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Existence of Solutions: Investigating Fredholm Integral Equations via a Fixed-Point Theorem

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Abstract: Integral equations, which are defined as "the equation containing an unknown function under the integral sign", have many applications of real-world problems. The second type of Fredholm integral equations is generally used in radiation transfer theory, kinetic theory of gases, and neutron transfer theory. A special case of these equations, known as the quadratic Chandrasekhar integral equation, given by $x(s) = 1 + \lambda x(s) \int_0^1 \frac{s}{t+s} x(t)dt$, can be very often encountered in many applications, where x is the function to be determined, λ is a parameter, and $t, s \in [0,1]$. In this paper, using a fixed-point theorem, the existence conditions for the solution of Fredholm integral equations of the form $\chi(l) = \varrho(l) + \chi(l) \int_p^q k(l,z)(V\chi)(z)dz$ are investigated in the space $C_\omega[p,q]$, where χ is the unknown function to be determined, V is a given operator, and ϱ, k are two given functions. Moreover, certain important applications demonstrating the applicability of the existence theorem presented in this paper are provided.

Keywords: Fredholm integral equations; quadratic Chandrasekhar integral equation; Hölder space, tempered modulus of continuity; the space $C_\omega(X)$; fixed-point theorem

MSC: 45G10; 45B05; 47H10

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1. Introduction

Fredholm integral equations are a class of integral equations named after the Swedish mathematician Erik Ivar Fredholm. These equations are of the form

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\phi(t)dt,$$

where $\phi(x)$ is the unknown function to be determined, $f(x)$ is a given function, $K(x,t)$ is the kernel function, and λ is a parameter, often referred to as the Fredholm parameter.

Fredholm integral equations arise in various areas of mathematics and physics, including potential theory, signal processing, and quantum mechanics. They represent a wide range of problems where an unknown function is defined in terms of its integral over some interval or domain.

One of the key aspects of Fredholm integral equations is the study of their solvability and properties of their solutions. Depending on the properties of the kernel function $K(x,t)$ and the interval of integration, solutions to Fredholm integral equations may exhibit different behaviors, including uniqueness, existence, and convergence properties.

Fredholm integral equations have been extensively studied, and various numerical and analytical methods have been developed to solve them. These methods include—but are not limited to—Fredholm's alternative, Green's functions, eigenfunction expansions, and numerical quadrature techniques.

Overall, Fredholm integral equations play a significant role in mathematical analysis and have applications in diverse fields, making them an essential topic of study in both pure and applied mathematics.

Fractional integral and differential equations are becoming increasingly vital for modeling real-world scenarios in physics, mechanics, and related fields. They provide a powerful framework for describing complex phenomena that traditional integer-order calculus struggles to capture adequately.

Quadratic integral equations are a specific class of integral equations where the unknown function appears squared within the integral. They are represented by equations of the form

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\phi(t)^2 dt.$$

These equations arise in various fields of science and engineering, particularly in problems where phenomena exhibit quadratic dependencies. Examples include nonlinear wave propagation, population dynamics, and certain types of chemical reaction kinetics.

The study of quadratic integral equations involves investigating the existence, uniqueness, and certain properties of their solutions. Analytical techniques such as iteration methods, Fredholm alternative, and fixed-point theorems are often employed. Numerical methods may also be used for practical solutions.

Understanding quadratic integral equations is crucial for modeling nonlinear phenomena accurately and finding solutions to a wide range of problems in diverse scientific and engineering applications.

Quadratic integral equations, in particular, have significant applications in fields like radiative transfer, neutron transport, and kinetic theory of gases. They arise naturally in these contexts and have been extensively studied for their existence properties and solution behavior.

Recent developments in fractional calculus, including Riemann–Liouville, Caputo, and Hadamard approaches, have further enriched our understanding and application of these equations. Researchers have been exploring various types of quadratic integral equations and extending their study to fractional versions like Urysohn type, Erdélyi–Kober type, and Hadamard types.

The existence, local attractivity, and stability of solutions to these fractional quadratic integral equations are crucial aspects of the research. Additionally, recent works have focused on studying the solvability of quadratic integral equations of Fredholm type in spaces of functions satisfying specific continuity conditions, such as the Hölder condition. These investigations contribute to both theoretical understanding and practical applications of these equations in diverse fields [1–25].

Moreover, the mathematical description of many processes in engineering, biological and physical sciences give rise to quadratic integral equations. For instance, research in radiative transfer theory and in kinetic theory of gases leads to the quadratic integral equation (see [15,16]).

$$x(t) = 1 + tx(t)\int_0^1 \frac{\Phi(\tau)}{t+\tau}x(\tau)d\tau,$$

Furthermore, some articles using simulant techniques have been applied to vibrations in thermoelasticity and a micropolar porous body [21,22]. That is, these articles are applied to a micropolar porous body, including voidage time derivative among the independent constitutive variables.

Quadratic integral equations are frequently applicable in neutron transport, radiative transfer and traffic theories and in kinetic theory of gases [16–18]. Particularly, the Chandrasekhar-type quadratic integral equation, which is defined as

$$x(s) = 1 + \lambda x(s)\int_0^1 \frac{s}{t+s}x(t)dt,$$

can be very frequently encountered in many applications [16].

Moreover, certain biology and queuing theory research problems lead to the following nonlinear integral equation [20]:

$$y(t) = f(t) + \frac{y(t)}{\Gamma(\alpha)} \int_0^t \frac{u(\tau, y(\tau))}{(t-\tau)^{1-\alpha}} d\tau,\ t \in [0,T], \alpha \in (0,1].$$

The aim of current paper is to prove an existence theorem for the nonlinear quadratic integral equation of the following form in the space $C_\omega[p,q]$, where ω is a modulus of continuity (see Section 2):

$$\chi(l) = \varrho(l) + \chi(l) \int_p^q k(l,z)(V\chi)(z)dz, \tag{1}$$

where V is a given operator and $\varrho,\ k$ THE GAP BEFORE k IS REMOVED are two given functions.

By using a sufficient condition for the relative compactness in the space of functions with moduli of continuity and the classical Schauder fixed-point theorem, we derive a new existence result (see Theorem 4).

In Section 2, general definitions and theorems are given. The third section provides the main result of the paper and shows that there is at least one solution of the investigated equation. In the last section, two applications are given to support our main result.

2. Preliminaries

Let us give the notations, definitions and theorems that we will use in the paper .

Definition 1 ([8])**.** *A nondecreasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a modulus of continuity if $\omega(0) = 0$ and $\omega(\epsilon) > 0$ for $\epsilon > 0$.*

The space of continuous functions on $[p,q]$ with the sup norm

$$\|\chi\|_\infty = \sup\{|\chi(l)| : l \in [p,q]\}$$

is denoted by $C[p,q]$ for $\chi \in C[p,q]$. Let $([p,q],d)$ be a given bounded metric space and $C_\omega[p,q]$ be the set of all real functions defined on $[p,q]$ such that their growths are tempered by the modulus of continuity ω with respect to d. A function $\chi = \chi(l)$ is in the set $C_\omega[p,q]$ if there exists a constant $H > 0$ satisfying

$$|\chi(l) - \chi(m)| \leqslant H\omega(d(l,m)) \tag{2}$$

for all $l, m \in [p,q]$. Also, $C_\omega[p,q]$ is a linear subspace of $C[p,q]$.

The least possible constant H_χ^ω for which the inequality (2) is satisfied is given as

$$H_\chi^\omega = \sup\left\{ \frac{|\chi(l) - \chi(m)|}{\omega(d(l,m))} : l, m \in [p,q], l \neq m \right\}, \tag{3}$$

where $\chi \in C_\omega[p,q]$.

The norm on the space $C_\omega[p,q]$ is

$$\|\chi\|_\omega = |\chi(p)| + \sup\left\{ \frac{|\chi(l) - \chi(m)|}{\omega(d(l,m))} : l, m \in [p,q], l \neq m \right\} \tag{4}$$

for $\chi \in C_\omega[p,q]$. By (3) and (4), we can write

$$\|\chi\|_\omega = |\chi(p)| + H_\chi^\omega.$$

The space $C_\omega[p,q]$ depends on the metric d and continuity modulus ω.

If we take $d(l,m) = |l-m|$ and $\omega(\varepsilon) = \varepsilon^{\alpha}$ for $0 < \alpha < 1$, the space $C_{\omega}[p,q]$ becomes the space $H_{\alpha}[p,q]$ (i.e., Hölder Space). In [8], the authors proved that $(C_{\omega}[p,q], \|\cdot\|_{\omega})$ is a Banach space.

Lemma 1 ([25])**.** *The following relation is satisfied for each $\chi \in C_{\omega}[p,q]$:*

$$\|\chi\|_{\infty} \leqslant \max\{1, \omega(diam[p,q])\}\|\chi\|_{\omega},$$

where $diam[p,q]$ denotes the diameter of the metric space $[p,q]$.

Lemma 2 ([25])**.** *The following relation is satisfied for each $\chi \in C_{\omega}[p,q]$. Suppose that $\omega_2(d(l,m)) \leqslant L\omega_1(d(l,m))$ for all $l,m \in [p,q]$, where $L > 0$. Then*

$$C_{\omega_2}[p,q] \subset C_{\omega_1}[p,q] \subset C[p,q].$$

Furthermore, the following expression is satisfied for any $\chi \in C_{\omega_2}[p,q]$:

$$\|\chi\|_{\omega_1} \leqslant \max\{1, L\}\|\chi\|_{\omega_2}.$$

Remark 1 ([25])**.** *Let $\lim_{\varepsilon \to 0} \frac{\omega_2(\varepsilon)}{\omega_1(\varepsilon)} = 0$ then, there exists a number $M > 0$ satisfying*

$$\omega_2(d(l,m)) \leqslant M\omega_1(d(l,m))$$

for all $l,m \in [p,q]$ and so imbedding relations in Lemma 2 and the inequality

$$\|\chi\|_{\omega_1} \leqslant \max\{1, M\}\|\chi\|_{\omega_2}.$$

also hold for any $\chi \in C_{\omega_2}[p,q]$.

Theorem 1 (Theorem 5 in [8])**.** *Let (X,d) be a compact metric space and*

$$\lim_{\varepsilon \to 0} \frac{\omega_2(\varepsilon)}{\omega_1(\varepsilon)} = 0,$$

where ω_1, ω_2 are moduli of continuity being continuous at zero. Then, if A is a bounded subset of the space $C_{\omega_2}(X)$, the set A is relatively compact in the space $C_{\omega_1}(X)$.

Theorem 2 ([25])**.** *Let $\lim_{\varepsilon \to 0} \frac{\omega_2(\varepsilon)}{\omega_1(\varepsilon)} = 0$. Then the set $B_r^{\omega_2} \in C_{\omega_1}[p,q]$ given by $B_r^{\omega_2} = \left\{\chi \in C_{\omega_2}[p,q] : \|\chi\|_{\omega_2} \leqslant r\right\}$ is compact.*

Theorem 3 (Schauder's fixed-point theorem [23])**.** *Let $T : \Omega \to \Omega$ be a continuous mapping, where Ω is a nonempty, compact and convex subset of a Banach space $(X, \|\cdot\|)$, then T has at least one fixed point in Ω.*

3. Existence Theorem

In this part, we provide sufficient conditions that guarantee the Equation (1) has at least one solution in $C_{\omega_1}[p,q]$.

Let $([p,q], d)$ be a compact metric space and

$$\lim_{\varepsilon \to 0} \frac{\omega_2(\varepsilon)}{\omega_1(\varepsilon)} = 0,$$

where $\omega_1,\ \omega_2$ are moduli of continuity being continuous at zero. We use the hypotheses presented below:

(i) $\varrho \in C_{\omega_2}[p,q]$,

(ii) The continuous function $k : [p,q] \times [p,q] \to \mathbb{R}$ satisfies the tempered by the modulus of continuity with respect to the first variable, that is, there exists a constant k_{ω_2} such that

$$|k(l,z) - k(m,z)| \leqslant k_{\omega_2}\omega_2(d(l,m))$$

for any $l, m, z \in [p,q]$.

(iii) Let $V : C_{\omega_2}[p,q] \to C[p,q]$ be a continuous operator on $C_{\omega_2}[p,q]$ with respect to norm $\|.\|_{\omega_1}$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function, the following inequality is satisfied for each $\chi \in C_{\omega_2}[p,q]$:

$$\|V\chi\|_\infty \leqslant f(\|\chi\|_{\omega_2}).$$

(iv) There exists a positive solution $r = r_0$ of the inequality

$$\overline{P} + [\eta_2 K + K + \eta_2 k_{\omega_2}(q-p)]f(r)r \leqslant r,$$

where $\overline{P}$, K and η_2 are the constants such that $\|\varrho\|_{\omega_2} \leqslant \overline{P}$,

$$\sup\left\{\int_p^q |k(l,z)|dz : l \in [p,q]\right\} \leqslant K$$

and

$$\eta_2 = \max\{1, \omega_2(\mathrm{diam}[p,q])\}.$$

Theorem 4. *The Equation (1) has at least one solution belonging to the space $C_{\omega_1}[p,q]$ under the assumptions (i)–(iv).*

Proof. We set the operator $\Lambda \in C_{\omega_2}[p,q]$ as follows:

$$(\Lambda\chi)(l) = \varrho(l) + \chi(l)\int_p^q k(l,z)(V\chi)(z)dz,\ l \in [p,q].$$

We first show that Λ transforms the space $C_{\omega_2}[p,q]$ into itself. For arbitrarily fixed $\chi \in C_{\omega_2}[p,q]$ and $l, m \in [p,q]$, taking into account the assumptions, we obtain that

$$\begin{aligned}
(\Lambda\chi)(l) - (\Lambda\chi)(m) &= \varrho(l) + \chi(l)\int_p^q k(l,z)(V\chi)(z)dz \\
&\quad -\varrho(m) - \chi(m)\int_p^q k(m,z)(V\chi)(z)dz \\
&= \varrho(l) - \varrho(m) + [\chi(l) - \chi(m)]\int_p^q k(l,z)(V\chi)(z)dz \\
&\quad +\chi(m)\int_p^q [k(l,z) - k(m,z)](V\chi)(z)dz
\end{aligned}$$

which implies that

$$
\begin{aligned}
&\frac{|(\Lambda\chi)(l)-(\Lambda\chi)(m)|}{\omega_2(d(l,m))} \\
\leqslant\ & \frac{|\varrho(l)-\varrho(m)|}{\omega_2(d(l,m))}+\frac{|\chi(l)-\chi(m)|}{\omega_2(d(l,m))}\int_p^q |k(l,z)||(V\chi)(z)|dz \\
&+|\chi(m)|\int_p^q \frac{|k(l,z)-k(m,z)|}{\omega_2(d(l,m))}|(V\chi)(z)|dz \\
\leqslant\ & \frac{|\varrho(l)-\varrho(m)|}{\omega_2(d(l,m))}+\frac{|\chi(l)-\chi(m)|}{\omega_2(d(l,m))}\|V\chi\|_\infty\int_p^q |k(l,z)|dz \\
&+\|\chi\|_\infty\|V\chi\|_\infty\int_p^q \frac{k_{\omega_2}\omega_2(d(l,m))}{\omega_2(d(l,m))}dz \\
\leqslant\ & H_\varrho^{\omega_2}+H_\chi^{\omega_2}\|V\chi\|_\infty K+\|\chi\|_\infty\|V\chi\|_\infty k_{\omega_2}(q-p) \qquad (5)
\end{aligned}
$$

for all $l,m\in[p,q]$ and $l\neq m$. By using the fact that $H_\chi^\omega\leqslant\|\chi\|_\omega$ and Lemma 1, we infer from (5) that

$$
\begin{aligned}
\frac{|(\Lambda\chi)(l)-(\Lambda\chi)(m)|}{\omega_2(d(l,m))} \leqslant\ & H_\varrho^{\omega_2}+\|V\chi\|_\infty\|\chi\|_{\omega_2}K \\
&+\eta_2\|V\chi\|_\infty\|\chi\|_{\omega_2}k_{\omega_2}(q-p) \\
\leqslant\ & H_\varrho^{\omega_2}+f\left(\|\chi\|_{\omega_2}\right)\|\chi\|_{\omega_2}K \\
&+f\left(\|\chi\|_{\omega_2}\right)\eta_2\|\chi\|_{\omega_2}k_{\omega_2}(q-p), \qquad (6)
\end{aligned}
$$

where

$$\eta_2=\max\{1,\omega_2(\mathrm{diam}[p,q])\}.$$

From (6), we have $\Lambda\chi\in C_{\omega_2}[p,q]$. This proves that the operator Λ maps the space $C_{\omega_2}[p,q]$ into itself.

Also, we derive that

$$
\begin{aligned}
|(\Lambda\chi)(p)| =\ & \left|\varrho(p)+\chi(p)\int_p^q k(p,z)(V\chi)(z)dz\right| \\
\leqslant\ & |\varrho(p)|+|\chi(p)|\int_p^q |k(p,z)||(V\chi)(z)|dz \\
\leqslant\ & |\varrho(p)|+\|\chi\|_\infty\|V\chi\|_\infty K \\
\leqslant\ & |\varrho(p)|+(\max\{1,\omega_2(\mathrm{diam}[p,q])\})\|\chi\|_{\omega_2}\|V\chi\|_\infty K \\
\leqslant\ & |\varrho(p)|+\eta_2\|\chi\|_{\omega_2}f\left(\|\chi\|_{\omega_2}\right)K. \qquad (7)
\end{aligned}
$$

By using definition of the norm $\|\Lambda\chi\|_{\omega_2}$, (6) and (7), we can write

$$
\begin{aligned}
\|\Lambda\chi\|_{\omega_2} =\ & |(\Lambda\chi)(p)|+\sup\left\{\frac{|(\Lambda\chi)(l)-(\Lambda\chi)(m)|}{\omega_2(d(l,m))}: l,m\in[p,q], l\neq m\right\} \\
\leqslant\ & |\varrho(p)|+f\left(\|\chi\|_{\omega_2}\right)\eta_2\|\chi\|_{\omega_2}K \\
&+H_\varrho^{\omega_2}+f\left(\|\chi\|_{\omega_2}\right)\|\chi\|_{\omega_2}K \\
&+f\left(\|\chi\|_{\omega_2}\right)\eta_2\|\chi\|_{\omega_2}k_{\omega_2}(q-p) \\
\leqslant\ & \|\varrho\|_{\omega_2}+[\eta_2K+K+\eta_2k_{\omega_2}(q-p)]f\left(\|\chi\|_{\omega_2}\right)\|\chi\|_{\omega_2} \qquad (8)
\end{aligned}
$$

for any $\chi \in C_{\omega_2}[p,q]$. So, if we take χ in $B_{r_0}^{\omega_2}$ then by (8) and the assumption (iv) we get the following inequality:

$$\|\Lambda\chi\|_{\omega_2} \leqslant \overline{P} + [\eta_2 K + K + \eta_2 k_{\omega_2}(q-p)]f(r_0)r_0 \leqslant r_0.$$

So $\Lambda\chi \in B_{r_0}^{\omega_2}$. Thus, Λ transforms the ball

$$B_{r_0}^{\omega_2} = \{\chi \in C_{\omega_2}[p,q] : \|\chi\|_{\omega_2} \leqslant r_0\}$$

into itself. That is, $\Lambda : B_{r_0}^{\omega_2} \to B_{r_0}^{\omega_2}$. Next, we will show that the operator Λ is continuous on $B_{r_0}^{\omega_2}$ with respect to norm $\|\cdot\|_{\omega_1}$. To carry this out, we fix $\psi \in B_{r_0}^{\omega_2}$ and an arbitrary $\varepsilon > 0$. Since the operator $V : C_{\omega_2}[p,q] \to C[p,q]$ is continuous on $C_{\omega_2}[p,q]$ with respect to norm $\|.\|_{\omega_1}$, there is a positive number δ such that the estimate

$$\|V\chi - V\psi\|_\infty < \frac{\varepsilon}{2(KL + L(q-p)\eta_2 k_{\omega_2} + K\eta_2)r_0}$$

is satisfied for all $\chi \in B_{r_0}^{\omega_2}$, where $\|\chi - \psi\|_{\omega_1} < \delta$, where δ is a number satisfying the following inequality:

$$0 < \delta < \frac{\varepsilon}{2(K + \eta_1 L(q-p)k_{\omega_2} + K\eta_1)f(r_0)},$$

where

$$\max\{1, \omega_1(\mathrm{diam}[p,q])\} = \eta_1.$$

By using the equality

$$\begin{aligned}
& (\Lambda\chi)(l) - (\Lambda\psi)(l) - ((\Lambda\chi)(m) - (\Lambda\psi)(m)) \\
= \; & \chi(l)\int_p^q k(l,z)(V\chi)(z)dz - \psi(l)\int_p^q k(l,z)(V\psi)(z)dz \\
& -\chi(m)\int_p^q k(m,z)(V\chi)(z)dz + \psi(m)\int_p^q k(m,z)(V\psi)(z)dz \\
= \; & [\chi(l) - \psi(l)]\int_p^q k(l,z)(V\chi)(z)dz + \psi(l)\int_p^q k(l,z)[(V\chi)(z) - (V\psi)(z)]dz \\
& -[\chi(m) - \psi(m)]\int_p^q k(m,z)(V\chi)(z)dz - \psi(m)\int_p^q k(m,z)[(V\chi)(z) - (V\psi)(z)]dz \\
= \; & \{[\chi(l) - \psi(l)] - [\chi(m) - \psi(m)]\}\int_p^q k(l,z)(V\chi)(z)dz \\
& +[\chi(m) - \psi(m)]\int_p^q [k(l,z) - k(m,z)](V\chi)(z)dz \\
& +[\psi(l) - \psi(m)]\int_p^q k(l,z)[(V\chi)(z) - (V\psi)(z)]dz \\
& +\psi(m)\int_p^q [k(l,z) - k(m,z)][(V\chi)(z) - (V\psi)(z)]dz,
\end{aligned}$$

we obtain that

$$
\begin{aligned}
&\frac{|[(\Lambda\chi)(l)-(\Lambda\psi)(l)]-[(\Lambda\chi)(m)-(\Lambda\psi)(m)]|}{\omega_1(d(l,m))}\\
\leqslant\ & \frac{1}{\omega_1(d(l,m))}|[\chi(l)-\psi(l)]-[\chi(m)-\psi(m)]|\left|\int_p^q k(l,z)(V\chi)(z)dz\right|\\
&+\frac{1}{\omega_1(d(l,m))}|\chi(m)-\psi(m)|\left|\int_p^q [k(l,z)-k(m,z)](V\chi)(z)dz\right|\\
&+\frac{1}{\omega_1(d(l,m))}\left|[\psi(l)-\psi(m)]\int_p^q k(l,z)[(V\chi)(z)-(V\psi)(z)]dz\right|\\
&+\frac{1}{\omega_1(d(l,m))}\left|\psi(m)\int_p^q [k(l,z)-k(m,z)][(V\chi)(z)-(V\psi)(z)]dz\right|
\end{aligned}
$$

which yields that

$$
\begin{aligned}
&\frac{|[(\Lambda\chi)(l)-(\Lambda\psi)(l)]-[(\Lambda\chi)(m)-(\Lambda\psi)(m)]|}{\omega_1(d(l,m))}\\
\leqslant\ & \frac{|[\chi(l)-\psi(l)]-[\chi(m)-\psi(m)]|}{\omega_1(d(l,m))}\|V\chi\|_\infty\int_p^q |k(l,z)|dz\\
&+\|\chi-\psi\|_\infty\|V\chi\|_\infty\int_p^q \frac{|k(l,z)-k(m,z)|}{\omega_1(d(l,m))}dz\\
&+\frac{|\psi(l)-\psi(m)|}{\omega_1(d(l,m))}\left|\int_p^q k(l,z)[(V\chi)(z)-(V\psi)(z)]dz\right|\\
&+|\psi(m)|\left|\int_p^q \frac{k(l,z)-k(m,z)}{\omega_1(d(l,m))}[(V\chi)(z)-(V\psi)(z)]dz\right|\\
\leqslant\ & H_{\chi-\psi}^{\omega_1}\|V\chi\|_\infty K+\eta_1\|\chi-\psi\|_{\omega_1}\|V\chi\|_\infty\int_p^q \frac{|k(l,z)-k(m,z)|}{\omega_1(d(l,m))}dz\\
&+\frac{|\psi(l)-\psi(m)|}{\omega_1(d(l,m))}\int_p^q |k(l,z)||(V\chi)(z)-(V\psi)(z)|dz\\
&+|\psi(m)|\int_p^q \frac{|k(l,z)-k(m,z)|}{\omega_1(d(l,m))}|(V\chi)(z)-(V\psi)(z)|dz
\end{aligned}
\tag{9}
$$

for all $l,m\in[p,q]$ with $l\neq m$. The estimates

$$\|\chi\|_\infty\leqslant\max\{1,\omega_2(\mathrm{diam}[p,q])\}\|\chi\|_{\omega_2}$$

and

$$H_\chi^\omega\leqslant\|\chi\|_\omega$$

hold from Lemma 1 and definition of the norm $\|\chi\|_\omega$. By (9), we derive the following estimate:

$$
\begin{aligned}
&\frac{|[(\Lambda\chi)(l)-(\Lambda\psi)(l)]-[(\Lambda\chi)(m)-(\Lambda\psi)(m)]|}{\omega_1(d(l,m))}\\
\leqslant\ & K\|V\chi\|_\infty H_{\chi-\psi}^{\omega_1}+\eta_1\|\chi-\psi\|_{\omega_1}\|V\chi\|_\infty\int_p^q \frac{k_{\omega_2}\omega_2(d(l,m))}{\omega_1(d(l,m))}dz\\
&+\frac{|\psi(l)-\psi(m)|}{\omega_2(d(l,m))}\frac{\omega_2(d(l,m))}{\omega_1(d(l,m))}\int_p^q |k(l,z)||(V\chi)(z)-(V\psi)(z)|dz\\
&+|\psi(m)|\int_p^q \frac{k_{\omega_2}\omega_2(d(l,m))}{\omega_1(d(l,m))}|(V\chi)(z)-(V\psi)(z)|dz\\
\leqslant\ & K\|V\chi\|_\infty\|\chi-\psi\|_{\omega_1}+\eta_1L(q-p)k_{\omega_2}\|V\chi\|_\infty\|\chi-\psi\|_{\omega_1}\\
&+KLH_\psi^{\omega_2}\|V\chi-V\psi\|_\infty+L(q-p)k_{\omega_2}\|\psi\|_\infty\|V\chi-V\psi\|_\infty,
\end{aligned}
\tag{10}
$$

Then, by the assumption (iii), Lemma 1 and (10), we obtain the following inequality:

$$\begin{aligned} & \frac{|[(\Lambda\chi)(l)-(\Lambda\psi)(l)]-[(\Lambda\chi)(m)-(\Lambda\psi)(m)]|}{\omega_1(d(l,m))} \\ \leqslant\ & Kf\left(\|\chi\|_{\omega_2}\right)\|\chi-\psi\|_{\omega_1}+\eta_1 L(q-p)k_{\omega_2}f\left(\|\chi\|_{\omega_2}\right)\|\chi-\psi\|_{\omega_1} \\ & +KL\|\psi\|_{\omega_2}\|V\chi-V\psi\|_\infty+L(q-p)\eta_2 k_{\omega_2}\|\psi\|_{\omega_2}\|V\chi-V\psi\|_\infty \\ \leqslant\ & Kf(r_0)\delta+\eta_1 L(q-p)k_{\omega_2}f(r_0)\delta+KLr_0\|V\chi-V\psi\|_\infty \\ & +L(q-p)\eta_2 k_{\omega_2}r_0\|V\chi-V\psi\|_\infty. \end{aligned} \tag{11}$$

On the other hand,

$$\begin{aligned} (\Lambda\chi)(p)-(\Lambda\psi)(p) &= \chi(p)\int_p^q k(p,z)(V\chi)(z)dz \\ &\quad -\psi(p)\int_p^q k(p,z)(V\psi)(z)dz \\ &= \chi(p)\int_p^q k(p,z)[(V\chi)(z)-(V\psi)(z)]dz \\ &\quad +[\chi(p)-\psi(p)]\int_p^q k(p,z)(V\psi)(z)dz \end{aligned}$$

and hence

$$\begin{aligned} & |(\Lambda\chi)(p)-(\Lambda\psi)(p)| \\ \leqslant\ & |\chi(p)|\int_p^q |k(p,z)||(V\chi)(z)-(V\psi)(z)|dz \\ & +|\chi(p)-\psi(p)|\int_p^q |k(p,z)||(V\psi)(z)|dz \\ \leqslant\ & K\|\chi\|_\infty\|V\chi-V\psi\|_\infty+K\|V\psi\|_\infty\|\chi-\psi\|_\infty \\ \leqslant\ & K\|\chi\|_{\omega_2}\eta_2\|V\chi-V\psi\|_\infty+K\|V\psi\|_\infty\eta_1\|\chi-\psi\|_{\omega_1} \\ \leqslant\ & K\eta_2\|\chi\|_{\omega_2}\|V\chi-V\psi\|_\infty+K\eta_1 f(\|\psi\|_{\omega_2})\|\chi-\psi\|_{\omega_1} \\ \leqslant\ & K\eta_2 r_0\|V\chi-V\psi\|_\infty+K\eta_1 f(r_0)\delta. \end{aligned} \tag{12}$$

From (11) and (12), it follows that

$$\begin{aligned} & \|\Lambda\chi-\Lambda\psi\|_{\omega_1} \\ =\ & |(\Lambda\chi)(p)-(\Lambda\psi)(p)|+H^{\omega_1}_{\Lambda\chi-\Lambda\psi} \\ =\ & |(\Lambda\chi)(p)-(\Lambda\psi)(p)| \\ & +\sup\left\{\frac{|[(\Lambda\chi)(l)-(\Lambda\psi)(l)]-[(\Lambda\chi)(m)-(\Lambda\psi)(m)]|}{\omega_1(d(l,m))}: l,m\in[p,q], l\neq m\right\} \\ \leqslant\ & (K+\eta_1 L(q-p)k_{\omega_2}+K\eta_1)f(r_0)\delta \\ & +(KL+L(q-p)\eta_2 k_{\omega_2}+K\eta_2)r_0\|V\chi-V\psi\|_\infty \\ <\ & \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon. \end{aligned}$$

Therefore, the operator Λ is continuous at the point $\chi\in B^{\omega_2}_{r_0}$. We conclude that Λ is continuous on $B^{\omega_2}_{r_0}$ with respect to the norm $\|\cdot\|_{\omega_1}$. By Theorem 2, $B^{\omega_2}_{r_0}$ is compact in $C_{\omega_1}[p,q]$ and through Schauder fixed-point theorem proof is completed. □

4. Applications

In this section, we employed the regularization method combined with some of the proper well-known techniques to handle the Fredholm integral equations. Our approach has demonstrated reliability in tackling these challenging problems. To further illustrate the effectiveness of our method, we present two numerical applications that corroborate our fundamental theorem. Through these applications, we aim to reinforce our findings and foster a deeper, more abstract comprehension of the topic.

Application 1. Let n and $\hat{n}$ be two non-negative constants and $l \in [0,1]$. Consider the integral equation given below:

$$\chi(l) = \sqrt[3]{n\sin(\pi l) + \hat{n}} + \chi(l)\int_0^1 \sqrt[5]{5l^2+z}\sqrt{|\chi(z)|}\,dz. \tag{13}$$

Also, define the operator V as $(V\chi)(z) = \sqrt{|\chi(z)|}$ for all $l,z \in [0,1]$ and set $\varrho(l) = \sqrt[3]{n\sin(\pi l)+\hat{n}}$, $k(l,z) = \sqrt[5]{5l^2+z}$.

Additionally we choose $\omega_2(\varepsilon) = \varepsilon^{\frac{1}{3}}$, $\omega_1(\varepsilon) = \varepsilon^{\alpha}$ such that $0 < \alpha < 1/3$ and $d(l,m) = |l-m|$. Since the function $h(l) = \sqrt[3]{l}$ is concave for $h : \mathbb{R}^+ \to \mathbb{R}^+$, this function is subadditive from Lemma 4.4 in [24]. Moreover, if we take into account $|\sin\chi - \sin\psi| \leqslant |\chi - \psi|$ for $\chi, \psi \in \mathbb{R}$, we have

$$\begin{aligned} |\varrho(l) - \varrho(m)| &= \left|\sqrt[3]{n\sin\pi l + \hat{n}} - \sqrt[3]{n\sin\pi m + \hat{n}}\right| \\ &\leqslant \sqrt[3]{|n(\sin\pi l - \sin\pi m)|} \\ &\leqslant \sqrt[3]{n\pi}|l-m|^{\frac{1}{3}}. \end{aligned}$$

Thus,

$$\begin{aligned} \|\varrho\|_{\omega_2} &= |\varrho(0)| + \sup\left\{\frac{|\varrho(l)-\varrho(m)|}{\omega_2(d(l,m))} : l,m \in [0,1], l \neq m\right\} \\ &= \left|\sqrt[3]{\hat{n}}\right| + \sup\left\{\frac{|\varrho(l)-\varrho(m)|}{|l-m|^{\frac{1}{3}}} : l,m \in [0,1], l \neq m\right\} \\ &\leqslant \left|\sqrt[3]{\hat{n}}\right| + \sup\left\{\frac{\sqrt[3]{n\pi}|l-m|^{\frac{1}{3}}}{|l-m|^{\frac{1}{3}}} : l,m \in [0,1], l \neq m\right\} \\ &\leqslant \sqrt[3]{\hat{n}} + \sqrt[3]{n\pi} = \overline{P} \end{aligned}$$

which means that the condition (i) of Theorem 4 is fulfilled.

Further, we have

$$\begin{aligned} |k(l,z) - k(m,z)| &= \left|\sqrt[5]{5l^2+z} - \sqrt[5]{5m^2+z}\right| \\ &\leqslant \sqrt[5]{|5l^2 - 5m^2|} \\ &\leqslant \sqrt[5]{10}|l-m|^{\frac{1}{5}} \end{aligned}$$

for all $l,m \in [0,1]$. The condition (ii) of Theorem 4 holds with $k_{\omega_2} = \sqrt[5]{10}$, $d(l,m) = |l-m|$ and $\omega_2(d(l,m)) = |l-m|^{1/5}$.

Since

$$\begin{aligned} \sup\left\{\int_0^1 |k(l,z)|dz : l \in [0,1]\right\} &= \sup\left\{\int_0^1 \left|\sqrt[5]{5l^2+z}\right|dz : l \in [0,1]\right\} \\ &= \frac{5}{6}(5l^2+z)^{6/5}\Big|_0^1 \\ &\leqslant \frac{5}{6}(6\sqrt[5]{6} - 5\sqrt[5]{5}), \end{aligned}$$

the constant K can be taken as $K = \frac{5}{6}(6\sqrt[5]{6} - 5\sqrt[5]{5})$.

Since
$$\eta_2 = \max\{1, \omega_2(\mathrm{diam}[0,1])\} = \max\left\{1, 1^{\frac{1}{5}}\right\} = 1,$$
we have
$$|(V\chi)(z)| = \sqrt{|\chi(z)|} \leqslant \sqrt{\|\chi\|_\infty} \leqslant \sqrt{\eta_2\|\chi\|_{\omega_2}} = \sqrt{\|\chi\|_{\omega_2}} \tag{14}$$
for all $\chi \in C_{\omega_2}[0,1]$ and $z \in [0,1]$. (14) yields that
$$\|V\chi\|_\infty \leqslant \sqrt{\|\chi\|_{\omega_2}}$$
for all $\chi \in C_{\omega_2}[0,1]$. Therefore, V is an operator from $C_{\omega_2}[0,1]$ into $C[0,1]$ and we can choose the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ as $f(\chi) = \sqrt{\chi}$. This function is non-decreasing and verifies the assumption (iii).

Now, we prove that V is continuous on $C_{\omega_2}[0,1]$ with $\|.\|_{\omega_1}$. Let $\psi \in C_{\omega_2}[0,1]$ and $\varepsilon > 0$. Also, let $\chi \in C_{\omega_2}[0,1]$ be an arbitrary function satisfying $\|\chi - \psi\|_{\omega_1} < \delta$ such that $0 < \delta \leqslant \varepsilon^2$.

The inequality
$$|(V\chi)(z) - (V\psi)(z)| = \sqrt{|\chi(z) - \psi(z)|} \leqslant \sqrt{\|\chi - \psi\|_\infty} \leqslant \sqrt{\eta_1\|\chi - \psi\|_{\omega_1}}$$
holds for all $\chi, \psi \in C_{\omega_2}[0,1]$ and $z \in [0,1]$. Since
$$\eta_1 = \max\{1, \omega_1(\mathrm{diam}[0,1])\} = \max\{1, 1^\alpha\} = 1,$$
we have
$$\|V\chi - V\psi\|_\infty \leqslant \sqrt{\|\chi - \psi\|_{\omega_1}}$$
for all $\chi, \psi \in C_{\omega_2}[0,1]$.

Hence, V is continuous at the point $\psi \in C_{\omega_2}[0,1]$ with $\|.\|_{\omega_1}$ since ψ is arbitrarily selected.
$$\overline{P} + [\eta_2 K + K + \eta_2 k_{\omega_2}(q-p)]f(r)r \leqslant r$$
is equivalent to
$$\sqrt[3]{\hat{n}} + \sqrt[3]{n\pi} + \left[\frac{5}{3}\left(6\sqrt[5]{6} - 5\sqrt[5]{5}\right) + \sqrt[5]{10}(1-0)\right] r\sqrt{r} \leqslant r. \tag{15}$$

If we take the constants n and $\hat{n}$ as suitable, then there exists a positive number r_0 satisfying (15). For instance, if we select $n = 0$ and $\hat{n} = 0$, then (15) holds for $r = r_0 \in (0, 0.0517272]$.

Therefore, we show the existence of the solution for Equation (13) in the space $C_{\omega_1}[0,1]$ via Theorem 4.

Application 2. Consider the following quadratic integral equation:
$$\chi(l) = \frac{1}{100}\sqrt{\arctan(l-1)} + \chi(l)\int_1^e \frac{\sqrt[3]{\ln l + \ln z}}{z} \sin\chi(z)dz, \tag{16}$$
where $l, z \in [1, e]$.

Set $\varrho(l) = \frac{1}{100}\sqrt{\arctan(l-1)}$, $k(l,z) = \frac{\sqrt[3]{\ln l + \ln z}}{z}$, $(V\chi)(z) = \sin\chi(z)$ and $\omega_2(\varepsilon) = \varepsilon^{\frac{1}{3}}$, $\omega_1(\varepsilon) = \varepsilon^\alpha$ for $0 < \alpha < 1/3$, $d(l,m) = |\ln l - \ln m|$.

Since the function $h:\mathbb{R}^+\to\mathbb{R}^+$ defined by $h(\chi)=\sqrt{\chi}$ is concave and $h(0)=0$, this function is subadditive from Remark 4.5 in [24] ref IS CHANGED AS cite. If we consider $|\arctan\chi-\arctan\psi|\leqslant|\chi-\psi|$ for $\chi,\psi\in\mathbb{R}$, we can write

$$\begin{aligned}|\varrho(l)-\varrho(m)| &= \left|\frac{1}{100}\sqrt{\arctan(l-1)}-\frac{1}{100}\sqrt{\arctan(m-1)}\right| \\ &\leqslant \frac{1}{100}\sqrt{|\arctan(l-1)-\arctan(m-1)|} \\ &\leqslant \frac{1}{100}\sqrt{|l-m|}\end{aligned}$$

for all $l,m\in[1,e]$.

Let $l<m$, so there is $\xi\in(l,m)$ satisfying

$$|\ln l-\ln m|=\frac{1}{\xi}|l-m|$$

which yields that

$$|\ln l-\ln m|\geqslant\frac{1}{e}|l-m|$$

and hence

$$\frac{1}{|\ln l-\ln m|^{\frac{1}{3}}}\leqslant\frac{\sqrt[3]{e}}{|l-m|^{\frac{1}{3}}}$$

for all $l,m\in[1,e]$ and $l\neq m$.

Since

$$\begin{aligned}\|\varrho\|_{\omega_2} &= |\varrho(1)|+\sup\left\{\frac{|\varrho(l)-\varrho(m)|}{\omega_2(d(l,m))}:l,m\in[1,e],l\neq m\right\} \\ &= \sup\left\{\frac{|\varrho(l)-\varrho(m)|}{|\ln l-\ln m|^{\frac{1}{3}}}:l,m\in[1,e],l\neq m\right\} \\ &\leqslant \frac{\sqrt[3]{e}}{100}\sup\left\{|l-m|^{\frac{1}{6}}:l,m\in[1,e],l\neq m\right\} \\ &\leqslant \frac{1}{50}=\overline{P},\end{aligned}$$

assumption (i) of Theorem 4 is satisfied.

Further, we have

$$\begin{aligned}|k(l,z)-k(m,z)| &= \frac{1}{z}\left|\sqrt[3]{\ln l+\ln z}-\sqrt[3]{\ln m+\ln z}\right| \\ &\leqslant \frac{1}{z}\sqrt[3]{|\ln l-\ln m|} \\ &\leqslant |\ln l-\ln m|^{\frac{1}{3}} \\ &= \omega_2(d(l,m))\end{aligned}$$

for all $l,m,z\in[1,e]$. Condition (ii) of Theorem 4 holds with the constant $k_{\omega_2}=1$.

Also,

$$\begin{aligned}\sup\left\{\int_1^e|k(l,z)|dz:l\in[1,e]\right\} &= \sup\left\{\int_1^e\left|\frac{\sqrt[3]{\ln l+\ln z}}{z}\right|dz:l\in[1,e]\right\} \\ &\leqslant \int_1^e\frac{\sqrt[3]{1+\ln z}}{z}dz \\ &= \frac{3}{4}\left(\sqrt[3]{16}-1\right).\end{aligned}$$

So the constant K can be taken as $\frac{3}{4}\left(\sqrt[3]{16}-1\right)$.

Since
$$\eta_2 = \max\{1, \omega_2(\text{diam}[1,e])\} = \max\left\{1, (\ln e - \ln 1)^{\frac{1}{3}}\right\} = 1,$$
the estimate
$$|(V\chi)(z)| = |\sin\chi(z)| \leqslant |\chi(z)| \leqslant \|\chi\|_\infty \leqslant \eta_2\|\chi\|_{\omega_2} = \|\chi\|_{\omega_2}$$
holds for all $\chi \in C_{\omega_2}[1,e]$ and $z \in [1,e]$, and hence
$$\|V\chi\|_\infty = \sup_{z\in[0,1]} |(V\chi)(z)| \leqslant \|\chi\|_{\omega_2}.$$
Therefore, V is an operator from $C_{\omega_2}[1,e]$ into $C[1,e]$, and we can take the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ as $f(\chi) = \chi$. It is obvious that the function f is non-decreasing and satisfies assumption (iv).

Now, we show that the operator V is continuous on $C_{\omega_2}[1,e]$ with $\|.\|_{\omega_1}$. Let $\psi \in C_{\omega_2}[1,e]$ be arbitrarily selected, $\varepsilon > 0$ and $\chi \in C_{\omega_2}[1,e]$ be an arbitrary function, and inequality $\|\chi-\psi\|_{\omega_1} < \delta$ be satisfied such that $0 < \delta \leqslant \varepsilon$.

Then, for arbitrary $l, m \in [1,e]$, we obtain
$$|\sin\chi(z) - \sin\psi(z)| \leqslant |\chi(z)-\psi(z)| \leqslant \|\chi-\psi\|_\infty \leqslant \eta_1\|\chi-\psi\|_{\omega_1} = \|\chi-\psi\|_{\omega_1},$$
where
$$\eta_1 = \max\{1, \omega_1(\text{diam}[1,e])\} = \max\{1, (\ln e - \ln 1)^\alpha\} = 1.$$

Thus,
$$\|V\chi - V\psi\|_\infty = \sup_{z\in[0,1]} |\sin\chi(z) - \sin\psi(z)| \leqslant \|\chi-\psi\|_{\omega_1}$$
which implies
$$\|V\chi - V\psi\|_\infty < \delta \leqslant \varepsilon$$
is satisfied for all $\chi \in C_{\omega_2}[1,e]$. This proves that the operator V is continuous at the point $\psi \in C_{\omega_2}[1,e]$, and it is continuous on $C_{\omega_2}[1,e]$ with $\|.\|_{\omega_1}$ because ψ is arbitrarily selected.

Hypothesis (iv) of Theorem 4
$$\overline{P} + [\eta_2 K + K + \eta_2 k_{\omega_2}(q-p)]f(r)r \leqslant r$$
is equivalent to
$$\frac{1}{50} + \left[\frac{3}{2}\left(\sqrt[3]{16}-1\right) + (e-1)\right]r^2 \leqslant r. \tag{17}$$
The number r_0 chosen as $r_0 \in [0.0219212, 0.228201]$ satisfies inequality (17).

Therefore, we show the existence of solution for Equation (16) in the space $C_{\omega_1}[1,e]$ via Theorem 4.

5. Conclusions

In this paper, using a fixed-point theorem, the existence conditions for the solution of Fredholm integral equations of the form
$$\chi(l) = \varrho(l) + \chi(l)\int_p^q k(l,z)(V\chi)(z)dz$$
are investigated in the space $C_\omega[p,q]$. The main theorem is based on a useful technique. It is clear that this theorem is more general than many equations considered so far. By using a sufficient condition for the relative compactness in the space of functions with tempered moduli of continuity (see Theorem 2) and the classical Schauder fixed-point

theorem, we derive a new existence result (see Theorem 4). Fredholm integral equations are used in various scientific and engineering disciplines to model phenomena like heat transfer, population dynamics, and signal processing. The established results in this paper could be applied to problems in these fields where existence of solutions is crucial for model validity. The existence theorems provide a theoretical foundation for developing numerical methods to solve Fredholm integral equations. By guaranteeing the existence of a solution, these results can help guide the development of more robust and efficient numerical algorithms.

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