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Article

Global Existence of Chemotaxis-Navier–Stokes System with Logistic Source on the Whole Space $\mathbb{R}^2$

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Abstract: In this article, we study the Cauchy problem of the chemotaxis-Navier–Stokes system with the consumption and production of chemosignals with a logistic source. The parameters $\chi \neq 0$, $\xi \neq 0$, $\lambda > 0$ and $\mu > 0$. The system is a model that involves double chemosignals; one is an attractant consumed by the cells themselves, and the other is an attractant or a repellent produced by the cells themselves. We prove the global-in-time existence and uniqueness of the weak solution to the system for a large class of initial data on the whole space $\mathbb{R}^2$.

Keywords: chemotaxis; attraction; repulsion; Navier–Stokes; weak solutions; logistic source

MSC: Primary: 42B30; Secondary: 35Q30; 42B35; 35B40

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1. Introduction

The present paper is concerned with the following chemotaxis-Navier–Stokes system with the consumption and production of chemosignals with logistic source

$$\begin{cases} u_t + \kappa(u\cdot\nabla)u = \Delta u + \nabla\mathbb{P} + n\nabla\phi, \quad \nabla\cdot u = 0, \\ n_t + (u\cdot\nabla)n = \Delta n - \chi\nabla\cdot(n\nabla c) + \xi\nabla\cdot(n\nabla v) + \lambda n - \mu n^2, \\ v_t + (u\cdot\nabla)v = \Delta v - v + n, \\ c_t + u\cdot\nabla c = \Delta c - nc, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^2, t > 0$. The terms $n = n(x,t), c = c(x,t), v = v(x,t), u = u(x,t) = (u_1(x,t), u_2(x,t))$ and $\mathbb{P}$ denote the unknown density of amoebae, the unknown oxygen concentration, the unknown concentration of the chemical attractant, and the unknown fluid velocity field and the unknown pressure, respectively. The parameters $\chi \neq 0, \xi \neq 0$ and $\kappa \geq 0$. The terms $\lambda \geq 0$ and $\mu \geq 0$ reflect the rate of reproduction and death, respectively. We impose the following intial data

$$(u(x,0), n(x,0), v(x,0), c(x,0)) = (u_0, n_0, v_0, c_0).$$

The time-independent function $\phi = \phi(x)$ denotes the potential function produced by different physical mechanisms, e.g., the gravitational force or centrifugal force.

In some biological processes, chemotactic cells often interact with multiple chemotactic cues, either of which may be an attractant or a repellant, to produce variety of intricate patterns. It was pointed out in [1–4] that this phenomenon is widely present in many prototypical biological situations. As compared to the chemotaxis-fluid models involving just one chemical signal that is consumed or produced by the species themselves as mentioned above, chemotaxis-fluid models incorporating at least two different chemical signals seem much less understood. When $\chi > 0$, $\xi < 0$, $\lambda = \mu = 0$ and $\kappa = 1$, the system (1) becomes an attraction–attraction Navier–Stokes system, and the corresponding Cauchy problem admits global mild solutions with small initial data in some scaling invariant space [5]. When $\chi > 0$, $\xi > 0$ and $\lambda = \kappa = \mu = 0$, the system (1) becomes an attraction-repulsion-Stokes system; the global bounded classical solution and the large time behavior of the solution have been established in a smoothly bounded planar domain [6,7]. When $\kappa = 1$ and $\chi > 0$, $\xi > 0$, $\lambda = \mu = 0$, in [8] the corresponding attraction-repulsion Navier–Stokes system is proved to possess a unique global classical solution; however, the uniform boundedness and large time behavior of the solutions to this attraction-repulsion Navier–Stokes system can be achieved simultaneously in [9]. We refer to [10–22] for more details concerning some properties of chemotaxis-fluid models.

In this work, we shall focus on the Cauchy problem (1) with the logistic term in two dimensional. Here, the parameters $\chi \neq 0$, $\xi \neq 0$, $\lambda > 0$, $\mu > 0$ and $\kappa = 1$. Precisely, we shall consider the global-in-time existence and uniqueness of the weak solution to the system for a large class of initial data on the whole space $\mathbb{R}^2$.

We first state the assumptions on the initial data and introduce the following notation:

$$\begin{aligned} X_0 = \big\{ (n_0, v_0, u_0, c_0) \mid\ & n_0 \in L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2),\ \sqrt{1+|x|^2}n_0 \in L^1(\mathbb{R}^2),\ v_0 \in H^1(\mathbb{R}^2), \\ & u_0 \in L^2(\mathbb{R}^2),\ \nabla\sqrt{c_0} \in L^{2v}(\mathbb{R}^2),\ c_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2),\ n_0 \geq 0,\ v_0 \geq 0,\ c_0 \geq 0 \big\}. \end{aligned} \tag{2}$$

Now, we state our main theorem.

Theorem 1. *Let $(u_0,\ v_0,\ n_0,\ c_0) \in X_0$ and $\nabla\phi \in L^\infty(\mathbb{R}^2)$. Then, the solution of (1) possesses a unique global-in-time weak solution satisfying*

$$n \in (L^\infty_{loc}(\mathbb{R}^+; L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)) \cap L^2_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^2)))^2,$$

$$c \in L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)) \cap L^\infty_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^2)) \cap L^2_{loc}(\mathbb{R}^+; H^2(\mathbb{R}^2)),$$

$$u \in L^\infty_{loc}(\mathbb{R}^+; L^2(\mathbb{R}^2)) \cap L^2_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^2))$$

and

$$v \in L^\infty_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^2)) \cap L^2_{loc}(\mathbb{R}^+; H^2(\mathbb{R}^2)) \cap L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^2)).$$

We mention that our results may be generalized to a bounded domain by slightly modifying our proof and adding reasonable boundary conditions. Compared with [5,9], we can obtain the existence and uniqueness of weak solutions to (1) in the whole space $\mathbb{R}^2$.

Notation. We will set $\partial_t = \frac{\partial}{\partial t}$ and $\partial_i = \frac{\partial}{\partial x_i}$ for $i = 1, 2$ and denote all the partial derivatives ∂_β with multi-index β satisfying $|\beta| = k$ by ∇^k $(k \geq 0)$. We adopt the convention that the nonessential constant C may change from line to line, and $C(a_1, a_2, ..., a_k)$ means a constant C depending on $a_1, a_2, ..., a_k$. Given two quantities A and B, we denote $B \lesssim A$ as $B \leq CA$. We often label $\|(a,b)\|_X = \|a\|_X + \|b\|_X$.

2. Preliminaries

In the following, we would like to present some preliminaries. We begin with recalling the well-known estimate for the product of two functions.

Lemma 1 ([22])**.** *Let $s > 0$. Then, there exists a constant $C > 0$ such that for all $u, v \in L^\infty(\mathbb{R}^2) \cap H^s(\mathbb{R}^2)$,*

$$\|uv\|_{H^s(\mathbb{R}^2)} \leq C(\|v\|_{H^s(\mathbb{R}^2)}\|u\|_{L^\infty(\mathbb{R}^2)} + \|u\|_{H^s(\mathbb{R}^2)}\|v\|_{L^\infty(\mathbb{R}^2)}).$$

A key point of obtaining direct compactness results is the so-called Aubin–Lions lemma.

Lemma 2 (Aubin–Lions [23,24]). *Let $1 < \alpha < \infty, 1 \leq \beta \leq \infty$ and X_0, X_1, X_2 be reflexive separable Banach spaces such that $X_0 \hookrightarrow\hookrightarrow, X_1$ and $X_1 \hookrightarrow X_2$. Then,*

$$\{u \in L^\alpha(0,T;X_0); \partial_t u \in L^\beta(0,T;X_2)\} \hookrightarrow L^\alpha(0,T;X_1).$$

Definition 1. *For $T > 0$, let $Q_T = \mathbb{R}^2 \times T$. For initial data, we say that (n,m,c,u) is a weak solution of system (1) if the following conditions hold:*

(i) $n(x,t) \geq 0, v(x,t) \geq 0, c(x,t) \geq 0$, $t \geq 0, x \in \mathbb{R}^2$, *and for any $T > 0$,*

$$\begin{cases} n \in (L^\infty([0,T];L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)) \cap L^2([0,T];H^1(\mathbb{R}^2)))^2, \\ v \in (L^\infty([0,T];L^2(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)) \cap L^2([0,T];H^2(\mathbb{R}^2)))^2, \\ c \in L^\infty([0,T];L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)) \cap L^2([0,T];H^2(\mathbb{R}^2)), \\ u \in L^\infty([0,T];L^2(\mathbb{R}^2)) \cap L^2([0,T];H^1(\mathbb{R}^2)). \end{cases}$$

(ii) *Moreover, for any $\psi \in C_0^\infty(\mathbb{R}^2;[0,T])$,*

$$\int_{Q_T} n_t \psi dxdt + \int_{Q_T} (u\nabla n - n(\lambda - \mu n))\psi dxdt = \int_{Q_T} (\chi n \nabla c - \xi n \nabla v - \nabla n)\nabla\psi dxdt,$$

$$\int_{Q_T} v_t \psi dxdt + \int_{Q_T} (u\nabla v + v - n)\psi dxdt = -\int_{Q_T} \nabla v \nabla \psi dxdt,$$

$$\int_{Q_T} c_t \psi dxdt + \int_{Q_T} (u\nabla c + nc)\psi dxdt = -\int_{Q_T} \nabla c \nabla \psi dxdt,$$

and for any $\tilde{\psi} \in C_0^\infty(\mathbb{R}^2;[0,T])$,

$$\int_{Q_T} u_t \tilde{\psi} dxdt + \int_{Q_T} (u\nabla u - (m+n)\nabla\phi)\tilde{\psi} dxdt = -\int_{Q_T} \nabla u \nabla \tilde{\psi} dxdt$$

as well as

$$\int_{Q_T} \nabla\tilde{\psi} \cdot u dxdt = 0.$$

If (v,n,c,u) is a weak solution of system (1) in $\mathbb{R}^2 \times (0,T)$ for any $T > 0$, then (v,n,c,u) is called a global-in-time weak solution.

Let $(f * g)(x) = \int_{\mathbb{R}^2} f(x-y)g(y)dy$. Weak solutions to (1), in the sense of Definition 1, will be constructed as limit objects from a family of appropriately regularized systems as follows:

$$\begin{cases} n_t^\epsilon + (u^\epsilon \cdot \nabla)n^\epsilon \\ = \Delta n^\epsilon - \chi\nabla \cdot (n^\epsilon \nabla(c^\epsilon * \rho^\epsilon)) + \xi\nabla \cdot (n^\epsilon \nabla(v^\epsilon * \rho^\epsilon)) + \lambda n^\epsilon - \mu n^\epsilon (n^\epsilon * \rho^\epsilon), \\ c_t^\epsilon + u^\epsilon \cdot \nabla c^\epsilon = \Delta c^\epsilon - (n^\epsilon * \rho^\epsilon)c^\epsilon, \\ v_t^\epsilon + u^\epsilon \cdot \nabla v^\epsilon = \Delta v^\epsilon - v^\epsilon + n^\epsilon * \rho^\epsilon, \\ u_t^\epsilon + (u^\epsilon \cdot \nabla)u^\epsilon = \Delta u^\epsilon + \nabla \mathbb{P}^\epsilon - (n^\epsilon \nabla\phi) * \rho^\epsilon, \quad \nabla \cdot u^\epsilon = 0, \\ (u_0^\epsilon, n_0^\epsilon, v_0^\epsilon, c_0^\epsilon) = (u_0 * \rho^\epsilon,\ n_0 * \rho^\epsilon,\ v_0 * \rho^\epsilon,\ c_0 * \rho^\epsilon), \end{cases} \tag{3}$$

where $\rho^\epsilon(x)$ is defined by the standard mollifier $\rho(x)$ satisfying $\int_{\mathbb{R}^2}\rho(x)dx$. We now state the global classical solution for the regularized system to the Cauchy problem (3).

Lemma 3 (Global existence for the regularized system)**.** *Let $\phi \in \dot{W}^{1,\infty}(\mathbb{R}^2)$,*

$$(n_0^\epsilon,\, v_0^\epsilon,\, c_0^\epsilon,\, u_0^\epsilon) \in (H^s(\mathbb{R}^2))^4$$

with $s > 1$ and $n_0^\epsilon,\, v_0^\epsilon,\, c_0^\epsilon \geq 0$. Then, system (3) has a unique global smooth solution

$$(n^\epsilon,\, v^\epsilon,\, c^\epsilon,\, u^\epsilon) \in (\mathcal{C}(\mathbb{R}^+; H^s(\mathbb{R}^2)))^4 \cap (L^2_{loc}(\mathbb{R}^+; H^{s+1}(\mathbb{R}^2)))^4.$$

Moreover, $n^\epsilon \geq 0$, $v^\epsilon \geq 0$, $c^\epsilon \geq 0$ for all $(x,t) \in \mathbb{R}^2 \times \mathbb{R}^+$.

Proof. The proof of Lemma 3 is standard, we can refer to (Proposition 3.1, [22]). □

3. A Priori Estimates for a Regularized System

In what follows, we let C denote some different constants, which depend at most on χ, λ, ξ, μ, $\|\nabla\phi\|_{L^\infty(\mathbb{R}^2)}$ and $\|c_0\|_{L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)}$, $\|v_0\|_{H^1(\mathbb{R}^2)}$, $\|v_0\|_{L^1(\mathbb{R}^2)}$, $\|n_0\|_{L^1(\mathbb{R}^2)}$, $\|u_0\|_{L^2(\mathbb{R}^2)}$. If there are no special explanations, they are independent of ϵ and t.

Proposition 1. *Assume that $(u_0, c_0, m_0, n_0) \in X_0$, and let $(u^\epsilon, n^\epsilon, c^\epsilon)$ be a unique classical solution to the system (3). Then, a positive constant C exists such that*

$$\begin{cases} \|c^\epsilon(t)\|_{L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)} \leq \|c_0\|_{L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)}, \\ \|c^\epsilon(t)\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \|\nabla c^\epsilon(\tau)\|^2_{L^2(\mathbb{R}^2)} d\tau \leq \|c_0\|^2_{L^2(\mathbb{R}^2)} \end{cases}$$

and

$$\begin{cases} \|n^\epsilon(t)\|_{L^1(\mathbb{R}^2)} + \mu \int_0^t \|n^\epsilon(t)\|^2_{L^2(\mathbb{R}^2)} dt \leq C\|n_0\|_{L^1(\mathbb{R}^2)} e^{\lambda t}., \\ \|v^\epsilon(t)\|_{L^1(\mathbb{R}^2)} \leq \|v_0\|_{L^1(\mathbb{R}^2)} + \dfrac{C(e^{\lambda t + t} - 1)\|n_0\|_{L^1(\mathbb{R}^2)}}{\lambda + 1} \end{cases}$$

as well as

$$\begin{cases} \|u^\epsilon\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \|\nabla u^\epsilon(\tau)\|^2_{L^2(\mathbb{R}^2)} d\tau \\ \quad \leq (\|u_0\|^2_{L^2(\mathbb{R}^2)} + C\mu^{-1} e^{\lambda t}\|n_0\|_{L^1(\mathbb{R}^2)}) \exp(Ct\|\nabla\phi\|^2_{L^\infty(\mathbb{R}^2)}), \\ \|v^\epsilon\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \|\nabla v^\epsilon\|^2_{L^2(\mathbb{R}^2)} d\tau + \int_0^t \|v^\epsilon\|^2_{L^2(\mathbb{R}^2)} d\tau \\ \quad \leq \|v_0\|^2_{L^2(\mathbb{R}^2)} + C\mu^{-1} e^{\lambda t}\|n_0\|_{L^1(\mathbb{R}^2)}, \\ \|\nabla v^\epsilon\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \|\nabla v^\epsilon\|^2_{L^2(\mathbb{R}^2)} d\tau + \int_0^t \|\Delta v^\epsilon\|^2_{L^2(\mathbb{R}^2)} d\tau \\ \quad \leq (\|\nabla v_0\|^2_{L^2(\mathbb{R}^2)} + Ce^t\|n_0\|_{L^1(\mathbb{R}^2)}) \\ \quad\quad \times \exp\Big((\|u_0\|^2_{L^2(\mathbb{R}^2)} + C\mu^{-1} e^{\lambda t}\|n_0\|_{L^1(\mathbb{R}^2)})^2 \exp(Ct\|\nabla\phi\|^2_{L^\infty(\mathbb{R}^2)})\Big). \end{cases}$$

Proof. We first show some priori estimates of n^ϵ, v^ϵ, and c^ϵ. By a direct integrating for, we have

$$\frac{d}{dt}\|n^\epsilon(t)\|_{L^1(\mathbb{R}^2)} + \mu \int_{\mathbb{R}^2} n^\epsilon(t)(n^\epsilon(t) * \rho^\epsilon) dx = \lambda\|n^\epsilon\|_{L^1(\mathbb{R}^2)}, \tag{4}$$

$$\frac{d}{dt}\|v^\epsilon(t)\|_{L^1(\mathbb{R}^2)} + \|v^\epsilon(t)\|_{L^1(\mathbb{R}^2)} = \|n^\epsilon * \rho^\epsilon\|_{L^1(\mathbb{R}^2)}, \tag{5}$$

$$\|c^{\epsilon}(t)\|_{L^1(\mathbb{R}^2)} + \int_0^t \|(n^{\epsilon}(\tau) * \rho^{\epsilon})c^{\epsilon}(\tau)\|_{L^1(\mathbb{R}^2)} d\tau = \|c_0\|_{L^1(\mathbb{R}^2)}. \tag{6}$$

From (4), with the aid of Gronwall's inequality, we have

$$\|n^{\epsilon}(t)\|_{L^1(\mathbb{R}^2)} + \mu \int_0^t \|n^{\epsilon}(t)\|^2_{L^2(\mathbb{R}^2)} dt \le C\|n_0\|_{L^1(\mathbb{R}^2)} e^{\lambda t}. \tag{7}$$

From (5) and (7), we have

$$\begin{aligned} \|v^{\epsilon}(t)\|_{L^1(\mathbb{R}^2)} &\le e^{-t}\|v_0\|_{L^1(\mathbb{R}^2)} + \int_0^t e^{\tau - t}\|n * \rho^{\epsilon}\|_{L^1(\mathbb{R}^2)} d\tau \\ &\le e^{-t}\|v_0\|_{L^1(\mathbb{R}^2)} + C\|n_0\|_{L^1(\mathbb{R}^2)} \int_0^t e^{\tau - t + \lambda\tau} d\tau \\ &\le e^{-t}\|v_0\|_{L^1(\mathbb{R}^2)} + \frac{C\|n_0\|_{L^1(\mathbb{R}^2)} e^{-t}}{\lambda + 1}(e^{\lambda t + t} - 1) \\ &\le \|v_0\|_{L^1(\mathbb{R}^2)} + \frac{C\|n_0\|_{L^1(\mathbb{R}^2)}}{\lambda + 1}(e^{\lambda t + t} - 1). \end{aligned} \tag{8}$$

From (6), the weak maximum principle gives rise to

$$\|c^{\epsilon}\|_{L^1(\mathbb{R}^2) \cap L^{\infty}(\mathbb{R}^2)} \le \|c_0\|_{L^1(\mathbb{R}^2) \cap L^{\infty}(\mathbb{R}^2)}. \tag{9}$$

Testing the third equation in (3) by n^{ϵ} and integrating it over $\mathbb{R}^2$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v^{\epsilon}\|^2_{L^2(\mathbb{R}^2)} + \|\nabla v^{\epsilon}\|^2_{L^2(\mathbb{R}^2)} + \|v^{\epsilon}\|^2_{L^2(\mathbb{R}^2)} \\ &= \int_{\mathbb{R}^2} (n^{\epsilon} * \rho^{\epsilon}) v^{\epsilon} dx \le C\|n^{\epsilon}\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2}\|v^{\epsilon}\|^2_{L^2(\mathbb{R}^2)}, \end{aligned}$$

from which, with aid of (8),

$$\begin{aligned} &\|v^{\epsilon}\|^2_{L^2(\mathbb{R}^2)} + 2\int_0^t \|\nabla v^{\epsilon}\|^2_{L^2(\mathbb{R}^2)} d\tau + \int_0^t \|v^{\epsilon}\|^2_{L^2(\mathbb{R}^2)} d\tau \\ &\le \|v_0\|^2_{L^2(\mathbb{R}^2)} + C\int_0^t \|n^{\epsilon}\|^2_{L^2(\mathbb{R}^2)} d\tau \le \|v_0\|^2_{L^2(\mathbb{R}^2)} + C\mu^{-1}e^{\lambda t}\|n_0\|_{L^1(\mathbb{R}^2)}. \end{aligned} \tag{10}$$

Testing the second equation in (3) by c^{ϵ} and integrating it over $\mathbb{R}^2$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|c^{\epsilon}\|^2_{L^2(\mathbb{R}^2)} + \|\nabla c^{\epsilon}\|^2_{L^2(\mathbb{R}^2)} = -\int_{\mathbb{R}^2} (n^{\epsilon} * \rho^{\epsilon})(c^{\epsilon})^2 dx,$$

from which,

$$\|c^{\epsilon}(t)\|^2_{L^2(\mathbb{R}^2)} + 2\int_0^t \|\nabla c^{\epsilon}(\tau)\|^2_{L^2(\mathbb{R}^2)} d\tau + 2\int_0^t \int_{\mathbb{R}^2} n^{\epsilon}(c^{\epsilon})^2 dx d\tau \le \|c_0\|^2_{L^2(\mathbb{R}^2)}. \tag{11}$$

On the other hand, we multiply u^{ϵ} with the fourth equation of (3) and apply the Gagliardo–Nirenberg inequality to deduce that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u^{\epsilon}\|^2_{L^2(\mathbb{R}^2)} + \|\nabla u^{\epsilon}\|^2_{L^2(\mathbb{R}^2)} \\ &= -\int_{\mathbb{R}^2} u^{\epsilon} \cdot n^{\epsilon} \nabla\phi dx \le \|u^{\epsilon}\|_{L^2(\mathbb{R}^2)}\|n^{\epsilon}\|_{L^2(\mathbb{R}^2)}\|\nabla\phi\|_{L^{\infty}(\mathbb{R}^2)} \\ &\le C\|u^{\epsilon}\|^2_{L^2(\mathbb{R}^2)}\|\nabla\phi\|^2_{L^{\infty}(\mathbb{R}^2)} + C\|n^{\epsilon}\|^2_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{12}$$

From the Gronwall inequality and (7), we have

$$\begin{aligned}&\|u^\epsilon\|_{L^2(\mathbb{R}^2)}^2+\int_0^t\|\nabla u^\epsilon(\tau)\|_{L^2(\mathbb{R}^2)}^2d\tau\\&\le(\|u_0\|_{L^2(\mathbb{R}^2)}^2+C\mu^{-1}e^{\lambda t}\|n_0\|_{L^1(\mathbb{R}^2)})\exp(Ct\|\nabla\phi\|_{L^\infty(\mathbb{R}^2)}^2).\end{aligned}\tag{13}$$

It follows from Hölder's inequality, Gagliardo–Nirenberg's inequality, and Young's inequality that

$$\begin{aligned}&\int_{\mathbb{R}^2}(u^\epsilon\cdot\nabla v^\epsilon)\Delta v^\epsilon dx\le\|u\|_{L^4(\mathbb{R}^2)}\|\nabla v\|_{L^4(\mathbb{R}^2)}\|\Delta v\|_{L^2(\mathbb{R}^2)}\\&\le C\|u\|_{L^4(\mathbb{R}^2)}\|\nabla v\|_{L^4(\mathbb{R}^2)}^{1/2}\|\Delta v\|_{L^2(\mathbb{R}^2)}^{3/2}\\&\le C\|u\|_{L^4(\mathbb{R}^2)}^4\|\nabla v\|_{L^2(\mathbb{R}^2)}^2+\frac{1}{4}\|\Delta v\|_{L^2(\mathbb{R}^2)}^2\\&\le C\|u\|_{L^2(\mathbb{R}^2)}^2\|\nabla u\|_{L^2(\mathbb{R}^2)}^2\|\nabla v\|_{L^2(\mathbb{R}^2)}^2+\frac{1}{4}\|\Delta v\|_{L^2(\mathbb{R}^2)}^2.\end{aligned}\tag{14}$$

Applying ∇ to the third equation of (3) gives

$$\partial_t\nabla v^\epsilon+\nabla(u^\epsilon\nabla v^\epsilon)-\Delta\nabla v^\epsilon=-\nabla v^\epsilon+\nabla(n^\epsilon*\rho^\epsilon).$$

Taking the L^2 inner product for above equality with ∇v^ϵ and applying (14), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla v^\epsilon\|_{L^2(\mathbb{R}^2)}^2+\|\nabla v^\epsilon\|_{L^2(\mathbb{R}^2)}^2+\|\Delta v^\epsilon\|_{L^2(\mathbb{R}^2)}^2\\&=\int_{\mathbb{R}^2}(u^\epsilon\cdot\nabla v^\epsilon)\Delta v^\epsilon dx+\int_{\mathbb{R}^2}\nabla(n^\epsilon*\rho^\epsilon)\nabla v^\epsilon dx\\&\le C\|u\|_{L^2(\mathbb{R}^2)}^2\|\nabla u\|_{L^2(\mathbb{R}^2)}^2\|\nabla v\|_{L^2(\mathbb{R}^2)}^2+\frac{1}{4}\|\Delta v\|_{L^2(\mathbb{R}^2)}^2+\frac{1}{4}\|\Delta v\|_{L^2(\mathbb{R}^2)}^2+C\|n\|_{L^2(\mathbb{R}^2)}^2,\end{aligned}$$

from which, from (7), from (13), and from the Gronwall inequality, we have

$$\begin{aligned}&\|\nabla v^\epsilon\|_{L^2(\mathbb{R}^2)}^2+\int_0^t\|\nabla v^\epsilon\|_{L^2(\mathbb{R}^2)}^2d\tau+\int_0^t\|\Delta v^\epsilon\|_{L^2(\mathbb{R}^2)}^2d\tau\\&\le(\|\nabla v_0\|_{L^2(\mathbb{R}^2)}^2+C\mu^{-1}e^{\lambda t}\|n_0\|_{L^1(\mathbb{R}^2)})\exp\Big(C\int_0^t\|u\|_{L^2(\mathbb{R}^2)}^2\|\nabla u\|_{L^2(\mathbb{R}^2)}^2dt\Big)\\&\le(\|\nabla v_0\|_{L^2(\mathbb{R}^2)}^2+Ce^t\|n_0\|_{L^1(\mathbb{R}^2)})\\&\quad\times\exp\Big((\|u_0\|_{L^2(\mathbb{R}^2)}^2+C\mu^{-1}e^{\lambda t}\|n_0\|_{L^1(\mathbb{R}^2)})^2\exp(Ct\|\nabla\phi\|_{L^\infty(\mathbb{R}^2)}^2)\Big).\end{aligned}$$

Combining this with (7), (8), (9), (10), (11), and (13) directly result. □

Proposition 2. *Suppose $v_0, n_0, c_0\ge 0$ and that $(u_0, c_0, v_0, n_0)\in X_0$ and $\nabla\phi\in L^\infty(\mathbb{R}^2)$. Let $(n^\epsilon, v^\epsilon, c^\epsilon, u^\epsilon)$ be the solutions to the model (3). Then, there a constant $C>0$ exists independently of ϵ such that*

$$\begin{aligned}&\int_{\mathbb{R}^2}\Big(n^\epsilon|\ln n^\epsilon|+|\nabla\sqrt{c^\epsilon}|^2\Big)dx+\int_0^t\int_{\mathbb{R}^2}\frac{|\nabla n^\epsilon|^2}{n^\epsilon}dxd\tau+\int_0^t\int_{\mathbb{R}^2}(n^\epsilon)^2\langle x\rangle dxd\tau\\&+\int_0^t\|\Delta\sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}^2+\int_{\mathbb{R}^2}|\nabla\sqrt{c^\epsilon}|^4(c^\epsilon)^{-1}dxd\tau\le Ce^{Ce^{Ct}}.\end{aligned}\tag{15}$$

Proof. Multiplying Equation $(3)_2$ by $\ln n^\epsilon$ and integrating over $\mathbb{R}^2$ and by prats, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2} n^\epsilon \ln n^\epsilon dx + \int_{\mathbb{R}^2} \frac{|\nabla n^\epsilon|^2}{n^\epsilon} dx \\
&= \int_{\mathbb{R}^2} \nabla n^\epsilon \nabla (c^\epsilon * \rho^\epsilon)) dx - \int_{\mathbb{R}^2} n^\epsilon \Delta (v^\epsilon * \rho^\epsilon)) dx + \int_{\mathbb{R}^2} n^\epsilon \ln n^\epsilon dx - \int_{\mathbb{R}^2} (n^\epsilon)^2 \ln n^\epsilon dx \\
&\leq \int_{\mathbb{R}^2} \nabla n^\epsilon \nabla (c^\epsilon * \rho^\epsilon)) dx + \epsilon \|n^\epsilon\|^2_{L^2(\mathbb{R}^2)} + C\|\Delta v^\epsilon\|^2_{L^2(\mathbb{R}^2)} + \int_{\mathbb{R}^2} n^\epsilon \ln n^\epsilon dx - \int_{\mathbb{R}^2} (n^\epsilon)^2 \ln n^\epsilon dx \\
&= \int_{\mathbb{R}^2} \nabla n^\epsilon \nabla (c^\epsilon * \rho^\epsilon)) dx + \epsilon \|n^\epsilon\|^2_{L^2(\mathbb{R}^2)} + C\|\Delta v^\epsilon\|^2_{L^2(\mathbb{R}^2)} + \int_{\mathbb{R}^2} n^\epsilon \ln n^\epsilon dx \\
&\quad - \int_{0<n^\epsilon \leq e^{-\langle x\rangle}} (n^\epsilon)^2 \ln n^\epsilon dx - \int_{e^{-\langle x\rangle}<n^\epsilon<1} (n^\epsilon)^2 \ln n^\epsilon dx - \int_{n^\epsilon \geq 1} (n^\epsilon)^2 \ln n^\epsilon dx \\
&= \int_{\mathbb{R}^2} \nabla n^\epsilon \nabla (c^\epsilon * \rho^\epsilon)) dx + C \|n^\epsilon\|^2_{L^2(\mathbb{R}^2)} + C\|\Delta v^\epsilon\|^2_{L^2(\mathbb{R}^2)} + \int_{\mathbb{R}^2} n^\epsilon \ln n^\epsilon dx \\
&\quad - \int_{0<n^\epsilon \leq e^{-\langle x\rangle}} (n^\epsilon)^2 \ln n^\epsilon dx - \int_{e^{-\langle x\rangle}<n^\epsilon<1} (n^\epsilon)^2 \ln n^\epsilon dx \\
&\leq \int_{\mathbb{R}^2} \nabla n^\epsilon \nabla (c^\epsilon * \rho^\epsilon)) dx + \epsilon \|n^\epsilon\|^2_{L^2(\mathbb{R}^2)} + C\|\Delta v^\epsilon\|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \int_{\mathbb{R}^2} n^\epsilon \ln n^\epsilon dx + \int_{\mathbb{R}^2} (n^\epsilon)^2 \langle x\rangle dx + C,
\end{aligned} \tag{16}$$

where $\langle x\rangle = \sqrt{1+x^2}$. Since we work in whole space $\mathbb{R}^2$, here we have to bound $\int_{\mathbb{R}^2} n^\epsilon \ln n^\epsilon$ from below. In order to do that, we have to control the behavior of n^ϵ as $|x| \to \infty$, similarly to [15,25]. To perform this task, we multiply $(3)_1$ by the smooth function $\langle x\rangle$; then, by integrating it over $\mathbb{R}^2$, by Young's inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2} \langle x\rangle n^\epsilon dx + \int_{\mathbb{R}^2} (n^\epsilon)^2 \langle x\rangle dx \\
&= \int_{\mathbb{R}^2} n^\epsilon u^\epsilon \nabla \langle x\rangle dx + \int_{\mathbb{R}^2} n^\epsilon \Delta \langle x\rangle dx + \int_{\mathbb{R}^2} n^\epsilon \cdot \nabla c^\epsilon \cdot \nabla \langle x\rangle dx^\epsilon \\
&\quad + \int_{\mathbb{R}^2} n^\epsilon \cdot \nabla v^\epsilon \cdot \nabla \langle x\rangle dx^\epsilon + \int_{\mathbb{R}^2} n^\epsilon \langle x\rangle dx \\
&\leq \|u^\epsilon\|_{L^2(\mathbb{R}^2)} \|n^\epsilon\|_{L^2(\mathbb{R}^2)} \|\nabla \langle x\rangle\|_{L^\infty(\mathbb{R}^2)} + \|\Delta \langle x\rangle\|_{L^\infty(\mathbb{R}^2)} \|n^\epsilon\|_{L^1(\mathbb{R}^2)} + \int_{\mathbb{R}^2} n^\epsilon \langle x\rangle dx \\
&\quad + \|\nabla c^\epsilon\|_{L^2(\mathbb{R}^2)} \|n^\epsilon\|_{L^2(\mathbb{R}^2)} \|\nabla \langle x\rangle\|_{L^\infty(\mathbb{R}^2)} + \|\nabla v^\epsilon\|_{L^2(\mathbb{R}^2)} \|n^\epsilon\|_{L^2(\mathbb{R}^2)} \|\nabla \langle x\rangle\|_{L^\infty(\mathbb{R}^2)} \\
&\leq C\Big(\|(u^\epsilon, \nabla c^\epsilon, \nabla v^\epsilon)\|^2_{L^2(\mathbb{R}^2)} + \|n^\epsilon\|_{L^1(\mathbb{R}^2)} \Big) + \epsilon \|n^\epsilon\|^2_{L^2(\mathbb{R}^2)} + \int_{\mathbb{R}^2} n^\epsilon \langle x\rangle dx.
\end{aligned}$$

Multiplying the above inequality by 2 and using (16), one obtains

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2} (n^\epsilon \ln n^\epsilon + 2\langle x\rangle n) dx + \int_{\mathbb{R}^2} \frac{|\nabla n^\epsilon|^2}{n^\epsilon} dx + \int_{\mathbb{R}^2} (n^\epsilon)^2 \langle x\rangle dx \\
&\leq \int_{\mathbb{R}^2} \nabla n^\epsilon \nabla (c^\epsilon * \rho^\epsilon)) dx + C\|\Delta v^\epsilon\|^2_{L^2(\mathbb{R}^2)} + 3\epsilon \|n^\epsilon\|^2_{L^2(\mathbb{R}^2)} \\
&\quad + C\Big(\|(u^\epsilon, \nabla c^\epsilon, \nabla v^\epsilon)\|^2_{L^2(\mathbb{R}^2)} + \|n^\epsilon\|_{L^1(\mathbb{R}^2)} \Big) + \int_{\mathbb{R}^2} n^\epsilon \ln n^\epsilon + 2n^\epsilon \langle x\rangle dx + C.
\end{aligned} \tag{17}$$

By the pointwise identity $\Delta c^\epsilon = 2|\nabla \sqrt{c^\epsilon}|^2 + 2\sqrt{c^\epsilon} \Delta \sqrt{c^\epsilon}$, the third equation of (3) thereupon turns into the relation

$$\partial_t \sqrt{c^\epsilon} - \Delta \sqrt{c^\epsilon} - (\sqrt{c^\epsilon})^{-1} |\nabla \sqrt{c^\epsilon}|^2 + u^\epsilon \cdot \nabla \sqrt{c^\epsilon} = -\frac{1}{2}\sqrt{c^\epsilon}(n^\epsilon * \rho^\epsilon). \tag{18}$$

Multiplying the above equation by $-\Delta\sqrt{c^\epsilon}$, by Young's inequality, we integrate by parts to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\sqrt{c^\epsilon}\|^2_{L^2(\mathbb{R}^2)}+\|\Delta\sqrt{c^\epsilon}\|^2_{L^2(\mathbb{R}^2)}\\
&=\int_{\mathbb{R}^2}u^\epsilon\cdot\nabla\sqrt{c^\epsilon}\cdot\Delta c^\epsilon dx+\int_{\mathbb{R}^2}\frac{1}{2}(n^\epsilon*\rho^\epsilon)\sqrt{c^\epsilon}\Delta\sqrt{c^\epsilon}x+I_2\\
&=-\int_{\mathbb{R}^2}\sqrt{c^\epsilon}(\nabla\sqrt{c^\epsilon}\cdot\nabla)u^\epsilon\cdot(\nabla\sqrt{c^\epsilon})\sqrt{c^\epsilon}^{-1}dx+\int_{\mathbb{R}^2}\frac{1}{2}(n^\epsilon*\rho^\epsilon)\sqrt{c^\epsilon}\Delta\sqrt{c^\epsilon}x+I_2\\
&\le 3\|c^\epsilon\|_{L^\infty(\mathbb{R}^2)}\|\nabla u^\epsilon\|^2_{L^2(\mathbb{R}^2)}+\frac{1}{12}\int_{\mathbb{R}^2}\frac{|\nabla\sqrt{c^\epsilon}|^4}{c^\epsilon}dx+\int_{\mathbb{R}^2}\frac{1}{2}(n^\epsilon*\rho^\epsilon)\sqrt{c^\epsilon}\Delta\sqrt{c^\epsilon}x+I_2\\
&\le C\|\nabla u^\epsilon\|^2_{L^2(\mathbb{R}^2)}+\frac{1}{12}\int_{\mathbb{R}^2}\frac{|\nabla\sqrt{c^\epsilon}|^4}{c^\epsilon}dx+I_2\\
&\quad-\int_{\mathbb{R}^2}\frac{1}{4}\nabla n^\epsilon\cdot\nabla(c^\epsilon*\rho^\epsilon)+\frac{1}{2}n^\epsilon*\rho^\epsilon|\nabla\sqrt{c^\epsilon}|^2dx,
\end{aligned}
\tag{19}
$$

where

$$
I_2=-\int_{\mathbb{R}^2}(c^\epsilon)^{-\frac{1}{2}}|\nabla\sqrt{c^\epsilon}|^2\Delta\sqrt{c^\epsilon}dx.
$$

For I_2, we have that

$$
\begin{aligned}
I_2&=-\sum_{i,j}\int_{\mathbb{R}^2}(\sqrt{c})^{-1}(\partial_j\sqrt{c})^2(\partial_{i,i}\sqrt{c})^2dx\\
&=-\sum_{i,j}\int_{\mathbb{R}^2}(\sqrt{c^\epsilon})^{-2}|\partial_i\sqrt{c^\epsilon}|^2|\partial_j\sqrt{c^\epsilon}|^2dx+2\sum_{i=j}\int_{\mathbb{R}^2}(\sqrt{c^\epsilon})^{-1}\partial_i\sqrt{c^\epsilon}\partial_{i,j}\sqrt{c^\epsilon}\partial_j\sqrt{c^\epsilon}dx\\
&\quad+2\sum_{i\neq j}\int_{\mathbb{R}^2}(\sqrt{c^\epsilon})^{-1}\partial_i\sqrt{c^\epsilon}\partial_{i,j}\sqrt{c^\epsilon}\partial_j\sqrt{c^\epsilon}dx\\
&=-\sum_{i,j}\int_{\mathbb{R}^2}(\sqrt{c^\epsilon})^{-2}|\partial_i\sqrt{c^\epsilon}|^2|\partial_j\sqrt{c^\epsilon}|^2dx-2I_2-2\sum_{i\neq j}\int_{\mathbb{R}^2}(\sqrt{c^\epsilon})^{-1}(\partial_j\sqrt{c^\epsilon})^2\partial_{i,i}\sqrt{c^\epsilon}dx\\
&\quad+2\sum_{i\neq j}\int_{\mathbb{R}^2}(\sqrt{c^\epsilon})^{-1}\partial_i\sqrt{c^\epsilon}\partial_{i,j}\sqrt{c^\epsilon}\partial_j\sqrt{c^\epsilon}dx,
\end{aligned}
\tag{20}
$$

from which, by Young's inequality, it follows that

$$
\begin{aligned}
I_2&=-\frac{1}{3}\sum_{i,j}\int_{\mathbb{R}^2}(\sqrt{c^\epsilon})^{-2}|\partial_i\sqrt{c}|^2|\partial_j\sqrt{c^\epsilon}|^2dx-\frac{2}{3}\sum_{i\neq j}\int_{\mathbb{R}^2}(\sqrt{c^\epsilon})^{-1}(\partial_j\sqrt{c^\epsilon})^2\partial_{i,i}\sqrt{c^\epsilon}dx\\
&\quad+\frac{2}{3}\sum_{i\neq j}\int_{\mathbb{R}^2}(\sqrt{c^\epsilon})^{-1}\partial_i\sqrt{c^\epsilon}\partial_{i,j}\sqrt{c^\epsilon}\partial_j\sqrt{c^\epsilon}dx\\
&\le-\frac{1}{3}\sum_{i,j}\int_{\mathbb{R}^2}(\sqrt{c^\epsilon})^{-2}|\partial_i\sqrt{c^\epsilon}|^2|\partial_j\sqrt{c^\epsilon}|^2dx\\
&\quad+\frac{1}{6}\sum_{i}\int_{\mathbb{R}^2}(\sqrt{c^\epsilon})^{-2}|\partial_i\sqrt{c^\epsilon}|^4dx+\frac{2}{3}\sum_{i}\int_{\mathbb{R}^2}(\partial_{i,i}\sqrt{c^\epsilon})^2dx\\
&\quad+\frac{1}{6}\sum_{i\neq j}\int_{\mathbb{R}^2}(\sqrt{c^\epsilon})^{-2}|\partial_i\sqrt{c^\epsilon}|^2|\partial_j\sqrt{c}|^2dx+\frac{2}{3}\sum_{i\neq j}\int_{\mathbb{R}^2}(\partial_{i,j}\sqrt{c^\epsilon})^2dx\\
&\le-\frac{1}{6}\sum_{i,j}\int_{\mathbb{R}^2}(\sqrt{c^\epsilon})^{-2}|\partial_i\sqrt{c^\epsilon}|^2|\partial_j\sqrt{c^\epsilon}|^2dx+\frac{2}{3}\sum_{i,j}\int_{\mathbb{R}^2}(\partial_{i,j}\sqrt{c^\epsilon})^2dx.
\end{aligned}
\tag{21}
$$

Plugging (21) into (19), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\sqrt{c^\epsilon}\|^2_{L^2(\mathbb{R}^2)}+\|\Delta\sqrt{c^\epsilon}\|^2_{L^2(\mathbb{R}^2)}+\int_{\mathbb{R}^2}(\frac{1}{2}n^\epsilon*\rho^\epsilon)|\nabla\sqrt{c^\epsilon}|^2dx\\
&\quad+\frac{1}{6}\sum_{i,j}\int_{\mathbb{R}^2}(\sqrt{c^\epsilon})^{-2}|\partial_i\sqrt{c}|^2|\partial_j\sqrt{c^\epsilon}|^2dx\\
&\le C\|\nabla u^\epsilon\|^2_{L^2(\mathbb{R}^2)}+\frac{1}{12}\int_{\mathbb{R}^2}\frac{|\nabla\sqrt{c^\epsilon}|^4}{c^\epsilon}dx-\int_{\mathbb{R}^2}\frac{1}{4}\nabla n^\epsilon\cdot\nabla(c^\epsilon*\rho^\epsilon)dx\\
&\quad+\frac{2}{3}\sum_{i,j}\int_{\mathbb{R}^2}(\partial_{i,j}\sqrt{c^\epsilon})^2dx.
\end{aligned}
\tag{22}
$$

Due to $\|\Delta\sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}=\|\nabla^2\sqrt{c^\epsilon}\|_{L^2(\mathbb{R}^2)}$, we have by multiplying (22) by 4 that

$$
\begin{aligned}
&2\frac{d}{dt}\|\nabla\sqrt{c^\epsilon}\|^2_{L^2(\mathbb{R}^2)}+2\int_{\mathbb{R}^2}n^\epsilon*\rho^\epsilon|\nabla\sqrt{c}|^2dx\\
&\quad+\frac{4}{3}\|\Delta\sqrt{c^\epsilon}\|^2_{L^2(\mathbb{R}^2)}+\frac{1}{3}\int_{\mathbb{R}^2}\frac{|\nabla\sqrt{c^\epsilon}|^4}{c^\epsilon}dx\\
&\le C\|\nabla u^\epsilon\|^2_{L^2(\mathbb{R}^2)}-\int_{\mathbb{R}^2}\nabla n^\epsilon\cdot\nabla(c^\epsilon*\rho^\epsilon)dx.
\end{aligned}
\tag{23}
$$

Summing up (17) and (23), we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}\Big(n^\epsilon\ln n^\epsilon+2\langle x\rangle n+2|\nabla\sqrt{c^\epsilon}|^2\Big)dx+\int_{\mathbb{R}^2}\frac{|\nabla n^\epsilon|^2}{n^\epsilon}dx+\int_{\mathbb{R}^2}(n^\epsilon)^2\langle x\rangle dx\\
&\quad+\frac{4}{3}\|\Delta\sqrt{c^\epsilon}\|^2_{L^2(\mathbb{R}^2)}+\frac{1}{3}\int_{\mathbb{R}^2}|\nabla\sqrt{c^\epsilon}|^4(c^\epsilon)^{-1}dx+2\int_{\mathbb{R}^2}n^\epsilon*\rho^\epsilon|\nabla\sqrt{c^\epsilon}|^2dx\\
&\le C\Big(\|n^\epsilon\|^2_{L^2(\mathbb{R}^2)}+\|\nabla u^\epsilon\|^2_{L^2(\mathbb{R}^2)}+\|\Delta v^\epsilon\|^2_{L^2(\mathbb{R}^2)}+\|(u^\epsilon,\nabla c^\epsilon,\nabla v^\epsilon)\|^2_{L^2(\mathbb{R}^2)}+\|n^\epsilon\|_{L^1(\mathbb{R}^2)}\Big)\\
&\quad+\int_{\mathbb{R}^2}\Big(n^\epsilon\ln n^\epsilon+2\langle x\rangle n+2|\nabla\sqrt{c^\epsilon}|^2\Big)dx,
\end{aligned}
\tag{24}
$$

from which, let

$$
U_0=\int_{\mathbb{R}^2}\Big(n_0\ln n_0+2\langle x\rangle n_0+2|\nabla\sqrt{c_0}|^2\Big)dx,
$$

we have

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\Big(n^\epsilon\ln n^\epsilon+2\langle x\rangle n+2|\nabla\sqrt{c^\epsilon}|^2\Big)dx+\int_0^t\int_{\mathbb{R}^2}\frac{|\nabla n^\epsilon|^2}{n^\epsilon}dxd\tau+\int_0^t\int_{\mathbb{R}^2}(n^\epsilon)^2\langle x\rangle dxd\tau\\
&\quad+\int_0^t\|\Delta\sqrt{c^\epsilon}\|^2_{L^2(\mathbb{R}^2)}+\int_{\mathbb{R}^2}|\nabla\sqrt{c^\epsilon}|^4(c^\epsilon)^{-1}dxd\tau+\int_0^t\int_{\mathbb{R}^2}n^\epsilon*\rho^\epsilon|\nabla\sqrt{c^\epsilon}|^2dxd\tau\\
&\le C\int_0^t\Big(\|(\Delta v^\epsilon,n^\epsilon,\nabla u^\epsilon,\nabla c^\epsilon,\nabla v^\epsilon)\|^2_{L^2(\mathbb{R}^2)}+\|u^\epsilon\|^2_{L^2(\mathbb{R}^2)}+\|n^\epsilon\|_{L^1(\mathbb{R}^2)}\Big)d\tau\\
&\quad+C\int_0^t\int_{\mathbb{R}^2}\Big(n^\epsilon\ln n^\epsilon+2\langle x\rangle n+2|\nabla\sqrt{c^\epsilon}|^2\Big)dxd\tau+CU_0.
\end{aligned}
\tag{25}
$$

By Proposition 1, one has

$$
\begin{aligned}
&\int_0^t\Big(\|(\Delta v^\epsilon,n^\epsilon,\nabla u^\epsilon,\nabla c^\epsilon,\nabla v^\epsilon)\|^2_{L^2(\mathbb{R}^2)}+\|u^\epsilon\|^2_{L^2(\mathbb{R}^2)}+\|n^\epsilon\|_{L^1(\mathbb{R}^2)}\Big)d\tau\\
&\le C(e^{Ce^{Ct}}+e^{Ct}+te^{Ct}+1)\le Ce^{Ce^{Ct}}.
\end{aligned}
\tag{26}
$$

By same reasoning for obtaining (2.27) in [15], we can easily obtain

$$
\begin{aligned}
\int_{\mathbb{R}^2}n^\epsilon|\ln n^\epsilon|dx&\le\int_{\mathbb{R}^2}n^\epsilon\ln n^\epsilon dx+4e^{-1}\int_{\mathbb{R}^2}e^{-\frac{1}{2}\langle x\rangle}dx+2\int_{\mathbb{R}^2}n^\epsilon\langle x\rangle dx\\
&\le\int_{\mathbb{R}^2}n^\epsilon\ln n^\epsilon dx+C+2\int_{\mathbb{R}^2}n^\epsilon\langle x\rangle dx,
\end{aligned}
\tag{27}
$$

from (26) and (27) and applying Grönwall's inequality to (25), we have

$$\begin{aligned}&\int_{\mathbb{R}^2}\Big(n^\epsilon|\ln n^\epsilon|+|\nabla\sqrt{c^\epsilon}|^2\Big)dx+\int_0^t\int_{\mathbb{R}^2}\frac{|\nabla n^\epsilon|^2}{n^\epsilon}dxd\tau+\int_0^t\int_{\mathbb{R}^2}(n^\epsilon)^2\langle x\rangle dxd\tau\\&+\int_0^t\|\Delta\sqrt{c^\epsilon}\|^2_{L^2(\mathbb{R}^2)}+\int_{\mathbb{R}^2}|\nabla\sqrt{c^\epsilon}|^4(c^\epsilon)^{-1}dxd\tau\\&\le Ce^{Ct}(CU_0+e^{Ce^{Ct}})+C\le Ce^{Ce^{Ct}}.\end{aligned}\tag{28}$$

The proof of Proposition 2 is completed. □

With the preparations of Propositions 1 and 2 at hand, we can further obtain a uniform estimate for the high regularity of $(n^\epsilon, v^\epsilon, c^\epsilon, u^\epsilon)$.

Proposition 3. *Assume that $v_0, n_0, c_0 \ge 0, (u_0, c_0, v_0, n_0) \in X_0$ and $\nabla\phi \in L^\infty(\mathbb{R}^2)$. Let $(n^\epsilon, v^\epsilon, c^\epsilon, u^\epsilon)$ be a solution of system (3). Then, a constant $C > 0$ exists independently of ϵ such that*

$$\|c^\epsilon\|_{L^\infty_t H^1}+\|c^\epsilon\|_{L^2_t H^2}\le Ce^{Ct}$$

as well as

$$\|n^\epsilon\|^2_{L^2(\mathbb{R}^2)}+\int_0^t\Big(\|\nabla n^\epsilon\|^2_{L^2(\mathbb{R}^2)}+\|n^\epsilon\|^3_{L^3(\mathbb{R}^2)}\Big)ds\le Ce^{Ce^{Ce^{Ct}}}.$$

Proof. Now, by the Cauchy–Schwarzed inequality, Propositions 1 and 2, we have

$$\begin{aligned}&\|c^\epsilon\|_{L^\infty_t(L^2(\mathbb{R}^2))}+\|\nabla c^\epsilon\|_{L^\infty_t(L^2(\mathbb{R}^2))}=\|c^\epsilon\|_{L^\infty_t(L^2(\mathbb{R}^2))}+2\|\sqrt{c^\epsilon}\nabla\sqrt{c^\epsilon}\|_{L^\infty_t(L^2(\mathbb{R}^2))}\\&\le\|c^\epsilon\|_{L^\infty_t(L^2(\mathbb{R}^2))}+2\|c^\epsilon\|^{\frac12}_{L^\infty_t(L^\infty(\mathbb{R}^2))}\|\nabla\sqrt{c^\epsilon}\|_{L^\infty_t(L^2(\mathbb{R}^2))}\\&\le\|c_0\|_{L^2(\mathbb{R}^2)}+C\|c_0\|^{\frac12}_{L^\infty(\mathbb{R}^2)}e^{Ct}\le Ce^{Ct}.\end{aligned}\tag{29}$$

Using identity $\Delta c^\epsilon = 2\sqrt{c^\epsilon}\Delta\sqrt{c^\epsilon}+2|\nabla\sqrt{c^\epsilon}|^2$ again, it follows from Propositions 1 and 2 that

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^2}|\Delta c^\epsilon|^2dxd\tau\le 8\int_0^t\int_{\mathbb{R}^2}c^\epsilon|\Delta\sqrt{c}|^2+|\Delta\sqrt{c^\epsilon}|^2dxd\tau\\&\le 8\|c_0\|_{L^\infty(\mathbb{R}^2)}\int_0^t\|\Delta\sqrt{c^\epsilon}\|^2_{L^2(\mathbb{R}^2)}+\|(c^\epsilon)^{-1}|\nabla\sqrt{c^\epsilon}|^4\|_{L^1(\mathbb{R}^2)}d\tau\le Ce^{Ct}.\end{aligned}\tag{30}$$

Using Proposition 1, we have

$$\int_0^t\|c^\epsilon\|^2_{L^2(\mathbb{R}^2)}d\tau+\int_0^t\|\nabla c^\epsilon\|^2_{L^2(\mathbb{R}^2)}d\tau\le t\|c_0\|^2_{L^2(\mathbb{R}^2)}+\frac12\|c_0\|^2_{L^2(\mathbb{R}^2)}.\tag{31}$$

Then, owing to (29)–(31), we have

$$\|c^\epsilon\|_{L^\infty_t H^1}+\|c^\epsilon\|_{L^2_t H^2}\le Ce^{Ct}.\tag{32}$$

Testing the first equation in (3) against n^ϵ and using Young's inequality yields

$$\begin{aligned}&\frac12\frac{d}{dt}\|n^\epsilon\|^2_{L^2(\mathbb{R}^2)}+\|\nabla n^\epsilon\|^2_{L^2(\mathbb{R}^2)}+\mu\int_{\mathbb{R}^2}(n^\epsilon)^2(n^\epsilon*\rho^\epsilon)dx\\&=\lambda\|n^\epsilon\|^2_{L^2(\mathbb{R}^2)}-\frac{\chi}{2}\int_{\mathbb{R}^2}\Delta(c*\rho^\epsilon)(n^\epsilon)^2dx+\frac{\xi}{2}\int_{\mathbb{R}^2}\Delta(v*\rho^\epsilon)(n^\epsilon)^2dx\\&\le\|n^\epsilon\|^2_{L^2(\mathbb{R}^2)}+\frac12\|(\Delta c^\epsilon,\Delta v^\epsilon)\|_{L^2(\mathbb{R}^2)}\|n^\epsilon\|^2_{L^4(\mathbb{R}^2)}\\&\le\|n^\epsilon\|^2_{L^2(\mathbb{R}^2)}+\frac12\|(\Delta c^\epsilon,\Delta v^\epsilon)\|_{L^2(\mathbb{R}^2)}\|n^\epsilon\|_{L^2(\mathbb{R}^2)}\|\nabla n^\epsilon\|_{L^2(\mathbb{R}^2)}\\&\le C\Big(1+\|(\Delta c^\epsilon,\Delta v^\epsilon)\|^2_{L^2(\mathbb{R}^2)}\Big)\|n^\epsilon\|^2_{L^2(\mathbb{R}^2)}+\frac12\|\nabla n^\epsilon\|^2_{L^2(\mathbb{R}^2)},\end{aligned}$$

on the basis of which, (32), Proposition 1, and Grönwall's inequality, it follows that

$$\begin{aligned}
&\|n^\epsilon\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \Big(\|\nabla n^\epsilon\|^2_{L^2(\mathbb{R}^2)} + \|(n^\epsilon)^2(n^\epsilon * \rho^\epsilon)\|_{L^1(\mathbb{R}^2)}\Big) ds \\
&\le \|n_0\|^2_{L^2(\mathbb{R}^2)} \exp\Big(C\int_0^t \Big(1 + \|(\Delta c^\epsilon, \Delta v^\epsilon)\|^2_{L^2(\mathbb{R}^2)}\Big) ds\Big) \qquad (33)\\
&\le \|n_0\|^2_{L^2(\mathbb{R}^2)} \exp\Big(Ct + Ce^{Ct} + Ce^{Ce^{Ct}}\Big) \le Ce^{Ce^{Ce^{Ct}}}.
\end{aligned}$$

Collecting the above inequality with (32), we can thereby complete the proof of Proposition 3. □

Furthermore, using the regularized equations, and the uniform estimates obtained above, we can directly obtain the following proposition.

Proposition 4. *Assume that v_0, $n_0\, c_0 \ge 0$, $(u_0, c_0, v_0, n_0) \in X_0$ and $\nabla\phi \in L^\infty(\mathbb{R}^2)$. Let $(n^\epsilon,\, m^\epsilon,\, c^\epsilon,\, u^\epsilon)$ be a solution of system (3). Then,*

$$\|(\partial_t u^\epsilon, \partial_t n^\epsilon, \partial_t c^\epsilon)\|_{L^2_t(H^{-1}(\mathbb{R}^2))} + \|\partial_t v^\epsilon\|_{L^2_t L^2(\mathbb{R}^2)} \le C(t).$$

Proof. We will prove the uniform boundedness for $\partial_t n^\epsilon, \partial_t v^\epsilon, \partial_t c^\epsilon$ and $\partial_t u^\epsilon$. Making use of (3) and the Gagliardo–Nirenberg inequality

$$\|f\|^2_{L^4_t(L^4(\mathbb{R}^2))} \le \|f\|_{L^\infty_t(L^2(\mathbb{R}^2))}\|\nabla f\|_{L^2_t(L^2(\mathbb{R}^2))}$$

and $H^{s+\epsilon}(\mathbb{R}^2) \hookrightarrow H^s(\mathbb{R}^2)$ for any $\epsilon > 0$, one can readily obtain that

$$\begin{aligned}
\|n^\epsilon_t\|_{L^2_t H^{-1}} &\lesssim \|u^\epsilon\cdot n^\epsilon\|_{L^2_t(L^2(\mathbb{R}^2))} + \|n^\epsilon\|_{L^2_t(H^1(\mathbb{R}^2))} + \|n^\epsilon\nabla(c^\epsilon * \rho^\epsilon))\|_{L^2_t(L^2(\mathbb{R}^2))} \\
&\quad + \|n^\epsilon\nabla(c^\epsilon * \rho^\epsilon))\|_{L^2_t(L^2(\mathbb{R}^2))} + \|n^\epsilon(1-n^\epsilon)\|_{L^2_t(L^2(\mathbb{R}^2))} \\
&\lesssim \|u^\epsilon\|^2_{L^4_t(L^4(\mathbb{R}^2))} + \|n^\epsilon\|^2_{L^4_t(L^4(\mathbb{R}^2))} + \|n^\epsilon\|_{L^2_t H^1} + \|\nabla c^\epsilon\|^2_{L^4_t(L^4(\mathbb{R}^2))} + \|\nabla v^\epsilon\|^2_{L^4_t(L^4(\mathbb{R}^2))} \\
&\lesssim \|u^\epsilon\|_{L^\infty_t(L^2(\mathbb{R}^2))}\|u^\epsilon\|_{L^2_t(H^1(\mathbb{R}^2))} + \|n^\epsilon\|_{L^\infty_t(L^2(\mathbb{R}^2))}\|n^\epsilon\|_{L^2_t(H^1(\mathbb{R}^2))} + \|n^\epsilon\|_{L^2_t(H^1(\mathbb{R}^2))} \\
&\quad + \|\nabla c^\epsilon\|_{L^\infty_t(L^2(\mathbb{R}^2))}\|\Delta c^\epsilon\|_{L^2_t(H^2(\mathbb{R}^2))} + \|\nabla v^\epsilon\|_{L^\infty_t(L^2(\mathbb{R}^2))}\|\Delta v^\epsilon\|_{L^2_t(H^2(\mathbb{R}^2))},
\end{aligned}$$

$$\begin{aligned}
\|c^\epsilon_t\|_{L^2_t(H^{-1}(\mathbb{R}^2))} &\lesssim \|c^\epsilon\|^2_{L^4_t(L^4(\mathbb{R}^2))} + \|u^\epsilon\|^2_{L^4_t(L^4(\mathbb{R}^2))} + \|c^\epsilon\|_{L^2_t(H^1(\mathbb{R}^2))} + \|n^\epsilon\|^2_{L^4_t(L^4(\mathbb{R}^2))} \\
&\lesssim \|c^\epsilon\|_{L^\infty_t(L^2(\mathbb{R}^2))}\|c^\epsilon\|_{L^2_t(H^1(\mathbb{R}^2))} + \|c^\epsilon\|_{L^2_t(H^1(\mathbb{R}^2))} \\
&\quad + \|u^\epsilon\|_{L^\infty_t L^2}\|u^\epsilon\|_{L^2_t(H^1(\mathbb{R}^2))} + \|n^\epsilon\|_{L^\infty_t(L^2(\mathbb{R}^2))}\|n^\epsilon\|_{L^2_t(H^1(\mathbb{R}^2))}
\end{aligned}$$

and

$$\begin{aligned}
\|u^\epsilon_t\|_{L^2_t(H^{-1}(\mathbb{R}^2))} &\lesssim \|u^\epsilon\|^2_{L^4_t(L^4(\mathbb{R}^2))} + \|u^\epsilon\|_{L^2_t(H^1(\mathbb{R}^2))} + \|n^\epsilon\|_{L^2_t(L^2(\mathbb{R}^2))}\|\nabla\phi\|_{L^\infty(\mathbb{R}^2)} \\
&\lesssim \|u^\epsilon\|_{L^\infty_t(L^2(\mathbb{R}^2))}\|\nabla u^\epsilon\|_{L^2_t(H^1(\mathbb{R}^2)} + \|u^\epsilon\|_{L^2_t(H^1(\mathbb{R}^2))} \\
&\quad + \|n^\epsilon\|_{L^2_t(L^2(\mathbb{R}^2))}\|\nabla\phi\|_{L^\infty(\mathbb{R}^2)}
\end{aligned}$$

as well as

$$\begin{aligned}
\|v^\epsilon_t\|_{L^2_t(L^2(\mathbb{R}^2))} &\lesssim \|v^\epsilon\|^2_{L^4_t(L^4(\mathbb{R}^2))} + \|u^\epsilon\|^2_{L^4_t(L^4(\mathbb{R}^2))} + \|v^\epsilon\|_{L^2_t(H^1(\mathbb{R}^2))} + \|n^\epsilon\|_{L^2_t(H^{-1}(\mathbb{R}^2))} \\
&\lesssim \|v^\epsilon\|_{L^\infty_t(L^2(\mathbb{R}^2))}\|v^\epsilon\|_{L^2_t(H^1(\mathbb{R}^2))} + \|v^\epsilon\|_{L^2_t(H^1(\mathbb{R}^2))} + \|n^\epsilon\|_{L^2_t(H^1(\mathbb{R}^2))} \\
&\quad + \|u^\epsilon\|_{L^\infty_t(L^2(\mathbb{R}^2)}\|u^\epsilon\|_{L^2_t(H^1(\mathbb{R}^2))}.
\end{aligned}$$

With aid of Proposition 1, Proposition 3 produces the following inequalities:

$$\|\partial_t u^\epsilon\|_{L_t^2 L^2(\mathbb{R}^2)}, \ \|\partial_t n^\epsilon\|_{L_t^2 L^2(\mathbb{R}^2)}, \ \|\partial_t c^\epsilon\|_{L_t^2 L^2(\mathbb{R}^2)} \leq C(t)$$

and $\|\partial_t v^\epsilon\|_{L_t^2 L^2(\mathbb{R}^2)} \leq C(t)$. This completes the proof of Proposition 4. □

4. Proof of Theorem 1

This section mainly deals with the proof of Theorem 1. In this subsection, we shall extract a suitable subsequence from $(n^\epsilon,\, v^\epsilon,\, c^\epsilon,\, u^\epsilon)$ with the help of a priori energy estimates such that it is convergent, and the corresponding limit triple $(n,\, v,\, c,\, u)$ will be a global weak solution of system (1).

- **Existence.**

Taking advantage of Propositions 1–4, we can achieve that

$$\|n^\epsilon\|_{L_t^2(H^1(\mathbb{R}^2))} + + \|n^\epsilon\|_{L_t^\infty(L^2(\mathbb{R}^2))} + \|n^\epsilon\|_{L_t^\infty(L^1(\mathbb{R}^2))} \leq C(t),$$

$$\|c^\epsilon\|_{L^1((\mathbb{R}^2))\cap L^\infty((\mathbb{R}^2))} \leq \|c_0^\epsilon\|_{L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)}, \quad \|c\|_{L_t^\infty(H^1(\mathbb{R}^2))} + \|c\|_{L_t^2(H^2(\mathbb{R}^2))} \leq C(t)$$

and

$$\|u^\epsilon\|_{L_t^\infty(L^2(\mathbb{R}^2))} + \|u^\epsilon\|_{L_t^2(H^1(\mathbb{R}^2))} \leq C(t)$$

as well as

$$\|v^\epsilon(t)\|_{L^1(\mathbb{R}^2)} \leq \|(n_0, v_0)\|_{L^1(\mathbb{R}^2)}, \quad \|v^\epsilon\|_{L_t^\infty(H^1(\mathbb{R}^2))} + \|v^\epsilon\|_{L_t^2(H^2(\mathbb{R}^2))} \leq C(t).$$

We thus have the following bounds uniformly with ϵ:

$$n^\epsilon \in (L^\infty_{loc}(\mathbb{R}^+; L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)) \cap L^2_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^2)),$$

$$c^\epsilon \in L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)) \cap L^\infty_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^2)) \cap L^2_{loc}(\mathbb{R}^+; H^2(\mathbb{R}^2)),$$

$$u^\epsilon \in L^\infty_{loc}(\mathbb{R}^+; L^2(\mathbb{R}^2)) \cap L^2_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^2))$$

and

$$v^\epsilon \in L^\infty_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^2)) \cap L^2_{loc}(\mathbb{R}^+; H^2(\mathbb{R}^2)) \cap L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^2).$$

The Aubin–Lions compactness Lemma 2 proves that $\partial_t n^\epsilon, \partial_t u^\epsilon, \partial_t v^\epsilon$ are bounded in $L^2_{loc}(\mathbb{R}^+; H^{-1}(\mathbb{R}^2))$ and $\partial_t c^\epsilon$ is bounded in $L^2_{loc}(\mathbb{R}^+; L^2(\mathbb{R}^2))$. Since $L^2(\mathbb{R}^2)$ is locally compactly embedded $H^s(\mathbb{R}^2)$ and $H^s(\mathbb{R}^2)$ is continuously embedded in $H^{-1}(\mathbb{R}^2)$ with $s \in (-1, 0)$, by a compactness argument, we thus deduce that a function exists such that for all $\tilde{\phi} \in \mathcal{D}$, the sequence $(\tilde{\phi} v^\epsilon, \tilde{\phi} n^\epsilon, \tilde{\phi} c^\epsilon, \tilde{\phi} u^\epsilon)$ converges (up to a subsequence independent of $\tilde{\phi}$) to $(\tilde{\phi} v, \tilde{\phi} n, \tilde{\phi} c, \tilde{\phi} u)$ in $C(\mathbb{R}^+; H^s(\mathbb{R}^2))$. Therefore, in the sense of distribution $(v^\epsilon, n^\epsilon, c^\epsilon, u^\epsilon)$ converges to (v, n, c, u) when $\epsilon \to 0$. Taking advantage of Fatou's Lemma, we have

$$n \in (L^\infty_{loc}(\mathbb{R}^+; L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)) \cap L^2_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^2)))^2,$$

$$c \in L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)) \cap L^\infty_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^2)) \cap L^2_{loc}(\mathbb{R}^+; H^2(\mathbb{R}^2)),$$

$$u \in L^\infty_{loc}(\mathbb{R}^+; L^2(\mathbb{R}^2)) \cap L^2_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^2))$$

and

$$v \in L^\infty_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^2)) \cap L^2_{loc}(\mathbb{R}^+; H^2(\mathbb{R}^2)) \cap L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^2).$$

Hence, we readily obtain that (v, n, c, u) is a global-in-time weak solution of system (1).

- **Uniqueness.**

Let (v_1, n_1, c_1, u_1) and (v_2, n_2, c_2, u_2) be two solutions of system (1) associated with the same initial data (v_0, n_0, c_0, u_0). Let $\mathbf{V} = v_1 - v_2, \mathbf{N} = n_1 - n_2, \mathbf{C} = c_1 - c_2$, and $\mathbf{U} = u_1 - u_2$. From Definition 1, we have for any $t > 0$,

$$\begin{aligned}
&\int_{Q_t} \partial_s \mathbf{N} \psi dxds + \int_{Q_t} \nabla \mathbf{N} \cdot \nabla \psi dxds + \int_{Q_t} (\mathbf{U} \cdot \nabla n_1 + u_2 \nabla \mathbf{N}) \psi dxds \\
&= -\xi \int_{Q_t} (\mathbf{N} \nabla v_1 + n_2 \nabla \mathbf{V}) \nabla \psi dxds \\
&+ \chi \int_{Q_t} (\mathbf{N} \nabla c_1 + n_2 \nabla \mathbf{C}) \nabla \psi dxds + \int_{Q_t} (\lambda \mathbf{N} - \mu \mathbf{N}(n_1 + n_2)) \psi dxds.
\end{aligned}$$

Taking $\psi = \mathbf{N}$, we have

$$\begin{aligned}
&\frac{1}{2} \|\mathbf{N}\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \|\nabla \mathbf{N}(s)\|^2_{L^2(\mathbb{R}^2)} ds \\
&\le -\int_{Q_t} (\mathbf{U} \cdot \nabla n_1 + u_2 \nabla \mathbf{N}) \mathbf{N} dxds + \int_{Q_t} (\chi \mathbf{N} \nabla c_1 + \chi n_2 \nabla \mathbf{C} - \xi \mathbf{N} \nabla v_1 - \xi n_2 \nabla \mathbf{V}) \nabla \mathbf{N} dxds \\
&\quad + \int_{Q_t} (\lambda \mathbf{N} - \mu \mathbf{N}(n_1 + n_2)) \mathbf{N} dxds = I + II + III.
\end{aligned} \tag{34}$$

Using Hölder's inequality, Young's inequality, and the Gagliardo–Nirenberg interpolation inequality, by virtue of $\nabla \cdot u_2 = \nabla \cdot \mathbf{U} = 0$, we deduce

$$\begin{aligned}
I &= \int_{Q_t} n_1 \mathbf{U} \cdot \nabla \mathbf{N} dxds \\
&\le C \int_0^t \|n_1\|_{L^2(\mathbb{R}^2)} \|\nabla n_1\|_{L^2} \|\mathbf{U}\|_{L^2(\mathbb{R}^2)} \|\nabla \mathbf{U}\|_{L^2(\mathbb{R}^2)} + \frac{1}{10} \|\nabla \mathbf{N}\|^2_{L^2(\mathbb{R}^2)} ds \\
&\le C \int_0^t \|n_1\|^2_{L^2(\mathbb{R}^2)} \|\nabla n_1\|^2_{L^2(\mathbb{R}^2)} \|\mathbf{U}\|^2_{L^2(\mathbb{R}^2)} ds + \int_0^t \frac{1}{10} \|\nabla \mathbf{U}\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{10} \|\nabla \mathbf{N}\|^2_{L^2(\mathbb{R}^2)} ds
\end{aligned} \tag{35}$$

and

$$\begin{aligned}
II &\le C \int_0^t \|\mathbf{N}\|^2_{L^4(\mathbb{R}^2)} \|\nabla c_1\|^2_{L^4(\mathbb{R}^2)} + C \|n_2\|^2_{L^4(\mathbb{R}^2)} \|\nabla \mathbf{C}\|^2_{L^4(\mathbb{R}^2)} ds \\
&\quad + \int_0^t C \|\mathbf{N}\|^2_{L^4(\mathbb{R}^2)} \|\nabla v_1\|^2_{L^4(\mathbb{R}^2)} + C \|n_2\|^2_{L^4(\mathbb{R}^2)} \|\nabla \mathbf{V}\|^2_{L^4(\mathbb{R}^2)} ds + \int_0^t \frac{1}{20} \|\nabla \mathbf{N}\|^2_{L^2(\mathbb{R}^2)} ds \\
&\le \int_0^t C \|\mathbf{N}\|^2_{L^2(\mathbb{R}^2)} \|\nabla c_1\|^2_{L^2(\mathbb{R}^2)} \|\Delta c_1\|^2_{L^2(\mathbb{R}^2)} + \|n_2\|^2_{L^2(\mathbb{R}^2)} \|\nabla n_2\|^2_{L^2(\mathbb{R}^2)} \|\nabla \mathbf{C}\|^2_{L^2} ds \\
&\quad + \int_0^t C \|\mathbf{N}\|^2_{L^2(\mathbb{R}^2)} \|\nabla v_1\|^2_{L^2(\mathbb{R}^2)} \|\Delta v_1\|^2_{L^2(\mathbb{R}^2)} + \|n_2\|^2_{L^2(\mathbb{R}^2)} \|\nabla n_2\|^2_{L^2(\mathbb{R}^2)} \|\nabla \mathbf{V}\|^2_{L^2} ds \\
&\quad + \int_0^t \frac{1}{10} \|\nabla \mathbf{N}\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{10} \|\Delta \mathbf{C}\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{10} \|\Delta \mathbf{V}\|^2_{L^2(\mathbb{R}^2)} ds
\end{aligned} \tag{36}$$

as well as

$$\begin{aligned}
III &\le \int_0^t C \|\mathbf{N}\|^2_{L^2(\mathbb{R}^2)} + C \|(n_1, n_2)\|_{L^2(\mathbb{R}^2)} \|\mathbf{N}\|^2_{L^4(\mathbb{R}^2)} ds \\
&\le \int_0^t C \|\mathbf{N}\|^2_{L^2(\mathbb{R}^2)} + C \|(n_1, n_2)\|_{L^2} \|\mathbf{N}\|_{L^2(\mathbb{R}^2)} \|\nabla \mathbf{N}\|_{L^2} ds \\
&\le \int_0^t C \|\mathbf{N}\|^2_{L^2(\mathbb{R}^2)} + C \|(n_1, n_2)\|^2_{L^2(\mathbb{R}^2)} \|\mathbf{N}\|^2_{L^2} ds + \int_0^t \frac{1}{10} \|\nabla \mathbf{N}\|^2_{L^2(\mathbb{R}^2)} ds.
\end{aligned} \tag{37}$$

Substituting (35), (36), and (37) into (34), one has

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{N}\|^2_{L^2(\mathbb{R}^2)}+\int_0^t\|\nabla\mathbf{N}\|^2_{L^2(\mathbb{R}^2)}ds\\
&\leq\int_0^t\frac{1}{10}\|(\nabla\mathbf{U},\Delta\mathbf{C},\Delta\mathbf{V})\|^2_{L^2}+\frac{3}{10}\|\nabla\mathbf{N}\|^2_{L^2(\mathbb{R}^2)}ds\\
&\quad+C\int_0^t\left(\|\nabla c_1\|^2_{L^2(\mathbb{R}^2)}\|\Delta c_1\|^2_{L^2(\mathbb{R}^2)}+\|(n_1,n_2)\|^2_{L^2(\mathbb{R}^2)}+1\right.\\
&\quad\left.+\|(n_1,n_2)\|^2_{L^2(\mathbb{R}^2)}\|(\nabla n_1,\nabla n_2)\|^2_{L^2(\mathbb{R}^2)}\right)\|(\mathbf{U},\mathbf{N},\nabla\mathbf{C},\nabla\mathbf{V})\|^2_{L^2(\mathbb{R}^2)}ds,
\end{aligned}\tag{38}$$

from which we have

$$\begin{aligned}
&\|\mathbf{N}\|^2_{L^2(\mathbb{R}^2)}+\int_0^t\|\nabla\mathbf{N}\|^2_{L^2(\mathbb{R}^2)}ds\\
&\leq\int_0^t\frac{3}{5}\|(\nabla\mathbf{N},\nabla\mathbf{U},\Delta\mathbf{C},\Delta\mathbf{V})\|^2_{L^2}ds\\
&\quad+C\int_0^t\left(\|\nabla c_1\|^2_{L^2(\mathbb{R}^2)}\|\Delta c_1\|^2_{L^2(\mathbb{R}^2)}+\|(n_1,n_2)\|^2_{L^2(\mathbb{R}^2)}+1\right.\\
&\quad\left.+\|(n_1,n_2)\|^2_{L^2(\mathbb{R}^2)}\|(\nabla n_1,\nabla n_2)\|^2_{L^2(\mathbb{R}^2)}\right)\|(\mathbf{U},\mathbf{N},\nabla\mathbf{C},\nabla\mathbf{V})\|^2_{L^2(\mathbb{R}^2)}ds.
\end{aligned}\tag{39}$$

Next from Definition 1, we have

$$\begin{aligned}
&\int_{Q_t}\partial_s\mathbf{C}\psi dxds+\int_{Q_t}(\mathbf{U}\cdot\nabla c_1+u_2\nabla\mathbf{C})\psi dxds+\int_{Q_t}\nabla\mathbf{C}\cdot\nabla\psi dxds\\
&=\int_{Q_t}(-\mathbf{N}c_1-n_2\mathbf{C})\psi dxds.
\end{aligned}$$

Taking $\psi=\mathbf{C}$, we have

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{C}\|^2_{L^2(\mathbb{R}^2)}+\int_0^t\|\nabla\mathbf{C}\|^2_{L^2(\mathbb{R}^2)}ds\\
&\leq\int_0^t\|\mathbf{U}\|_{L^2(\mathbb{R}^2)}\|\mathbf{C}\|_{L^4(\mathbb{R}^2)}\|\nabla c_1\|_{L^4(\mathbb{R}^2)}\\
&\quad+\|\mathbf{N}\|_{L^2(\mathbb{R}^2)}\|c_1\|_{L^\infty(\mathbb{R}^2)}\|\mathbf{C}\|_{L^2(\mathbb{R}^2)}+\|n_2\|_{L^2(\mathbb{R}^2)}\|\mathbf{C}\|^2_{L^4(\mathbb{R}^2)}ds\\
&\leq\int_0^t C\|\mathbf{U}\|^2_{L^2(\mathbb{R}^2)}\|\nabla c_1\|_{L^2(\mathbb{R}^2)}\|\Delta c_1\|_{L^2(\mathbb{R}^2)}+C\|(\mathbf{C},\mathbf{N})\|^2_{L^2(\mathbb{R}^2)}ds\\
&\quad+\int_0^t C\|n_2\|^2_{L^2(\mathbb{R}^2)}\|\mathbf{C}\|^2_{L^2(\mathbb{R}^2)}+\frac{1}{2}\|\nabla\mathbf{C}\|^2_{L^2(\mathbb{R}^2)}ds,
\end{aligned}\tag{40}$$

from which we have

$$\begin{aligned}
&\|\mathbf{C}\|^2_{L^2(\mathbb{R}^2)}+\int_0^t\|\nabla\mathbf{C}\|^2_{L^2(\mathbb{R}^2)}ds\\
&\leq C\int_0^t\|(\mathbf{U},\mathbf{N},\mathbf{C})\|^2_{L^2(\mathbb{R}^2)}\left(1+\|n_2\|^2_{L^2(\mathbb{R}^2)}+\|\nabla c_1\|^2_{L^2(\mathbb{R}^2)}+\|\Delta c_1\|^2_{L^2(\mathbb{R}^2)}\right)ds.
\end{aligned}\tag{41}$$

Since

$$\begin{aligned}
&\int_{Q_t}\partial_s\partial_i\mathbf{C}\psi dxds+\int_{Q_t}(\partial_i(\mathbf{U}\cdot\nabla c_1)+u_2\nabla\partial_i\mathbf{C}+\partial_i u_2\nabla\mathbf{C})\psi dxds+\int_{Q_t}\nabla\partial_i\mathbf{C}\cdot\nabla\psi dxds\\
&=-\int_{Q_t}\partial_i(\mathbf{N}c_1+n_2\mathbf{C})\psi dxds.
\end{aligned}$$

Letting $\psi=-\partial_i\mathbf{C}$ and summming over i, we infer that

$$
\begin{aligned}
&\frac{1}{2}\|\nabla \mathbf{C}\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \|\Delta \mathbf{C}\|^2_{L^2(\mathbb{R}^2)} ds \\
&= \int_{Q_t} (\mathbf{U}\cdot\nabla c_1)\Delta \mathbf{C} dxds - \int_{Q_t} (\nabla \mathbf{C}\cdot\nabla) u_2 \cdot \nabla \mathbf{C} dxds \\
&\quad + \int_{Q_t} (\mathbf{N} c_1 + n_2 \mathbf{C})\Delta \mathbf{C} dxds = \widetilde{II}_1 + \widetilde{II}_2 + \widetilde{II}_3.
\end{aligned} \tag{42}
$$

Using Hölder's inequality, Young's inequality, and the Gagliardo–Nirenberg interpolation inequality, one has

$$
\begin{aligned}
\widetilde{II}_1 &\le \int_0^t \frac{1}{10}\|\Delta \mathbf{C}\|^2_{L^2(\mathbb{R}^2)} + C\|\mathbf{U}\|^2_{L^4(\mathbb{R}^2)}\|\nabla c_1\|^2_{L^4(\mathbb{R}^2)} ds \\
&\le \int_0^t \frac{1}{10}\|\Delta \mathbf{C}\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{10}\|\nabla \mathbf{U}\|^2_{L^2(\mathbb{R}^2)} + C\|\mathbf{U}\|^2_{L^2(\mathbb{R}^2)}\|\nabla c_1\|^2_{L^2(\mathbb{R}^2)}\|\Delta c_1\|^2_{L^2(\mathbb{R}^2)} ds
\end{aligned} \tag{43}
$$

and

$$
\begin{aligned}
\widetilde{II}_2 &\le \int_0^t \|\nabla \mathbf{C}\|^2_{L^4(\mathbb{R}^2)}\|\nabla u_2\|_{L^2(\mathbb{R}^2)} ds \\
&\le \int_0^t \frac{1}{10}\|\Delta \mathbf{C}\|^2_{L^2(\mathbb{R}^2)} + C\|\mathbf{C}\|^2_{L^2(\mathbb{R}^2)}\|\nabla u_2\|^2_{L^2(\mathbb{R}^2)} ds
\end{aligned} \tag{44}
$$

as well as

$$
\begin{aligned}
\widetilde{II}_3 &\le \int_0^t \|\mathbf{N}\|_{L^2(\mathbb{R}^2)}\|\Delta \mathbf{C}\|_{L^2(\mathbb{R}^2)}\|c_1\|_{L^\infty(\mathbb{R}^2)} + \|n_2\|_{L^4(\mathbb{R}^2)}\|\Delta \mathbf{C}\|_{L^2(\mathbb{R}^2)}\|\mathbf{C}\|_{L^4(\mathbb{R}^2)} ds \\
&\le \int_0^t C\|\mathbf{N}\|^2_{L^2(\mathbb{R}^2)}\|c_1\|^2_{L^\infty(\mathbb{R}^2)} + \frac{1}{10}\|\Delta \mathbf{C}\|^2_{L^2(\mathbb{R}^2)} ds \\
&\quad + \int_0^t \|\nabla n_2\|^2_{L^2(\mathbb{R}^2)}\|n_2\|^2_{L^2(\mathbb{R}^2)}\|\mathbf{C}\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{10}\|\nabla \mathbf{C}\|^2_{L^2(\mathbb{R}^2)} ds.
\end{aligned} \tag{45}
$$

Substituting (43), (44), and (45) into (42), we arrive at

$$
\begin{aligned}
&\frac{1}{2}\|\nabla \mathbf{C}\|^2_{L^2(\mathbb{R}^2)} + \frac{7}{10}\int_0^t \|\Delta \mathbf{C}\|^2_{L^2(\mathbb{R}^2)} ds \\
&\le C\int_0^t (\|\nabla c_1\|^2_{L^2(\mathbb{R}^2)}\|\Delta c_1\|^2_{L^2(\mathbb{R}^2)} + \|\nabla u_2\|^2_{L^2(\mathbb{R}^2)} + 1 + \|\nabla n_2\|^2_{L^2(\mathbb{R}^2)}\|n_2\|^2_{L^2(\mathbb{R}^2)}) \\
&\quad \times \|(\mathbf{C},\mathbf{U},\mathbf{N})\|^2_{L^2(\mathbb{R}^2)} ds + \frac{1}{10}\int_0^t \|(\nabla \mathbf{C}, \nabla \mathbf{U})\|^2_{L^2(\mathbb{R}^2)} ds.
\end{aligned} \tag{46}
$$

From Definition 1, it follows that

$$
\int_{Q_t} \partial_s \mathbf{U}\tilde{\psi} dxdt + \int_{Q_t} (\mathbf{U}\cdot\nabla u_1 + u_2\nabla \mathbf{U} - \mathbf{N}\nabla\phi)\tilde{\psi} dxds = -\int_{Q_t} \nabla \mathbf{U}\nabla\tilde{\psi} dxds
$$

as well as

$$
\int_{Q_t} \nabla\tilde{\psi}\cdot u dxdt = 0.
$$

Taking $\tilde{\psi} = \mathbf{U}$ yields

$$
\begin{aligned}
&\frac{1}{2}\|\mathbf{U}\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \|\nabla \mathbf{U}\|^2_{L^2(\mathbb{R}^2)} ds \\
&= \int_{Q_t} (\mathbf{U}\cdot\nabla \mathbf{U})\cdot u_1 dxds + \int_{Q_t} \mathbf{N}\nabla\phi\cdot \mathbf{U} dxds \\
&\le C\int_0^t \|\mathbf{U}\|^2_{L^2(\mathbb{R}^2)}(\|u_1\|^2_{L^2(\mathbb{R}^2)}\|\nabla u_1\|^2_{L^2(\mathbb{R}^2)} + 1) ds \\
&\quad + \frac{1}{2}\int_0^t \|\nabla \mathbf{U}\|^2_{L^2(\mathbb{R}^2)} ds + C\int_0^t \|\mathbf{N}\|^2_{L^2(\mathbb{R}^2)} ds,
\end{aligned} \tag{47}
$$

which implies that

$$\begin{aligned}&\|\mathbf{U}\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \|\nabla \mathbf{U}\|^2_{L^2(\mathbb{R}^2)} ds \\ &\le \int_0^t C\|(\mathbf{U},\mathbf{N})\|^2_{L^2(\mathbb{R}^2)}(\|u_1\|^2_{L^2(\mathbb{R}^2)}\|\nabla u_1\|^2_{L^2(\mathbb{R}^2)} + 1)ds.\end{aligned} \tag{48}$$

From Definition 1, we have for any $t > 0$,

$$\begin{aligned}&\int_{Q_t} \partial_s \mathbf{V}\psi dxds + \int_{Q_t}(\mathbf{U}\cdot\nabla v_1 + u_2\nabla\mathbf{V})\psi dxds + \int_{Q_t}\nabla\mathbf{V}\cdot\nabla\psi dxds \\ &= \int_{Q_t}(\mathbf{N}-\mathbf{V})\psi dxds.\end{aligned}$$

Taking $\psi = \mathbf{V}$, we have

$$\begin{aligned}&\frac{1}{2}\|\mathbf{V}\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \|\nabla\mathbf{V}\|^2_{L^2(\mathbb{R}^2)}ds \\ &\le \int_0^t C\|\mathbf{U}\|^2_{L^2(\mathbb{R}^2)}\|\nabla v_1\|_{L^2(\mathbb{R}^2)}\|\Delta v_1\|_{L^2(\mathbb{R}^2)} + C\|\mathbf{V}\|^2_{L^2(\mathbb{R}^2)}ds \\ &\quad + \int_0^t C\|\nabla u_2\|^2_{L^2(\mathbb{R}^2)}\|u_2\|^2_{L^2(\mathbb{R}^2)}\|\mathbf{V}\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2}\|\nabla\mathbf{V}\|^2_{L^2(\mathbb{R}^2)}ds,\end{aligned} \tag{49}$$

from which we have

$$\begin{aligned}&\|\mathbf{V}\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \|\nabla\mathbf{V}\|^2_{L^2(\mathbb{R}^2)}ds \\ &\le \int_0^t C\|(\mathbf{U},\mathbf{V})\|^2_{L^2(\mathbb{R}^2)}(\|u_2\|^2_{L^2(\mathbb{R}^2)}\|\nabla u_2\|^2_{L^2(\mathbb{R}^2)} + \|\nabla v_1\|_{L^2(\mathbb{R}^2)}\|\Delta v_1\|_{L^2(\mathbb{R}^2)} + 1)ds.\end{aligned} \tag{50}$$

When $i = 1, 2$, we also have

$$\begin{aligned}&\int_{Q_t}\partial_s\partial_i\mathbf{V}\psi dxds + \int_{Q_t}(\partial_i(\mathbf{U}\cdot\nabla v_1) + u_2\nabla\partial_i\mathbf{V} + \partial_i u_2\nabla\mathbf{V})\psi dxds \\ &+ \int_{Q_t}\nabla\partial_i\mathbf{V}\cdot\nabla\psi dxds = \int_{Q_t}\partial_i(\mathbf{N}-\mathbf{V})\psi dxds.\end{aligned}$$

Letting $\psi = \partial_i\mathbf{V}$ and summming over i, we infer that

$$\begin{aligned}&\frac{1}{2}\|\nabla\mathbf{V}\|^2_{L^2(\mathbb{R}^2)} + \int_0^t\|\Delta\mathbf{V}\|^2_{L^2(\mathbb{R}^2)}ds + \int_0^t\|\nabla\mathbf{V}\|^2_{L^2(\mathbb{R}^2)}ds \\ &= \int_{Q_t}(\mathbf{U}\cdot\nabla v_1)\Delta\mathbf{V}dxds - \int_{Q_t}(\nabla\mathbf{V}\cdot\nabla)u_2\cdot\nabla\mathbf{V}dxds - \int_{Q_t}\mathbf{N}\Delta\mathbf{V}dxds \\ &\le C\int_0^t \|\mathbf{N}\|^2_{L^2(\mathbb{R}^2)} + \|\mathbf{U}\|^2_{L^4(\mathbb{R}^2)}\|\nabla v_1\|^2_{L^4(\mathbb{R}^2)} + \|\nabla u_2\|_{L^2(\mathbb{R}^2)}\|\nabla\mathbf{V}\|^2_{L^4(\mathbb{R}^2)}ds \\ &\quad + \frac{1}{20}\int_0^t \|\Delta\mathbf{V}\|^2_{L^2(\mathbb{R}^2)} \\ &\le C\int_0^t \|(\mathbf{N},\mathbf{U},\nabla\mathbf{V})\|^2_{L^2(\mathbb{R}^2)}(1 + \|\nabla v_1\|^2_{L^2(\mathbb{R}^2)}\|\Delta v_1\|^2_{L^2(\mathbb{R}^2)} + \|\nabla u_2\|^2_{L^2(\mathbb{R}^2)})ds \\ &\quad + \frac{1}{10}\int_0^t \|\Delta\mathbf{V}\|^2_{L^2(\mathbb{R}^2)} + \|\nabla\mathbf{U}\|^2_{L^2(\mathbb{R}^2)}ds.\end{aligned} \tag{51}$$

Collecting (39), (42), (46), (50), (42), and (51) leads to

$$\tilde{E}(t) + \int_0^t \tilde{F}(s)ds \le \int_0^t \tilde{G}(s)\tilde{E}(s)ds, \tag{52}$$

with

$$\tilde{E}(t) = \|(\mathbf{U}, \mathbf{V}, \mathbf{N}, \mathbf{C})(t)\|^2_{L^2(\mathbb{R}^2)} + \|(\nabla\mathbf{C}, \nabla\mathbf{V})\|^2_{L^2(\mathbb{R}^2)},$$

$$\tilde{F}(s) = \|(\nabla\mathbf{U}, \nabla\mathbf{V}, \nabla\mathbf{N}, \nabla\mathbf{C})(s)\|^2_{L^2(\mathbb{R}^2)} + \|(\Delta\mathbf{C}(s), \Delta\mathbf{C}(s))\|^2_{L^2(\mathbb{R}^2)},$$

and

$$\tilde{G}(s) = C(\|(n_1, n_2, u_1, u_2, \nabla c_1, \nabla v_1)\|^2_{L^2}\|(\nabla n_1, \nabla n_2, \nabla u_1, \nabla u_2, \Delta c_1, \nabla v_1)\|^2_{L^2} + \|(\nabla u_2, \nabla c_1, \Delta c_1, n_1, n_2)\|^2_{L^2} + 1).$$

Applying Lemmas 1–3, we have

$$\int_0^t \tilde{G}(s)ds \leq Ce^{Ce^{Ct}} + Ce^{Ct} + Ct,$$

from which and (52), applying Grönwall's inequality, we have

$$\tilde{E}(t) \leq \tilde{E}(t=0)\exp\left\{\int_0^t \tilde{G}(s)ds\right\},$$

from which we conclude that $(\mathbf{U}, \mathbf{V}, \mathbf{N}, \mathbf{C}) = 0$, and we thus complete the proof of uniqueness.

5. Conclusions

We introduce the notion of a weak solution and establish both the existence and uniqueness of such a weak solution for a large class of initial data on the whole space $\mathbb{R}^2$.

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