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On the Unified Concept of Generalizations of Λ-Sets

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Abstract: In this paper, we propose a unified concept encompassing generalizations of two types of families defined based on Levine's notions of generalized closed sets and Maki's Λ sets. The methods used in this investigation are described in my previous work, where a unified concept of general closedness is presented. From a methodology point of view, the present concept is symmetric to the previous. In generalizing open subsets, one can use the two methods. According to the first one, the family of Levine's generalization is used as some base to build the family of closed subsets of the new topology. In the second method, the family of open subsets is extended, in the same way, as the family of closed subsets in the classic Levine's method. The results obtained in this general conception easily extend and imply well-known theorems of this area of investigation. In the literature on this issue, many versions of generalizations of Λ-sets have been investigated. The tools used in this paper enabled us to prove that there exist at most 10 generalizations of these types, and we show the relationships between them in the graph. As a result, it turns out that some generalizations investigated in the literature are trivial.

Keywords: generalizedclosed set; closure operators; regular sets; lattice; α-open; pre-open; γ-open; semi-open; β-open

1. Introduction

The idea of generalized closed subsets in topological spaces introduced by Levine [1] was extended in many manners and used in many topics of mathematical study.

In the literature on the issue, there are many definitions of various kinds of generalized closed sets obtained by using the subsets more general than the open sets or operators that are more general than Kuratowski's operator [2–5]. Many works in the literature concern the problem of unification of the generalizations mentioned above [6–11].

The concept of Maki's Λ-sets, defined in 1986, can be treated as symmetric to the concept of Levine's generalized closed subsets in a topological space. In the succeeding years, this concept has been generalized in many versions [12–15], which are not embedded into some unified theory.

In this work, we propose a general concept encompassing both the generalizations based on Levine's notion of generalized closed sets and the notion of Λ-sets. An overview of such a theory concerning only the generalized closed sets was recently proposed in [9], where the notion of the closure operator [16] is basic. Here, we propose an analogical approach.

In a preliminary study of our paper, we present an overview of basic concepts related to the closure operators generated from arbitrary families of subsets. In Lemmas 2, 3 and 4, some important general conditions of closure operators are formulated. Those conditions are used for the natural definitions of some topologies designated by arbitrary families of subsets. We will discuss this issue in Section 3. We formulate the necessary and sufficient properties of the families $\overline{\mathcal{B}}$ guaranteeing that the operators $\overline{(\dots)}^{\mathcal{B}}$ determine topologies. We summarize it in Remark 8. In Definition 3, we state the unified concept of generalized closed sets, introduced in [9]. This concept's basic properties are helpful for our investigations in

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Sections 4 and 5. The most fundamental results of this concept are stated in Lemma 5 and Theorem 1.

In Section 4.1, we investigate the properties that imply the family of type $\mathcal{B} \triangleleft \mathcal{K}$ is the topology. In Section 4.2, we present an investigation of the axiom $T_{\frac{1}{2}}$ in terms of the theory of families of type $\mathcal{B} \triangleleft \mathcal{K}$.

In Section 5.1, based on a general concept stated in this work, we explain the nature of well-known results concerning the axiom $T_{\frac{1}{2}}$ and families of sets of type Λ-set [17]. We proved what is a maximal number of types of generalizations based on the notion of Λ-sets, and we show the relationships between them in the graph.

2. Preliminaries

In 1970, N. Levine [1] introduced the concept of general closed sets, called a subset A of a space topological (X, τ) generalized closed, shortly g-closed, if

$$\mathrm{cl}(A) \subseteq U, \text{ for every open set } U \text{ such that } A \subseteq U,$$

where $\mathrm{cl}(A)$ denotes the closure of A.

A subset $A \subset X$ is called generalized open [1] in a topological space (X, τ) if $X \setminus A$ is generalized closed.

Remark 1. *According to the unified concept of generalized closedness presented in [9], the standard definition of the family of all g-closed subsets in a topological space (X, τ) can be treated as a family specified by the pair (ϕ, ϕ^*) of the operators defined by*

$$\phi(A) = cl(A) \text{ and } \phi^*(A) = \overline{A}^{\tau},$$

where $\overline{A}^{\tau}$ is given by $\bigcap \{U \in \tau : A \subseteq U\}$ for $A \subseteq X$. Namely, the family of subsets $A \subseteq X$ such that

$$\phi(A) \subseteq \phi^*(A).$$

Let us recall a definition of the operator use in this concept.

A function $\phi : \mathcal{P}(X) \to \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the family of all subset of X, and satisfying the following properties is called a closure operator [16]:

(i) ϕ is isotone: $A \subseteq B$ implies $\phi(A) \subseteq \phi(B)$ for all $A, B \subseteq X$;
(ii) ϕ is extensive: $A \subseteq \phi(A)$ for all $A \subseteq X$;
(iii) ϕ is idempotent: $\phi(\phi(A)) = \phi(A)$ for all $A \subseteq X$.

A pair (X, ϕ), where ϕ satisfies the above properties, is called a closure system.

A subset $A \subset X$ is called closed with respect to ϕ if it is a fixed point of ϕ. The family of all such subsets will be denoted by $\mathcal{C}^{\phi}$, i.e.,

$$\mathcal{C}^{\phi} = \{A \subseteq X : \phi(A) = A\}.$$

The properties (i) and (iii) imply that the family $\mathcal{C}^{\phi}$ is closed under arbitrary intersections. Cohn [18] have shown that

$$\phi(A) = \bigcap \{B \in \mathcal{C}^{\phi} : A \subseteq B\}$$

for any $A \subseteq X$.

In our investigation, we will use the operators specified by the families $\mathcal{B} \subseteq \mathcal{P}(X)$.

For convenience, we use the notation $\mathcal{A}^{[c]}$ to describe the following family

$$\{A \subseteq X : X \setminus A \in \mathcal{A}\}.$$

Definition 1. *Given family $\mathcal{B} \subseteq \mathcal{P}(X)$, such that $\emptyset, X \in \mathcal{B}$, we define the operators $\overline{(...)}^{\mathcal{B}} : \mathcal{P}(X) \to \mathcal{P}(X)$ and $\mathcal{B}.int(...) : \mathcal{P}(X) \to \mathcal{P}(X)$ by*

$$\overline{A}^{\mathcal{B}} = \bigcap \{B \in \mathcal{B} : A \subseteq B\} \text{ and}$$

$$\mathcal{B}.int(A) = \bigcup \{U : U \in \mathcal{B}^{[c]} \quad \text{and} \quad U \subseteq A\} \text{ or equivalently}$$

$$\mathcal{B}.int(A) = X \setminus \overline{X \setminus A}^{\mathcal{B}} \text{ for any } A \subseteq X.$$

Remark 2. *In a topological space (X, τ), the convention used in Definition 1 leads to the following notation of the closure ($cl(...)$) (resp. interior $int(...)$) operator*

$$cl(A) = \overline{A}^{\mathcal{C}} \text{ (resp. } int(A) = \mathcal{C}.int(A)) \text{ for } A \subseteq X,$$

where $\mathcal{C}$ is the family of all closed sets in (X, τ).

Lemma 1. *In reference [9], the operator $\overline{(...)}^{\mathcal{B}}$ is a closure operator, for any family $\mathcal{B} \subseteq \mathcal{P}(X)$.*

Lemma 2. *In reference [9], the following equivalence holds, for any family $\mathcal{B} \subseteq \mathcal{P}(X)$ and a subset $A \subseteq X$:*

$$x \in \overline{A}^{\mathcal{B}} \text{ if and only if } U \cap A \neq \emptyset \text{ for every } U \in \mathcal{B}^{[c]} \text{ such that } x \in U.$$

We will also consider an operator $\overline{(...)} : \mathcal{P}^2(X) \to \mathcal{P}^2(X)$, and given the below, we associated the operators of type $\overline{(...)}^{\mathcal{B}}$, where $\mathcal{B} \subseteq \mathcal{P}(X)$.

Definition 2. *We define the following operators $\overline{(...)}, int(...) : \mathcal{P}^2(X) \to \mathcal{P}^2(X)$ by*

$$\overline{\mathcal{B}} = \{A \subseteq X : \overline{A}^{\mathcal{B}} = A\} \text{ and}$$

$$int(\mathcal{B}) = \{A \subseteq X : \mathcal{B}.int(A) = A\}$$

for any family $\mathcal{B} \subseteq \mathcal{P}^2(X)$, such that $\emptyset, X \in \mathcal{B}$.

Remark 3. *Directly from Definition 2, we have*

$$(\overline{\mathcal{B}})^{[c]} = int(\mathcal{B}).$$

Remark 4. *According to the Cohn result cited above, the families $\mathcal{B}$ and $\overline{\mathcal{B}}$ are equivalent in the sense that they define the same closure operator, i.e.,*

(i) $\overline{A}^{\mathcal{B}} = \overline{A}^{\overline{\mathcal{B}}}$ *for all $A \subseteq X$. As a result, we have*
(ii) $\overline{\overline{\mathcal{B}}} = \overline{\mathcal{B}}$.

The particular relevance to our investigation is the following property of the closure operator $\overline{(\dots)}$:

Lemma 3. *For any family $\mathcal{B} \subseteq \mathcal{P}(X)$, the family $\overline{\mathcal{B}}$ is closed under arbitrary intersection.*

Proof. Let us take a family $\{A_s\}_{s \in S} \subseteq \overline{\mathcal{B}}$. According to the isotone, $\overline{\bigcap\limits_{s \in S} A_s}^{\mathcal{B}} \subseteq \bigcap\limits_{s \in S} \overline{A_s}^{\mathcal{B}}$ and by the assumption, $\bigcap\limits_{s \in S} \overline{A_s}^{\mathcal{B}} = \bigcap\limits_{s \in S} A_s$. So, $\overline{\bigcap\limits_{s \in S} A_s}^{\mathcal{B}} = \bigcap\limits_{s \in S} A_s$ which means that $\bigcap\limits_{s \in S} A_s \in \overline{\mathcal{B}}$. □

Lemma 4. *If a family $\mathcal{B} \subseteq \mathcal{P}(X)$ is closed under finite (arbitrary) unions, then $\overline{\mathcal{B}}$ is also closed under finite (arbitrary) unions.*

Proof. Let us take a family $\{A_s\}_{s \in S} \subseteq \overline{\mathcal{B}}$. We will prove that $\bigcup\limits_{s \in S} A_s \in \overline{\mathcal{B}}$, i.e., $\bigcup\limits_{s \in S} \overline{A_s}^{\mathcal{B}} = \bigcup\limits_{s \in S} A_s$. By the assumption, $\overline{A_s}^{\mathcal{B}} = A_s$ for all $s \in S$. So, $\bigcup\limits_{s \in S} \overline{A_s}^{\mathcal{B}} = \bigcup\limits_{s \in S} A_s$.

We prove that

$$\bigcup_{s\in S}\overline{A_s}^{\mathcal{B}} = \overline{\bigcup_{s\in S} A_s}^{\mathcal{B}}. \qquad (\star)$$

According to the issotonity of the operator $\overline{(...)}^{\mathcal{B}}$, it is enough to show that $\overline{\bigcup_{s\in S} A_s}^{\mathcal{B}} \subseteq \bigcup_{s\in S}\overline{A_s}^{\mathcal{B}}$. Assume, that $x \notin \bigcup_{s\in S}\overline{A_s}^{\mathcal{B}}$, i.e, by Lemma 2, for every $s \in S$, there exists $U_s \in \mathcal{B}^{[c]}$ such that $x \in U_s$ and $U_s \cap A_s = \emptyset$. Let us put $U = \bigcap_{s\in S} U_s$. Then, $U \cap A_s = \emptyset$ for all $s \in S$, i.e.,

$$U \cap \bigcup_{s\in S} A_s = \emptyset \text{ and } x \in U. \qquad (\star\star)$$

Of course, $X \setminus U = \bigcup_{s\in S} X \setminus U_s$ and $X \setminus U_s \in \mathcal{B}$ for all $s \in S$, and by our assumption, we have $X \setminus U \in \mathcal{B}$, i.e., $U \in \mathcal{B}^{[c]}$. Consequently, according to $(\star\star)$ and Lemma 2, $x \notin \overline{\bigcup_{s\in S} A_s}^{\mathcal{B}}$ which ends the proof on the arbitrary unions. The second part of the proof precedes in the same way. $\square$

Based on the notion of a closure operator generated by a family, we study an extended concept of generalized closedness.

Definition 3. *In reference [9], given a pair $(\mathcal{B}, \mathcal{K})$ of families of subsets of X such that X, $\emptyset \in \mathcal{B} \cap \mathcal{K}$, we define $\mathcal{B} \lhd \mathcal{K}$ to be the following collection:*

$$\{A \subseteq X : \overline{A}^{\mathcal{B}} \subseteq \overline{A}^{\mathcal{K}}\}.$$

The family $\mathcal{B} \lhd \mathcal{K}$ we will call a generalization of $\mathcal{B}$ by the family $\mathcal{K}$. The collections of all families of type $\mathcal{B} \lhd \mathcal{K}$ will be denoted by $\Gamma(X)$.

Remark 5. *According to Remark 1, in a topological space (X, τ), the family of all g-closed subsets is specified by the pair $\left(\overline{(...)}^{\mathcal{C}}, \overline{(...)}^{\tau}\right)$ or equivalently by the pair $(\mathcal{C}, \tau)$. So, $\mathcal{C} \lhd \tau$ is strictly the family of g-closed subsets, and the elements of the family $(\mathcal{C} \lhd \tau)^{[c]}$ are g-open subsets in the space (X, τ).*

Remark 6. *It is evident that $A \in \mathcal{B} \lhd \mathcal{K}$ if and only if*

$$\overline{A}^{\mathcal{B}} \subseteq U \text{ for every } U \in \mathcal{K} \text{ such that } A \subseteq U.$$

The following lemma presents the basic properties of families of the form $\mathcal{B} \lhd \mathcal{K}$.

Lemma 5. *In reference [9], the following conditions hold for any family $\mathcal{B} \lhd \mathcal{K} \in \Gamma(X)$ and any subset $A \subseteq X$:*

(i) *If $\mathcal{K} \subseteq \mathcal{B}$, then $\mathcal{B} \lhd \mathcal{K} = \mathcal{P}(X)$;*
(ii) *$\mathcal{B} \lhd \mathcal{K} \subseteq \mathcal{B}^* \lhd \mathcal{K}$ for any $\mathcal{B}^* \subseteq \mathcal{P}(X)$ such that $\mathcal{B} \subseteq \mathcal{B}^*$;*
(iii) *$\mathcal{B} \lhd \mathcal{K} \supset \mathcal{B} \lhd \mathcal{K}^*$ for any $\mathcal{K}^* \subseteq \mathcal{P}(X)$ such that $\mathcal{K} \subseteq \mathcal{K}^*$;*
(iv) *$\overline{\mathcal{B}} \subseteq \mathcal{B} \lhd \mathcal{K}$;*
(v) *$\overline{\mathcal{B}} = \bigcap\{\mathcal{B} \lhd \mathcal{K} : \mathcal{K} \subseteq \mathcal{P}(X)\} = \mathcal{B} \lhd \mathcal{P}(X)$;*
(vi) *If $\mathcal{B} \subseteq \mathcal{K}$, then $A \in \mathcal{B} \lhd \mathcal{K}$ if and only if $\overline{A}^{\mathcal{B}} = \overline{A}^{\mathcal{K}}$;*
(vii) *$\mathcal{B} \lhd \mathcal{K} = \overline{\mathcal{B}} \lhd \mathcal{K} = \mathcal{B} \lhd \overline{\mathcal{K}} = \overline{\mathcal{B}} \lhd \overline{\mathcal{K}}$ for all $\mathcal{K} \subseteq \mathcal{P}(X)$.*

Theorem 1. *In reference [9] , for any family $\mathcal{K} \subseteq \mathcal{P}(X)$ and a point $x \in X$, the following statements hold:*

(i) *$X \setminus \{x\} \in \mathcal{K}$ or $X \setminus \{x\} \in \mathcal{B} \lhd \mathcal{K}$ for any family $\mathcal{B} \subseteq \mathcal{P}(X)$.*
(ii) *$X \setminus \{x\} \in \overline{\mathcal{K}}$ or $X \setminus \{x\} \in \overline{\mathcal{B}}$ for every $x \in X$, implies that $\mathcal{B} \lhd \mathcal{K} = \overline{\mathcal{B}}$.*

3. Topologies Determined by Families via the Closure Operators

We would like to indicate the sufficient assumptions for the families $\mathcal{B} \subseteq \mathcal{P}(X)$ that imply that the operators $\overline{(\dots)}^{\mathcal{B}}$ determine topologies. As Lemma 3 says, any family of type $\overline{\mathcal{B}}$ is always closed under arbitrary intersections. So, first let us consider the assumption that the families of type $\overline{\mathcal{B}}$ are closed under finite unions (of course, then the families $\mathcal{B}$ do not have to have this property). Then, it turns out that this assumption is sufficient, and what is more, it is necessary as the theorem below shows. Next, in Lemma 6, we will consider the assumption that the families of type $\overline{\mathcal{B}}$ are closed under arbitrary unions.

Theorem 2. *For any family $\mathcal{B} \subseteq \mathcal{P}(X)$, the following properties are equivalent:*
(i) $\overline{\mathcal{B}}$ *is closed under finite unions;*
(ii) $\overline{(...)}^{\mathcal{B}} : \mathcal{P}(X) \to \mathcal{P}(X)$ *is a Kuratowski closure operator.*

Proof. $(i) \Rightarrow (ii)$ It is enough to prove that, for all subsets A_1, A_2 of X, $\overline{A_1 \cup A_2}^{\mathcal{B}} \subseteq \overline{A_1}^{\mathcal{B}} \cup \overline{A_2}^{\mathcal{B}}$. Assume to the contrary that $x \in \overline{A_1 \cup A_2}^{\mathcal{B}}$, $x \notin \overline{A_1}^{\mathcal{B}}$ and $x \notin \overline{A_2}^{\mathcal{B}}$ for some $x \in X$. According to Remark 4 (i), it is equivalent to $x \notin \overline{A_1}^{\overline{\mathcal{B}}}$ and $\notin \overline{A_2}^{\overline{\mathcal{B}}}$. Then, based on Lemma 2, there exist $U_1 \in (\overline{\mathcal{B}})^{[c]}$ and $U_2 \in (\overline{\mathcal{B}})^{[c]}$ such that $x \in U_1 \cap U_2$, $U_1 \cap A_1 = \emptyset$ and $U_2 \cap A_2 = \emptyset$. Consequently, $U_1 \cap U_2 \cap A_1 = \emptyset$ and $U_1 \cap U_2 \cap A_2 = \emptyset$. So, $U_1 \cap U_2 \cap (A_1 \cup A_2) = \emptyset$. Because of the assumption about $\overline{\mathcal{B}}$, the family $(\overline{\mathcal{B}})^{[c]}$ is closed under finite intersection, therefore $U_1 \cap U_2 \in (\overline{\mathcal{B}})^{[c]}$. Consequently, using again the Lemma 2, we have proven that $x \notin \overline{A_1 \cup A_2}^{\overline{\mathcal{B}}}$, i.e., according to Remark 4 (i), we obtain $x \notin \overline{A_1 \cup A_2}^{\mathcal{B}}$ which gives a contradiction.

To prove the converse implication, let $A_1, A_2 \in \overline{\mathcal{B}}$. We will show that $A_1 \cup A_2 \in \overline{\mathcal{B}}$, i.e., $\overline{A_1 \cup A_2} = A_1 \cup A_2$. Since $\overline{A_1}^{\mathcal{B}} = A_1$ and $\overline{A_2}^{\mathcal{B}} = A_2$, assuming (ii), i.e., $\overline{A_1 \cup A_2}^{\mathcal{B}} = \overline{A_1}^{\mathcal{B}} \cup \overline{A_2}^{\mathcal{B}}$, we obtain $\overline{A_1 \cup A_2}^{\mathcal{B}} = A_1 \cup A_2$. $\square$

The following corollary follows immediately from the above theorem.

Corollary 1. *For any family $\mathcal{B} \subseteq \mathcal{P}(X)$, τ is a topology generated by the operator $\overline{(\dots)}^{\mathcal{B}}$ if and only if $\tau = (\overline{\mathcal{B}})^{[c]}$. Of course, $\overline{\mathcal{B}}$ is the family of all closed sets in the topological space (X, τ).*

Remark 7. *If a family $\mathcal{B} \subseteq \mathcal{P}(X)$ is closed under arbitrary unions, then according to Lemmas 3 and 4, the families $\overline{\mathcal{B}}$, $(\overline{\mathcal{B}})^{[c]}$ are closed under arbitrary unions and arbitrary intersections. So, they are topologies.*

The assumption that the family $\overline{\mathcal{B}}$ is closed under arbitrary unions, i.e., a stronger assumption than in the above result leads to the following lemma. Before we state it, let us recall what we mean by an Alexandroff topology if it is closed under arbitrary intersections [19].

Lemma 6. *For any family $\mathcal{B} \subseteq \mathcal{P}(X)$, the following properties are equivalent:*
(i) $\overline{\mathcal{B}}$ *is closed under arbitrary unions;*
(ii) $\overline{\mathcal{B}}$ *is an Alexandroff topology;*
(iii) $(\overline{\mathcal{B}})^{[c]}$ *is an Alexandroff topology.*

Proof. $(i) \Rightarrow (ii)$. Since $\overline{\mathcal{B}}$ is closed under arbitrary unions, then according to Lemma 3, it is additionally closed under arbitrary intersections. So $\overline{\mathcal{B}}$ is an Alexandroff topology. Of course, the family $(\overline{\mathcal{B}})^{[c]}$ also satisfies those two properties. So, (ii) implies (iii). Now, let us note that since $\left((\overline{\mathcal{B}})^{[c]}\right)^{[c]} = \overline{\mathcal{B}}$, the property *(iii)* implies that $\overline{\mathcal{B}}$ is closed under arbitrary unions, i.e., the property (i). $\square$

Since, according to Lemma 3, for any $\mathcal{B} \subseteq \mathcal{P}(X)$ the family $\overline{\mathcal{B}}$ is closed under arbitrary intersections, we know that the family $(\overline{\mathcal{B}})^{[c]}$ is closed under arbitrary unions. Then, by Lemma 4, the family $\overline{(\overline{\mathcal{B}})^{[c]}}$ is also closed under arbitrary unions and of course, is closed under arbitrary intersections. So, the above lemma implies the following corollary.

Corollary 2. *For any family $\mathcal{B} \subseteq \mathcal{P}(X)$, the family $\overline{(\overline{\mathcal{B}})^{[c]}}$ and equivalently $\left(\overline{(\overline{\mathcal{B}})^{[c]}}\right)^{[c]}$ is an Alexandroff topology.*

Remark 8. *In summarizing, let us pay attention to the fact that there are three cases of topologies determined by a family $\mathcal{B} \subseteq \mathcal{P}(X)$ via the closure operators $\overline{(...)}^{\mathcal{B}}$, namely*

(i) If $\overline{\mathcal{B}}$ is closed under finite unions, we have the topology $(\overline{\mathcal{B}})^{[c]}$.

(ii) If $\overline{\mathcal{B}}$ is closed under arbitrary unions, we have two Alexandroff topologies $\overline{\mathcal{B}}$ and $(\overline{\mathcal{B}})^{[c]}$.

(iii) Without any assumptions about the family $\overline{\mathcal{B}}$, we have the following two Alexandroff topologies $\overline{(\overline{\mathcal{B}})^{[c]}}$ and $\left(\overline{(\overline{\mathcal{B}})^{[c]}}\right)^{[c]}$.

4. Topologies Related to Generalized Closed Sets the Levine-Type Generalizations of Closedness

4.1. Topologies Specified by the Families of Type $\mathcal{B} \triangleleft \mathcal{K}$

The generalizations of type $\mathcal{B} \triangleleft \mathcal{K}$, given in Definition 3, retain some properties of $\mathcal{B}$ which we used in Theorem 2 and Lemma 6. We will show it in the following lemma and Theorem 3.

Lemma 7. *For any family $\mathcal{B} \subseteq \mathcal{P}(X)$, $\overline{\mathcal{B}}$ is closed under finite (arbitrary) unions if and only if the family $\mathcal{B} \triangleleft \mathcal{K}$ is closed under finite (arbitrary) unions for any $\mathcal{K} \subseteq \mathcal{P}(X)$.*

Proof. Let us assume that $A_1, A_2 \in \mathcal{B} \triangleleft \mathcal{K}$. We will prove that $A_1 \cup A_2 \in \mathcal{B} \triangleleft \mathcal{K}$, i.e., $\overline{A_1 \cup A_2}^{\mathcal{B}} \subseteq \overline{A_1 \cup A_2}^{\mathcal{K}}$. Because of the assumption, we have $\overline{A_1}^{\mathcal{B}} \subseteq \overline{A_1}^{\mathcal{K}}$ and $\overline{A_2}^{\mathcal{B}} \subseteq \overline{A_2}^{\mathcal{K}}$. Consequently, $\overline{A_1}^{\mathcal{B}} \cup \overline{A_2}^{\mathcal{B}} \subseteq \overline{A_1}^{\mathcal{K}} \cup \overline{A_2}^{\mathcal{K}}$ and, since $\overline{A_1}^{\mathcal{K}} \cup \overline{A_2}^{\mathcal{K}} \subseteq \overline{A_1 \cup A_2}^{\mathcal{K}}$, using Theorem 2, we have $\overline{A_1 \cup A_2}^{\mathcal{B}} \subseteq \overline{A_1 \cup A_2}^{\mathcal{K}}$.

Now, let us take a family $\{A_s\}_{s \in S} \subseteq \mathcal{B} \triangleleft \mathcal{K}$. Then, $\overline{A_s}^{\mathcal{B}} \subseteq \overline{A_s}^{\mathcal{K}}$ for all $s \in S$, and consequently, $\bigcup\limits_{s \in S} \overline{A_s}^{\mathcal{B}} \subseteq \bigcup\limits_{s \in S} \overline{A_s}^{\mathcal{K}} \subseteq \overline{\bigcup\limits_{s \in S} A_s}^{\mathcal{K}}$.

Thus, it is enough to show that $\overline{\bigcup\limits_{s \in S} A_s}^{\mathcal{B}} \subseteq \bigcup\limits_{s \in S} \overline{A_s}^{\mathcal{B}}$. Let us assume to the contrary that $x \in \overline{\bigcup\limits_{s \in S} A_s}^{\mathcal{B}}$ and $x \notin \bigcup\limits_{s \in S} \overline{A_s}^{\mathcal{B}}$ for some $x \in X$. Then, for every $s \in S$, $x \notin \overline{A_s}^{\mathcal{B}}$ or equivalently by Remark 4 (i), $x \notin \overline{A_s}^{\overline{\mathcal{B}}}$. According to Lemma 2, for every $s \in S$, there exists $U_s \in (\overline{\mathcal{B}})^{[c]}$ such that $x \in U_s$ and $A_s \subseteq X \setminus U_s$. Consequently, $\bigcup\limits_{s \in S} A_s \subseteq \bigcup\limits_{s \in S} X \setminus U_s = X \setminus \bigcap\limits_{s \in S} U_s$ and of course $x \in \bigcap\limits_{s \in S} U_s$.

Since $U_s \in (\overline{\mathcal{B}})^{[c]}$, i.e., $X \setminus U_s \in \overline{\mathcal{B}}$ for every $s \in S$, then by the assumption, $\bigcup (X \setminus U_s) \in \overline{\mathcal{B}}$ or equivalently $X \setminus \bigcap\limits_{s \in S} U_s \in \overline{\mathcal{B}}$. This finishes the proof that $\bigcap\limits_{s \in S} U_s \in (\overline{\mathcal{B}})^{[c]}$ and $\left(\bigcup\limits_{s \in S} A_s\right) \cap \left(\bigcap\limits_{s \in S} U_s\right) = \emptyset$, so $x \notin \overline{\bigcup\limits_{s \in S} A_s}^{\overline{\mathcal{B}}}$ or equivalently $x \notin \overline{\bigcup\limits_{s \in S} A_s}^{\mathcal{B}}$ which gives a contradiction and finishes the proof of the Lemma.

To prove the converse implication it is enough to use Lemma 5 (v). □

According to Lemma 4, the above lemma implies the following.

Theorem 3. *For any family $\mathcal{B} \subseteq \mathcal{P}(X)$ the following properties are equivalent:*

(i) $\overline{\mathcal{B}}$ *is closed under finite unions;*

(ii) $\overline{(\ldots)}^{\mathcal{B} \triangleleft \mathcal{K}} : \mathcal{P}(X) \to \mathcal{P}(X)$ *is a Kuratowski closure operator for any* $\mathcal{K} \subseteq \mathcal{P}(X)$.

Proof. $(i) \Rightarrow (ii)$ Let us take a family $\mathcal{B} \triangleleft \mathcal{K}$ for some $\mathcal{K} \subseteq \mathcal{P}(X)$, such that $\overline{\mathcal{B}}$ is closed under finite unions. Lemma 7 implies that $\mathcal{B} \triangleleft \mathcal{K}$ is also closed under finite unions, and consequently, by Lemma 4, the family $\overline{\mathcal{B} \triangleleft \mathcal{K}}$ is closed under finite unions too. So, according to Theorem 2, this condition is equivalent to the fact that the function $\overline{(\ldots)}^{\mathcal{B} \triangleleft \mathcal{K}} : \mathcal{P}(X) \to \mathcal{P}(X)$ is a Kuratowski closure operator.

To prove the converse implication, we use the family $\mathcal{K} = \mathcal{P}(X)$ in the assumption (*ii*). Since, according to Lemma 5 (*v*), $\mathcal{B} \triangleleft \mathcal{K} = \overline{\mathcal{B}}$, we know that $\overline{(\ldots)}^{\overline{\mathcal{B}}}$ is a Kuratowski closure operator. Consequently, by Remark 4 (*i*), the function $\overline{(\ldots)}^{\mathcal{B}}$ is a Kuratowski closure operator, and using Theorem 2, we obtain the property (i). □

Theorems 2 and 3 imply the following.

Corollary 3. *The following properties are equivalent for any family $\mathcal{B} \subseteq \mathcal{P}(X)$:*

(i) $\overline{\mathcal{B}}$ *is closed under finite unions;*

(ii) $\overline{\mathcal{B} \triangleleft \mathcal{K}}$ *is closed under finite unions for all* $\mathcal{K} \subseteq \mathcal{P}(X)$*;*

(iii) $\overline{(\ldots)}^{\mathcal{B}} : \mathcal{P}(X) \to \mathcal{P}(X)$ *is a Kuratowski closure operator;*

(iv) $\overline{(\ldots)}^{\mathcal{B} \triangleleft \mathcal{K}} : \mathcal{P}(X) \to \mathcal{P}(X)$ *is a Kuratowski closure operator for any* $\mathcal{K} \subseteq \mathcal{P}(X)$.

Directly from the above theorem and Lemma 4, we obtain the following.

Corollary 4. *If $\mathcal{B} \subseteq \mathcal{P}(X)$ is a family closed under finite unions, then the function $\overline{(\ldots)}^{\mathcal{B} \triangleleft \mathcal{K}}$: $\mathcal{P}(X) \to \mathcal{P}(X)$ is a Kuratowski closure operator for any $\mathcal{K} \subseteq \mathcal{P}(X)$.*

From Lemmas 7 and 4, and Corollary 1, we have the following.

Corollary 5. *For any family $\mathcal{B} \subseteq \mathcal{P}(X)$, $\overline{\mathcal{B}}$ is closed under finite unions if and only if the family $\left(\overline{\mathcal{B} \triangleleft \mathcal{K}}\right)^{[c]}$ is a topology for any $\mathcal{K} \subseteq \mathcal{P}(X)$.*

In the case of the assumption that the family is closed under arbitrary unions, we obtain the following corollary directly from Lemmas 6 and 7.

Corollary 6. *For any family $\mathcal{B} \subseteq \mathcal{P}(X)$ such that $\overline{\mathcal{B}}$ is closed under arbitrary unions, the family $\overline{\mathcal{B} \triangleleft \mathcal{K}}$ (equivalently $\left(\overline{\mathcal{B} \triangleleft \mathcal{K}}\right)^{[c]}$) is an Alexandroff topology for any $\mathcal{K} \subseteq \mathcal{P}(X)$.*

Since, for any family $\mathcal{B} \subseteq \mathcal{P}(X)$, according to Lemma 3, the family $\overline{\mathcal{B}}$ is closed under arbitrary intersections, then the family $(\overline{\mathcal{B}})^{[c]}$ is closed under arbitrary unions. Consequently, by Lemma 4, the family $\overline{(\overline{\mathcal{B}})^{[c]}}$ is also closed under arbitrary unions, and applying Corollary 6, we obtain the following corollary.

Corollary 7. *For any family $\mathcal{B} \subseteq \mathcal{P}(X)$, the family $\overline{(\overline{\mathcal{B}})^{[c]} \triangleleft \mathcal{K}}$ (equivalently $\left(\overline{(\overline{\mathcal{B}})^{[c]} \triangleleft \mathcal{K}}\right)^{[c]}$) is an Alexandroff topology for any $\mathcal{K} \subseteq \mathcal{P}(X)$.*

Remark 9. *In summarizing the results described in Corollaries 5, 6 and 7, analogously as in Remark 8, let us note some facts:*

(i) *In the case of the family $\mathcal{B}$ for that $\overline{\mathcal{B}}$ is closed under finite unions, we have the topology* $\left(\overline{\mathcal{B} \triangleleft \mathcal{K}}\right)^{[c]}$.

(ii) *If $\overline{\mathcal{B}}$ is closed under arbitrary unions, we have two Alexandroff topologies $\overline{\mathcal{B} \triangleleft \mathcal{K}}$ and* $\left(\overline{\mathcal{B} \triangleleft \mathcal{K}}\right)^{[c]}$.

(iii) *Without any assumptions about the family $\overline{\mathcal{B}}$, we have the following two Alexandroff topologies* $\overline{\left(\overline{\mathcal{B}}\right)^{[c]} \triangleleft \mathcal{K}}$ *and* $\left(\overline{\left(\overline{\mathcal{B}}\right)^{[c]} \triangleleft \mathcal{K}}\right)^{[c]}$.

4.2. The Results about $T_{\frac{1}{2}}$ Separation Axiom

The type of topologies considered in Corollary 3 above are strictly related to $T_{\frac{1}{2}}$ separation axiom as we will show in Theorem 4 below. So, let us recall the definition of $T_{\frac{1}{2}}$.

Definition 4. *In reference [1], a topological space (X, τ) is $T_{\frac{1}{2}}$ if every general closed set is closed, i.e., according in our convention, $\tau^{[c]} = \tau^{[c]} \triangleleft \tau$.*

Remark 10. *Dunham proves in [20] (Theorem 2.6) that the property $T_{\frac{1}{2}}$ is equivalent to every singleton in X being either open or closed.*

Let us note that, in general, for any family $\mathcal{B} \subseteq \mathcal{P}(X)$ such that $\overline{\mathcal{B}}$ is closed under finite unions, according to Corollary 1, we can consider the topology $\left(\overline{\mathcal{B}}\right)^{[c]}$ and, according to Corollary 5, we have the family of topologies $\left(\overline{\mathcal{B} \triangleleft \mathcal{K}}\right)^{[c]}$, for arbitrary $\mathcal{K} \subseteq \mathcal{P}(X)$. We will be interested in the topologies of this type, where $\mathcal{K} \subseteq \left(\overline{\mathcal{B}}\right)^{[c]}$. For convenience, we will use the following notation:

$$T(\mathcal{B}) = \{\left(\overline{\mathcal{B} \triangleleft \mathcal{K}}\right)^{[c]} : \mathcal{K} \subseteq \left(\overline{\mathcal{B}}\right)^{[c]}\}.$$

Theorem 4. *For any family $\mathcal{B} \subseteq \mathcal{P}(X)$ such that $\overline{\mathcal{B}}$ is closed under finite unions, the followings hold:*

(i) *Every topology from the family $T(\mathcal{B})$ satisfies the axiom $T_{\frac{1}{2}}$.*

(ii) $\left(\overline{\mathcal{B}}\right)^{[c]} \subseteq \bigcap T(\mathcal{B}) = \left(\overline{\mathcal{B} \triangleleft \left(\overline{\mathcal{B}}\right)^{[c]}}\right)^{[c]}$.

(iii) *The topology $\left(\overline{\mathcal{B}}\right)^{[c]}$ satisfies the axiom $T_{\frac{1}{2}}$ if and only if* $\left(\overline{\mathcal{B}}\right)^{[c]} = \left(\overline{\mathcal{B} \triangleleft \left(\overline{\mathcal{B}}\right)^{[c]}}\right)^{[c]}$.

Proof. We will show that for each $x \in X$ and any $\mathcal{K} \subseteq \mathcal{P}(X)$, either $\{x\} \in \left(\overline{\mathcal{B} \triangleleft \mathcal{K}}\right)^{[c]}$ or $\{x\} \in \overline{\mathcal{B} \triangleleft \mathcal{K}}$. So, let us assume that $\{x_0\} \notin \left(\overline{\mathcal{B} \triangleleft \mathcal{K}}\right)^{[c]}$ for some $x_0 \in X$ and $\mathcal{K} \subseteq \left(\overline{\mathcal{B}}\right)^{[c]}$. Then, $X \setminus \{x_0\} \notin \overline{\mathcal{B} \triangleleft \mathcal{K}}$, consequently $X \setminus \{x_0\} \notin \mathcal{B} \triangleleft \mathcal{K}$, and according to Theorem 1, we have $X \setminus \{x_0\} \in \mathcal{K}$. So, $X \setminus \{x_0\} \in \left(\overline{\mathcal{B}}\right)^{[c]}$, i.e., $\{x_0\} \in \overline{\mathcal{B}}$, and then by Lemma 5 (*iv*), we obtain $\{x_0\} \in \overline{\mathcal{B} \triangleleft \mathcal{K}}$ which finishes the proof (*i*).

Now, it is clear that, by Lemma 5 (*iv*) and Remark 4 (*ii*), we have $\left(\overline{\mathcal{B}}\right)^{[c]} \subseteq \left(\overline{\mathcal{B} \triangleleft \mathcal{K}}\right)^{[c]}$ for any $\mathcal{K} \subseteq \mathcal{P}(X)$. So,

$$\left(\overline{\mathcal{B}}\right)^{[c]} \subseteq \left(\overline{\mathcal{B} \triangleleft \left(\overline{\mathcal{B}}\right)^{[c]}}\right)^{[c]}. \qquad (\star)$$

Since, using (*iii*) from the same Lemma, for all $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{P}(X)$, the inclusion $\mathcal{K}_1 \subseteq \mathcal{K}_2$ implies $\left(\overline{\mathcal{B} \triangleleft \mathcal{K}_1}\right)^{[c]} \supset \left(\overline{\mathcal{B} \triangleleft \mathcal{K}_2}\right)^{[c]}$, and we obtain $\left(\overline{\mathcal{B} \triangleleft \mathcal{K}}\right)^{[c]} \supset \left(\overline{\mathcal{B} \triangleleft \left(\overline{\mathcal{B}}\right)^{[c]}}\right)^{[c]}$ for all $\mathcal{K} \subseteq \left(\overline{\mathcal{B}}\right)^{[c]}$, i.e., $\bigcap T(\mathcal{B}) \supset \left(\overline{\mathcal{B} \triangleleft \left(\overline{\mathcal{B}}\right)^{[c]}}\right)^{[c]}$.

Additionally, $\left(\overline{\mathcal{B} \triangleleft \left(\overline{\mathcal{B}}\right)^{[c]}}\right)^{[c]} \in T(\mathcal{B})$, so we have proven the equality:

$\left(\overline{\mathcal{B}\triangleleft(\overline{\mathcal{B}})^{[c]}}\right)^{[c]} = \bigcap T(\mathcal{B})$, and according to $(\star)$, we obtain (ii).

For the proof of (iii), assume that $\left(X, (\overline{\mathcal{B}})^{[c]}\right)$ is $T_{\frac{1}{2}}$, i.e., every general closed sets is closed or shorter $\overline{\mathcal{B}} = \overline{\mathcal{B}} \triangleleft (\overline{\mathcal{B}})^{[c]}$. According to Lemma 5 (vii) and Remark 4 (ii), we have $\overline{\mathcal{B}} = \overline{\mathcal{B}} \triangleleft (\overline{\mathcal{B}})^{[c]} = \mathcal{B} \triangleleft (\overline{\mathcal{B}})^{[c]} = \overline{\mathcal{B} \triangleleft (\overline{\mathcal{B}})^{[c]}}$ and consequently $(\overline{\mathcal{B}})^{[c]} = \left(\overline{\mathcal{B}\triangleleft(\overline{\mathcal{B}})^{[c]}}\right)^{[c]}$. The inverse implication follows directly from (i) and the fact that $(\overline{\mathcal{B} \triangleleft \mathcal{K}})^{[c]} \in T(\mathcal{B})$, where we can take $\mathcal{K} = (\overline{\mathcal{B}})^{[c]}$. □

In summarizing, let us note that by starting from a family $\mathcal{B}$ such that $\overline{\mathcal{B}}$ is closed under finite unions, we obtain the topology $(\overline{\mathcal{B}})^{[c]}$ as is shown in Corollary 1. Next, we use the family $(\overline{\mathcal{B}})^{[c]}$ to construct the topology $\left(\overline{\mathcal{B}\triangleleft(\overline{\mathcal{B}})^{[c]}}\right)^{[c]}$. It begs the question: does the repetition of this operation on the topology $\left(\overline{\mathcal{B}\triangleleft(\overline{\mathcal{B}})^{[c]}}\right)^{[c]}$ lead to the other topologies? It turns out that it does not. Namely, we have the following.

Remark 11. *For any $\mathcal{B} \subseteq \mathcal{P}(X)$ such that $\overline{\mathcal{B}}$ is closed under finite unions, we have*

$$\left(\overline{\left(\mathcal{B}\triangleleft(\overline{\mathcal{B}})^{[c]}\right)\triangleleft\left(\overline{\mathcal{B}\triangleleft(\overline{\mathcal{B}})^{[c]}}\right)^{[c]}}\right)^{[c]} = \left(\overline{\mathcal{B}\triangleleft(\overline{\mathcal{B}})^{[c]}}\right)^{[c]}$$

Indeed, the family of all general closed sets in the topological space $\left(X, \left(\overline{\mathcal{B}\triangleleft(\overline{\mathcal{B}})^{[c]}}\right)^{[c]}\right)$ is equal to $\left(\mathcal{B}\triangleleft(\overline{\mathcal{B}})^{[c]}\right)\triangleleft\left(\overline{\mathcal{B}\triangleleft(\overline{\mathcal{B}})^{[c]}}\right)^{[c]}$. Since, the topology $\left(\overline{\mathcal{B}\triangleleft(\overline{\mathcal{B}})^{[c]}}\right)^{[c]}$ satisfies the axiom $T_{\frac{1}{2}}$, then

$$\left(\mathcal{B}\triangleleft(\overline{\mathcal{B}})^{[c]}\right)\triangleleft\left(\overline{\mathcal{B}\triangleleft(\overline{\mathcal{B}})^{[c]}}\right)^{[c]} = \mathcal{B}\triangleleft(\overline{\mathcal{B}})^{[c]}.$$

Consequently,

$$\overline{\left(\mathcal{B}\triangleleft(\overline{\mathcal{B}})^{[c]}\right)\triangleleft\left(\overline{\mathcal{B}\triangleleft(\overline{\mathcal{B}})^{[c]}}\right)^{[c]}} = \overline{\mathcal{B}\triangleleft(\overline{\mathcal{B}})^{[c]}}$$

and finally,

$$\left(\overline{\left(\mathcal{B}\triangleleft(\overline{\mathcal{B}})^{[c]}\right)\triangleleft\left(\overline{\mathcal{B}\triangleleft(\overline{\mathcal{B}})^{[c]}}\right)^{[c]}}\right)^{[c]} = \left(\overline{\mathcal{B}\triangleleft(\overline{\mathcal{B}})^{[c]}}\right)^{[c]}$$

Corollary 8. *A topological space (X, τ) is $T_{\frac{1}{2}}$ if and only if there exists a family $\mathcal{B} \subseteq \mathcal{P}(X)$ such that $\overline{\mathcal{B}}$ is closed under finite unions and $\tau \in T(\mathcal{B})$.*

Proof. For a topology τ that satisfies the property $T_{\frac{1}{2}}$, by definition, we have $\tau^{[c]} = \tau^{[c]} \triangleleft \tau$. So, it is enough to take $\mathcal{B} = \tau^{[c]}$ and $\mathcal{K} = \tau$ and then, $\mathcal{B} = \mathcal{B} \triangleleft \mathcal{K}$, so $\overline{\mathcal{B}} = \overline{\mathcal{B} \triangleleft \mathcal{K}}$. It implies that $(\overline{\mathcal{B}})^{[c]} = (\overline{\mathcal{B} \triangleleft \mathcal{K}})^{[c]}$, and since $\overline{\mathcal{B}} = \mathcal{B}$, we have $\tau = (\overline{\mathcal{B} \triangleleft \mathcal{K}})^{[c]}$. The inverse implication follows immediately from Theorem 4. □

If $\mathcal{B} \subseteq \mathcal{P}(X)$ is a family such that $\overline{\mathcal{B}}$ is closed under arbitrary unions then, according to Remark 8 (i), $(\overline{\mathcal{B}})^{[c]}$ is a topology. However, in the case when $\overline{\mathcal{B}}$ is closed under arbitrary union, we have the two topologies $\overline{\mathcal{B}}$ and $(\overline{\mathcal{B}})^{[c]}$.

Remark 12. *The topologies* $\left(\overline{\mathcal{B}}\right)^{[c]}$ *and* $\overline{\mathcal{B}}$ *are closely related to the separation axiom* $T_{\frac{1}{2}}$*. Namely, since the family of all closed subsets in the space* $(X, \overline{\mathcal{B}})$ *is equal to the family of all open subsets in the space* $\left(X, \left(\overline{\mathcal{B}}\right)^{[c]}\right)$*, directly from the definition of* $T_{\frac{1}{2}}$*, for any family* $\mathcal{B} \subseteq \mathcal{P}(X)$ *such that* $\overline{\mathcal{B}}$ *is a topology, the following conditions are equivalent:*

(i) $\overline{\mathcal{B}}$ *is* $T_{\frac{1}{2}}$*;*

(ii) $\left(\overline{\mathcal{B}}\right)^{[c]}$ *is* $T_{\frac{1}{2}}$*.*

The below result gives some sufficient conditions to satisfies the property $T_{\frac{1}{2}}$ of the topologies $\overline{\mathcal{B}}$ and $\left(\overline{\mathcal{B}}\right)^{[c]}$.

Theorem 5. *Let* $\mathcal{B} \subseteq \mathcal{P}(X)$ *be a family such that* $\overline{\mathcal{B}}$ *is closed under arbitrary unions and* $\mathcal{B}^{[c]} = \mathcal{B}^{[c]} \triangleleft \mathcal{B}$*, then the topological spaces* $(X, \overline{\mathcal{B}})$ *and* $\left(X, \left(\overline{\mathcal{B}}\right)^{[c]}\right)$ *are* $T_{\frac{1}{2}}$*.*

Proof. According to Theorem 1 (*i*), the assumption implies that $X \setminus \{x\} \in \mathcal{B}$ or $X \setminus \{x\} \in \mathcal{B}^{[c]}$ for all $x \in X$. Consequently, for every $x \in X$, we have

$$X \setminus \{x\} \in \overline{\mathcal{B}} \text{ or } X \setminus \{x\} \in \left(\overline{\mathcal{B}}\right)^{[c]} \qquad (\star)$$

since $\mathcal{B} \subseteq \overline{\mathcal{B}}$ by Remark 1. Because the family $\overline{\mathcal{B}}$ is closed under arbitrary unions and simultaneously, by Lemma 3, it is closed under arbitrary intersections, the family $\left(\overline{\mathcal{B}}\right)^{[c]}$ satisfies also these two conditions, and hence, according to Remark 2, we have $\overline{\left(\overline{\mathcal{B}}\right)^{[c]}} = \left(\overline{\mathcal{B}}\right)^{[c]}$. Then, the property $(\star)$ can be formulated as $X \setminus \{x\} \in \overline{\mathcal{B}}$ or $X \setminus \{x\} \in \overline{\left(\overline{\mathcal{B}}\right)^{[c]}}$. Now, using Theorem 1 (*ii*), where we take $\mathcal{B}$ instead of $\mathcal{K}$ and $\left(\overline{\mathcal{B}}\right)^{[c]}$ instead of $\mathcal{B}$, we obtain $\left(\overline{\mathcal{B}}\right)^{[c]} \triangleleft \mathcal{B} = \overline{\left(\overline{\mathcal{B}}\right)^{[c]}} = \left(\overline{\mathcal{B}}\right)^{[c]}$, but according to Lemma 5 (vii), $\left(\overline{\mathcal{B}}\right)^{[c]} \triangleleft \mathcal{B} = \left(\overline{\mathcal{B}}\right)^{[c]} \triangleleft \overline{\mathcal{B}}$. So, the topological space $(X, \overline{\mathcal{B}})$ is $T_{\frac{1}{2}}$.

If we take $\left(\overline{\mathcal{B}}\right)^{[c]}$ instead of $\mathcal{K}$ in Theorem 1 (*ii*), then we obtain $\mathcal{B} \triangleleft \left(\overline{\mathcal{B}}\right)^{[c]} = \overline{\mathcal{B}} \triangleleft \left(\overline{\mathcal{B}}\right)^{[c]} = \overline{\mathcal{B}}$. This proves that the topological space $\left(X, \left(\overline{\mathcal{B}}\right)^{[c]}\right)$ is $T_{\frac{1}{2}}$. □

5. Some Well-Known Particular Cases

In the concept presented above, we have used as the basic notion the closure operator given in [16] that does not need to be the Kuratowski closure operator. In the applications of those operators, we use the families which designate them.

In this chapter, we use the general concept for the case of the Kuratowski closure operator so in the topological spaces. Given a topological space (X, τ), we investigate the two procedures of a generalization of τ. The first, initiated by [20], uses the notion of generalized closed sets, where the author has introduced in this way topology τ^*, which is bigger than τ. In the second procedure, in [21], the family of the topology τ is used as a family designated by the closure operator that designates the family of subsets called Λ-sets, which is also a topology. In this chapter, we will show that those two procedures are some particular cases of the general concept introduced above.

5.1. Topologies Defined in Terms of Generalized Closed Sets

Given a topological space (X, τ), let us denote by $\tau^{[c]}$ the family of all closed subsets of X. Then, $\tau^{[c]} \triangleleft \tau$ is the family of all general closed sets in (X, τ) (see Remark 5). Of course, the family $\tau^{[c]}$ is closed under all finite unions, and according to Remark 1, we have $\overline{\tau^{[c]}} = \tau^{[c]}$.

Lemma 6 implies that any family of the form $\tau^{[c]} \triangleleft \mathcal{K}$, where $\mathcal{K} \subseteq \mathcal{P}(X)$, is closed under finite unions. So, as a direct corollary of this lemma, where $\mathcal{K} = \tau$, we obtain the following well-known property of the family of generalized closed sets.

Theorem 6. *In reference [1], the union of two general closed sets is general closed.*

In [20], (Definition 3.1), the family $\mathcal{C} \triangleleft \tau$ is denoted by $\mathcal{D}$. Then, the operator c^* is defined by $c^*(A) = \bigcap\{A : E \subseteq A \in \mathcal{D}\}$ for $A \subseteq X$, i.e., according to our convention, by $c^*(A) = \overline{A}^{\mathcal{C}\triangleleft\tau}$. So, Theorem 3 implies the following result.

Corollary 9. *In reference [20] (Theorem 3.5), the operator c^* is a Kuratowski closure operator on X.*

The author in [20] (Definition 3.6) had defined the topology $\tau^* = \{U \subseteq X : c^*(X \setminus U) = X \setminus U\}$ generated by the operator c^*. Let us note that according to Corollary 3, we have the following:

Remark 13. *For any topological space (X, τ), the following interpretation holds:*

$$\tau^* = (\overline{\mathcal{C} \triangleleft \tau})^{[c]}.$$

Indeed, $U \in \tau^*$ means, by definition, that $c^*(X \setminus U) = X \setminus U$ which is equivalent to the equality $\overline{X \setminus U}^{\mathcal{C}\triangleleft\tau} = X \setminus U$. By Definition 2, we have $X \setminus U \in \overline{\mathcal{C} \triangleleft \tau}$, i.e., $U \in (\overline{\mathcal{C} \triangleleft \tau})^{[c]}$.

It is clear that every topology τ on X can be described by $\tau = (\overline{\mathcal{B}})^{[c]}$ where $\overline{\mathcal{B}}$ is closed under finite unions, it is enough to take $\mathcal{B} = \mathcal{C}$. So, according to the above Remark and having regard to Remark 1, we have

$$\tau^* = \left(\overline{\mathcal{B} \triangleleft (\overline{\mathcal{B}})^{[c]}}\right)^{[c]}.$$

This equality is helpful in the justification of Corollaries 11, 12 and 13.
Remark 9 implies the following result.

Corollary 10. *In reference [20] (Theorem 3.7), a topological space (X, τ) is $T_{\frac{1}{2}}$ if and only if $\tau = \tau^*$.*

The following result is a simple conclusion from Corollary 8 and Remark 12.

Corollary 11. *In reference [20] (Theorem 3.10), for any topological space (X, τ), the space (X, τ^*) is $T_{\frac{1}{2}}$.*

Directly from the equality presented in Remark 10, we obtain the following conclusion.

Corollary 12. *In reference [20] (Corollary 3.12), for any topological space (X, τ), we have $(\tau^*)^* = \tau^*$.*

The author in [9] investigated the generalizations of Levine's g-closedness (compare Remarks 1 and 5) by using different types of families wider than $\mathcal{C}$ or $\mathcal{T}$.

Let us recall the definition of these families. As usual, $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ denote the closure and the interior of A in (X, τ), respectively.

For a topological space (X, τ), a subset $A \subseteq X$ is called α-open [22], semi-open [23], pre-open [24], γ-open [25] and β-open [26] if

$A \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$;
$A \subseteq \mathrm{cl}(\mathrm{int}(A))$;
$A \subseteq \mathrm{int}(\mathrm{cl}(A))$;
$A \subseteq \mathrm{cl}(\mathrm{int}(A)) \cup \mathrm{int}(\mathrm{cl}(A))$;
$A \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$, respectively.

The family of all such subsets will be denoted by $\mathcal{O}^\alpha$, $\mathcal{O}^s$, $\mathcal{O}^p$, $\mathcal{O}^\gamma$ and $\mathcal{O}^\beta$, respectively.

The complement $A = X \setminus B$ of an α-open, semi-open, pre-open, γ-open and β-open, $B \subseteq X$ is called α-closed, semi-closed, pre-closed, γ-closed and β-closed, respectively, i.e., satisfies the following conditions:

$\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) \subseteq A$;
$\mathrm{int}(\mathrm{cl}(A)) \subseteq A$;
$\mathrm{cl}(\mathrm{int}(A)) \subseteq A$;
$\mathrm{cl}(\mathrm{int}(A)) \cap \mathrm{int}(\mathrm{cl}(A)) \subseteq A$;
$\mathrm{int}(\mathrm{cl}(\mathrm{int}(A))) \subseteq A$, respectively.

The family of all such subsets will be denoted by $\mathcal{C}^\alpha$ (resp. $\mathcal{C}^s$, $\mathcal{C}^p$, $\mathcal{C}^\gamma$, $\mathcal{C}^\beta$).

Using the notation adopted in our article, according to Definition 1, one can consider the closure operators and interior operators designated by the families τ, $\mathcal{O}^\alpha$, $\mathcal{O}^s$, $\mathcal{O}^p$, $\mathcal{O}^\gamma$ and $\mathcal{O}^\beta$ denoted by $\overline{A}^\tau$, $\overline{A}^{\mathcal{O}^\alpha}$, $\overline{A}^{\mathcal{O}^s}$, $\overline{A}^{\mathcal{O}^p}$, $\overline{A}^{\mathcal{O}^\gamma}$ and $\overline{A}^{\mathcal{O}^\beta}$, respectively, and accordingly $\tau.\mathrm{int}(A)$, $\mathcal{O}^\alpha.\mathrm{int}(A)$, $\mathcal{O}^s.\mathrm{int}(A)$, $\mathcal{O}^p.\mathrm{int}(A)$, $\mathcal{O}^\gamma.\mathrm{int}(A)$, $\mathcal{O}^\beta.\mathrm{int}(A)$ for any $A \subseteq X$.

In this paper, the following families were considered:

$$\{\mathcal{B} \triangleleft \mathcal{K} : (\mathcal{B}, \mathcal{K}) \in \{\mathcal{C}, \mathcal{C}^\alpha, \mathcal{C}^s, \mathcal{C}^p, \mathcal{C}^\gamma, \mathcal{C}^\beta\} \times \{\tau, \mathcal{O}^\alpha, \mathcal{O}^p, \mathcal{O}^s, \mathcal{O}^\gamma, \mathcal{O}^\beta\}\}$$

Some of these generalizations are trivial. In particular, the following equalities are useful in our investigations.

Lemma 8. *In reference [9], (Corollary 3.4) in a topological space (X, τ), the following hold:*

(i) $\mathcal{C}^p \triangleleft \mathcal{O}^\beta = \mathcal{C}^p \triangleleft \mathcal{O}^\gamma = \mathcal{C}^p \triangleleft \mathcal{O}^s = \mathcal{C}^p \triangleleft \mathcal{O}^p = \mathcal{C}^p \triangleleft \mathcal{O}^\alpha = \mathcal{C}^p$;
(ii) $\mathcal{C}^\gamma \triangleleft \mathcal{O}^\beta = \mathcal{C}^\gamma \triangleleft \mathcal{O}^\gamma = \mathcal{C}^\gamma \triangleleft \mathcal{O}^s = \mathcal{C}^\gamma \triangleleft \mathcal{O}^p = \mathcal{C}^\gamma \triangleleft \mathcal{O}^\alpha = \mathcal{C}^\gamma$;
(iii) $\mathcal{C}^\beta \triangleleft \mathcal{O}^\beta = \mathcal{C}^\beta \triangleleft \mathcal{O}^\gamma = \mathcal{C}^\beta \triangleleft \mathcal{O}^s = \mathcal{C}^\beta \triangleleft \mathcal{O}^p = \mathcal{C}^\beta \triangleleft \mathcal{O}^\alpha = \mathcal{C}^\beta$.

If one takes $\mathcal{B} = \mathcal{O}^p$ (resp. $\mathcal{O}^\gamma$, $\mathcal{O}^\beta$), it is easy to see that, by the above lemma, equality $\mathcal{B}^{[c]} = \mathcal{B}^{[c]} \triangleleft \mathcal{B}$ is true. So, Theorem 5 implies the following results:

Corollary 13. *For any topological space (X, τ), the following pairs of topological spaces are $T_{\frac{1}{2}}$:*

(i) $(X, \overline{\mathcal{O}^p})$, $\left(X, \left(\overline{\mathcal{O}^p}\right)^{[c]}\right)$ *[13] (Proposition 4.1 (1));*
(ii) $(X, \overline{\mathcal{O}^\gamma})$, $\left(X, \left(\overline{\mathcal{O}^\gamma}\right)^{[c]}\right)$ *[27] (Proposition 2.3 (1));*
(iii) $\left(X, \overline{\mathcal{O}^\beta}\right)$, $\left(X, \left(\overline{\mathcal{O}^\beta}\right)^{[c]}\right)$ *[21] (Theorem 1).*

5.2. Topologies Defined in Terms of Λ-Sets

In a topological space (X, τ), subset A is called Λ-set [17] if $A = \bigcap\{U \subseteq X : A \subseteq U$ and $U \in \tau\}$. Many authors [13,15,21,27] have extended this notion by using families bigger than the topology.

Remark 14. *In the studies, concerning the concepts related to Λ-sets, the authors have used some operators, which we will denote in our convention, as follows:*

(i) $\Lambda(A) = \overline{A}^\tau$*, and* $V(A) = \tau.int(A)$ *[17];*
(ii) $\Lambda_\alpha(A) = \overline{A}^{\mathcal{O}^\alpha}$*, and* $V_\alpha(A) = \mathcal{O}^\alpha.int(A)$ *[15];*
(iii) $\Lambda_s(A) = \overline{A}^{\mathcal{O}^s}$*, and* $V_s(A) = \mathcal{O}^s.int(A)$ *[15];*
(iv) $\Lambda_p(A) = \overline{A}^{\mathcal{O}^p}$*, and* $V_p(A) = \mathcal{O}^p.int(A)$ *[13];*
(v) $\Lambda_b(A) = \overline{A}^{\mathcal{O}^\gamma}$*, and* $V_b(A) = \mathcal{O}^\gamma.int(A)$ *[27];*
(vi) $\Lambda_\beta(A) = \overline{A}^{\mathcal{O}^\beta}$*, and* $V_\beta(A) = \mathcal{O}^\beta.int(A)$ *[21] for* $A \subseteq X$.

Let us recall the definitions of families designated by the above operators.

Definition 5. *A set $A \subseteq X$ in a topological space (X, τ) is called*

(i) Λ-set [17] if $A = \overline{A}^{\tau}$ (resp. V-set [17] if $A = \tau.int(A)$);

(ii) Λ_α-set [15] if $A = \overline{A}^{\mathcal{O}^\alpha}$ (resp.V_α-set [15] if $A = \mathcal{O}^\alpha.int(A)$);

(iii) Λ_s-set [15] if $A = \overline{A}^{\mathcal{O}^s}$ (resp. V_s-set [15] if $A = \mathcal{O}^s.int(A)$);

(iv) Λ_p-set [13] if $A = \overline{A}^{\mathcal{O}^p}$ (resp. V_p-set [13] if $A = \mathcal{O}^p.int(A)$);

(v) Λ_b-set [27] if $A = \overline{A}^{\mathcal{O}^\gamma}$ (resp. V_b-set [27] if $A = \mathcal{O}^\gamma.int(A)$);

(vi) Λ_β-set [21] if $A = \overline{A}^{\mathcal{O}^\beta}$ (resp. V_β-set [21] if $A = \mathcal{O}^\beta.int(A)$).

Remark 15. *The above definitions one can write in the form corresponding to Definition 2. Precisely, a subset $A \subseteq X$ in a topological space (X, τ) is called*

(i) Λ-set if $A \in \overline{\tau}$ (resp. V-set if $A \in int(\tau)$);

(ii) Λ_α-set if $A \in \overline{\mathcal{O}^\alpha}$ (resp.V_α-set if $A \in int(\mathcal{O}^\alpha)$);

(iii) Λ_s-set if $A \in \overline{\mathcal{O}^s}$ (resp. V_s-set if $A \in int(\mathcal{O}^s)$);

(iv) Λ_p-set if $A \in \overline{\mathcal{O}^p}$ (resp. V_p-set if $A \in int(\mathcal{O}^p)$),;

(v) Λ_b-set if $A \in \overline{\mathcal{O}^\gamma}$ (resp. V_b-set if $A \in int(\mathcal{O}^\gamma)$);

(vi) Λ_β-set if $A \in \overline{\mathcal{O}^\beta}$ (resp. V_β-set if $A \in int(\mathcal{O}^\beta)$).

Remark 16. *Let us note that, according to Remark 2, the operation $int(\dots)$ can be represented by the operation $\overline{(\dots)}$. Namely, we have*

$$(1)\ (\overline{\tau})^{[c]} = int(\tau), \quad (3)\ (\overline{\mathcal{O}^s})^{[c]} = int(\mathcal{O}^s), \quad (5)\ (\overline{\mathcal{O}^\gamma})^{[c]} = int(\mathcal{O}^\gamma),$$
$$(2)\ (\overline{\mathcal{O}^\alpha})^{[c]} = int(\mathcal{O}^\alpha), \quad (4)\ (\overline{\mathcal{O}^p})^{[c]} = int(\mathcal{O}^p), \quad (6)\ \left(\overline{\mathcal{O}^\beta}\right)^{[c]} = int(\mathcal{O}^\beta).$$

Since, the family τ (resp. $\mathcal{O}^\alpha, \mathcal{O}^s, \mathcal{O}^p, \mathcal{O}^\gamma, \mathcal{O}^\beta$) is closed under arbitrary unions, then the family $\overline{\tau}$ (resp. $\overline{\mathcal{O}^\alpha}, \overline{\mathcal{O}^s}, \overline{\mathcal{O}^p}, \overline{\mathcal{O}^\gamma}, \overline{\mathcal{O}^\beta}$) also satisfies this condition. So, according to Lemma 6, we obtain the corollary below.

In the literature on the subject, the families $\overline{\tau}, \overline{\mathcal{O}^\alpha}, \overline{\mathcal{O}^s}, \overline{\mathcal{O}^p}, \overline{\mathcal{O}^\gamma}, \overline{\mathcal{O}^\beta}$ (resp. $(\overline{\tau})^{[c]}$, $(\overline{\mathcal{O}^\alpha})^{[c]}$, $(\overline{\mathcal{O}^s})^{[c]}$, $(\overline{\mathcal{O}^p})^{[c]}$, $(\overline{\mathcal{O}^\gamma})^{[c]}$, $\left(\overline{\mathcal{O}^\beta}\right)^{[c]}$) are denoted as $\tau^\Lambda, \tau^{\Lambda_\alpha}, \tau^{\Lambda_s}, \tau^{\Lambda_p}, \tau^{\Lambda_b}, \tau^{\Lambda_\beta}$ (resp. $\tau^V, \tau^{V_\alpha}, \tau^{V_s}, \tau^{V_p}, \tau^{V_b}, \tau^{V_\beta}$).

Lemmas 3, 4 and 6 imply the following:

Corollary 14. *For any topological space (X, τ), the topologies in the following pairs are*

(i) $\left(\overline{\tau}, (\overline{\tau})^{[c]}\right)$;

(ii) $\left(\overline{\mathcal{O}^\alpha}, (\overline{\mathcal{O}^\alpha})^{[c]}\right)$ [14] (Remark 3.4);

(iii) $\left(\overline{\mathcal{O}^s}, (\overline{\mathcal{O}^s})^{[c]}\right)$;

(iv) $\left(\overline{\mathcal{O}^p}, (\overline{\mathcal{O}^p})^{[c]}\right)$, [13] (Remark 2.5);

(v) $\left(\overline{\mathcal{O}^\gamma}, (\overline{\mathcal{O}^\gamma})^{[c]}\right)$ [27] (Proposition 2.1);

(vi) $\left(\overline{\mathcal{O}^\beta}, \left(\overline{\mathcal{O}^\beta}\right)^{[c]}\right)$ [21] (Remark 4).

For the same reason as the above, we obtain the following:

Corollary 15. *For any topological space (X, τ), the topologies in the following pairs are Aleksandroff:*

(i) $\left(\overline{\tau}, (\overline{\tau})^{[c]}\right)$;

(ii) $\left(\overline{\mathcal{O}^\alpha}, (\overline{\mathcal{O}^\alpha})^{[c]}\right)$;

(iii) $\left(\overline{\mathcal{O}^s}, (\overline{\mathcal{O}^s})^{[c]}\right)$;

(iv) $\left(\overline{\mathcal{O}^p}, (\overline{\mathcal{O}^p})^{[c]}\right)$ [13] (Remark 2.5);

(v) $\left(\overline{\mathcal{O}^\gamma}, (\overline{\mathcal{O}^\gamma})^{[c]}\right)$ [27] (Remark 2.2);

(vi) $\left(\overline{\mathcal{O}^\beta}, \left(\overline{\mathcal{O}^\beta}\right)^{[c]}\right)$ [21] (Remark 4).

Since every one of the families τ, $\mathcal{O}^\alpha$, $\mathcal{O}^s$, $\mathcal{O}^p$, $\mathcal{O}^\gamma$, $\mathcal{O}^\beta$ are closed under arbitrary unions, the equality $(\star)$ showed in the proof of Lemma 4 implies the following result:

Corollary 16. *For any family $\{A_s : s \in S\}$, the following equalities are true:*

1. $\bigcup_{s\in S} \overline{A_s}^{\tau} = \overline{\bigcup_{s\in S} A_s}^{\tau}$;
2. $\bigcup_{s\in S} \overline{A_s}^{\mathcal{O}^\alpha} = \overline{\bigcup_{s\in S} A_s}^{\mathcal{O}^\alpha}$ [14] (Proposition 3.1 (e));
3. $\bigcup_{s\in S} \overline{A_s}^{\mathcal{O}^p} = \overline{\bigcup_{s\in S} A_s}^{\mathcal{O}^p}$ [13] (Lemma 2.1 (5));
4. $\bigcup_{s\in S} \overline{A_s}^{\mathcal{O}^s} = \overline{\bigcup_{s\in S} A_s}^{\mathcal{O}^s}$ [15] (Proposition 1.2 (d));
5. $\bigcup_{s\in S} \overline{A_s}^{\mathcal{O}^\gamma} = \overline{\bigcup_{s\in S} A_s}^{\mathcal{O}^\gamma}$ [27] (Lemma 2.1 (5));
6. $\bigcup_{s\in S} \overline{A_s}^{\mathcal{O}^\beta} = \overline{\bigcup_{s\in S} A_s}^{\mathcal{O}^\beta}$ [21] (Proposition 1 (e)).

5.3. Generalizations of Λ-Type Sets

The notion of generalized Λ-sets in topological space was initiated by H.Maki in 1986 as follows. A subset A in topological space (X, τ) is called the generalized Λ-sets (V-sets) if $\Lambda(A) \subseteq F$ whenever $A \subseteq F$ and F is closed (resp. $U \subseteq V(A)$ whenever $U \subseteq A$ and U is open).

Remark 17. *The property of being a generalized Λ-set in a topological space (X, τ) can be described in the following equivalent ways:*

(i) *For every closed subset $F \subseteq X$, $A \subseteq F$ implies $\Lambda(A) \subseteq F$;*

(ii) $\Lambda(A) \subseteq \overline{A}^{\mathcal{C}}$, *(see Remark 14);*

(iii) $\overline{A}^{\tau} \subseteq \overline{A}^{\mathcal{C}}$;

(iv) $A \in \overline{\tau} \triangleleft \mathcal{C}$.

So, the family of all generalized Λ-sets in a topological space (X, τ), according to the Definition 3, is a generalization of the family $\overline{\tau}$, i.e., Λ-sets, by the family $\mathcal{C}$, i.e., $\overline{\tau} \triangleleft \mathcal{C}$. Analogously, we can describe corresponding equivalences for generalized V-sets. Before that, let us notice the equivalences of the following equalities for $A \subseteq X$:

- $\mathcal{C}.\mathrm{int}(A) \subseteq \tau.int(A)$;
- $\overline{X \setminus A}^{\tau} \subseteq \overline{X \setminus A}^{\mathcal{C}}$;
- $X \setminus A \in \tau \triangleleft \mathcal{C}$;
- $A \in (\overline{\tau} \triangleleft \mathcal{C})^{[c]}$.

So, we have the following:

Remark 18. *The property of being a generalized V-set in a topological space (X, τ) can be described in the following equivalent ways:*

(i) *for every open subset $U \subseteq X$, $U \subseteq A$ implies $U \subseteq V(A)$;*

(ii) $\mathcal{C}.int(A) \subseteq V(A)$;

(iii) $\mathcal{C}.int(A) \subseteq \tau.int(A)$;

(iv) $A \in (\overline{\tau} \triangleleft \mathcal{C})^{[c]}$.

Using the convention justified in Remarks 1 and 2, let us recall the definitions of other generalizations like Λ sets investigated in the literature:

Definition 6. *A set $A \subseteq X$ in a topological space (X, τ) is called*

(i) *Generalized Λ_α-set if $A \in \overline{\mathcal{O}^\alpha} \triangleleft \mathcal{C}^\alpha$ (resp. generalized V_α-set if $A \in \left(\overline{\mathcal{O}^\alpha} \triangleleft \mathcal{C}^\alpha\right)^{[c]}$);*

(ii) *Generalized Λ_s-set [15] if $A \in \overline{\mathcal{O}^s} \triangleleft \mathcal{C}^s$ (resp. generalized V_s-set [15] if $A \in \left(\overline{\mathcal{O}^s} \triangleleft \mathcal{C}^s\right)^{[c]}$);*

(iii) *Generalized Λ_p-set [13] if $A \in \overline{\mathcal{O}^p} \triangleleft \mathcal{C}^p$ (resp. generalized V_p-set [13] if $A \in \left(\overline{\mathcal{O}^p} \triangleleft \mathcal{C}^p\right)^{[c]}$);*

(iv) *Generalized Λ_b-set [27] if $A \in \overline{\mathcal{O}^\gamma} \triangleleft \mathcal{C}^\gamma$ (resp. generalized V_b-set [27] if $A \in \left(\overline{\mathcal{O}^\gamma} \triangleleft \mathcal{C}^\gamma\right)^{[c]}$);*

(v) *Generalized Λ_β-set [21] if $A \in \overline{\mathcal{O}^\beta} \triangleleft \mathcal{C}^\beta$ (resp. generalized V_β-set [21] if $A \in \left(\overline{\mathcal{O}^\beta} \triangleleft \mathcal{C}^\beta\right)^{[c]}$).*

Remark 19. *According to Lemma 5 (vii) in the above definition and Remark 17, the lines $\overline{(\ldots)}$ over the letters $\overline{\tau}$, $\overline{\mathcal{O}^\alpha}$, $\overline{\mathcal{O}^p}$, $\overline{\mathcal{O}^s}$, $\overline{\mathcal{O}^\gamma}$, $\overline{\mathcal{O}^\beta}$ can be omitted.*

So, in a topological space (X, τ), we consider the following families:

$$g.\Lambda(\tau) = \{\mathcal{B} \triangleleft \mathcal{K} : (\mathcal{B}, \mathcal{K}) \in \{\tau, \mathcal{O}^\alpha, \mathcal{O}^p, \mathcal{O}^s, \mathcal{O}^\gamma, \mathcal{O}^\beta\} \times \{\mathcal{C}, \mathcal{C}^\alpha, \mathcal{C}^s, \mathcal{C}^p, \mathcal{C}^\gamma, \mathcal{C}^\beta\}\}.$$

It begs the question, are all these generalizations non-trivial, i.e., do not satisfy the equation $\overline{\mathcal{B}} = \mathcal{B} \triangleleft \mathcal{K}$? The answer to this problem is in the theorem below.

Before that, we recall some properties of singletons in topological spaces. In [28], it is shown that every singleton $\{x\}$ in a topological space (X, τ) is either nowhere dense or pre-open (Jankovic–Reilly decomposition [4], [29]). Since, the nowhere denseness of $\{x\}$ implies that $X \setminus \{x\}$ is α-open then for every singleton $\{x\}$, the subset $X \setminus \{x\}$ is either α-open or pre-closed.

Similarly, every singleton $\{x\}$ of a topological space (X, τ) is either open or has an empty interior [7] (Lemma 2.4). The property $Int(\{x\}) = \emptyset$ implies that $X \setminus \{x\}$ is pre-open. So, for every singleton $\{x\}$, the subset $X \setminus \{x\}$ is either pre-open or closed.

Formally, we can write it as follows.

Remark 20. *In a topological space (X, τ), the following properties are true for all $x \in X$:*

(i) *$X \setminus \{x\} \in \mathcal{C}^p$ or $X \setminus \{x\} \in \mathcal{O}^\alpha$;*

(ii) *$X \setminus \{x\} \in \mathcal{C}$ or $X \setminus \{x\} \in \mathcal{O}^p$.*

Lemma 9. *In a topological space (X, τ), the following equalities are true:*

(i) *$\mathcal{O}^\alpha \triangleleft \mathcal{C}^\alpha = \mathcal{O}^\alpha \triangleleft \mathcal{C}^s$,*

(ii) *$\mathcal{O}^s \triangleleft \mathcal{C}^\alpha = \mathcal{O}^s \triangleleft \mathcal{C}^s$.*

Since $\mathcal{C}^\alpha \subseteq \mathcal{C}^s$ then, according to Lemma 5 (iii), it is enough to show that $\mathcal{O}^\alpha \triangleleft \mathcal{C}^\alpha \subseteq \mathcal{O}^\alpha \triangleleft \mathcal{C}^s$. Let $A \in \mathcal{O}^\alpha \triangleleft \mathcal{C}^\alpha$, i.e., $\overline{A}^{\mathcal{O}^\alpha} \subseteq \overline{A}^{\mathcal{C}^\alpha}$ and assume, to the contrary, that $A \notin \mathcal{O}^\alpha \triangleleft \mathcal{C}^s$, i.e., $\overline{A}^{\mathcal{O}^\alpha} \nsubseteq \overline{A}^{\mathcal{C}^s}$. It means that, for some $x \in X$, we have $x \in \overline{A}^{\mathcal{O}^\alpha}$ and $x \notin \overline{A}^{\mathcal{C}^s}$. Thus, by Lemma 2, for every $K \in \mathcal{C}^\alpha$ such that $x \in K$, $K \cap A \neq \emptyset$. Let us take the family $\mathcal{K} = \{K \in \mathcal{C}^\alpha : x \in K\}$.

Obviously, $\bigcap \mathcal{K} = \overline{\{x\}}^{\mathcal{C}^\alpha} \in \mathcal{C}^\alpha$ because the family $\mathcal{C}^\alpha$ is closed under arbitrary intersection. Consequently, $\overline{\{x\}}^{\mathcal{C}^\alpha} \cap A \neq \emptyset$, i.e., $\left(\{x\} \cup \overline{\mathrm{int}\overline{\{x\}}}\right) \cap A \neq \emptyset$. Thus, $\overline{\mathrm{int}\overline{\{x\}}} \cap A \neq \emptyset$ because of $x \notin \overline{A}^{\mathcal{C}^s}$, i.e., $x \notin A \cup \mathrm{int}\overline{A}$. Since $x \notin \mathrm{int}\overline{A}$, then $x \in \overline{X \setminus \overline{A}}$ which implies, $\overline{\{x\}} \subseteq X \setminus \overline{A}$ and consequently $\mathrm{int}\overline{\{x\}} \subseteq X \setminus \overline{\mathrm{int}\overline{A}}$. It is clear, that $x \in \overline{\mathrm{int}\overline{\{x\}}}$, because $\mathrm{int}\overline{\{x\}} \neq \emptyset$. As a result, we obtain $x \notin A \cup \overline{\mathrm{int}\overline{A}}$, i.e., $x \notin \overline{A}^{\mathcal{C}^\alpha}$ and we have a contradiction.

The proof of the (ii) proceeds analogously, it is enough to use $\overline{\{x\}}^{\mathcal{C}^s} = \{x\} \cup \mathrm{int}\overline{\{x\}}$ instead of $\overline{\{x\}}^{\mathcal{C}^\alpha}$ and the fact that $\overline{A}^{\mathcal{C}^s} \subseteq \overline{A}^{\mathcal{C}^\alpha}$.

Theorem 7. *For any topological space* (X, τ)*, there exists at most* 10 *generalizations of* Λ*-type presented in the following graph.*

$$\begin{tikzcd}
 & & & & & \overline{\mathcal{O}^{\beta}} \\
 & & & & \overline{\mathcal{O}^{p}} \arrow[r] & \overline{\mathcal{O}^{\gamma}} \arrow[u] \\
 & & & \tau \lhd \mathcal{C} \arrow[r] & \mathcal{O}^{\alpha} \lhd \mathcal{C} \arrow[r] \arrow[u] & \mathcal{O}^{s} \lhd \mathcal{C} \arrow[u] \\
 & & \tau \lhd \mathcal{C}^{s} \arrow[r] & \tau \lhd \mathcal{C}^{\alpha} \arrow[r] \arrow[u] & \mathcal{O}^{\alpha} \lhd \mathcal{C}^{\alpha} \arrow[r] \arrow[u] & \mathcal{O}^{s} \lhd \mathcal{C}^{\alpha} \arrow[u] \\
\overline{\tau} \arrow[r] & \tau \lhd \mathcal{C}^{\beta} \arrow[r] & \tau \lhd \mathcal{C}^{\gamma} \arrow[r] \arrow[u] & \tau \lhd \mathcal{C}^{p} \arrow[r] \arrow[u] & \overline{\mathcal{O}^{\alpha}} \arrow[r] \arrow[u] & \overline{\mathcal{O}^{s}} \arrow[u]
\end{tikzcd}$$

Proof. Firstly, we will show that the following equalities are true:

(A) $\overline{\mathcal{O}^{\alpha}} = \mathcal{O}^{\alpha} \lhd \mathcal{C}^{p} = \mathcal{O}^{\alpha} \lhd \mathcal{C}^{\gamma} = \mathcal{O}^{\alpha} \lhd \mathcal{C}^{\beta}$;
(B) $\overline{\mathcal{O}^{s}} = \mathcal{O}^{s} \lhd \mathcal{C}^{p} = \mathcal{O}^{s} \lhd \mathcal{C}^{\gamma} = \mathcal{O}^{s} \lhd \mathcal{C}^{\beta}$;
(C) $\overline{\mathcal{O}^{p}} = \mathcal{O}^{p} \lhd \mathcal{C} = \mathcal{O}^{p} \lhd \mathcal{C}^{\alpha} = \mathcal{O}^{p} \lhd \mathcal{C}^{s} = \mathcal{O}^{p} \lhd \mathcal{C}^{p} = \mathcal{O}^{p} \lhd \mathcal{C}^{\gamma} = \mathcal{O}^{p} \lhd \mathcal{C}^{\beta}$;
(D) $\overline{\mathcal{O}^{\gamma}} = \mathcal{O}^{\gamma} \lhd \mathcal{C} = \mathcal{O}^{\gamma} \lhd \mathcal{C}^{\alpha} = \mathcal{O}^{\gamma} \lhd \mathcal{C}^{s} = \mathcal{O}^{\gamma} \lhd \mathcal{C}^{p} = \mathcal{O}^{\gamma} \lhd \mathcal{C}^{\gamma} = \mathcal{O}^{\gamma} \lhd \mathcal{C}^{\beta}$;
(E) $\overline{\mathcal{O}^{\beta}} = \mathcal{O}^{\beta} \lhd \mathcal{C} = \mathcal{O}^{\beta} \lhd \mathcal{C}^{\alpha} = \mathcal{O}^{\beta} \lhd \mathcal{C}^{s} = \mathcal{O}^{\beta} \lhd \mathcal{C}^{p} = \mathcal{O}^{\beta} \lhd \mathcal{C}^{\gamma} = \mathcal{O}^{\beta} \lhd \mathcal{C}^{\beta}$.

Using the property (i) of above remark and Theorem 1 (ii), we obtain for the pair $(\mathcal{O}^{\alpha}, \mathcal{C}^{p})$, the equality

$$\overline{\mathcal{O}^{\alpha}} = \mathcal{O}^{\alpha} \lhd \mathcal{C}^{p}.$$

Because of the relationships among the families that we use here, precisely,

$$\begin{tikzcd}
 & & \mathcal{O}^{p} \arrow[dr] & & \\
\tau \arrow[r] & \mathcal{O}^{\alpha} \arrow[ur] \arrow[dr] & & \mathcal{O}^{\gamma} \arrow[r] & \mathcal{O}^{\beta} \\
 & & \mathcal{O}^{s} \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \mathcal{C}^{p} \arrow[dr] & & \\
\mathcal{C} \arrow[r] & \mathcal{C}^{\alpha} \arrow[ur] \arrow[dr] & & \mathcal{C}^{\gamma} \arrow[r] & \mathcal{C}^{\beta} \\
 & & \mathcal{C}^{s} \arrow[ur] & &
\end{tikzcd}$$

For the same reason, we obtain the following equalities, analogously to the above one, for the following pairs

$$(\mathcal{B}, \mathcal{K}) \in \{\mathcal{O}^{\alpha}\mathcal{O}^{s}, \mathcal{O}^{p}, \mathcal{O}^{\gamma}, \mathcal{O}^{\beta}\} \times \{\mathcal{C}^{p}, \mathcal{C}^{\gamma}, \mathcal{C}^{\beta}\}.$$

(1) $\overline{\mathcal{O}^{\alpha}} = \mathcal{O}^{\alpha} \lhd \mathcal{C}^{p} = \mathcal{O}^{\alpha} \lhd \mathcal{C}^{\gamma} = \mathcal{O}^{\alpha} \lhd \mathcal{C}^{\beta}$;
(2) $\overline{\mathcal{O}^{p}} = \mathcal{O}^{p} \lhd \mathcal{C}^{p} = \mathcal{O}^{p} \lhd \mathcal{C}^{\gamma} = \mathcal{O}^{p} \lhd \mathcal{C}^{\beta}$;
(3) $\overline{\mathcal{O}^{s}} = \mathcal{O}^{s} \lhd \mathcal{C}^{p} = \mathcal{O}^{s} \lhd \mathcal{C}^{\gamma} = \mathcal{O}^{s} \lhd \mathcal{C}^{\beta}$;
(4) $\overline{\mathcal{O}^{\gamma}} = \mathcal{O}^{\gamma} \lhd \mathcal{C}^{p} = \mathcal{O}^{\gamma} \lhd \mathcal{C}^{\gamma} = \mathcal{O}^{\gamma} \lhd \mathcal{C}^{\beta}$;
(5) $\overline{\mathcal{O}^{\beta}} = \mathcal{O}^{\beta} \lhd \mathcal{C}^{p} = \mathcal{O}^{\beta} \lhd \mathcal{C}^{\gamma} = \mathcal{O}^{\beta} \lhd \mathcal{C}^{\beta}$.

Using the property (*ii*) of Remark 17 and Theorem 1 (*ii*), we obtain

$$\overline{\mathcal{O}^p} = \mathcal{O}^p \triangleleft \mathcal{C}.$$

For the same way as in the previous part of the proof, using the above graph, we know that such equality is true for all the following pairs.

$$(\mathcal{B}, \mathcal{K}) \in \{\mathcal{O}^p, \mathcal{O}^\gamma, \mathcal{O}^\beta\} \times \{\mathcal{C}, \mathcal{C}^\alpha, \mathcal{C}^s, \mathcal{C}^p, \mathcal{C}^\gamma, \mathcal{C}^\beta\}.$$

So, we have the following

(6) $\overline{\mathcal{O}^p} = \mathcal{O}^p \triangleleft \mathcal{C} = \mathcal{O}^p \triangleleft \mathcal{C}^\alpha = \mathcal{O}^p \triangleleft \mathcal{C}^s = \mathcal{O}^p \triangleleft \mathcal{C}^p = \mathcal{O}^p \triangleleft \mathcal{C}^\gamma = \mathcal{O}^p \triangleleft \mathcal{C}^\beta$;

(7) $\overline{\mathcal{O}^\gamma} = \mathcal{O}^\gamma \triangleleft \mathcal{C} = \mathcal{O}^\gamma \triangleleft \mathcal{C}^\alpha = \mathcal{O}^\gamma \triangleleft \mathcal{C}^s = \mathcal{O}^\gamma \triangleleft \mathcal{C}^p = \mathcal{O}^\gamma \triangleleft \mathcal{C}^\gamma = \mathcal{O}^\gamma \triangleleft \mathcal{C}^\beta$;

(8) $\overline{\mathcal{O}^\beta} = \mathcal{O}^\beta \triangleleft \mathcal{C} = \mathcal{O}^\beta \triangleleft \mathcal{C}^\alpha = \mathcal{O}^\beta \triangleleft \mathcal{C}^s = \mathcal{O}^\beta \triangleleft \mathcal{C}^p = \mathcal{O}^\beta \triangleleft \mathcal{C}^\gamma = \mathcal{O}^\beta \triangleleft \mathcal{C}^\beta$.

Let us note, as a side note, that these equalities imply (2), (4) and (5), respectively.

Finally, the equalities (6), (7) and (8) mean (C), (D) and (E), respectively. Furthermore, the properties (1) and (3) mean (A) and (B), respectively, and the proof is complete. □

Remark 21. *The equalities* (C) *i* (D) *of Theorem 7 imply*

(i) $\overline{\mathcal{O}^p} = \mathcal{O}^p \triangleleft \mathcal{C}^p$ *see Proposition 3.1 in [13],*

(ii) $\overline{\mathcal{O}^\gamma} = \mathcal{O}^\gamma \triangleleft \mathcal{C}^\gamma$ *see Proposition 3.1 in [27] and Proposition 3.2 [12].*

Remark 22. *The equality (E) in the above theorem implies in particular that*

$$\overline{\mathcal{O}^\beta} = \mathcal{O}^\beta \triangleleft \mathcal{C}^\beta.$$

So, the generalizations Λ_β*-sets and* V_β*-sets are trivial. It means that the converse of (a) and (b) of Proposition 10 in [21] is true contrary to the author's opinion. Of course, Example 7 in [21] is subject to error.*

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