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Editorial

Advance in Topology and Functional Analysis in Honour of María Jesús Chasco's 65th Birthday

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1. Introduction

We are honoured to present this Special Issue of *Axioms* with the title "Topology and Functional Analysis" to showcase recent work on this and related topics and to provide an opportunity for María Jesús Chasco's friends and colleagues to pay tribute to her mathematical career on the occasion of her 65th birthday. This issue includes significant papers dealing with topological groups, topological semi-groups and topological vector spaces. Many of them exploit the rich and fruitful interaction between topology and functional analysis, which has been a wellspring of powerful mathematical ideas and development since the early stages of both disciplines.

One of the many important programs originating within this framework can be described as borrowing some of the tools and concepts of topological vector space theory to study the structure and duality properties of Abelian topological groups. Such a viewpoint turns out to be particularly useful, for instance, when dealing with Pontryagin duality and reflexivity outside the class of locally compact groups. It should be noted that the notion of convexity admits a counterpart in the field of Abelian topological groups. Inspired by the Hahn–Banach theorem, Vilenkin introduced in [1] the notion of a quasi-convex subset of an Abelian topological group, which immediately led to the definition of locally quasi-convex groups. With these objects at hand, it is natural to extend well-known theorems from the class of locally convex spaces to the broader class of locally quasi-convex groups.

María Jesús Chasco completed her doctoral dissertation under the direction of Antonio Plans while she was working as a high-school chair. Her first research was in Hilbert space theory, and the defense of her thesis took place at the University of Zaragoza in 1985.

Soon after, she obtained a position as a Professor at the Department of Mathematics at the Engineering School of the University of Vigo, where she remained for 7 years. She was involved in multiple collaborations with her colleagues in the Department of Mathematics, and became director of the department for some time. She encouraged visits from numerous professors from other countries who contributed to creating a fruitful scientific environment, which attracted students to attend the university to write their doctoral dissertations. Concretely, she was coadvisor of the theses of Ricardo Vidal and Xabier Domínguez.

In 1997, she obtained a professorship at the University of Navarre, in Pamplona, her native town, where she has remained ever since. Her brilliant work there spanned fruitful research, pedagogical tasks that included advising the theses of Hugo Bello and Carlos Bejines and administrative positions as the Vice Dean of the Faculty of Sciences.

As a university professor, she has enjoyed the interaction with her students, who recognize her generous dedication to them and her unconditional availability to help solve their problems.

Mª Jesús has a special quality of discovering beautiful problems in mathematics and sharing ideas with colleagues. She has published more than 40 papers, mainly in topological groups, but also in functional analysis dealing with locally convex spaces and Banach spaces. Her favourite topic is duality in topological groups. She knows how to extract the best from her colleagues and discover talent wherever it is. She can easily make a team work, being a loyal friend that always speaks the truth.

She also has deep convictions which guide her actions, not falling into relativism or preconceived opinions. Friendship is one of her highest values. She enjoys travelling, art, nature and beauty wherever it is. Thus, we expect to keep sharing her enthusiastic attitude to life, her friendship and new results in mathematics for many years.

2. Overview of the Published Papers

Large-scale topology or, in other words, the study of coarse structures, is currently an important area of topology, with essential geometric and combinatorial connections [2]. In the natural coarse structures associated with any given group, the discrete subsets are exactly the so-called thin subsets: a subset X of a group G is called thin if given any finite subset F of G, one has both $Fx \cap Fy = \emptyset$ and $xF \cap yF = \emptyset$ for all but finitely many different $x, y \in X$. The paper *On factoring groups into thin subsets* (Contribution 1 by I. Protasov) is devoted to the proof of the following factorization result: Every Hausdorff nondiscrete countably infinite topological group G has two thin subsets A and B such that every $g \in G$ can be uniquely expressed as a product $g = ab$ with $a \in A$ and $b \in B$.

The contribution *Factoring continuous characters defined on subgroups of products of topological groups* (Contribution 2 by M. Tkachenko) mainly deals with extensions of characters defined on such subgroups to the whole product space. For precompact Abelian subgroups (without requiring the Hausdorff property of the spaces involved), a nice result is obtained, which includes a factorization theorem (Theorem 4). It is well known that factorization theorems constitute a powerful tool to study continuity of functions defined on products of topological spaces. The author provides interesting examples and poses the problem of whether precompact subgroups of products of paratopological Abelian groups are dually embedded (Problem 1).

In *A distinguished subgroup of compact Abelian groups* (Contribution 3), D. Dikranjan, W. Lewis, P. Loth and A. Mader consider the family of all subgroups Δ of a compact abelian group G that are compact, totally disconnected and such that G/Δ is a torus. The sum of all the subgroups with these properties is a functorial subgroup $\Delta(G)$ that is dense, zero-dimensional and such that the quotient $G/\Delta(G)$ is torsion-free and divisible. Using these ideas, the authors survey and extend earlier results on the resolution theorem for compact Abelian groups and about minimal groups.

Aggregation operators are an essential tool in science and engineering due to the ubiquitous necessity of combining several input values into a single value. In the paper *On self-aggregations of min-subgroups* (Contribution 4 by C. Bejines, S. Ardanza-Trevijano and J. Elorza), the authors study the preservation of the min-subgroup structure under aggregation functions. Min-subgroups of a group G are fuzzy sets μ with domain G satisfying $\mu(x) = \mu(x^{-1})$ and $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for any $x, y \in G$. P. Das proved that a fuzzy set of G is a min-subgroup of G if and only if all its nonempty level sets are subgroups of G. He also introduced a natural equivalence relation between min-subgroups in terms of their level sets. The main result of this paper is the following: If G is a group and $A : [0,1]^n \to [0,1]$ is an aggregation function, then $A(\mu, \ldots, \mu)$ and μ induce the same level sets for every min-subgroup μ of G if and only if A is strictly increasing on its diagonal; that is, $A(x, \ldots, x) < A(y, \ldots, y)$ whenever $x < y$. From this result, it follows that for any min-subgroup μ of G, $A(\mu, \ldots, \mu)$ and μ belong to the same Das class whenever A is a strict t-norm or a strict t-conorm.

The notion of boundedness is a fundamental tool in many fields of mathematics, particularly in the framework of topological vector spaces. In 1959, Hejcman dealt with boundedness in uniform spaces and in topological groups [3]. The article *Bounded sets in topological spaces* (Contribution 5 by C. Bors, M. Ferrer and S. Hernández) approaches this notion for a topological space subject to the action of a monoid. They define the G-bounded sets of the topological space X, where G is a monoid that acts on X. This notion can be the seed of important developments, taking into account that the topic of actions of groups on topological spaces is nowadays a challenging one. In this paper, the authors prove, among other properties, that for a metrizable separable G-space X, the G-bounded subsets of X are completely determined by the G-bounded subsets of any dense subspace.

The paper *Distinguished property in tensor products and weak* dual spaces* (Contribution 6 by S. López-Alfonso, M. López-Pellicer and S. Moll-López) deals with locally convex spaces. Recall that a locally convex space E is distinguished if its strong dual E'_β is a barrelled space or equivalently if for every bounded subset M in $(E'_\beta)'_\beta$ there is a bounded set N in E such that $M \subset N^{\circ\circ}$. The notion of distinguished Fréchet spaces was already defined by Dieudonné and Schwartz. Here, the authors obtain distinguished properties of injective tensor products $L_p(X) \bigotimes_\varepsilon E$, where $L_p(X)$ denotes the dual space of the classical space $C_p(X)$ (the space of continuous real functions over a topological space X, endowed with the pointwise convergence topology) and E denotes a locally convex space. By imposing conditions either on X or on E, they are able to find many classes of distinguished spaces of the above-mentioned form.

The article *Aspects of differential calculus related to infinite-dimensional vector bundles and Poisson vector spaces* (Contribution 7 by H. Glöckner) deals with infinite-dimensional differential calculus. Among other questions, the differentiability properties of operator-valued maps and compositions with hypocontinuous k-linear mappings are investigated. A wide scope of applications is provided. In the field of infinite-dimensional vector bundles, these results are used to construct new bundles from given ones, such as dual bundles, topological tensor products, infinite direct sums and completions under suitable hypotheses. Another field of applications is in the class of locally convex Poisson spaces, a class defined by the author in earlier work. Roughly speaking, locally convex Poisson vector spaces are locally convex spaces E such that $E \times E$ is a $k_{\mathbb{R}}$-space, and a "Poisson bracket"—a more restrictive notion than that of a Lie bracket—is defined for the dual space of E. The differentiability results are used in this context to prove the continuity of the Poisson bracket and the continuity of the passage from a function to the associated Hamiltonian vector field.

Pro-Lie groups, which are defined as projective limits of finite-dimensional Lie groups, have been the subject of many fruitful investigations in recent years. In the article *Advances in the theory of compact groups and pro-Lie groups in the last quarter century* (Contribution 8), K. Hofmann and S. Morris provide a masterful summary that motivates and contextualizes their own contributions and those of others. Particular attention is paid to structure theorems for pro-Lie groups that are connected, almost connected or Abelian. The authors also explore the connection between pro-Lie groups and linear algebra, thereby identifying a new approach to the Hochschild–Tanaka duality of compact groups. This is one of several areas that they mention as ripe for further study.

Exploiting the connection between topological vector spaces and topological groups that were mentioned in the Introduction, an analogue of the Mackey–Arens theorem for the class of topological groups is considered in [4], a paper which initiated an extensive literature on this topic. The Mackey topology for a topological Abelian group G is defined as the finest locally quasi-convex topology which admits the same character group as G, and G is said to be a Mackey group if its original topology coincides with its Mackey topology. The class of Mackey groups includes all locally compact groups, as well as all complete metrizable ones [4]; however, there are topological Abelian groups which do not admit a Mackey topology, as was proven in [5,6]. Thus, the natural counterpart of the Mackey–Arens theorem does not hold for Abelian topological groups. As a result, it is natural to ask what is the relationship between the properties of "being a Mackey space" and "being a Mackey group" in the field of locally convex spaces. In [7], it was shown that a metrizable locally convex space might not be a Mackey group, a fact that disproved a conjecture stated in [8], (8.1). It is well known that a metrizable locally convex space carries its Mackey topology. In the paper *Normed spaces which are not Mackey groups* (Contribution 9 by S. Gabriyelyan), it is further proven that even a normed space may fail to be a Mackey group.

In an infinite dimensional topological vector space, the closed convex hull of a compact set might not be compact. Krein's theorem is an important result in this line, which can be formulated as follows: "If E is a complete locally convex space, then the closed convex hull of a weakly compact subset of E is again weakly compact". In the paper *Krein's theorem in the context of topological Abelian groups* (Contribution 10 by T. Borsich, X. Domínguez and E. Martín-Peinador), the authors interpret this result in the class of locally quasi-convex Abelian topological groups, analyze the resulting concepts and properties and expose an obstruction to the generalization of Krein's theorem to this wider context. In fact, if G denotes the family of null sequences of a compact metrizable connected group X, G has a natural group structure provided by that of $X^{\mathbb{N}}$. Under the uniform topology of $X^{\mathbb{N}}$, G becomes a complete metrizable locally quasi-convex topological group. However, the corresponding weak topology on G does not satisfy Krein's property. In other words, there exist weakly compact subsets of G whose quasi-convex hulls are not weakly compact .

A sequence (x_n) of elements of a locally convex space X is said to be absolutely summable if $\sum_n p(x_n) < \infty$ whenever p is a continuous seminorm on X or, equivalently, the Minkowski functional of an absolutely convex neighborhood of zero in X. This definition can be carried over to an arbitrary topological Abelian group G via Kaplan's generalization of Minkowski functionals. The same functionals can be then invoked to endow the group $\ell^1(G)$ of all absolutely summable sequences in G with a natural group topology. These concepts and constructions can be applied to a wide range of situations, and they often provide illuminating generalizations of the normed or the topological vector space setting. The article *On the group of absolutely summable sequences* (Contribution 11 by L. Außenhofer) contains quite a few of these generalizations; among other results, it is shown here that $\ell^1(G)$ is a Pontryagin reflexive group if G is either reflexive and metrizable or an LCA group, and $\ell^1(G)$ has the Schur property if and only if G has it.

The classical theorems by Dirichlet and Riemann on the convergence of a series of real terms can be partially generalized to much wider contexts, giving rise to a rich theory which is still being developed in a relevant way. The paper *Permutations, signs and sum ranges* (Contribution 12 by S. Chobanyan, X. Domínguez, V. Tarieladze and R. Vidal) consists mostly of a detailed survey of the advances in the sum range problem from its first formulations to the present day, including some results by M. J. Chasco and the first named author.

Along the same lines, in the paper *Series with commuting terms in topologized semigroups* (Contribution 13 by A. Castejón, E. Corbacho and V. Tarieladze), the authors present a version of the Riemann–Dirichlet unconditional convergence theorem for topologized semigroups.

The contribution *An expository lecture of María Jesús Chasco on some applications of Fubini's theorem* (Contribution 14 by A. Castejón, M. J. Chasco, E. Corbacho and V. Rodríguez de Miguel) is an elegant and powerful piece of mathematical exposition at the advanced undergraduate level, based on a masterclass given by M. J. Chasco at the University of Vigo. It contains a remarkable presentation of the Brunn–Minkowski and isoperimetric inequalities as consequences of Fubini's theorem, as well as some estimations of volumes of sections of n-dimensional balls.

3. Conclusions

The authors of the fourteen papers in this volume include friends, colleagues and collaborators of María Jesús Chasco. Ranging over many different branches of mathematics, the papers reflect the breadth of her mathematical interests. Several deal with various aspects of topological groups, but we expect that this volume will also interest specialists in general topology, functional analysis, algebra, geometry and number theory. Their quality and depth make them a fitting tribute for María Jesús Chasco's 65th birthday.

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