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Article

An Extension of the Fréchet Distribution and Applications

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Abstract: This paper presents the Slash-Exponential-Fréchet distribution, which is an expanded version of the Fréchet distribution. Through its stochastic representation, probability distribution function, moments and other relevant features are obtained. Evidence supports that the updated model displays a lighter right tail than the Fréchet model and is more flexible as for skewness and kurtosis. Results on maximum likelihood estimators are given. Our proposition's applicability is demonstrated through a simulation study and the evaluation of two real-world datasets.

Keywords: Fréchet distribution; slash exponential distribution; slash-exponential Fréchet; kurtosis

MSC: 60E05; 62E15; 62F10

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1. Introduction

The French mathematician Maurice René Fréchet introduced the Fréchet (Fr) distribution in the 1920s as a maximum value distribution, see Fréchet [1] or Fisher and Tippet [2]. It is also recognized as the inverse Weibull distribution and this is a specific instance of what is commonly referred to as Generalized Extreme Value (GEV) distributions, (see Gumbel [3] Embrechts et al. [4], Resnick [5], or Haan and Ferreira [6]). Since its origin, this model has been applied in different areas, such as reliability, life testing, extreme events, rainfall, wind speeds, among others. Information regarding the theoretical characteristics and uses of Fréchet distribution can be found in the works of Kotz and Nadarajah [7], Gupta et al. [8], Coles [9], and Ramos et al. [10]. Results using the term inverse Weibull distribution can be seen in Calabria and Pulcini [11], Maswadah [12], and Salman [13], and from a Bayesian point of view in Abbas and Tang [14]. These papers along with the references appearing therein, show the interest of Fréchet model. In this paper a new extension of the Fréchet distribution is proposed, which is based on the Fréchet and the Slash-Exponential distribution recently introduced in Punathumparambath [15]. Next, properties of these models are given, which will allow us to reach our end. So, a continuous random variable (rv) X follows a Fréchet distribution if its probability density function (pdf) is

$$f_X(x;\lambda) = \lambda x^{-(1+\lambda)} \exp\left(-x^{-\lambda}\right), \quad x > 0, \tag{1}$$

where $\lambda > 0$ is a shape parameter. Equation (1) is denoted $X \sim Fr(\lambda)$. Some properties of interest are listed in next lemma.

Lemma 1. *(a) Let $X \sim Fr(\lambda)$. Then its cumulative distribution function (cdf) is*

$$F_X(x;\lambda) = \exp\left(-x^{-\lambda}\right), \quad x > 0. \tag{2}$$

(b) Let $V \sim Exp(1)$. Then $V^{-\frac{1}{\lambda}} \sim Fr(\lambda)$.

(c) *If $X \sim Fr(\lambda)$, then the rth-moment of X exists for $r < \lambda$ and*

$$\mathbb{E}(X^r) = \Gamma\Big(1 - \frac{r}{\lambda}\Big), \tag{3}$$

where $\Gamma(\cdot)$ denotes the gamma function.

(d) *Let $X \sim Fr(\lambda)$. Then the pdf of X is unimodal with mode $m = \left(\frac{\lambda}{1+\lambda}\right)^{1/\lambda}$ or decreasing (for λ close to zero), the hazard function is always unimodal, see [10].*

The gamma distribution will be used throughout this paper. In Remark 1, its pdf and cdf are given.

Remark 1. *If a rv T adheres to a gamma distribution, represented as $T \sim Ga(a,b)$, it implies that its pdf is as such*

$$g(t;a,b) = \frac{b^a}{\Gamma(a)}\, t^{a-1}e^{-bt}, \quad t > 0, \tag{4}$$

with $a, b > 0$. The cdf of T is

$$G(z;a,b) = \int_0^z g(t;a,b)dt = \frac{\gamma(a,bz)}{\Gamma(a)}\ , \tag{5}$$

where $\gamma(a,bz) = \int_0^{bz} u^{a-1}e^{-u}du$ is the (lower) incomplete gamma function, (Abramowitz and Stegun [16]).

Next, the basis of our proposal is introduced, that is, the Slash-Exponential distribution. As origin of the Slash-Exponential model, the Generalized Exponential distribution can be cited (Gupta and Kundu [17,18]). Astorga et al. [19] proposed an extension called Slashed Generalized Exponential, which contains as a submodel, the Slash Exponential. Details can be seen in Punathumparambath [15].

Definition 1. *A rv Y follows a Slash-Exponential distribution with shape parameter $\alpha > 0$, $Y \sim SE(\alpha)$, if its pdf is presented by*

$$f_Y(y;\alpha) = \alpha^2 y^{-(1+\alpha)}\Gamma(\alpha)G(y;1+\alpha,1), \quad y > 0, \tag{6}$$

where $G(\cdot)$ denotes the cdf of the gamma distribution given in (5).

The following lemma presents some properties of Slash-Exponential distribution.

Lemma 2. (a) *Let $Y \sim SE(\alpha)$. Then the cdf is*

$$F_Y(y;\alpha) = 1 - \alpha y^{-\alpha}\Gamma(\alpha)G(y;1+\alpha,1) - \exp(-y), \qquad y > 0.$$

(b) *If $Y \sim SE(\alpha)$ then the survival and hazard rate functions are as follows*

$$S_Y(y;\alpha) = \alpha y^{-\alpha}\Gamma(\alpha)G(y;1+\alpha,1) + \exp(-y), \qquad y > 0,$$

$$h_Y(y;\alpha) = \frac{\alpha^2 y^{-(1+\alpha)}\Gamma(\alpha)G(y;1+\alpha,1)}{\alpha y^{-\alpha}\Gamma(\alpha)G(y;1+\alpha,1) + \exp(-y)}, \qquad y > 0.$$

(c) *Let $Y \sim SE(\alpha)$. Then the rth-moment of Y exists for $\alpha > r$ and*

$$\mathbb{E}(Y^r) = \frac{\alpha}{\alpha - r} r!.$$

The objective of this article is to present a novel expansion of the Fréchet distribution taking as starting point the Slash-Exponential distribution defined in (6). Specifically, by proceeding similarly to property b given in Lemma 1, an updated extension of the Fréchet distribution has been achieved. This is called Slash-Exponential-Fréchet (SEFr) model. The SEFr distribution can be used in different fields where the Fréchet distribution is of common use, with the advantage of being a model more flexible as for skewness and kurtosis than the Fréchet model. On the other hand, the SFEr model can be used as an alternative to the Slash Fréchet distribution proposed in Castillo et al. [20] (model with two shape parameters), since the proposed model incorporates a scale parameter, resulting a distribution with greater flexibility. As novelty of this paper, we highlight that, the new model has a lighter right tail than the baseline Fréchet distribution, this is a new feature in the field of applications of slash methodology. The Slash distribution is statistically characterized as the ratio formed by two independent rv's, with one conforming to a standard normal distribution and the other representing a power of a uniform distribution. Therefore, it can be mentioned that Y possesses a slash distribution if it can be represented as such:

$$Y = \frac{X_1}{X_2}$$

where $X_1 \sim N(0,1)$ and $X_2 \sim Beta(q,1)$, X_1 is independent of X_2 and $q > 0$. In light of the groundbreaking studies by Rogers and Tukey [21], Andrews et al. [22] on slash distribution; multivariate slash models proposed in Gómez et al. [23] or Arslan and Genc [24], slash methodology has been proven to be useful to increase the weight of tails of a baseline distribution. See for instance Reyes et al. [25], Del Castillo [26], Zörnig [27], Olmos et al. [28] for the half-normal and generalized half-normal, Astorga et al. [19] for the generalized exponential distribution, Barranco-Chamorro et al. [29] for the Rayleigh distribution, Barrios et al. [30] for the power half-normal, Gui [31] for the Lindley, and Castillo et al. [32] for weighted Lindley, among others. However, in this paper, the slash methodology is applied in such a way, that we get a class of distributions more flexible than Fréchet baseline distribution as for skewness and kurtosis.

The structure of this paper is as follows. In Section 2, the stochastic representation and relevant properties are given. These are: pdf, mode, cdf, survival and hazard rate function, right tail behaviour, approximations to Fréchet model, moments, Shannon entropy, and other properties of interest in reliability. Section 3 is devoted to inferential results. Maximum Likelihood (ML) estimation method is studied in detail. A simulation study is conducted to evaluate the effective performance of ML estimators. In Section 4, a pair of practical applications are presented where our suggestion is benchmarked against other rival models, demonstrating its superior efficacy. The concluding remarks can be found in Section 5.

2. Results for the SEFr Distribution

First, the stochastic representation of the Slash-Exponential-Fréchet distribution is given as

$$X = \sigma Y^{-1/\lambda}, \tag{7}$$

where $Y \sim SE(\alpha)$ with $\sigma, \lambda, \alpha > 0$. Equation (7) is denoted by $X \sim SEFr\,(\sigma, \lambda, \alpha)$.

In Equation (7), $\sigma > 0$ is a scale parameter, whereas $\lambda > 0$ and $\alpha > 0$ are shape parameters.

Next proposition provides the pdf of the SEFr model.

Proposition 1. *Let $X \sim SEFr\,(\sigma, \lambda, \alpha)$. Then the pdf of X is given by*

$$f_X(x; \sigma, \lambda, \alpha) = \frac{\lambda \alpha^2 \Gamma(\alpha)}{\sigma^{\lambda\alpha}} x^{\lambda\alpha - 1} G\left(\left(\frac{\sigma}{x}\right)^{\lambda}; 1+\alpha, 1\right), \quad x > 0, \tag{8}$$

where $\sigma, \lambda, \alpha > 0$ and $G\,(z; a, b)$ denotes the cdf of the gamma distribution introduced in (5).

Proof. By employing the representation outlined in (7) and the rv transformation method, the result is obtained. □

Given that σ is a scale parameter, we can assume $\sigma = 1$ without any loss of generality. Figure 1 provides illustrative plots for varying values of α and λ. These plots suggest that, like the Fréchet model, the SEFr distribution is unimodal or decreasing. These appreciations are proved in Proposition 2.

Proposition 2. *Let $X \sim SEFr\,(1, \lambda, \alpha)$. Then the mode of X is given as the solution of*

$$e^{x^{-\lambda}} x^{\lambda(\alpha+1)} \gamma(\alpha+1, x^{-\lambda}) = \frac{\lambda}{\lambda\alpha - 1}, \quad \textit{provided that } \lambda\alpha > 1\,. \tag{9}$$

For $\lambda\alpha \leq 1$, the pdf of X is strictly decreasing.

Proof. Straightforward, studying the first derivative of (8) with respect to x. □

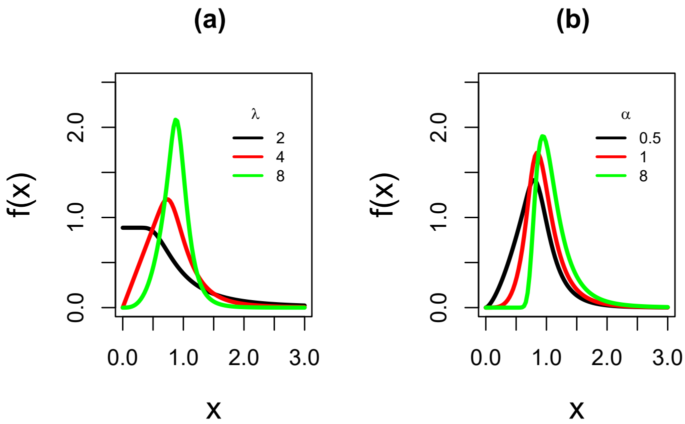

Figure 1. pdf of $SEFr\,(1, \lambda, \alpha)$: (**a**) for $\alpha = 0.5$ and $\lambda \in \{2, 4, 8\}$, (**b**) for $\lambda = 5$ and $\alpha \in \{0.5, 1, 8\}$.

2.1. Properties

Proposition 3. *Let $X \sim SEFr\,(\sigma, \lambda, \alpha)$. Then, the cdf of X is provided by*

$$F_X(x; \sigma, \lambda, \alpha) = \alpha\Gamma(\alpha)\left(\frac{x}{\sigma}\right)^{\lambda\alpha} G\left(\left(\frac{\sigma}{x}\right)^{\lambda}; 1+\alpha, 1\right) + \exp\left\{-\left(\frac{\sigma}{x}\right)^{\lambda}\right\}, \quad x > 0\,. \tag{10}$$

Proof. By definition

$$F_X(x; \sigma, \lambda, \alpha) = \frac{\lambda\alpha^2\Gamma(\alpha)}{\sigma^{\lambda\alpha}} \int_0^x t^{\lambda\alpha - 1} G\left(\left(\frac{\sigma}{t}\right)^{\lambda}, 1+\alpha, 1\right) dt.$$

By employing the methods of integration by parts, it can be shown that

$$\begin{aligned}
u &= G\left(\left(\frac{\sigma}{t}\right)^{\lambda}; 1+\alpha, 1\right) \Rightarrow \\
du &= \frac{-\lambda\sigma^{\lambda}}{t^{\lambda+1}} g\left(\left(\frac{\sigma}{t}\right)^{\lambda}, 1+\alpha, 1\right) dt = \frac{-\lambda\sigma^{\lambda}}{t^{\lambda+1}\Gamma(1+\alpha)}\left(\frac{\sigma}{t}\right)^{\lambda\alpha} \exp\left\{-\left(\frac{\sigma}{t}\right)^{\lambda}\right\} dt\,, \\
dv &= t^{\lambda\alpha - 1} \Rightarrow v = \frac{t^{\lambda\alpha}}{\lambda\alpha},
\end{aligned}$$

where $g(\cdot)$ denotes the pdf of the gamma distribution. Then,

$$\begin{aligned}
F_X(x; \sigma, \lambda, \alpha) = {}& \frac{\lambda\alpha^2\Gamma(\alpha)}{\sigma^{\lambda\alpha}} \\
& \times \left[G\left(\left(\frac{\sigma}{t}\right)^{\lambda}; 1+\alpha, 1\right)\frac{x^{\lambda\alpha}}{\lambda\alpha} + \frac{\sigma^{\lambda+\lambda\alpha}}{\alpha\Gamma(1+\alpha)} \int_0^x t^{-\lambda-1} \exp\left\{-\left(\frac{\sigma}{t}\right)^{\lambda}\right\}\right] dt,
\end{aligned}$$

and making the substitution $w = \left(\frac{\sigma}{t}\right)^{\lambda}$, the proposed result is obtained. □

Corollary 1. *Let $X \sim SEFr\,(\sigma, \lambda, \alpha)$. Then the survival function and hazard rate function of X are provided by*

$$
\begin{aligned}
S(x) &= 1 - \alpha\Gamma(\alpha)\left(\frac{x}{\sigma}\right)^{\lambda\alpha} G\left(\left(\frac{\sigma}{x}\right)^{\lambda}; 1+\alpha, 1\right) - \exp\left\{-\left(\frac{\sigma}{x}\right)^{\lambda}\right\}, \\
h(x) &= \frac{\lambda\alpha^2\Gamma(\alpha)x^{\lambda\alpha-1} G\left(\left(\frac{\sigma}{x}\right)^{\lambda}; 1+\alpha, 1\right)}{\sigma^{\lambda\alpha}\left[1 - \alpha\Gamma(\alpha)\left(\frac{x}{\sigma}\right)^{\lambda\alpha} G\left(\left(\frac{\sigma}{x}\right)^{\lambda}; 1+\alpha, 1\right) - \exp\left\{-\left(\frac{\sigma}{x}\right)^{\lambda}\right\}\right]}.
\end{aligned}
$$

Proof. It follows by applying $S(x) = 1 - F_X(x)$ and $h(x) = f_X(x)/S(x)$. □

As an illustration, plots for the cdf, survival function, and hazard rate function in the SEFr $(1, \lambda, 0.5)$ model are given in Figure 2.

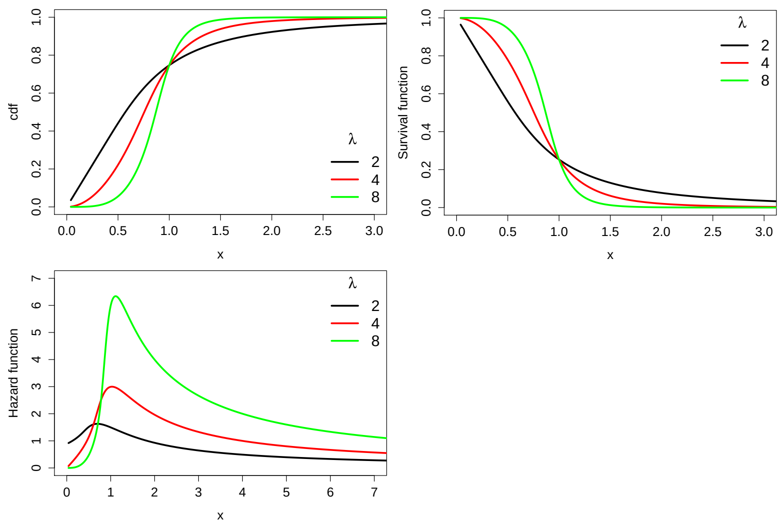

Figure 2. The cdf, survival function and hazard rate function for the SEFr $(1, \lambda, 0.5)$ model can be visualized through their respective plots.

Next we focus on studying the right tail of SEFr model. Specifically, in Proposition 4, it is proven that the right tail of SEFr model is lighter than the right tail of baseline Fréchet model. Moreover, in Proposition 5, it is proven that the SEFr model tends to a Fréchet model when $\alpha \to \infty$.

Proposition 4. *Let $X \sim SEFr\,(1, \lambda, \alpha)$ and $F \sim Fr\,(\lambda)$ with survival functions S_X and S_F, respectively. Then*

$$S_X(t) \leq S_F(t), \qquad \forall t > 0. \tag{11}$$

That is, for all fixed $\alpha > 0$ the right tail of SEFr $(1, \lambda, \alpha)$ model is lighter than the right tail in the Fr (λ) distribution.

Proof. From Corollary 1 and (10), note that $\forall t > 0$

$$
\begin{aligned}
S_X(t) &= 1 - \exp\left\{-t^{-\lambda}\right\} - \alpha\Gamma(\alpha)t^{\lambda\alpha}G(t^{-\lambda}; 1+\alpha, 1) \\
&= S_F(t) - \alpha\Gamma(\alpha)t^{\lambda\alpha}G(t^{-\lambda}; 1+\alpha, 1)\,.
\end{aligned}
$$

Since $\alpha > 0$ and $G(\cdot)$ is the cdf of a gamma distribution, we have that $S_X(t) \leq S_F(t)$, $\forall t > 0$. □

For $\lambda = 5$ and increasing values of α, the property given in Proposition 4 is illustrated in Table 1.

Table 1. Some probabilities on right tails of SEFr and Fr distributions.

α	$P[X>1]$	$P[X>1.5]$	$P[X>2]$	$P[X>2.5]$
$SEFr\,(1,5,0.5)$	0.253	0.042	0.010	0.003
$SEFr\,(1,5,1)$	0.368	0.063	0.016	0.005
$SEFr\,(1,5,8)$	0.587	0.110	0.027	0.009
$Fr\,(5)$	0.632	0.123	0.031	0.010

Moreover we have that the cdf of a $SEFr\,(1,\lambda,\alpha)$ approaches to the $Fr\,(\lambda)$ distribution when $\alpha \to \infty$. This appreciation is formalised in next proposition.

Proposition 5. *Let $X_\alpha \sim SEFr\,(1,\lambda,\alpha)$. If $\alpha \to \infty$, then X_α converges in distribution to a rv $F \sim Fr(\lambda)$.*

Proof. Let $X_\alpha \sim SEFr\,(1,\lambda,\alpha)$. By using the stochastic representation given in (7) and the definition of the Slash Exponential proposed in [15], we can write $X_\alpha = U_1^{-\frac{1}{\lambda}}\,U^{\frac{1}{\alpha\lambda}}$ where $U_1 \sim Exp(1)$ and $U \sim U(0,1)$ independent. Whereas $U^{\frac{1}{\alpha\lambda}} \xrightarrow{a.s} 1$ if $\alpha \to \infty$ where $a.s$ denotes for almost surely convergence, then we have,

$$U^{\frac{1}{\alpha\lambda}} \xrightarrow{P} 1, \qquad \alpha \to \infty,$$

where P signifies the convergence in probability. By applying Slutsky's lemma, see for instance Lehmann [33], it follows that

$$U_1^{-\frac{1}{\lambda}}\,U^{\frac{1}{\alpha\lambda}} \xrightarrow{D} U_1^{-\frac{1}{\lambda}} = F \sim Fr(\lambda)\,,$$

where D denotes convergence in distribution. □

Note that Proposition 5 establishes that the for large α, the SEFr model can be approximated by a Fréchet distribution.

2.2. Moments

Proposition 6. *Let $X \sim SEFr\,(\sigma,\lambda,\alpha)$. Then, for r a positive integer, $E[X^r]$ exists, if and only if $r<\lambda$, and in this case,*

$$E[X^r] = \sigma^r \frac{\alpha\lambda}{\alpha\lambda + r}\Gamma\left(1-\frac{r}{\lambda}\right).$$

Proof. Through the utilization of the stochastic representation provided in (7)

$$E[X^r] = \sigma^r E\left[U_1^{-r/\lambda}\right] E\left[U^{\frac{r}{\alpha\lambda}}\right] \tag{12}$$

where $U_1 \sim Exp(1)$ and $U \sim U(0,1)$ independent. Therefore, we have the proposed result. □

The subsequent corollary is a direct result of Proposition 6, as indicated

Corollary 2. *Let $X \sim SEFr\,(\sigma,\lambda,\alpha)$. Then*

1. $E[X] = \sigma\frac{\alpha\lambda}{\alpha\lambda+1}\Gamma\left(1-\frac{1}{\lambda}\right)$ *, provided that $\lambda > 1$.*
2. $Var[X] = \sigma^2\alpha\lambda\left[\frac{1}{\alpha\lambda+2}\Gamma\left(1-\frac{2}{\lambda}\right) - \frac{\alpha\lambda}{(\alpha\lambda+1)^2}\Gamma^2\left(1-\frac{1}{\lambda}\right)\right]$*, provided that $\lambda > 2$.*

3. *Let $\mu_j = E[X^j]$. Then, the skewness, $\sqrt{\beta_1}$, and kurtosis, β_2, coefficients can be obtained by using*

$$\begin{aligned}\sqrt{\beta_1} &= \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3}{(\mu_2 - \mu_1^2)^{\frac{3}{2}}}, \quad \lambda > 3,\\ \beta_2 &= \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_2\mu_1^2 - 3\mu_1^4}{(\mu_2 - \mu_1^2)^2}, \quad \lambda > 4.\end{aligned}$$

In Figures 3 and 4, plots and contour plots for the skewness and kurtosis coefficients in the SEFr $(1, \lambda, \alpha)$ model are given. These plots suggest that for fixed α and an increasing value of λ the skewness and kurtosis coefficients decrease quickly. On the other hand, for λ fixed and decreasing values of α, the skewness and kurtosis coefficients decrease quite slower.

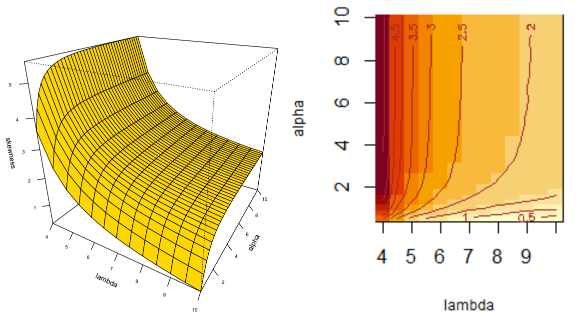

Figure 3. Plot and contour plot of the skewness coefficient in the SEFr $(1, \lambda, \alpha)$ model.

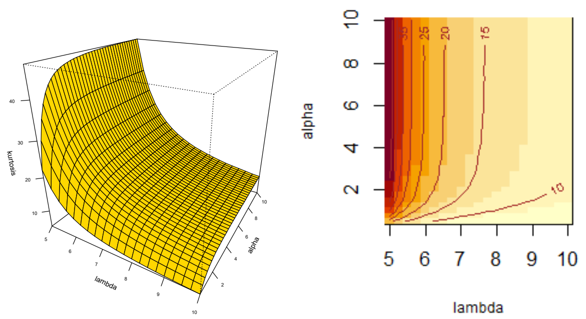

Figure 4. Plot and contour plot of the kurtosis coefficient in the SEFr $(1, \lambda, \alpha)$ model.

Shannon Entropy.

Recall that the Shannon entropy of a continuous rv X, $H(X)$, is defined as

$$H(X) = -E[\log f_X(X)] \tag{13}$$

where log denotes neperian logarithm and f_X is the pdf of X. A useful result which can be used to get the entropy of a rv is given in Lemma 3, c.f. Jones [34] (p. 137) or Awad [35].

Lemma 3. *If X is a continuous rv and $Z = g(X)$ is one-to-one transformation, then the relationship between the Shannon entropies of X and Z is*

$$H(Z) = H(X) + E\left[\log\left|\frac{dZ}{dX}\right|\right]. \tag{14}$$

Proposition 7. *Let $X \sim SEFr\,(\sigma, \lambda, \alpha)$. Then the Shannon entropy of X, $H(X)$, can be obtained as*

$$H(X) = \log\left(\frac{\sigma}{\lambda\alpha}\right) + \left(\alpha - \frac{1}{\lambda}\right)E[\log Y] - E[\log\ \gamma(1+\alpha, Y)], \tag{15}$$

where $Y \sim SE(\alpha)$.

Proof. Taking into account the definition of the SEFr model given in (7) and (14)

$$H(X) = H(Y) + \log\left(\frac{\sigma}{\lambda}\right) - \left(\frac{1+\lambda}{\lambda}\right) E[\log Y] \tag{16}$$

with $Y \sim SE(\alpha)$. Finally, by applying (13) to $Y \sim SE(\alpha)$, (15) follows. □

2.3. Other Properties of Interest in Reliability

In this subsection additional properties of interest in reliability are given. Specifically, these are: the reverse hazard rate function, the stress-strength parameter and the pdf of the order statistics.

The reverse hazard rate function.

The reverse hazard rate function, $r(x)$, has received increasing interest in reliability field. This function is defined as the quotient of the pdf to its cdf, $r(x) = f(x)/F(x)$. Roughly speaking, $r(x)$ gives the probability of a recent failure occurring given that a failure has already taken place, additional details can be seen in Block et al. [36]. Its expression for the $SEFr\,(\sigma, \lambda, \alpha)$ model is given next.

Corollary 3. *Let* $X \sim SEFr\,(\sigma, \lambda, \alpha)$. *Then the reverse hazard rate function of X is provided by*

$$r(x) \quad = \quad \frac{\lambda \alpha^2 \Gamma(\alpha) x^{\lambda\alpha - 1} G\left(\left(\frac{\sigma}{x}\right)^{\lambda}, 1+\alpha, 1\right)}{\sigma^{\lambda\alpha}\left[\alpha\Gamma(\alpha)\left(\frac{x}{\sigma}\right)^{\lambda\alpha} G\left(\left(\frac{\sigma}{x}\right)^{\lambda}, 1+\alpha, 1\right) + \exp\left\{-\left(\frac{\sigma}{x}\right)^{\lambda}\right\}\right]}.$$

Proof. It follows by applying (8) and (10). □

The stress-strength parameter.

Proposition 8. *Let X and Z independent Slash-Exponential-Fréchet rv's,* $X \sim SEFr\,(\sigma_x, \lambda_x, \alpha_x)$, $Z \sim SEFr\,(\sigma_z, \lambda_z, \alpha_z)$, *and R, the stress-strength parameter. Then*

$$R = \frac{\lambda_x \alpha_x}{\sigma_x} \int_0^{\infty} \left(\frac{s}{\sigma_x}\right)^{\lambda_x \alpha_x - 1} \gamma\left(1+\alpha_x,\ \left(\frac{\sigma_x}{s}\right)^{\lambda_x}\right) \left\{\left(\frac{s}{\sigma_z}\right)^{\lambda_z \alpha_z} \gamma\left(1+\alpha_z,\ \left(\frac{\sigma_z}{s}\right)^{\lambda_z}\right) + exp\left\{-\left(\frac{\sigma_z}{s}\right)^{\lambda_z}\right)\right\} ds\ .$$

Proof. It follows from the fact that $R = P[Z < X]$, see [37], by using (8) and (10) . □

Corollary 4. *Let X and Z independent Slash-Exponential-Fréchet rv's with the same scale parameter,* $X \sim SEFr\,(\sigma, \lambda_x, \alpha_x)$, $Z \sim SEFr\,(\sigma, \lambda_z, \alpha_z)$. *Then*

$$R = \lambda_x \alpha_x \int_0^{\infty} t^{\lambda_x \alpha_x - 1} \gamma\left(1+\alpha_x, t^{-\lambda_x}\right) \left\{t^{\lambda_z \alpha_z} \gamma\left(1+\alpha_z, t^{-\lambda_z}\right) + \exp(-t^{-\lambda_z})\right\} dt\ .$$

Proof. It is immediate making the change of variable $t = s/\sigma$ in the expression of R given in Proposition 8. □

Order Statistics.

Let $X_1, \ldots, X_n$ be independent identically distributed rv's, $X_i \sim SEFr\,(\sigma, \lambda, \alpha)$. Let us consider the order statistics from these n rv's, denoted by $X_{r:n}$, $r = 1, \ldots, n$. Then the pdf of $X_{r:n}$ is

$$f_{r:n}(x) = \frac{n!}{(r-1)!\ (n-r)!} f(x) \{F(x)\}^{r-1} \{1 - F(x)\}^{n-r}, \qquad r = 1, \ldots, n\ ,$$

where $f(\cdot)$ and $F(\cdot)$ are the pdf and cdf of the $SEFr\,(\sigma, \lambda, \alpha)$ model given in (8) and (10).

For $r = 1$ the pdf of the minimum is obtained and for $r = n$ the maximum. We highlight that the distribution of the maximum is not SEFr.

3. Inference

3.1. ML Estimators

Given $X_1, X_2, \ldots, X_n$ a random sample of size n from $SEFr(\sigma, \lambda, \alpha)$, then from (8), the log-likelihood function is given by

$$\begin{aligned} \ell(\boldsymbol{\theta}) \quad &\propto \quad n \log\, \lambda + n \log\, \alpha - n\lambda\, \alpha \log(\sigma) + (\lambda\, \alpha - 1) \sum_{i=1}^{n} \log(x_i) \\ &+ \quad \sum_{i=1}^{n} \log\, \gamma\left(\alpha + 1,\, \left(\frac{x_i}{\sigma}\right)^{-\lambda}\right), \end{aligned} \tag{17}$$

where $\boldsymbol{\theta} = (\sigma, \lambda, \alpha)$, $\propto$ means proportional to.

The components of the score vector are procured by performing partial derivatives with respect to σ, λ, and α, $S(\boldsymbol{\theta}) = \left(\frac{\partial \ell}{\partial \sigma}, \frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \alpha}\right)$, these are

$$\begin{aligned} \frac{\partial \ell}{\partial \sigma} &= -\frac{n\lambda\alpha}{\sigma} + \left(\frac{\lambda}{\sigma}\right) \sum_{i=1}^{n} \frac{\left(\frac{x_i}{\sigma}\right)^{-\lambda(\alpha+1)} \exp\left[-\left(\frac{x_i}{\sigma}\right)^{-\lambda}\right]}{\gamma\left(1+\alpha,\, \left(\frac{x_i}{\sigma}\right)^{-\lambda}\right)}, \\ \frac{\partial \ell}{\partial \lambda} &= \frac{n}{\lambda} - n\alpha \log(\sigma) + \alpha \sum_{i=1}^{n} \log(x_i) + \sum_{i=1}^{n} \frac{\log\left(\frac{\sigma}{x_i}\right) \left(\frac{x_i}{\sigma}\right)^{-\lambda(\alpha+1)} \exp\left[-\left(\frac{x_i}{\sigma}\right)^{-\lambda}\right]}{\gamma\left(1+\alpha,\, \left(\frac{x_i}{\sigma}\right)^{-\lambda}\right)}, \\ \frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} - n\lambda \log(\sigma) + \lambda \sum_{i=1}^{n} \log(x_i) + \sum_{i=1}^{n} \frac{I\left(\alpha+1,\, \left(\frac{x_i}{\sigma}\right)^{-\lambda}\right)}{\gamma\left(1+\alpha,\, \left(\frac{x_i}{\sigma}\right)^{-\lambda}\right)}, \end{aligned}$$

where $I(a,\, v) = \int_0^v t^{a-1} \log(t) e^{-t} dt,\;\; a > 0$, and $v > 0$. It can be seen in Milgram [38] that $I(a,\, v)$ is related to the generalized integro-exponential function when $v = \infty$.

The MLE of $\boldsymbol{\theta}$, represented as $\widehat{\boldsymbol{\theta}}$, can be secured by finding the solution to the equation $S(\boldsymbol{\theta}) = \mathbf{0}$ through numerical methodologies, for instance, the Newton-Rapson algorithm. In another way, the MLEs can be directly secured by optimizing the log-likelihood function as provided in (17) and using the "`BFGS`" method of the "`optim`" subroutine in R software [39]. The "`BFGS`" method is a limited-memory quasi-Newton method for approximating the Hessian matrix of the target distribution. It is worth mentioning that the parameter vector $\boldsymbol{\theta} = (\sigma, \lambda, \alpha)$ can be easily obtained, thanks to the properties of the pdf f. The smooth and continuous nature of the function f, along with the existence and finiteness of its first and second derivatives, ensure that the equation $S(\boldsymbol{\theta}) = \mathbf{0}$ has roots. These roots correspond to the MLEs of the vector $\boldsymbol{\theta}$. By employing relevant calculus techniques, it is possible to verify that the solutions correspond to a maximum.

3.2. Observed Fisher Information Matrix

We can estimate the asymptotic variance of the MLEs, denoted as $\widehat{\boldsymbol{\theta}} = (\widehat{\sigma}, \widehat{\lambda}, \widehat{\alpha})$, using the Fisher information matrix. The Fisher information matrix, denoted as $\mathcal{I}(\boldsymbol{\theta})$, is calculated as the negative expectation of the second derivative of the log-likelihood function, $\ell(\boldsymbol{\theta})$, with respect to $\boldsymbol{\theta}$. Under regularity conditions, the MLEs are asymptotically normal. In other words, as the sample size n approaches infinity, the distribution of $\mathcal{I}(\boldsymbol{\theta})^{-1/2}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)$ converges to a standard trivariate normal distribution, denoted as $N_3(\mathbf{0}_3, \boldsymbol{I}_3)$. To obtain $\mathcal{I}(\boldsymbol{\theta})$, we calculate the elements of the matrix $-\partial^2 \ell(\boldsymbol{\theta}) / \partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^{\top}$. Specifically, the elements are given by $I_{\sigma\sigma} = -\partial^2 \ell(\boldsymbol{\theta}) / \partial \sigma^2$, $I_{\sigma\lambda} = -\partial^2 \ell(\boldsymbol{\theta}) / \partial \sigma \partial \lambda$, and so on.

Typically, it is difficult to obtain an exact form for the expected value of previous expressions. Consequently, an estimate of the covariance matrix of MLEs, denoted as $\mathcal{I}(\boldsymbol{\theta})^{-1}$, can be obtained by evaluating the previous elements at the MLE $\widehat{\boldsymbol{\theta}}$, this is called the observed information matrix, and is denoted as $I(\widehat{\boldsymbol{\theta}})$.

$$I(\widehat{\boldsymbol{\theta}}) = -\partial^2 \ell(\boldsymbol{\theta}) / \partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}.$$

The estimation of the asymptotic variances for $\widehat{\sigma}$, $\widehat{\lambda}$, and $\widehat{\alpha}$ is done by calculating the diagonal elements of $I(\widehat{\boldsymbol{\theta}})^{-1}$. The standard errors are then obtained by taking the square root of these asymptotic variances. For more information on the theoretical results used in this subsection, please refer to [40,41].

It is of interest to recall that the normality of maximum likelihood estimators is not satisfied for the case of small sample size. In this case, confidence interval estimates of parameters can be obtained by applying generalized estimation methods such as those described in Wang et al. [42] and Luo et al. [43].

3.3. Simulation Study

A simulation study has been carried out to assess the performance of ML estimators of σ, λ and α in the SEFr model. 1000 samples were generated with sample sizes $n = 50$, 100 and 200. The different possibilities for the parameters σ, λ and α can be seen in Table 2. Next Algorithm 1 to generate values of $X \sim SEFr\,(\sigma, \lambda, \alpha)$ is proposed.

Algorithm 1: To simulate values from the $X \sim SEFr(\sigma, \lambda, \alpha)$.

Step 1: Generate $X_1 \sim Exp(1)$ and $X_2 \sim Uniform(0, 1)$.
Step 2: Compute $Y = \frac{X_1}{X_2^{1/\alpha}}$.
Step 3: Compute $X = \sigma Y^{-1/\lambda}$.

In Table 2, the bias, Standard Error (SE), Root of Mean Squared Error (RMSE), and the empirical Coverage Probability (CP) to 95% for the asymptotic intervals based on ML estimators are given. Note that the bias, SE, and RMSE decrease when the sample size n increases. These facts suggest that the MLE are consistent. Moreover, the empirical CP's also approach to the nominal 95% when n increases.

Table 2. Bias, SE, RMSE, and empirical CP for the simulation results in SEFr model.

True Value				n = 50				n = 100				n = 200			
σ	λ	α	Estimator	Bias	SE	RMSE	CP	Bias	SE	RMSE	CP	Bias	SE	RMSE	CP
1.5	2	0.5	$\hat{\sigma}$	0.029	0.305	0.347	90.5	0.009	0.208	0.230	90.8	0.002	0.149	0.153	93.8
			$\hat{\lambda}$	0.316	0.682	2.067	94.0	0.099	0.356	0.407	95.1	0.037	0.241	0.262	93.3
			$\hat{\alpha}$	0.073	0.374	0.726	90.5	0.015	0.133	0.144	93.7	0.007	0.091	0.096	93.5
		1.2	$\hat{\sigma}$	−0.037	0.253	0.291	90.6	−0.025	0.177	0.196	93.8	−0.006	0.122	0.124	94.9
			$\hat{\lambda}$	0.124	0.404	0.598	92.8	0.027	0.259	0.278	94.1	0.024	0.181	0.185	95.1
			$\hat{\alpha}$	1.523	7.629	4.558	91.2	0.427	1.460	1.795	94.8	0.077	0.328	0.436	93.8
	4	0.5	$\hat{\sigma}$	−0.010	0.149	0.176	89.1	−0.004	0.106	0.109	93.3	−0.003	0.074	0.077	93.9
			$\hat{\lambda}$	0.512	1.222	3.100	92.9	0.187	0.722	0.854	94.8	0.084	0.483	0.521	94.7
			$\hat{\alpha}$	0.104	0.443	0.715	89.9	0.019	0.137	0.155	93.5	0.008	0.092	0.094	94.1
		1.2	$\hat{\sigma}$	−0.019	0.132	0.140	92.4	−0.014	0.089	0.097	95.3	−0.004	0.061	0.062	94.8
			$\hat{\lambda}$	0.256	0.810	1.023	93.4	0.096	0.527	0.583	93.1	0.049	0.362	0.368	95.2
			$\hat{\alpha}$	1.227	6.490	4.114	91.1	0.397	1.452	2.013	92.4	0.066	0.317	0.410	94.5

Table 2. *Cont.*

True Value			Estimator	n = 50				n = 100				n = 200			
σ	λ	α		Bias	SE	RMSE	CP	Bias	SE	RMSE	CP	Bias	SE	RMSE	CP
2	3	0.7	$\hat{\sigma}$	−0.018	0.242	0.262	90.9	−0.005	0.170	0.183	92.9	−0.002	0.120	0.125	94.3
			$\hat{\lambda}$	0.259	0.733	0.946	94.6	0.108	0.472	0.518	95.3	0.065	0.323	0.343	95.6
			$\hat{\alpha}$	0.180	0.842	1.153	92.0	0.052	0.252	0.434	94.0	0.013	0.133	0.144	93.4
		1	$\hat{\sigma}$	−0.028	0.234	0.272	91.2	−0.009	0.161	0.172	95.2	−0.002	0.111	0.111	95.7
			$\hat{\lambda}$	0.180	0.631	0.820	92.2	0.068	0.416	0.439	95.3	0.040	0.286	0.297	94.8
			$\hat{\alpha}$	0.889	4.261	3.233	91.3	0.164	0.595	0.912	93.5	0.036	0.222	0.266	94.6
	5	0.7	$\hat{\sigma}$	−0.016	0.147	0.166	91.7	0.003	0.103	0.109	94.4	0.001	0.072	0.074	94.5
			$\hat{\lambda}$	0.442	1.238	1.679	93.5	0.218	0.797	0.885	95.7	0.105	0.538	0.561	96.1
			$\hat{\alpha}$	0.197	0.898	1.110	91.8	0.024	0.204	0.234	93.1	0.009	0.132	0.145	93.8
		1	$\hat{\sigma}$	−0.025	0.147	0.165	92.9	−0.008	0.096	0.101	93.6	−0.005	0.067	0.067	94.7
			$\hat{\lambda}$	0.267	1.045	1.305	91.7	0.117	0.690	0.737	94.6	0.081	0.480	0.513	94.0
			$\hat{\alpha}$	0.974	6.529	3.370	92.4	0.152	0.561	0.858	93.9	0.046	0.222	0.254	95.2
3	1.5	0.3	$\hat{\sigma}$	0.170	0.957	1.184	88.1	0.054	0.684	0.733	92.1	0.020	0.475	0.495	94.5
			$\hat{\lambda}$	0.535	0.966	3.147	93.9	0.112	0.350	0.423	94.4	0.055	0.227	0.251	95.0
			$\hat{\alpha}$	0.013	0.123	0.152	88.9	0.005	0.083	0.098	92.1	0.002	0.057	0.059	94.6
		0.9	$\hat{\sigma}$	−0.033	0.689	0.766	90.6	−0.014	0.487	0.526	92.8	−0.006	0.340	0.338	95.0
			$\hat{\lambda}$	0.104	0.327	0.390	94.4	0.039	0.215	0.249	93.9	0.023	0.148	0.155	94.7
			$\hat{\alpha}$	0.622	2.755	2.598	91.7	0.142	0.544	0.807	93.4	0.034	0.189	0.209	95.2
	3.5	0.3	$\hat{\sigma}$	0.022	0.394	0.461	89.9	0.007	0.287	0.306	92.5	0.001	0.204	0.202	95.3
			$\hat{\lambda}$	0.923	1.649	4.378	93.2	0.289	0.832	1.182	95.2	0.093	0.524	0.547	95.5
			$\hat{\alpha}$	0.015	0.123	0.164	88.3	0.004	0.081	0.089	92.1	0.003	0.057	0.057	94.5
		0.9	$\hat{\sigma}$	−0.021	0.301	0.352	89.7	−0.005	0.209	0.232	93.2	−0.003	0.145	0.150	93.3
			$\hat{\lambda}$	0.280	0.794	1.344	93.7	0.123	0.509	0.549	95.5	0.064	0.346	0.370	95.1
			$\hat{\alpha}$	0.681	3.211	2.885	90.6	0.100	0.438	0.728	94.9	0.022	0.184	0.202	93.8

4. Applications

In this section, we provide applications to two actual datasets. In each setting the fit provided by the SEFr model is compared to the Fréchet (Fr), Slashed Quasi-Gamma (SQG), and Slash Fréchet (SFr). The criteria for comparison are: the Akaike Information Criterion (AIC), given by Akaike [44], and the Bayesian Information Criterion (BIC) proposed by Schwarz [45]. Next the pdf's of SQG and SFr are given.

1. Slashed Quasi-Gamma, $X \sim SQG(\beta, \theta, q)$, introduced in [46]. Its pdf is:

$$f_X(x; \beta, \theta, q) = \frac{q\beta^q x^{-(q+1)}}{\Gamma\left(\frac{1}{10}\right)} \Gamma\left(\frac{q}{\theta} + \frac{1}{10}\right) F\left(\left(\frac{x}{\beta}\right)^{\theta}, \frac{q}{\theta} + \frac{1}{10}, 1\right), \quad x > 0, \tag{18}$$

   where $\beta > 0, \theta > 0$, and $q > 0$.

2. Slash Fréchet, $X \sim SFr(\lambda, q)$, introduced in [20]. Its pdf is:

$$f_X(x; \lambda, q) = \frac{q}{x^{q+1}} \Gamma\left(1 - \frac{q}{\lambda}, x^{-\lambda}\right), \quad x > 0, \tag{19}$$

   where $\lambda > q > 0$ and $\Gamma(a, t) = \int_t^{\infty} w^{a-1} e^{-w} dw$ is the upper incomplete gamma function.

4.1. Application 1 (Patients with Bladder Cancer)

The remission times (in months) of 128 patients with bladder cancer are considered as first application. This dataset was first studied by Lee and Wang [47]. Table 3 provides the descriptive summaries. These are: sample mean, sample standard deviation (S), sample skewness coefficient ($\sqrt{b_1}$) and sample kurtosis coefficient (b_2). Also, the boxplot is given in Figure 5. We highlight the high values of the skewness and kurtosis coefficients, and the existence of possible outliers in this dataset.

Table 3. Descriptive Summary for Patients with Bladder Cancer dataset.

n	$\overline{x}$	S	$\sqrt{b_1}$	b_2
128	9.366	10.508	3.287	18.483

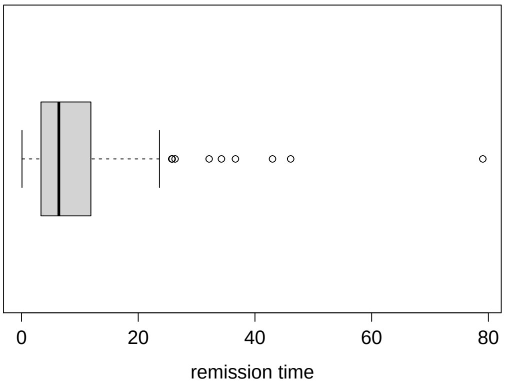

Figure 5. Boxplot for remission times of patients with bladder cancer.

The results for the different models under consideration are listed in Table 4, that is, the estimates of parameters in the Fr, SFr, SQG, and SEFr models, along with their standard errors, log-likelihood, AIC, and BIC. It can be seen that the SEFr model achieves the lower values of AIC and BIC, and therefore it provides the best fit to this dataset.

Table 4. Estimates, SE in parentheses, log-likelihood, AIC, and BIC values for the remission times of patients with bladder cancer dataset.

Parameters	$Fr\,(SE)$	$SFr\,(SE)$	$SQG\,(SE)$	$SEFr\,(SE)$
$\widehat{\sigma}$	-	-	-	9.9436 (1.3592)
$\widehat{\lambda}$	0.6726 (0.0479)	0.9242 (0.0688)	-	1.8586 (0.2660)
$\widehat{\alpha}$	-	-	-	0.6329 (0.1558)
$\widehat{\beta}$	-	-	7.7993 (0.9893)	-
$\widehat{\theta}$	-	-	10.8627 (1.3211)	-
$\widehat{q}$	-	0.9623 (0.1302)	1.5211 (0.2282)	-
log-likelihood	−481.0559	−448.1104	−411.7342	−410.0634
AIC	964.1118	900.2208	829.4683	826.1268
BIC	966.9638	905.9249	838.0244	834.6829

The histogram and the fitted pdf's for the previously mentioned distributions are given in Figure 6. In Figure 7, the corresponding QQ-plots can be seen. Both graphical summaries support our conclusions about the good fit of SEFr model to this dataset.

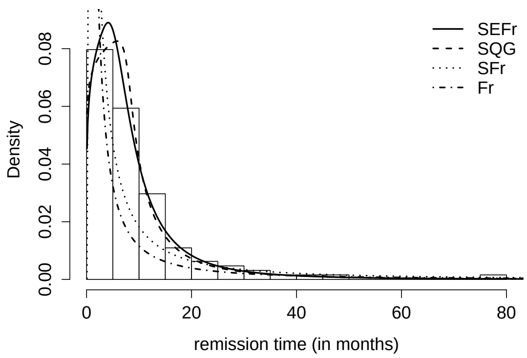

Figure 6. Fitted pdf for the remission times of patients with bladder cancer dataset in the Fr and SFr, SQG, and SEFr distributions.

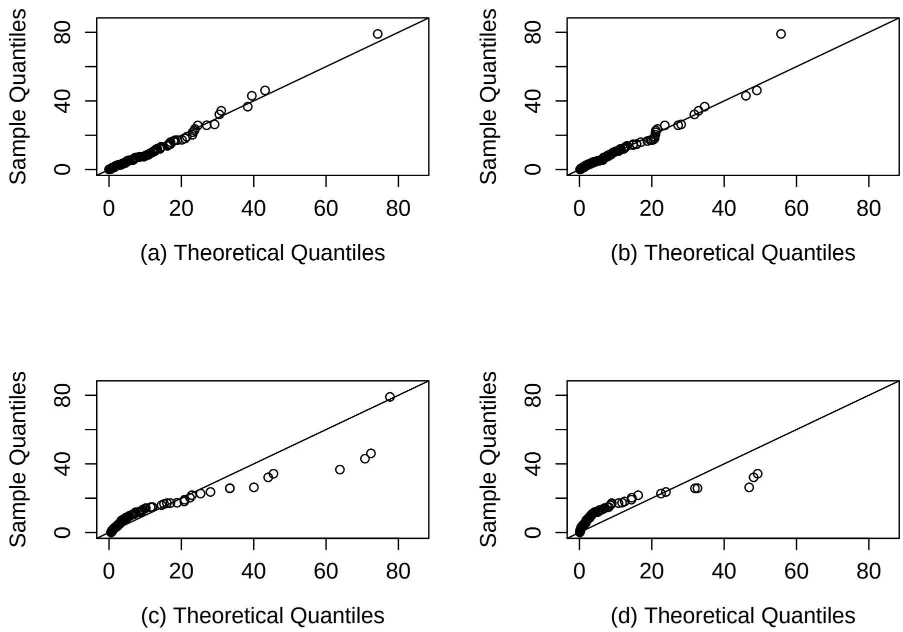

Figure 7. QQ plots for the remission times of patients with bladder cancer dataset: (**a**) SEFr Model; (**b**) SQG model; (**c**) SFr model; (**d**) Fr model.

4.2. Application 2 (Air Conditioning System Failures)

The second dataset was first studied in Proschan [48]. It consists of the time interval (in hours) between successive failures of the air-conditioning system of Boeing 720 number 7912. Table 5 provides the descriptive summary and Figure 8 the corresponding boxplot. Note the presence of outliers and that a model with a long right tail seems appropriate. These summaries suggest that the proposed distributions can be good candidates for modelling this dataset.

Table 5. Descriptive statistics for the Air Conditioning System Failures dataset.

n	$\overline{x}$	s	$\sqrt{b_1}$	b_2
30	59.6333	71.8996	1.6914	4.9595

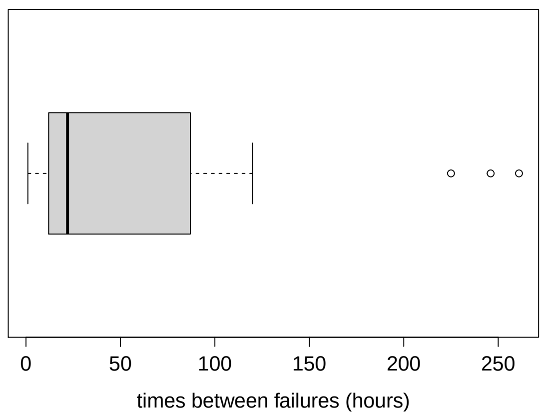

Figure 8. Boxplot of the Air Conditioning System Failures dataset.

The results for the fitted models are given in Table 6. According to AIC and BIC, the SEFr distribution provides a better fit than the other ones, since its AIC and BIC are the smallest ones.

Table 6. Estimates of parameters, SE in parentheses, log-likelihood, AIC and BIC values for Air Conditioning System Failures.

Parameters	$Fr\,(SE)$	$SFr\,(SE)$	$SQG\,(SE)$	$SEFr\,(SE)$
$\widehat{\sigma}$	-	-	-	38.5732 (13.9807)
$\widehat{\lambda}$	0.3924 (0.0601)	0.9508 (0.3089)	-	1.0968 (0.2273)
$\widehat{\alpha}$	-	-	-	1.1344 (0.5565)
$\widehat{\beta}$	-	-	14.7397 (3.1532)	-
$\widehat{\theta}$	-	-	14.3648 (4.2492)	-
$\widehat{q}$	-	0.4067 (0.0960)	0.7165 (0.1559)	-
log-likelihood	−177.5930	−163.9272	−153.0741	−152.3953
AIC	357.1859	331.8543	312.1481	310.7905
BIC	358.5871	334.6567	316.3517	314.9941

The histogram for this dataset is given in Figure 9 along with the fitted pdf's for Fr, SFr, SQG, and SEFr models. The QQ-plots are given in Figure 10. it can also be verified that the SEFr model outperforms other rival models in terms of fitting better. Hence, based on all these summaries and plot interpretations, it can be inferred that the SEFr model outperforms the Fr, SFr, and SQG distributions in fitting this dataset.

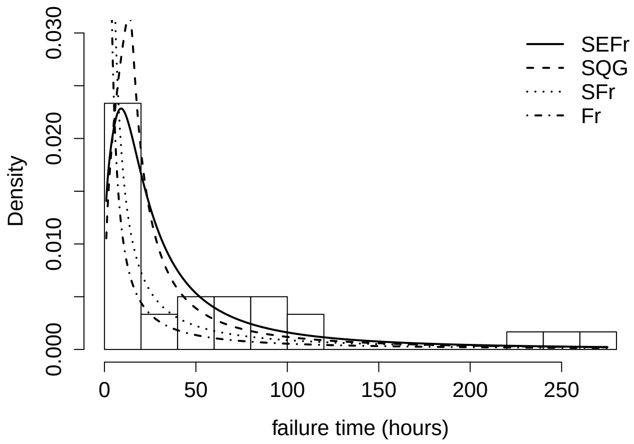

Figure 9. Fitted Density for Air Conditioning System Failures dataset in the Fr and SFr, SQG, and SEFr distributions.

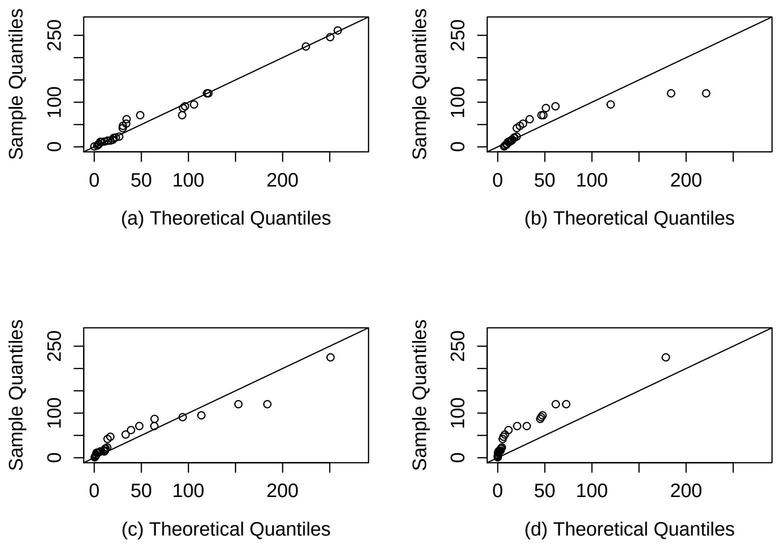

Figure 10. QQ plots for Air Conditioning System Failures dataset: (**a**) SEFr Model; (**b**) SQG model; (**c**) SFr model; (**d**) Fr model.

5. Conclusions

The Slash-Exponential-Fréchet distribution has been introduced. This model is an extension of the Fréchet, from which it inherits its main features, and moreover, it is more flexible as for its skewness and kurtosis. The following points have been considered:

- The stochastic representation of the new model in terms of the Slash-Exponential is given. In this way, an additional shape parameter is added to Fréchet model.
- Closed expressions for the pdf and cdf are given, therefore also for the survival and hazard rate function.
- It is shown that the new model is unimodal or decreasing. It is proven that if the new shape parameter tends to infinity then the SEFr approaches to Fréchet model.
- Closed expressions are given for the moments, with particular interest on skewness and kurtosis coefficients.
- We highlight that the new model presents less kurtosis than the basal Fréchet distribution. For the best of our understanding, it is the first time in literature that, as result of applying slash methodology, the new model exhibits a lighter right tail and less kurtosis compared to basal model.
- Maximum likelihood method has been proposed to estimate the parameters in the model. Score equations and the observed Fisher information matrix are studied.
- A simulation study has been carried out. There, bias, standard error, RMSE and empirical coverage probability for MLEs have been obtained for increasing sample size. The good asymptotic properties of MLEs can be seen.
- Two real applications are included where the SEFr model is compared to Fr, Slashed Quasi-Gamma and Slash-Fréchet. By using AIC and BIC, it has been seen that the new model provides a better fit compared to others.

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