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Generalized Limit Theorem for Mellin Transform of the Riemann Zeta-Function

Antanas Laurinčikas [1,*,†] and Darius Šiaučiūnas [2,†]

[1] Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University, Naugarduko Str. 24, LT-03225 Vilnius, Lithuania
[2] Institute of Regional Development, Šiauliai Academy, Vilnius University, Vytauto Str. 84, LT-76352 Šiauliai, Lithuania; darius.siauciunas@sa.vu.lt
* Correspondence: antanas.laurincikas@mif.vu.lt
† These authors contributed equally to this work.

Abstract: In the paper, we prove a limit theorem in the sense of the weak convergence of probability measures for the modified Mellin transform $\mathcal{Z}(s)$, $s = \sigma + it$, with fixed $1/2 < \sigma < 1$, of the square $|\zeta(1/2 + it)|^2$ of the Riemann zeta-function. We consider probability measures defined by means of $\mathcal{Z}(\sigma + i\varphi(t))$, where $\varphi(t)$, $t \geqslant t_0 > 0$, is an increasing to $+\infty$ differentiable function with monotonically decreasing derivative $\varphi'(t)$ satisfying a certain normalizing estimate related to the mean square of the function $\mathcal{Z}(\sigma + i\varphi(t))$. This allows us to extend the distribution laws for $\mathcal{Z}(s)$.

Keywords: modified Mellin transform; Riemann zeta-function; weak convergence of probability measures

MSC: 11M06

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Citation: Laurinčikas, A.; Šiaučiūnas, D. Generalized Limit Theorem for Mellin Transform of the Riemann Zeta-Function. *Axioms* **2024**, *13*, 251. https://doi.org/10.3390/axioms13040251

Academic Editors: Dongkyu Lim, Adem Kilicman and Arjun Kumar Rathie

Received: 13 March 2024
Revised: 6 April 2024
Accepted: 8 April 2024
Published: 10 April 2024

1. Introduction

Let $s = \sigma + it$ be a complex variable. One of the most important objects of the classical analytic number theory is the Riemann zeta-function $\zeta(s)$, which is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}.$$

Moreover, the function $\zeta(s)$ has analytic continuation to the region $\mathbb{C} \setminus \{1\}$, and the point $s = 1$ is its simple pole with residue 1. The first value distribution results for $\zeta(s)$ with real s were obtained by Euler. Riemann was the first mathematician who began to study [1] $\zeta(s)$ with complex variables, proved the functional equation for $\zeta(s)$, obtained its analytic continuation, proposed a means of using $\zeta(s)$ for the investigation of the asymptotic prime number distribution law

$$\pi(x) = \sum_{p \leqslant x} 1, \quad x \to \infty,$$

and stated some hypotheses on $\zeta(s)$. The most important hypothesis, now called the Riemann hypothesis, states that all zeros of $\zeta(s)$ in the region $\sigma \geqslant 0$ are located on the line $\sigma = 1/2$. Riemann's ideas concerning $\pi(x)$ were correct, and Hadamard [2] and de la Vallée Poussin [3], using them, independently proved that

$$\lim_{x \to \infty} \pi(x) \left(\int_2^x \frac{du}{\log u} \right)^{-1} = 1.$$

However, the Riemann hypothesis remains open at present; it is among the seven Millennium Problems of mathematics [4]. In the theory of $\zeta(s)$, there are other important problems. One of them is connected to the asymptotics of moments

$$M_k(\sigma,T) \stackrel{\text{def}}{=} \int_0^T |\zeta(\sigma+it)|^{2k}\,\mathrm{d}t, \quad k>0,\ \sigma \geqslant \frac{1}{2},$$

as $T \to \infty$. For example, at the moment the asymptotics of $M_k(\sigma,T)$, $\sigma = 1/2$ is known only for $k=1$ and $k=2$; see [5]. For the investigation of $M_k(\sigma,T)$, Motohashi proposed (see [6,7]) to use the modified Mellin transforms

$$\mathcal{Z}_k(s) = \int_1^\infty \left|\zeta\left(\frac{1}{2}+ix\right)\right|^{2k} x^{-s}\,\mathrm{d}x, \quad k \in \mathbb{N}.$$

Let $g(x)$ be a certain function, e.g., $g(x)x^{\sigma-1} \in L(0,\infty)$, and

$$G(s) = \int_0^\infty g(x)x^{s-1}\,\mathrm{d}x.$$

Then, using the Mellin inverse formula leads to the following equality (see [8]):

$$\int_1^\infty g\left(\frac{x}{T}\right)\left|\zeta\left(\frac{1}{2}+ix\right)\right|^{2k}\mathrm{d}x = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(s)T^s\mathcal{Z}_k(s)\,\mathrm{d}s$$

with a certain $c>1$. This shows that a suitable choice of the function $g(x)$ reduces investigations of $M_k(1/2,T)$ to those of properties of $\mathcal{Z}_k(s)$. The latter assertion inspired the creation of the analytic theory of the functions $\mathcal{Z}_k(s)$.

In this paper, we limit ourselves to the probabilistic value distribution of the function $\mathcal{Z}(s) \stackrel{\text{def}}{=} \mathcal{Z}_1(s)$ only. Before this, we recall some known results of the function $\mathcal{Z}(s)$.

Let $\gamma = 0.577\ldots$ denote the Euler constant and $E(T)$ be defined by

$$\int_0^T \left|\zeta(\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t = T\log\frac{T}{2\pi} + (2\gamma-1)T + E(T).$$

Moreover, let

$$F(t) = \int_1^T E(t)\,\mathrm{d}t - \pi T \quad \text{and} \quad F_1(T) = \int_1^T F(t)\,\mathrm{d}t.$$

The analytic behavior of the function $\mathcal{Z}(s)$ was described in [9] and forms the following theorem.

Theorem 1 ([9])**.** *The function $\mathcal{Z}(s)$ is analytically continuable to the region $\sigma > -3/4$, except the point $s=1$, which is a double pole, and*

$$\mathcal{Z}(s) = \frac{1}{(s-1)^2} + \frac{2\gamma-\log 2\pi}{s-1} - E(1) + \pi(s+1) + s(s+1)(s+2)\int_1^\infty F_1(x)x^{-s-3}\,\mathrm{d}x.$$

Moreover, the estimates

$$\mathcal{Z}(\sigma+it) \ll_\varepsilon t^{1-\sigma+\varepsilon}, \quad 0 \leqslant \sigma \leqslant 1,\ t \geqslant t_0 > 0,$$

and

$$\int_1^T |\mathcal{Z}(\sigma+it)|^2 \, \mathrm{d}t \ll_\varepsilon \begin{cases} T^{3-4\sigma+\varepsilon} & \text{if } 0 \leqslant \sigma \leqslant 1/2, \\ T^{2-2\sigma+\varepsilon} & \text{if } 1/2 \leqslant \sigma \leqslant 1, \end{cases} \tag{1}$$

are valid.

Here and in what follows, ε is an arbitrary fixed positive number that is not always the same, and the notation $x \ll_\varepsilon y$, $x \in \mathbb{C}$, $y > 0$, means that there is a constant $c = c(\varepsilon) > 0$ such that $|x| \leqslant cy$.

In [10], Bohr proposed to characterize the asymptotic behavior of the Riemann zeta-function by using a probabilistic approach. This idea is acceptable because the value distribution of $\zeta(s)$ is quite chaotic. Denote by $\mathrm{J}A$ the Jordan measure of the set $A \subset \mathbb{R}$. Then, Bohr, jointly with Jessen, roughly speaking, obtained in [11,12] that, for $\sigma > 1/2$ and every rectangle $R \subset \mathbb{C}$ with edges parallel to the axes, there exists a limit

$$\lim_{T\to\infty} \mathrm{J}\{t \in [0,T] : \zeta(\sigma+it) \in R\}.$$

In modern terminology, the Bohr–Jessen theorem is stated as a limit theorem on weakly convergent probability measures. Let $\mathcal{B}(\mathbb{X})$ stand for the Borel σ-field of the space $\mathbb{X}$ (in general, topological), and let P_n, $n \in \mathbb{N}$, and P be probability measures defined on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. By this definition, P_n converges weakly to P as $n \to \infty$ ($P_n \xrightarrow[n\to\infty]{\mathrm{w}} P$) if

$$\lim_{n\to\infty} \int_{\mathbb{X}} g \, \mathrm{d}P_n = \int_{\mathbb{X}} g \, \mathrm{d}P$$

for every real continuous bounded function g on $\mathbb{X}$. Let $\mathbf{L}A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then, the modern version of the Bohr–Jessen theorem is of the following form: for every fixed $\sigma > 1/2$, there exists a probability measure P_σ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ such that

$$\frac{1}{T}\mathbf{L}\{t \in [0,T] : \zeta(\sigma+it) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P_σ as $T \to \infty$.

The first probabilistic limit theorems for the function $\mathcal{Z}(s)$ were discussed in [13]. For $A \in \mathcal{B}(\mathbb{C})$, set

$$Q_{T,\sigma}(A) = \frac{1}{T}\mathbf{L}\{t \in [0,T] : \mathcal{Z}(\sigma+it) \in A\}.$$

Assuming that $\sigma > 1/2$, it was obtained that there is a probability measure Q_σ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ such that $Q_{T,\sigma} \xrightarrow[T\to\infty]{\mathrm{w}} Q_\sigma$. On the other hand, for every $\kappa > 0$, we have

$$\frac{1}{T}\mathbf{L}\{t \in [0,T] : |\mathcal{Z}(\sigma+it)| \geqslant \kappa\} \leqslant \frac{1}{\kappa T}\int_0^T |\mathcal{Z}(\sigma+it)| \, \mathrm{d}t \leqslant \frac{1}{\kappa}\left(\frac{1}{T}\int_0^T |\mathcal{Z}(\sigma+it)|^2 \, \mathrm{d}t\right)^{1/2}.$$

This, together with Theorem 1, implies that, for $1/2 < \sigma < 1$,

$$\lim_{T\to\infty} \frac{1}{T}\mathbf{L}\{t \in [0,T] : |\mathcal{Z}(\sigma+it)| \geqslant \kappa\} = 0.$$

The latter equality remains valid also for $\sigma > 1$. Thus, the limit measure Q_σ is degenerated at the point $s = 0$. In order to avoid this situation, we propose to consider $\mathcal{Z}(\sigma + i\varphi(t))$ with a certain function $\varphi(t)$. Moreover, it is more convenient to deal with $t \in [T, 2T]$ because, in this case, additional restrictions for $\varphi(t)$ with $t = 0$ are not needed.

Denote

$$I_\sigma(T) = \int_1^T |\mathcal{Z}(\sigma+it)|^2\, dt.$$

We suppose that $\varphi(t)$ is a positive increasing to $+\infty$ differentiable function with a monotonically decreasing derivative, such that

$$\frac{I_{\sigma-\varepsilon}(\varphi(T))}{\varphi'(T)} \ll T, \quad T \to \infty.$$

The class of such functions $\varphi(t)$ is denoted by W_σ. Consider the weak convergence for

$$P_{T,\sigma}(A) = \frac{1}{T}\mathbf{L}\{t \in [T,2T] : \mathcal{Z}(\sigma+i\varphi(\tau)) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}).$$

In this case, we have, by $\varepsilon \to 0$, that

$$\frac{I_\sigma(\varphi(T))}{\varphi'(T)} \ll T,$$

and

$$\frac{1}{T}\int_T^{2T} |\mathcal{Z}(\sigma+i\varphi(t))|^2\, dt = \frac{1}{T}\int_{\varphi(T)}^{\varphi(2T)} \frac{1}{\varphi'(t)}|\mathcal{Z}(\sigma+iu)|^2\, du \leqslant \frac{1}{T\varphi'(2T)} I_\sigma(\varphi(2T)) \ll 1 \tag{2}$$

for $\varphi(t) \in W_\sigma$. Thus, we cannot claim that the limit measure for $P_{T,\sigma}$ is degenerated at zero.

Now, we state a limit theorem for $P_{T,\sigma}$.

Theorem 2. *Assume that $\sigma \in (1/2, 1)$ is a given fixed number, and $\varphi(t) \in W_\sigma$. Then, on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists a probability measure P_σ such that $P_{T,\sigma} \xrightarrow[T\to\infty]{w} P_\sigma$.*

In virtue of Theorem 1, we see that

$$I_{\sigma-\varepsilon}(T) \ll T^{\alpha_\sigma}$$

with certain $0 < \alpha_\sigma < 1$. Take $\varphi(t) = (\log t)^{\beta_\sigma}$, $t \geqslant 2$, $\beta_\sigma > 0$. Then, $\varphi'(t)$ is decreasing, and

$$\frac{I_{\sigma-\varepsilon}(\varphi(T))}{\varphi'(T)} \ll T(\log T)^{\alpha_\sigma\beta_\sigma+1-\beta_\sigma} \ll T$$

if we choose

$$\beta_\sigma = (1-\alpha_\sigma)^{-1}.$$

This shows that $(\log T)^{\beta_\sigma}$ is an element of the class W_σ.

Theorem 2 shows that the asymptotic behavior of the function $\mathcal{Z}(s)$ on vertical lines is governed by a certain probabilistic law, and this confirms the chaos in its value distribution. Moreover, Theorem 2 is an example of the application of probability methods in analysis. Thus, it continues a tradition initiated in works [11,12] and developed by Selberg [14], Joyner [15], Bagchi [16], Korolev [17,18], Kowalski [19], Lamzouri, Lester and Radziwill [20,21], Steuding [22], and others; see also a survey paper [23]. We note that a generalization of Theorem 2 for the functional spaces can be applied for approximation problems of some classes of functions.

We divide the proof of Theorem 2 into several parts. First, we discuss weak convergence on a certain group. The second part is devoted to the case related to a integral. Further, we consider a measure defined by an absolutely convergent improper integral. In the last part, Theorem 2 is proven. For proofs of all assertions on weak convergence, the notions of relative compactness as well as of tightness and convergence in distribution are employed.

2. Fourier Transform Method

Let $b > 1$ be a fixed finite number, and

$$\mathbb{I}_b = \prod_{x\in[1,b]} \{s \in \mathbb{C} : |s| = 1\}.$$

The Cartesian product $\mathbb{I}_b$ consists of all functions $\mathfrak{i} : [1,b] \to \{s \in \mathbb{C} : |s| = 1\}$. On $\mathbb{I}_b$, the product topology and operation of pointwise multiplication can be defined. This reduces $\mathbb{I}_b$ to a compact topological group. We will give a limit lemma for probability measures on $(\mathbb{I}_b, \mathcal{B}(\mathbb{I}_b))$.

For $A \in \mathcal{B}(\mathbb{I}_b)$, put

$$V_{T,b}(A) = \frac{1}{T}\mathbf{L}\Big\{t \in [T,2T] : \Big(x^{-i\varphi(t)} : x \in [1,b]\Big) \in A\Big\}.$$

Lemma 1. *Suppose that the function $\varphi(t)$ has a monotonically decreasing derivative $\varphi'(t)$ such that*

$$(\varphi'(T))^{-1} = o(T), \quad T \to \infty. \tag{3}$$

Then $V_{T,b}$ converges weakly to a certain probability measure V_b as $T \to \infty$.

Proof. We use the Fourier transform approach. Denote the elements of $\mathbb{I}_b$ by $\mathfrak{i} = \{i_x : x \in [1,b]\}$. Then, the Fourier transform $f_{T,b}(\underline{k})$, $\underline{k} = (k_x : k_x \in \mathbb{Z},\ x \in [1,b])$ of the measure $V_{T,b}$ is the integral

$$f_{T,b}(\underline{k}) = \int\limits_{\mathbb{I}_b} \left(\prod_{x\in[1,b]} i_x^{k_x}\right) \mathrm{d}V_{T,b},$$

where only a finite number of integers k_x are not zeros. Therefore, the definition of $V_{T,b}$ yields

$$f_{T,b}(\underline{k}) = \frac{1}{T}\int\limits_T^{2T} \left(\prod_{x\in[1,b]} x^{-ik_x\varphi(t)}\right) \mathrm{d}t = \frac{1}{T}\int\limits_T^{2T} \exp\left\{-i\varphi(t) \sum_{x\in[1,b]} k_x \log x\right\} \mathrm{d}t. \tag{4}$$

For brevity, let $A_b(\underline{k}) = \sum_{k\in[1,b]} k_x \log x$. Then, the second mean value theorem, (4), and (3) give

$$\begin{aligned}\mathrm{Re} f_{T,b}(\underline{k}) &= \frac{1}{T}\int\limits_T^{2T} \cos(\varphi(t)A_b(\underline{k}))\,\mathrm{d}t = \frac{1}{A_b(\underline{k})T}\int\limits_T^{2T} \frac{1}{\varphi'(t)}\,\mathrm{d}\sin(\varphi(t)A_b(\underline{k}))\\ &\ll \frac{1}{|A_b(\underline{k})|}\frac{1}{\varphi'(2T)T} = o(1), \quad T \to \infty,\end{aligned}$$

provided that $A_b(\underline{k}) \neq 0$. Clearly, the same estimate holds for $\mathrm{Im} f_{T,b}(\underline{k})$. Hence, for $A_b(\underline{k}) \neq 0$,

$$\lim_{T\to\infty} f_{T,b}(\underline{k}) = 0. \tag{5}$$

Obviously,

$$f_{T,b}(\underline{k}) = 1$$

if $A_b(\underline{k}) = 0$. This and (5) show that

$$V_{T,b} \xrightarrow[T\to\infty]{\mathrm{w}} V_b,$$

where V_b is a probability measure on $(\mathbb{I}_b, \mathcal{B}(\mathbb{I}_b))$ defined by the Fourier transform

$$f_b(\underline{k}) = \begin{cases} 1 & \text{if } A_b(\underline{k}) = 0, \\ 0 & \text{if } A_b(\underline{k}) \neq 0. \end{cases}$$

□

Now, we will apply Lemma 1 for the measure defined by means of a certain finite sum. Let $\theta > 1/2$ be a fixed number, and, for $x, y \in [1, \infty)$,

$$u(x, y) = \exp\left\{ -\left(\frac{x}{y}\right)^\theta \right\}.$$

Moreover, we use the notation $\widehat{\zeta}(t) = |\zeta(1/2 + it)|^2$. Consider the nth integral sum

$$U_{n,b,y}(\sigma + i\varphi(t)) = \frac{b-1}{n} \sum_{l=1}^{n} \widehat{\zeta}(a_l) u(a_l, y) a_l^{-\sigma - i\varphi(t)}, \quad n \in \mathbb{N},$$

where $a_l \in [x_{l-1}, x_l]$ and $x_l = 1 + ((b-1)/n)l$.

For $A \in \mathcal{B}(\mathbb{C})$, set

$$P_{T,n,b,y}(A) = \frac{1}{T} \mathbf{L}\Big\{ t \in [T, 2T] : U_{n,b,y}(\sigma + i\varphi(t)) \in A \Big\}.$$

For simplicity, here and in the following, we omit the dependence on σ of some objects. Before the statement of the limit lemma for $P_{T,n,b,y}$, we will present some lower estimates for the mean square $I_\sigma(T)$. For this, we will apply the following general lemma from [8]. Let $\mathcal{F}(s)$ be the modified Mellin transform of $f(x)$, i.e.,

$$\mathcal{F}(s) = \int_1^\infty f(x) x^{-s} \, \mathrm{d}x.$$

Lemma 2 ([8], Lemma 5). *Let $f(x) \in C^\infty[2, \infty]$ be a real-valued function such that*

1°
$$\int_1^X \left| f^{(k)}(x) \right| \mathrm{d}x \ll_{\varepsilon,k} X^{1+\varepsilon}, \quad k \in \mathbb{N}_0;$$

2° *$\mathcal{F}(s)$ has analytic continuation to the half-plane $\sigma > 1/2$, except for a pole of order l at the point $s = 1$;*

3° *For $\sigma > 1/2$, $\mathcal{F}(s)$ is of polynomial growth in $|t|$.*

Then, for $1/2 < \sigma < 1$ and any fixed $\varepsilon > 0$,

$$\int_T^{2T} f^2(x) \, \mathrm{d}x \ll_\varepsilon \log^{l-1} T \int_{T/2}^{5T/2} |f(x)| \, \mathrm{d}x + T^{2\sigma - 1} \int_0^{T^{1+\varepsilon}} |\mathcal{F}(\sigma + it)|^2 \, \mathrm{d}t.$$

Lemma 3. *For $1/2 < \sigma < 1$, and any $\varepsilon > 0$, the estimate*

$$I_\sigma(T) \gg_\varepsilon T^{2-2\sigma-\varepsilon}$$

holds.

Proof. As usual, denote by $Z(t)$, $t \in \mathbb{R}$, the Hardy function, i.e.,

$$Z(t) = \zeta\left(\frac{1}{2} + it\right) \chi^{-1/2}\left(\frac{1}{2} + it\right),$$

where

$$\chi(s) = \frac{\zeta(s)}{\zeta(1-s)}.$$

It is well known that $Z(t)$ is a real-valued function satisfying $|Z(t)| = |\zeta(1/2+it)|$. Moreover, the estimate [8]

$$Z^{(k)}(t) \ll_k t^{-1/4}(\log T)^{k+1} + \sum_{m \leqslant \sqrt{t/(2\pi)}} m^{-1/2}\left(\log \frac{\sqrt{t/(2\pi)}}{m}\right)^k \tag{6}$$

holds. Take $f(x) = Z^2(x)$. Then, we have

$$\mathcal{F}(s) = \int_1^\infty Z^2(x)x^{-s}\,\mathrm{d}x = \int_1^\infty \left|\zeta\left(\frac{1}{2}+ix\right)\right|^2 x^{-s}\,\mathrm{d}x = \mathcal{Z}(s).$$

In view of Theorem 1 and (6), the function satisfies the hypotheses of Lemma 1 with $l = 2$. Thus, for $1/2 < \sigma < 1$,

$$\int_T^{2T} f^2(t)\,\mathrm{d}t = \int_T^{2T} \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \mathrm{d}t \ll_\varepsilon \log T \int_{T/2}^{5T/2} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \mathrm{d}t + T^{2\sigma-1} \int_0^{T^{1+\varepsilon}} |\mathcal{Z}(\sigma+it)|^2\,\mathrm{d}t.$$

Since [5]

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \mathrm{d}t = \frac{1}{2\pi^2} T\log^4 T + O(T\log^3 T)$$

and

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \mathrm{d}t \ll T\log T,$$

this implies

$$T\log^4 T \ll_\varepsilon T^{2\sigma-1} \int_0^{T^{1+\varepsilon}} |\mathcal{Z}(\sigma+it)|^2\,\mathrm{d}t.$$

Consequently,

$$I_\sigma(T) \gg_\varepsilon T^{(2-2\sigma)/(1+\varepsilon)} \gg_\varepsilon T^{2-2\sigma-\varepsilon}.$$

□

Lemma 4. *Assume that $\sigma \in (1/2,1)$ is a given fixed number, and $\varphi(t) \in W_\sigma$. Then, on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists a probability measure $P_{n,b,y}$ such that $P_{T,n,b,y} \xrightarrow[T\to\infty]{\mathrm{w}} P_{n,b,y}$.*

Proof. Lemma 3 implies that, for $\sigma \in (1/2,1)$, $I_\sigma(T) \to \infty$ as $T \to \infty$. Therefore, if $\varphi(t) \in W_\sigma$, then

$$\frac{1}{\varphi'(T)} \ll T I_\sigma^{-1}(\varphi(T)) = o(T)$$

as $T \to \infty$. Thus, the application of Lemma 1 is possible.

Consider the mapping $v_{n,b} : \mathbb{I}_b \to \mathbb{C}$ defined by

$$v_{n,b}(\mathfrak{i}) = \frac{b-1}{n} \sum_{l=1}^{n} \widehat{\zeta}(a_l) u(a_l, y) a_l^{-\sigma} i_{a_l}. \tag{7}$$

Since the latter sum is finite, and $\mathbb{I}_b$ is equipped with the product topology, the mapping $v_{n,b}$ is continuous. Moreover, in view of (7),

$$v_{n,b}\Big(x^{-i\varphi(t)} : x \in [1,b]\Big) = \frac{b-1}{n} \sum_{l=1}^{n} \widehat{\zeta}(a_l) u(a_l, y) a_l^{-\sigma - i\varphi(t)} = U_{n,b,y}(\sigma + i\varphi(t)).$$

Hence, for $A \in \mathcal{B}(\mathbb{C})$,

$$\begin{aligned} P_{T,n,b,y}(A) &= \frac{1}{T}\mathbf{L}\Big\{t \in [T, 2T] : v_{n,b}\Big(x^{-i\varphi(t)} : x \in [1,b]\Big) \in A\Big\} \\ &= \frac{1}{T}\mathbf{L}\Big\{t \in [T, 2T] : \Big(x^{-i\varphi(t)} : x \in [1,b]\Big) \in v_{n,b}^{-1}A\Big\} = V_{T,b}\Big(v_{n,b}^{-1}A\Big), \end{aligned} \tag{8}$$

where $V_{T,b}$ is from Lemma 1. The continuity of the mapping $v_{n,b}$ implies its $(\mathcal{B}(\mathbb{I}_b), \mathcal{B}(\mathbb{C}))$-measurability. Therefore, the mapping $v_{n,b}$ and any probability measure P on $(\mathbb{I}_b, \mathcal{B}(\mathbb{I}_b))$ define the unique probability measure $Pv_{n,b}^{-1}$ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ given by

$$Pv_{n,b}^{-1}(A) = P(v_{n,b}^{-1}A), \quad A \in \mathcal{B}(\mathbb{C}).$$

See Section 2 of [24]. Thus, by (8), we have $P_{T,n,b,y} = V_{T,b}v_{n,b}^{-1}$. Therefore, Lemma 1, the continuity of $v_{n,b}$, and the principle of the preservation of week convergence under continuity mappings (Theorem 5.1 of [24]) show that

$$P_{T,n,b,y} \xrightarrow[T\to\infty]{\mathrm{w}} P_{n,b,y},$$

where $P_{n,b,y} = V_b v_{n,b}^{-1}$, and V_b is the limit measure in Lemma 1. □

3. Limit Lemma for Integral

Denote

$$\mathcal{Z}_{b,y}(\sigma + i\varphi(t)) = \int_1^b \widehat{\zeta}(x) u(x, y) x^{-\sigma - i\varphi(t)}\, \mathrm{d}x,$$

and, for $A \in \mathcal{B}(\mathbb{C})$, set

$$P_{T,b,y}(A) = \frac{1}{T}\mathbf{L}\Big\{t \in [T, 2T] : \mathcal{Z}_{b,y}(\sigma + i\varphi(t)) \in A\Big\}.$$

In this section, we will prove the weak convergence for $P_{T,b,y}$ as $T \to \infty$. Before this, we recall some known probabilistic results. Let $\{Q\}$ be a certain family of probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. The family $\{Q\}$ is called tight if, for every $\delta > 0$, there is a compact set $K \subset \mathbb{X}$ such that

$$Q(K) > 1 - \delta$$

for all $Q \in \{Q\}$. The family $\{Q\}$ is said to be relatively compact if every sequence contains a subsequence weakly convergent to a certain probability measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. The Prokhorov theorem connects two above notions, and, for convenience, we state it as the following lemma.

Lemma 5. *If a family of probability measures is tight, then it is relatively compact.*

The proof of the lemma is given in [24], Theorem 5.1.

Moreover, we recall one useful property on convergence in distribution. Let ξ_n and ξ be $\mathbb{X}$-valued random elements defined on the probability space $(\Omega, \mathcal{F}, \mu)$ with distributions P_n and P, respectively. Then, ξ_n converges in distribution to ξ as $n \to \infty$ $\left(\xrightarrow[n\to\infty]{\mathcal{D}}\right)$ if

$$P_n \xrightarrow[n\to\infty]{\mathrm{w}} P.$$

Now, we state a lemma on convergence in distribution.

Lemma 6. *Assume that the metric space $(\mathbb{X}, d)$ is separable, and ξ_{nk}, ξ_n are $\mathbb{X}$-valued random elements defined on the same probability space $(\Omega, \mathcal{F}, \mu)$. Let*

$$\xi_{nk} \xrightarrow[n\to\infty]{\mathcal{D}} \xi_k$$

and

$$\xi_k \xrightarrow[k\to\infty]{\mathcal{D}} \xi.$$

If, for every $\delta > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mu\{d(\xi_{nk}, \eta_k) \geqslant \delta\} = 0,$$

then

$$\eta_n \xrightarrow[n\to\infty]{\mathcal{D}} \xi.$$

The lemma is proven in [24], Theorem 3.2.

Lemma 7. *Assume that $\sigma \in (1/2, 1)$ is a given fixed number, and $\varphi(t) \in W_\sigma$. Then, on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists a probability measure $P_{b,y}$ such that $P_{T,b,y} \xrightarrow[T\to\infty]{\mathrm{w}} P_{b,y}$.*

Proof. First, we will show that $\mathcal{Z}_{b,y}(\sigma + i\varphi(t))$ is close in a certain sense to $U_{n,b,y}(\sigma + i\varphi(t))$.

Let

$$J_{T,n} \stackrel{\text{def}}{=} \frac{1}{T} \int_T^{2T} \left| \mathcal{Z}_{b,y}(\sigma + i\varphi(t)) - U_{n,b,y}(\sigma + \varphi(t)) \right| \mathrm{d}t.$$

Clearly,

$$J_{T,n}^2 \leqslant \frac{1}{T} \int_T^{2T} \left| \mathcal{Z}_{b,y}(\sigma + i\varphi(t)) - U_{n,b,y}(\sigma + \varphi(t)) \right|^2 \mathrm{d}t. \tag{9}$$

We have

$$\begin{aligned}
&\int_T^{2T} \left| \mathcal{Z}_{b,y}(\sigma + i\varphi(t)) \right|^2 \mathrm{d}t \\
&\quad = \int_T^{2T} \left(\int_1^b \widehat{\zeta}(x) u(x,y) x^{-\sigma - i\varphi(t)} \, \mathrm{d}x \right) \left(\int_1^b \widehat{\zeta}(x) u(x,y) x^{-\sigma + i\varphi(t)} \, \mathrm{d}x \right) \mathrm{d}t \\
&\quad = T \int_1^b \int_{\substack{1 \\ x_1 = x_2}}^b \widehat{\zeta}(x_1) \widehat{\zeta}(x_2) u(x_1, y) u(x_2, y) x_1^{-\sigma} x_2^{-\sigma} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\
&\qquad + \int_1^b \int_{\substack{1 \\ x_1 \neq x_2}}^b \widehat{\zeta}(x_1) \widehat{\zeta}(x_2) u(x_1, y) u(x_2, y) x_1^{-\sigma} x_2^{-\sigma} \left(\int_T^{2T} \left(\frac{x_1}{x_2} \right)^{i\varphi(t)} \mathrm{d}t \right) \mathrm{d}x_1 \, \mathrm{d}x_2.
\end{aligned} \tag{10}$$

Since

$$\operatorname{Re}\int\limits_T^{2T}\left(\frac{x_1}{x_2}\right)^{i\varphi(t)}\mathrm{d}t=\left(\log\left(\frac{x_1}{x_2}\right)\right)^{-1}\int\limits_T^{2T}\frac{1}{\varphi'(t)}\mathrm{d}\sin\left(\varphi(t)\log\left(\frac{x_1}{x_2}\right)\right)\ll\left|\log\frac{x_1}{x_2}\right|^{-1}\frac{1}{\varphi'(2T)},$$

and the same bound is true for the imaginary part of the latter integral, we obtain by (10) that

$$\int\limits_T^{2T}\left|\mathcal{Z}_{b,y}(\sigma+i\varphi(t))\right|^2\mathrm{d}t=o(T),\quad T\to\infty. \tag{11}$$

Reasoning similarly, we find

$$\begin{aligned}\int\limits_T^{2T}\left|U_{n,b,y}(\sigma+i\varphi(t))\right|^2\mathrm{d}t&=T\left(\frac{b-1}{n}\right)^2\sum_{l=1}^n\widehat{\zeta}^{\,2}(a_l)u^2(a_l,y)a_l^{-2}\\&+O\left(\left(\frac{b-1}{n}\right)^2\sum_{\substack{l_1=1\\l_1\neq l_2}}^n\sum_{l_2=1}^n\widehat{\zeta}(a_{l_1})\widehat{\zeta}(a_{l_2})u(a_{l_1},y)u(a_{l_2},y)a_{l_1}^{-\sigma}a_{l_2}^{-\sigma}\left|\log\frac{a_{l_1}}{a_{l_2}}\right|^{-1}\right).\end{aligned} \tag{12}$$

Thus,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int\limits_T^{2T}\left|U_{n,b,y}(\sigma+i\varphi(t))\right|^2\mathrm{d}t=0. \tag{13}$$

By (9),

$$\begin{aligned}J_{T,n}^2\ll&\frac{1}{T}\left(\int\limits_T^{2T}\left|\mathcal{Z}_{b,y}(\sigma+i\varphi(t))\right|^2\mathrm{d}t+\left(\int\limits_T^{2T}\left|\mathcal{Z}_{b,y}(\sigma+i\varphi(t))\right|^2\mathrm{d}t\int\limits_T^{2T}\left|U_{n,b,y}(\sigma+i\varphi(t))\right|^2\mathrm{d}t\right)^{1/2}\right.\\&\left.+\int\limits_T^{2T}\left|U_{n,b,y}(\sigma+i\varphi(t))\right|^2\mathrm{d}t\right).\end{aligned}$$

Therefore, (11) and (13) yield

$$\lim_{n\to\infty}\limsup_{T\to\infty}J_{T,n}=0. \tag{14}$$

Now, we will deal with the sequence $\{P_{n,b,y}:n\in\mathbb{N}\}$. By (12), we have

$$\begin{aligned}&\sup_{s\in\mathbb{N}}\limsup_{T\to\infty}\frac{1}{T}\int\limits_T^{2T}\left|U_{n,b,y}(\sigma+i\varphi(t))\right|\mathrm{d}t\\&\ll\sup_{s\in\mathbb{N}}\limsup_{T\to\infty}\left(\frac{1}{T}\int\limits_T^{2T}\left|U_{n,b,y}(\sigma+i\varphi(t))\right|^2\mathrm{d}t\right)^{1/2}\\&\ll\sup_{n\in\mathbb{N}}\frac{b-1}{n}\left(\sum_{l=1}^n\widehat{\zeta}^{\,2}(a_l)u^2(a_l,y)a_l^{-2\sigma}\right)^{1/2}\leqslant C_{b,y,\sigma}<\infty\end{aligned} \tag{15}$$

because

$$\lim_{n\to\infty}\frac{b-1}{n}\sum_{l=1}^n\widehat{\zeta}^{\,2}(a_l)u^2(a_l,y)a_l^{-2\sigma}=\int\limits_1^b\widehat{\zeta}^{\,2}(x)u^2(x,y)x^{-2\sigma}\,\mathrm{d}x.$$

Take a random variable θ_T given on the probability space $(\Omega, \mathcal{F}, \mu)$ that is uniformly distributed on $[T, 2T]$. Consider the complex-valued random variables

$$x_{T,n.,b,y} = x_{T,n,b,y}(\sigma) = U_{n,b,y}(\sigma + i\varphi(\theta_T)),$$

and $x_{n,b,y}(\sigma)$ with the distribution $P_{n,b,y,\sigma}$. Then, rewrite the assertion of Lemma 4 in the form

$$x_{T,n,b,y} \xrightarrow[T\to\infty]{\mathcal{D}} x_{n,b,y}. \tag{16}$$

Fix $\delta > 0$. Then, in view of (15) and (16),

$$\begin{aligned} \mu\Big\{\Big|x_{n,b,y}(\sigma)\Big| > \delta^{-1}C_{b,y,\sigma}\Big\} &\leqslant \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \mu\Big\{\Big|x_{T,n,b,y}(\sigma)\Big| > \delta^{-1}C_{b,y,\sigma}\Big\} \\ &\leqslant \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{\delta}{C_{b,y,\sigma}} \int_T^{2T} \Big|U_{n,b,y}(\sigma + i\varphi(t))\Big| \,\mathrm{d}t \leqslant \delta. \end{aligned} \tag{17}$$

The set $K = \{s \in \mathbb{C} : |s| \leqslant \delta^{-1}C_{b,y,\sigma}\}$ is compact in $\mathbb{C}$. Moreover, by (17),

$$\mu\Big\{x_{n,b,y} \in K\Big\} = 1 - \mu\Big\{x_{n,b,y} \notin K\Big\} > 1 - \delta$$

for all $n \in \mathbb{N}$. This and the definition of $x_{n,b,y}$ show that, for all $n \in \mathbb{N}$,

$$P_{n,b,y,\sigma}(K) > 1 - \delta.$$

This means that the sequence $\{P_{n,b,y,\sigma} : n \in \mathbb{N}\}$ is tight. Therefore, by Lemma 5, it is relatively compact. Hence, there exists a subsequence $\{P_{n_l,b,y,\sigma}\} \subset \{P_{n,b,y,\sigma}\}$ and a probability measure $P_{b,y,\sigma}$ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ such that $P_{n_l,b,y,\sigma} \xrightarrow[l\to\infty]{\mathrm{w}} P_{b,y,\sigma}$. In other words,

$$x_{n_l,b,y} \xrightarrow[l\to\infty]{\mathcal{D}} P_{b,y,\sigma}.$$

This, (16), and (14) show that all hypotheses of Lemma 6 for $x_{T,n,b,y}$, $x_{n_l,b,y}$ and

$$y_{T,b,y} = y_{T,b,y}(\sigma) = \mathcal{Z}_{b,y}(\sigma + i\varphi(\theta_T))$$

are satisfied. Thus, we have

$$y_{T,b,y} \xrightarrow[T\to\infty]{\mathcal{D}} P_{b,y,\sigma},$$

which proves the lemma. $\square$

4. Case of Improper Integral

This section is devoted to a limit lemma for the integral

$$\mathcal{Z}_y(\sigma + i\varphi(t)) = \int_1^\infty \widehat{\zeta}(x)u(x,y)x^{-\sigma - i\varphi(t)}\,\mathrm{d}x.$$

It is well known that $\zeta(1/2 + ix) \ll (1 + |x|)^{1/6}$. Therefore, the integral for $\mathcal{Z}(\sigma + i\varphi(t))$ converges absolutely for $\sigma > \widehat{\sigma}$ with every finite $\widehat{\sigma}$.

For $A \in \mathcal{B}(\mathbb{C})$, let

$$P_{T,y,\sigma}(A) = \frac{1}{T}\mathbf{L}\big\{t \in [T, 2T] : \mathcal{Z}_y(\sigma + i\varphi(t)) \in A\big\}.$$

Lemma 8. *Assume that $\sigma \in (1/2, 1)$ is a given fixed number, and $\varphi(t) \in W_\sigma$. Then, there is a probability measure $P_{y,\sigma}$ on $(\mathbb{C}(\mathcal{B}(\mathbb{C}))$ such that $P_{T,y,\sigma} \xrightarrow[T\to\infty]{\mathrm{w}} P_{y,\sigma}$.*

Proof. We use a similar method as in the proof of Lemma 7. We begin with a mean value

$$J_{T,y} \stackrel{\text{def}}{=} \frac{1}{T}\int_0^T \Big| \mathcal{Z}_y(\sigma + i\varphi(t)) - \mathcal{Z}_{b,y}(\sigma + i\varphi(t)) \Big| \, \mathrm{d}t.$$

Clearly, the absolute convergence of the integral for $\mathcal{Z}_y(\sigma + i\varphi(t))$ shows that, for every fixed $y > 0$,

$$\begin{aligned} \mathcal{Z}_y(\sigma + i\varphi(t)) - \mathcal{Z}_{b,y}(\sigma + i\varphi(t)) &= \int_b^\infty \widehat{\zeta}(x) u(x,y) x^{-\sigma - i\varphi(t)} \, \mathrm{d}x \\ &\ll \int_b^\infty \widehat{\zeta}(x) u(x,y) x^{-\sigma} \, \mathrm{d}x = o_y(1) \end{aligned}$$

as $b \to \infty$. Hence, we obtain

$$\lim_{b\to\infty} \limsup_{T\to\infty} J_{T,y} = 0. \tag{18}$$

Let $y_{b,y}(\sigma)$ be the complex-valued random variable with distribution $P_{b,y,\sigma}$, and, in the notation of Lemma 7,

$$y_{T,b,y} = y_{T,b,y}(\sigma) = \mathcal{Z}_{b,y}(\sigma + i\varphi(\theta_T)).$$

Then, by Lemma 7,

$$y_{T,b,y} \xrightarrow[T\to\infty]{\mathcal{D}} y_{b,y}. \tag{19}$$

Moreover, in virtue of (11),

$$\sup_{b\geqslant 1} \limsup_{T\to\infty} \frac{1}{T}\int_T^{2T} \Big| \mathcal{Z}_{b,y}(\sigma + i\varphi(t)) \Big| \, \mathrm{d}t \leqslant C_{y,\sigma} < \infty.$$

This together with (19) gives, for $\delta > 0$,

$$\begin{aligned} \mu\Big\{ \big| y_{b,y} \big| > \delta^{-1} C_{y,\sigma} \Big\} &\leqslant \sup_{b\geqslant 1} \limsup_{T\to\infty} \mu\Big\{ \big| y_{b,y} \big| > \delta^{-1} C_{y,\sigma} \Big\} \\ &\leqslant \sup_{b\geqslant 1} \limsup_{T\to\infty} \frac{\delta}{C_{y,\sigma}} \int_T^{2T} \Big| \mathcal{Z}_{b,y}(\sigma + i\varphi(t)) \Big| \, \mathrm{d}t \leqslant \delta. \end{aligned}$$

Taking a set $K = \{s \in \mathbb{C} : |s| \leqslant \delta^{-1} C_{y,\sigma}\}$, from this, we deduce that

$$\mu\Big\{ y_{b,y} \in K \Big\} > 1 - \delta.$$

This implies that the family $\{P_{b,y,\sigma} : b \geqslant 1\}$ is tight. Therefore, in view of Lemma 5, it is relatively compact. Thus, there is a sequence $\{P_{b_l,y,\sigma}\}$ and a probability measure $P_{y,\sigma}$ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ such that

$$y_{b_l,y,\sigma} \xrightarrow[l\to\infty]{\mathcal{D}} P_{y,\sigma}.$$

This, (19), (18), and the application of Lemma 6 complete the proof of the lemma. □

5. Proof of Theorem 2

We recall that

$$u(x,y) = \exp\left\{ -\left(\frac{x}{y}\right)^{\theta} \right\}, \quad x, y \in [1, \infty),$$

with a fixed $\theta > 1/2$. For brevity, set

$$f(s,y) = \frac{1}{\theta}\Gamma\left(\frac{s}{\theta}\right) y^s,$$

where $\Gamma(s)$ is the Euler gamma-function. For the approximation of $\mathcal{Z}(\sigma + i\varphi(t))$ by $\mathcal{Z}_y(\sigma + \varphi(t))$, we use the representation

$$\mathcal{Z}_y(s) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \mathcal{Z}(s+z) f(z,y)\, \mathrm{d}z, \quad \frac{1}{2} < \sigma < 1, \tag{20}$$

obtained in [25], Lemma 9.

Lemma 9. *Under the hypotheses of Theorem 2,*

$$\lim_{y\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_{T}^{2T} \left| \mathcal{Z}(\sigma + i\varphi(t)) - \mathcal{Z}_y(\sigma + i\varphi(t)) \right| \mathrm{d}t = 0.$$

Proof. Let $\theta_1 = -\varepsilon$ and $\theta = 1/2 + \varepsilon$. The integrand in (20) has a double pole $z = 1 - s$ and a simple pole $z = 0$ lying in $\theta_1 < \mathrm{Re} z < \theta$. Therefore, by the residue theorem and (20), we have

$$\mathcal{Z}_y(s) - \mathcal{Z}(s) = \frac{1}{2\pi i} \int_{\theta_1 - i\infty}^{\theta_1 + i\infty} \mathcal{Z}(s+z) f(z,y)\, \mathrm{d}z + r_y(s),$$

where

$$r_y(s) = \operatorname*{Res}_{z=1-s} \mathcal{Z}(s) f(s,y). \tag{21}$$

From this, we obtain

$$\begin{aligned}
&\mathcal{Z}_y(\sigma + i\varphi(t)) - \mathcal{Z}(\sigma + i\varphi(t)) \\
&\quad = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{Z}(\sigma - \varepsilon + i\varphi(t) + i\tau) f(-\varepsilon + i\tau, y)\, \mathrm{d}\tau + r_y(\sigma + i\varphi(t)) \\
&\quad \ll \int_{-\infty}^{\infty} |\mathcal{Z}(\sigma - \varepsilon + i\varphi(t) + i\tau)| |f(-\varepsilon + i\tau, y)|\, \mathrm{d}\tau + \left| r_y(\sigma + i\varphi(t)) \right|.
\end{aligned}$$

Thus,

$$\frac{1}{T} \int_{T}^{2T} \left| \mathcal{Z}(\sigma + i\varphi(t)) - \mathcal{Z}_y(\sigma + i\varphi(t)) \right| \mathrm{d}t \ll I_{T,y},$$

where

$$I_{T,y} \overset{\text{def}}{=} \int\limits_{-\infty}^{\infty} \left(\frac{1}{T} \int\limits_{T}^{2T} |\mathcal{Z}(\sigma - \varepsilon + i\varphi(t) + i\tau)| \, \mathrm{d}t \right) |f(-\varepsilon + i\tau, y)| \, \mathrm{d}\tau$$
$$+ \frac{1}{T} \int\limits_{T}^{2T} \left| r_y(\sigma + i\varphi(t)) \right| \mathrm{d}t = I_{T,y}^{(1)} + I_{T,y}^{(2)}. \tag{22}$$

To estimate $I_{T,y}^{(1)}$, we observe that

$$\begin{aligned} \frac{1}{T} \int\limits_{T}^{2T} |\mathcal{Z}(\sigma - \varepsilon + i\varphi(t) + i\tau)| \, \mathrm{d}t &\leqslant \left(\frac{1}{T} \int\limits_{T}^{2T} |\mathcal{Z}(\sigma - \varepsilon + i\varphi(t) + i\tau)|^2 \, \mathrm{d}t \right)^{1/2} \\ &= \left(\frac{1}{T} \int\limits_{T}^{2T} |\mathcal{Z}(\sigma - \varepsilon + i\varphi(t) + i\tau)|^2 \, \frac{\varphi'(t)\mathrm{d}t}{\varphi'(t)} \right)^{1/2} \\ &\ll \left(\frac{1}{T\varphi'(2T + |\tau|)} \int\limits_{0}^{\varphi(2T+|\tau|)} |\mathcal{Z}(\sigma - \varepsilon + iu)|^2 \mathrm{d}u \right) \\ &\ll \left(\frac{I_{\sigma-\varepsilon}\varphi(2T + |\tau|)}{T\varphi'(2T + |\tau|)} \right)^{1/2} \\ &\ll \left(\frac{2T + |\tau|}{T} \right)^{1/2} \ll (1 + |\tau|)^{1/2}. \end{aligned} \tag{23}$$

For the gamma-function, the estimate

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\}, \quad c > 0, \tag{24}$$

is valid. Therefore,

$$f(-\varepsilon + i\tau, y) \ll y^{-\varepsilon} \exp\{-c_1|\tau|\}, \quad c_1 > 0.$$

This together with (23) leads to the bound

$$I_{T,y}^{(1)} \ll y^{-\varepsilon} \int\limits_{-\infty}^{\infty} (1 + |\tau|)^{1/2} \exp\{-c_1|\tau|\} \, \mathrm{d}\tau \ll y^{-\varepsilon}. \tag{25}$$

Let $a = 2\gamma - \log 2\pi$. In view of the formula for $\mathcal{Z}(s)$ in Theorem 1,

$$\begin{aligned} r_y(s) &= f'(1 - s, y) + af(1 - s, y) \\ &= \frac{1}{\theta^2}\Gamma'\left(\frac{1 - s}{\theta}\right) y^{1-s} + \frac{1}{\theta}\Gamma\left(\frac{1 - s}{\theta}\right) y^{1-s} \log y + \frac{a}{\theta}\Gamma\left(\frac{1 - s}{\theta}\right) y^{1-s} \\ &= \frac{y^{1-s}}{\theta}\Gamma\left(\frac{1 - s}{\theta}\right) \left(\frac{1}{\theta} \frac{\Gamma'}{\Gamma}\left(\frac{1 - s}{\theta}\right) + \log y + a \right). \end{aligned}$$

Hence, the estimates (24) and

$$\frac{\Gamma'}{\Gamma}(\sigma + it) \ll \log(|t| + 2)$$

yield

$$I_{T,y}^{(2)} \ll_\theta y^{1-\sigma} \log y \frac{1}{T} \int_T^{2T} \exp\left\{-\frac{c}{\theta}\varphi(t)\right\} \log \varphi(t)\, \mathrm{d}t$$
$$\ll_\theta y^{1-\sigma} \log y \exp\left\{-\frac{c}{2\theta}\varphi(T)\right\}.$$

This, (25), and (22) show that

$$I_{T,y} \ll_\delta y^{-\varepsilon} + y^{1-\sigma} \log y \exp\left\{-\frac{c}{2\theta}\varphi(T)\right\}.$$

Therefore,

$$\lim_{y\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_T^{2T} \left|\mathcal{Z}(\sigma + i\varphi(t)) - \mathcal{Z}_y(\sigma + i\varphi(t))\right| \mathrm{d}t = 0 \tag{26}$$

because $\varphi(T) \to \infty$ as $T \to \infty$. □

Now, we return to the limit measure $P_{y,\sigma}$ of Lemma 8.

Lemma 10. *Under the hypotheses of Theorem 2, the family $\{P_{y,\sigma} : y \in [1, \infty)\}$ is tight.*

Proof. We have

$$\frac{1}{T} \int_T^{2T} \left|\mathcal{Z}_y(\sigma + i\varphi(t))\right| \mathrm{d}t \leqslant \frac{1}{T} \int_T^{2T} \left|\mathcal{Z}(\sigma + i\varphi(t)) - \mathcal{Z}_y(\sigma + i\varphi(t))\right| \mathrm{d}t + \frac{1}{T} \int_T^{2T} \left|\mathcal{Z}(\sigma + i\varphi(t))\right| \mathrm{d}t.$$

Therefore, by (2) and (26),

$$\sup_{y\geqslant 1} \limsup_{T\to\infty} \frac{1}{T} \int_T^{2T} \left|\mathcal{Z}_y(\sigma + i\varphi(t))\right| \mathrm{d}t \leqslant C < \infty. \tag{27}$$

Let

$$z_{T,y} = z_{T,y}(\sigma) = \mathcal{Z}_y(\sigma + i\varphi(\theta_T)),$$

and $z_y = z_y(\sigma)$ be the complex-valued random variable with the distribution $P_{y,\sigma}$. Then, the statement of Lemma 8 can be written as

$$z_{T,y} \xrightarrow[T\to\infty]{\mathcal{D}} z_y. \tag{28}$$

From this and (27), we obtain that, for every $\delta > 0$,

$$\mu\left\{|z_y| > \delta^{-1}C\right\} \leqslant \sup_{y\geqslant 1} \limsup_{T\to\infty} \mu\left\{|z_{T,y}| > \delta^{-1}C\right\} \leqslant \frac{\delta}{TC} \int_T^{2T} \left|\mathcal{Z}_y(\sigma + i\varphi(t))\right| \mathrm{d}t \leqslant \delta.$$

This shows that, for $K = \{s \in \mathbb{C} : |s| \leqslant \delta^{-1}C\}$,

$$P_{y,\sigma}(K) \geqslant 1 - \delta,$$

and the lemma is proven. □

Proof of Theorem 2. Lemma 10 together with Lemma 5 implies that the family $\{P_{y,\sigma}\}$ is relatively compact. Therefore, there is a sequence $\{P_{y_k,\sigma}\} \subset \{P_{y,\sigma}\}$ weakly convergent to a certain probability measure P_σ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $k \to \infty$. This also can be written as

$$z_{y_k,\sigma} \xrightarrow[k\to\infty]{\mathcal{D}} P_\sigma. \tag{29}$$

Define one more random variable,

$$z_T = z_T(\sigma) = \mathcal{Z}(\sigma + i\varphi(\theta_T)).$$

Then, Lemma 9 implies, for every $\delta > 0$,

$$\lim_{k\to\infty} \limsup_{T\to\infty} \mu\{|z_T - z_{T,y_k}| > \delta\} \leqslant \lim_{k\to\infty} \limsup_{T\to\infty} \frac{1}{\delta T} \int_T^{2T} |\mathcal{Z}(\sigma + i\varphi(t)) - \mathcal{Z}_{y_k}(\sigma + i\varphi(t))| \, \mathrm{d}t = 0.$$

This, (28), and (29) together with Lemma 6 prove that

$$z_T \xrightarrow[T\to\infty]{\mathcal{D}} P_\sigma.$$

The theorem is proven. □

6. Conclusions

In the paper, we considered the asymptotic behavior of the modified Mellin transform of the square of the Riemann zeta-function by using a probabilistic approach. We proved a limit theorem on the weak convergence of probability measures defined by shifts $\mathcal{Z}(\sigma + i\varphi(t))$, $1/2 < \sigma < 1$, where $\varphi(t)$ is a differentiable increasing to infinity function with a monotonically decreasing derivative $\varphi'(t)$ satisfying a certain estimate connected to the mean square of the function $\mathcal{Z}(s)$. We expect that such normalization of the function $\mathcal{Z}(s)$ extends the class of limit distributions for $\mathcal{Z}(s)$. Our future plans are related to a similar theorem in the space of analytic functions.

Author Contributions: Conceptualization, A.L. and D.Š.; methodology, A.L. and D.Š.; investigation, A.L. and D.Š.; writing—original draft preparation, A.L. and D.Š. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

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