(M,\lambda)$ also exists, such that $deg(u)=deg(v)$ and u are H-invariant and H-coinvariant, respectively, then $J_\lambda(v)$ is a quotient module of M. In this case, if M is irreducible, then $J_\lambda(v)$ is 0 or isomorphic to M.*

Lemma 5. *The element $\sum\limits_{i=1}^{n}x_i\tilde{\otimes}\partial_{\lambda,i}\in M^\diamond\tilde{\otimes}END(A(M,\lambda))$ is H-invariant and H-coinvariant.*

Proof. There is no harm in replacing $\partial_{\lambda,i}$ by e_i by Proposition 2. So, it is equivalent to show that $\sum\limits_{i=1}^{n}x_i\tilde{\otimes}e_i$ is H-invariant and H-coinvariant.

By evaluating x_p, on the one hand,

$$\begin{aligned} \left(h \cdot (\sum_{i=1}^{n} x_i \tilde{\otimes} e_i)\right)(x_p) &= \sum_{i=1}^{n} h_1 \cdot x_i \tilde{\otimes} (h_2 \cdot e_i)(x_p) = \sum_{i=1}^{n} h_1 \cdot x_i \tilde{\otimes} e_i(S(h_2) \cdot x_p) \\ &= h_1 S(h_2) \cdot x_p \tilde{\otimes} 1_k = \varepsilon(h) x_p \tilde{\otimes} 1_k \\ &= \varepsilon(h) \sum_{i=1}^{n} x_i \tilde{\otimes} e_i(x_p), \end{aligned}$$

on the other hand,

$$\begin{aligned} \rho\left(\sum_{i=1}^{n} x_i \tilde{\otimes} e_i\right)(x_p) &= \left(\sum_{i=1}^{n} x_{i(0)} \tilde{\otimes} e_{i(0)} \otimes e_{i(1)} x_{i(1)}\right)(x_p) \\ &= \sum_{i=1}^{n} x_{i(0)} \tilde{\otimes} e_{i(0)}(x_p) \otimes e_{i(1)} x_{i(1)} \\ &= \sum_{i=1}^{n} x_{i(0)} \tilde{\otimes} e_i(x_{p(0)}) \otimes S^{-1}(x_{p(1)}) x_{i(1)} \\ &= x_{p(0)} \tilde{\otimes} 1_k \otimes S^{-1}(x_{p(2)}) x_{p(1)} \\ &= x_p \tilde{\otimes} 1_k \otimes 1_H = \left(\sum_{i=1}^{n} x_i \tilde{\otimes} e_i \otimes 1_H\right)(x_p). \end{aligned}$$

Hence, the lemma is proven. □

The following theorem is a generalization of Theorem 3.3(a) [6] and Theroem 3.9 [8].

Theorem 2. *The map*

$$\psi : A(M,\lambda)_q \to A(M,\lambda)_1 \tilde{\otimes} A(M,\lambda)_{q-1}, \ \ \psi(v) = \sum_{i=1}^{n} x_i \otimes \partial_{\lambda,i}(v),$$

is a morphism of Yetter–Drinfel'd modules.

Proof. It is easy to see that the following

$$\delta : END(A(M,\lambda)) \tilde{\otimes} A(M,\lambda) \to A(M,\lambda), \ \ \delta(f \tilde{\otimes} v) = f(v),$$

is a morphism of Yetter–Drinfel'd modules.

Then, for any $v \in A(M,\lambda)$ and $h \in H$, from Lemma 5, we obtain that

$$\begin{aligned} h \cdot \left(\sum_{i=1}^{n} x_i \tilde{\otimes} \partial_{\lambda,i}(v)\right) &= \sum_{i=1}^{n} h_1 \cdot x_i \tilde{\otimes} h_2 \cdot \partial_{\lambda,i}(v) = \sum_{i=1}^{n} h_1 \cdot x_i \tilde{\otimes} h_2 \cdot \partial_{\lambda,i}(h_3 \cdot v) \\ &= (id \tilde{\otimes} \delta)\left(\sum_{i=1}^{n} h_1 \cdot x_i \tilde{\otimes} h_2 \cdot \partial_{\lambda,i} \tilde{\otimes} h_3 \cdot v\right) \\ &= (id \tilde{\otimes} \delta)\left(\sum_{i=1}^{n} h_1 \cdot (x_i \tilde{\otimes} \partial_{\lambda,i}) \tilde{\otimes} h_2 \cdot v\right) \\ &= (id \tilde{\otimes} \delta)\left(\sum_{i=1}^{n} x_i \tilde{\otimes} \partial_{\lambda,i} \tilde{\otimes} h \cdot v\right) \\ &= \sum_{i=1}^{n} x_i \otimes \partial_{\lambda,i}(h \cdot v), \end{aligned}$$

thus, ψ is a left H module morphism.

Furthermore,

$$
\begin{aligned}
\rho\left(\sum_{i=1}^{n} x_i \otimes \partial_{\lambda,i}(v)\right) &= \sum_{i=1}^{n} x_{i(0)} \otimes \partial_{\lambda,i}(v)_{(0)} \otimes \partial_{\lambda,i}(v)_{(1)} x_{i(1)} \\
&= \sum_{i=1}^{n} x_{i(0)} \otimes \partial_{\lambda,i}(v_{(0)})_{(0)} \otimes v_{(2)} S^{-1}(v_{(1)}) \partial_{\lambda,i}(v_{(0)})_{(1)} x_{i(1)} \\
&= \sum_{i=1}^{n} x_{i(0)} \otimes \partial_{\lambda,i(0)}(v_{(0)}) \otimes v_{(1)} \partial_{\lambda,i(1)} x_{i(1)} \\
&= \sum_{i=1}^{n} x_i \otimes \partial_{\lambda,i}(v_{(0)}) \otimes v_{(1)},
\end{aligned}
$$

thus, ψ is a right H comodule morphism. Hence, ψ is a morphism of Yetter–Drinfel'd modules. □

6. Conclusions

It has been known that two kinds of homomorphisms of modules over groups and Lie algebras exist, established by Xi (see [6]), which play a key role in the proof of a conjecture of Yau (see [7]). In [8], the author showed that the two module homomorphisms could be generalized to the setting of quasi-triangular Hopf algebras. Following the above work, we furthermore generalized to the setting of quantum Yang–Baxter module algebra over Hopf algebra. This will also be useful in the problem of the decomposition of tensor products of modules and Yetter–Drinfel'd modules.

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