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Cramér Moderate Deviations for a Supercritical Galton–Watson Process with Immigration

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Abstract: Consider a supercritical Galton–Watson process with immigration $(X_n; n \geq 0)$. The Lotka–Nagaev estimator $\frac{X_{n+1}}{X_n}$ is a common estimator for the offspring mean. In this work, we used the Martingale method to establish several types of Cramér moderate deviation results for the Lotka–Nagaev estimator. To satisfy our needs, we employed the well-known Cramér approach for our proofs, which establishes the moderate deviation of the sum of the independent variables. Simultaneously, we provided a concrete example of its applicability in constructing confidence intervals.

Keywords: supercritical; Galton–Watson process; immigration; Lotka–Nagaev estimator; Cramér; moderate deviations

MSC: 60J27; 60J35

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Citation: Wang, J.; Peng, C. Cramér Moderate Deviations for a Supercritical Galton–Watson Process with Immigration. *Axioms* **2024**, *13*, 272. https://doi.org/10.3390/axioms13040272

Academic Editor: Stefano Isola

Received: 20 March 2024
Revised: 11 April 2024
Accepted: 16 April 2024
Published: 19 April 2024

1. Introduction

Consider a series of identically distributed (i.i.d) random variables with a mean of 0 and positive variance, σ^2, denoted as $(Z_i)_{i\geq 1}$. By $L_n = \sum_{i=1}^{n} Z_i$, denote the partial sums of $(Z_i)_{i\geq 1}$. Assume $E\big(\exp[C_0|Z_1|]\big) < \infty$ for some constant $C_0 > 0$. Cramér [1] established the following asymptotic moderate deviation for all $0 \leq x = o(n^{\frac{1}{2}})$:

$$\Big|\ln \frac{P(L_n > x\sigma\sqrt{n})}{1-\phi(x)}\Big| = O(\frac{1+x^3}{\sqrt{n}}) \tag{1}$$

as $n \to \infty$, where $\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(\frac{-t^2}{2})dt$ is the standard normal distribution. The moderate deviation results of type (1) are typically referred to as Cramér moderate deviations.

Let $(X_n)_{n\geq 0}$ be a series of random variables taking non-negative integer values, where X_n is the number of particles in the nth generation of the Galton–Watson process with immigration (i.e., GWIP). Then, a GWIP can be defined as follows:

$$X_0 = 1, X_{n+1} = \sum_{i=1}^{X_n} Z_i^n + Y_{n+1},\ for\ \ \forall n \geq 0, \tag{2}$$

where $(Z_i^n)_{i,n\geq 1}$ is a sequence of non-negative integer-valued random variables that are independently and identically distributed for $i, n \geq 1$ as follows:

$$P(Z_i^n = j) = p_j (j \geq 0).$$

Y_n represents the number of immigrant particles arriving in the nth generation, and $(Y_n)_{n\geq 1}$ is also a series of independently and identically distributed random variables. Simultaneously, we assume that branches and immigrants are independent of each other.

Indicate an individual's offspring mean by m, then

$$m = \mathrm{E}X_1. \tag{3}$$

By μ, denote the standard variance of X_1; then, we have

$$\mu^2 = \mathrm{E}(X_1 - m)^2 = \mathrm{Var}X_1. \tag{4}$$

To avoid triviality, and for convenience, we can take μ to be positive. A popular estimator for the offspring mean m is the Lotka–Nagaev estimator $\frac{X_{n+1}}{X_n}$. Since we will be assuming $p_0 = 0$ throughout this study, the Lotka–Nagaev estimator $\frac{X_{n+1}}{X_n}$ has a well-defined P-a.s.

In the literature, researchers have shown a great deal of interest in the topic of moderate and large deviations in the branching process. Athreya [2] researched large deviations for branching processes in the single-type case in 1994, investigating them for the normalized Lotka–Nagaev estimator. Moreover, Athreya et al. [3] extended these results to investigate the large deviation of different types of branching processes, including both supercritical and critical ones, showing geometric decay in the multi-type supercritical case and algebraic decay in the critical single-type case. Fleischmann and Wachtel [4] explored large deviations for sums of independent and identically distributed random variables for the Galoton–Watson process. Other relevant studies from the literature can be found in [5–11].

On the other hand, moderate deviations have been studied extensively for supercritical, subcritical and critical Galton–Watson processes. The Lotka–Nagaev estimator $\frac{Z_{n+1}}{Z_n}$ is a widely used estimator for offspring means. For instance, Ney and Vidyashanka [12] estimated the sharp rate for the large deviation behavior of the Lotka–Nagaev estimator. Bercu and Touati [13] demonstrated exponential inequalities for the Lotka–Nagaev estimator through self-normalized Martingale methods. Chen and Zhang [14] examined a nearly unstable sub-critical Galton–Watson process with immigration, exploring the moderate deviations for the total population. Doukhan et al. [15] focused on establishing Cramér moderate deviation results for the Lotka–Nagaev estimator in a supercritical Galton–Watson process using the Martingale method. Fan et al. [16] established self-normalized Cramér-type moderate deviations for a supercritical Galton–Watson process, specifically focusing on the Lotka–Nagaev estimator. Furthermore, Fan and Shao [17,18] introduced self-normalized Cramér-type moderate deviations and Berry–Esseen bounds for the Lotka–Nagaev estimator. These studies collectively contribute to the understanding and application of moderate deviations in various statistical scenarios.

Additionally, branching process models in a random environment have been introduced as an extension of the Galton–Watson process. Grama et al. [19] proved a Kesten–Stigum-type theorem for supercritical multitype branching processes in a random environment. Other influential contributions include [20–24]. Overall, these studies contribute to the understanding of moderate deviations in the context of Galton–Watson processes.

Different from the above studies, which mainly focus on the classical Galton–Watson process, in this paper, we research the same processes with immigration, introducing a range of Cramér moderate deviation outcomes for the Lotka–Nagaev estimator by employing a Martingale approach. For our proofs, we utilize the well-known Cramér method to demonstrate the moderate deviation of the sum of independent variables to satisfy our requirements. Moreover, we provide a further application as a by-product.

The remainder of this paper is structured as follows. In Section 2, we first outline some basic preliminaries and then provide the moderate deviations for a supercritical GWIP. On the basis of this, we present the Cramér moderate deviation results for a supercritical GWIP in Section 3, while Section 4 provides some applications for the moderate deviation results. Further discussions are presented in Section 5.

2. Preliminaries

For the purpose of engaging in discourse and facilitating a comprehensive analysis, we shall present the following lemmas.

Lemma 1. *Let $(\mathrm{X}_n)_{n\geq 0}$ be a GWI process. Then,*

$$\mathrm{P}(\mathrm{X}_n \leq n) \leq C_1 \exp(-C_0 n) \tag{5}$$

where $C_0 > 0$.

Proof. This is implied by (6.8) in [18],

$$\begin{aligned} \mathrm{P}(\mathrm{X}_n \leq n) &= \mathrm{P}(\sum_{i=1}^{\mathrm{X}_{n-1}} \mathrm{Z}_i \leq n - \mathrm{Y}_{n-1}) \\ &\leq \mathrm{P}(\sum_{i=1}^{\mathrm{X}_{n-1}} \mathrm{Z}_i \leq n) \\ &\leq C_1 \exp(-C_0 n). \end{aligned} \tag{6}$$

The proof is complete. □

Theorem 1. *Let $\mathrm{E}\exp(C_0|\mathrm{X}_1|) < \infty$ for some constant $C_0 > 0$. Then, for all $0 \leq x = o(n^{\frac{1}{2}})$,*

$$\Big| \ln \frac{\mathrm{P}(\mathrm{L}_n > x\mu\sqrt{n})}{1-\phi(x)} \Big| = O\Big(\frac{C_1 + C_2 x + C_3 x^2 + x^3}{\sqrt{n}}\Big), \tag{7}$$

where $\mathrm{L}_n = \sum_{i=1}^{n} \mathrm{X}_i$, $(\mathrm{X}_i)_{i\geq 1}$ are independent and identically distributed and centred random variables, $\mu^2 = \mathrm{Var}\mathrm{X}_1$.

Proof. According to the total probability formula, we derive the following comprehensive expression:

$$\begin{aligned} \mathrm{P}(\mathrm{L}_n > x\mu\sqrt{n}) &= \mathrm{P}(\sum_{i=1}^{n} \mathrm{Z}_i + \mathrm{Y}_n > x\mu\sqrt{n}) \\ &= \sum_{j=1}^{\infty} \mathrm{P}(\mathrm{Y}_n = j)\mathrm{P}(\sum_{i=1}^{n} \mathrm{Z}_i > x\mu\sqrt{n} - j) \\ &= \sum_{j=1}^{\infty} \mathrm{P}(\mathrm{Y}_n = j)\mathrm{P}\Big(\sum_{i=1}^{n} \mathrm{Z}_i > \mu\sqrt{n} \cdot \frac{x\mu\sqrt{n} - j}{\mu\sqrt{n}}\Big) \\ &=: \sum_{j=1}^{\infty} \mathrm{P}(\mathrm{Y}_n = j)\mathrm{P}_n(x - \frac{j}{\mu\sqrt{n}}). \end{aligned} \tag{8}$$

It can be inferred from (1) in [18] that

$$\mathrm{P}_n(x - \frac{j}{\mu\sqrt{n}}) < C_1\Big(1 - \phi(x - \frac{j}{\mu\sqrt{n}})\Big) \exp\Big(\frac{1 + [x - \frac{j}{\mu\sqrt{n}}]^3}{\sqrt{n}}\Big), \tag{9}$$

and

$$\mathrm{P}_n(x - \frac{j}{\mu\sqrt{n}}) > C_2\Big(1 - \phi(x - \frac{j}{\mu\sqrt{n}})\Big) \exp\Big(-\frac{1 + [x - \frac{j}{\mu\sqrt{n}}]^3}{\sqrt{n}}\Big). \tag{10}$$

Combine this with the following inequalities:

$$\frac{1}{\sqrt{2\pi}(1+x)}e^{\frac{-x^2}{2}} \leq 1-\phi(x) \leq \frac{1}{\sqrt{\pi}(1+x)}e^{\frac{-x^2}{2}}, x \geq 0 \tag{11}$$

Then, for $x \geq 0$,

$$\begin{aligned} \frac{1}{\sqrt{2\pi}(1+x-\frac{j}{\mu\sqrt{n}})} \exp\Big(\frac{-[x-\frac{j}{\mu\sqrt{n}}]^2}{2}\Big) &\leq 1-\phi(x-\frac{j}{\mu\sqrt{n}}) \\ &\leq \frac{1}{\sqrt{\pi}(1+x-\frac{j}{\mu\sqrt{n}})} \exp\Big(\frac{-[x-\frac{j}{\mu\sqrt{n}}]^2}{2}\Big). \end{aligned} \tag{12}$$

By leveraging expressions (11) and (12), in conjunction with the substitution $t = \frac{j}{\mu\sqrt{n}}$, we can further advance our analysis:

$$\frac{1-\phi(x-t)}{1-\phi(x)} < \frac{\sqrt{2}(1+x)}{1+x-t}\exp(xt-\frac{t^2}{2}), \tag{13}$$

and

$$\frac{1-\phi(x-t)}{1-\phi(x)} > \frac{(1+x)}{\sqrt{2}(1+x-t)}\exp(xt-\frac{t^2}{2}). \tag{14}$$

By (9) and (13), we obtain

$$\begin{aligned} \mathrm{P}(\sum_{i=1}^{n} \mathrm{Z}_i > x\mu\sqrt{n}-j) &< C_1\frac{\sqrt{2}(1+x)}{1+x-t}\Big(1-\phi(x)\Big)\exp\Big(\frac{1+[x-t]^3}{\sqrt{n}}+x\frac{j}{\mu\sqrt{n}}-\frac{j^2}{2\mu^2 n}\Big) \\ &< C_4\Big(1-\phi(x)\Big)\exp\Big(\frac{C_1+C_2x+C_3x^2+x^3}{\sqrt{n}}\Big) \end{aligned} \tag{15}$$

and

$$\mathrm{P}(\sum_{i=1}^{n} \mathrm{Z}_i > x\mu\sqrt{n}-j) > C_5\Big(1-\phi(x)\Big)\exp\Big(-\frac{C_1+C_2x+C_3x^2+x^3}{\sqrt{n}}\Big). \tag{16}$$

Finally, we can arrive at the following conclusion:

$$\Big|\ln\frac{\mathrm{P}(\mathrm{L}_n > x\mu\sqrt{n})}{1-\phi(x)}\Big| = O\Big(\frac{C_1+C_2x+C_3x^2+x^3}{\sqrt{n}}\Big). \tag{17}$$

□

3. Main Results

To establish our arguments and meet our analytical needs, we leveraged the renowned Cramér method to showcase the moderate deviation of the sum of independent variables.

In this section, our primary focus is on exploring the moderate deviations related to the weighted Lotka–Nagaev estimator utilizing the data $(\mathrm{X}_k)_{n_0 \leq k \leq n_0+n}$. Let $n, n_0 \in N$ and denote

$$\hat{m}_n = \frac{1}{\sum_{k=n_0}^{n_0+n-1}\sqrt{\mathrm{X}_k}+\frac{1}{\sqrt{\mathrm{X}_k}}}\sum_{k=n_0}^{n_0+n-1}\sqrt{\mathrm{X}_k}(\frac{\mathrm{X}_{k+1}}{\mathrm{X}_k}) \tag{18}$$

as the random weighted Lotka–Nagaev estimator. Typically, n_0 is set to 0. However, we will now explore the broader case where n_0 can be dependent on n. Denote

$$\mathrm{H}_{n_0,n} = \frac{1}{\mu\sqrt{n}} \sum_{k=n_0}^{n_0+n-1} \Big(\sqrt{\mathrm{X}_k}\,[\frac{\mathrm{X}_{k+1}}{\mathrm{X}_k} - m] - \frac{m}{\sqrt{\mathrm{X}_k}}\Big) \tag{19}$$

Subsequently, $\mathrm{H}_{n_0,n}$ can be reformulated as follows:

$$\mathrm{H}_{n_0,n} = \frac{\hat{m}_n - m}{\mu\sqrt{n}} \sum_{k=n_0}^{n_0+n-1} \Big(\sqrt{\mathrm{X}_k} + \frac{1}{\sqrt{\mathrm{X}_k}}\Big). \tag{20}$$

This provides an appropriate normalization of the error term $\hat{m}_n - m$. Verifying the following result is thus straightforward.

Lemma 2. $\mathrm{H}_{n_0,n}$ *is a standardized Martingale.*

Proof. It can be easily seen that

$$\begin{aligned} &\mathrm{E}\Big(\sqrt{\mathrm{X}_n}[\frac{\mathrm{X}_{n+1}}{\mathrm{X}_n} - m] - \frac{m}{\sqrt{\mathrm{X}_n}}\Big|\mathrm{X}_0, ..., \mathrm{X}_n\Big) \\ =&\mathrm{E}\Big(\sum_{i=1}^{\mathrm{X}_n}[Z_i - m] + Y_n|\mathrm{X}_0, ..., \mathrm{X}_n\Big) - \frac{m}{\sqrt{\mathrm{X}_n}} \\ =&\frac{\mathrm{E}(\mathrm{Y}_n)}{\sqrt{\mathrm{X}_n}} - \frac{m}{\sqrt{\mathrm{X}_n}} = 0 \end{aligned} \tag{21}$$

and

$$\mathrm{Var}\Big(\sqrt{\mathrm{X}_n}[\frac{\mathrm{X}_{n+1}}{\mathrm{X}_n} - m] - \frac{m}{\sqrt{\mathrm{X}_n}}\Big) = v^2. \tag{22}$$

Therefore, $\mathrm{H}_{n_0,n}$ is a standardized Martingale, and $(\mathrm{H}_{n_0,n})_{n\geq 1}$ hence represents the normalized process for the estimator, $\hat{m}_n$. □

We further derive the subsequent Cramér moderate deviation result concerning the Martingale, $\mathrm{H}_{n_0,n}$. Our analysis sheds light on the behavior of moderate deviations in relation to this crucial variable, giving valuable perspectives on the underlying stochastic process.

Denote

$$\mathrm{R}_n = \frac{\sqrt{\mathrm{X}_n}}{\mu}\Big(\frac{\mathrm{X}_{n+1}}{\mathrm{X}_n} - m\Big) - \frac{m}{\mu\sqrt{\mathrm{X}_n}}. \tag{23}$$

Theorem 2. *Let* $\mathrm{E}\big(\exp[k_0 Z_1]\big) < \infty$ *for a constant* $k_0 > 0$*. Then,*

$$\Big|\ln\frac{\mathrm{P}(\mathrm{R}_n > x)}{1 - \phi(x)}\Big| = O\Big(\frac{C_1 + C_2 x + C_3 x^2 + x^3}{\sqrt{n}}\Big). \tag{24}$$

Proof. By examining the definition of R_n, it becomes apparent that R_n can be reformulated as follows:

$$\mathrm{R}_n = \frac{\mathrm{X}_{n+1} - m\mathrm{X}_n - m}{\mu\sqrt{\mathrm{X}_n}} = \frac{1}{\mu\sqrt{\mathrm{X}_n}}\Big(\sum_{i=1}^{\mathrm{X}_n}[\mathrm{Z}_i{}^n - m] + \mathrm{Y}_n - m\Big). \tag{25}$$

By the total probability formula, we have

$$\begin{aligned}
P(R_n \geq x) &= \sum_{j=1}^{\infty} P(Y_n = j) P\Big(\frac{1}{\mu\sqrt{X_n}} \sum_{i=1}^{X_n} [Z_i{}^n - m] > x - \frac{j-m}{\mu\sqrt{X_n}}\Big) \\
&= \sum_{k=1}^{\infty} P(X_n = k) \sum_{j=1}^{\infty} P(Y_n = j) P\Big(\frac{1}{\mu\sqrt{k}} \sum_{i=1}^{k} [Z_i{}^n - m] > x - \frac{j-m}{\mu\sqrt{k}}\Big) \\
&=: \sum_{k=1}^{\infty} P(X_n = k) \sum_{j=1}^{\infty} P(Y_n = j) F_k(x - \frac{j-m}{\mu\sqrt{k}})
\end{aligned} \tag{26}$$

According to the results of the Galton–Watson process without immigration, one has

$$F_k(x - \frac{j-m}{\mu\sqrt{k}}) < C_1\Big(1 - \phi(x - \frac{j-m}{\mu\sqrt{k}})\Big) \exp\Big(\frac{1 + [x - \frac{j-m}{\mu\sqrt{k}}]^3}{\sqrt{n}}\Big), \tag{27}$$

$$F_k(x - \frac{j-m}{\mu\sqrt{k}}) > C_2\Big(1 - \phi(x - \frac{j-m}{\mu\sqrt{k}})\Big) \exp\Big(-\frac{1 + [x - \frac{j-m}{\mu\sqrt{k}}]^3}{\sqrt{n}}\Big), \tag{28}$$

This is a consequence of (12):

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}(1 + x - \frac{j-m}{\mu\sqrt{k}})} \exp\Big(\frac{-[x - \frac{j-m}{\mu\sqrt{k}}]^2}{2}\Big) &\leq 1 - \phi(x - \frac{j-m}{\mu\sqrt{k}}) \\
&\leq \frac{1}{\sqrt{\pi}(1 + x - \frac{j-m}{\mu\sqrt{k}})} \exp\Big(\frac{-[x - \frac{j-m}{\mu\sqrt{k}}]^2}{2}\Big)
\end{aligned} \tag{29}$$

Combining (12) and (29), we obtain

$$\begin{aligned}
\frac{1+x}{\sqrt{2}(1 + x - t_1)} \exp(x t_1 - \frac{t_1{}^2}{2}) &< \frac{1 - \phi(x - \frac{j-m}{\mu\sqrt{k}})}{1 - \phi(x)} \\
&< \frac{\sqrt{2}(1+x)}{1 + x - t_1} \exp(x t_1 - \frac{t_1{}^2}{2}), t_1 = \frac{j-m}{\mu\sqrt{k}}.
\end{aligned} \tag{30}$$

By (27) and (30), we have

$$\begin{aligned}
& P\Big(\frac{1}{\mu\sqrt{X_n}} \sum_{i=1}^{X_n} [Z_i{}^n - m] > x - \frac{j-m}{\mu\sqrt{X_n}}\Big) \\
& < C_4'\big(1 - \phi(x)\big) \exp\Big(\frac{1 + [x - t_1]^3}{\sqrt{n}} + x\frac{j-m}{\mu\sqrt{k}} - \frac{[j-m]^2}{2\mu^2 k}\Big) \\
& < C_4\big(1 - \phi(x)\big) \exp\Big(\frac{C_1 + C_2 x + C_3 x^2 + x^3}{\sqrt{n}}\Big)
\end{aligned} \tag{31}$$

and

$$P\Big(\frac{1}{\mu\sqrt{X_n}} \sum_{i=1}^{X_n} [Z_i{}^n - m] > x - \frac{j-m}{\mu\sqrt{X_n}}\Big) > C_5\big(1 - \phi(x)\big) \exp\Big(-\frac{C_1 + C_2 x + C_3 x^2 + x^3}{\sqrt{n}}\Big) \tag{32}$$

Finally, for all $k \geq 1$, we obtain the conclusion. □

Based on the above Theorem 2, we can directly obtain the following conclusion.

Remark 1. *Let* $\mathrm{E}\big(\exp[k_0 \mathrm{Z}_1]\big) < \infty$ *for a constant* $k_0 > 0$*. Then, the following equality*

$$\frac{\mathrm{P}(\mathrm{R}_n > x)}{1 - \phi(x)} = 1 + o(1). \tag{33}$$

naturally holds as $n \to \infty$ *for* $0 \le x = o(n^{\frac{1}{6}})$.

Theorem 3. *Let*

$$\mathrm{E}|\mathrm{Z}_1 + \mathrm{Y}_1 - m|^p \le \frac{1}{2} p!(p-1)^{-\frac{l}{2}} C^{p-2} E(\mathrm{Z}_1 + \mathrm{Y}_1 - m)^2, p \ge 2. \tag{34}$$

hold for some positive constant, c. Then, the following equalities

$$\Big| \ln \frac{\mathrm{P}(\mathrm{H}_{n_0,n} \ge x)}{1 - \phi(x)} \Big| = O\Big(\frac{\mathrm{C}_1 x^2 + x^3}{\sqrt{n}} + [\mathrm{C}_2 + \mathrm{C}_3 x] \frac{\ln n}{\sqrt{n}} \Big) \tag{35}$$

and

$$\Big| \ln \frac{\mathrm{P}(\mathrm{H}_{n_0,n} \le -x)}{1 - \phi(x)} \Big| = O\Big(\frac{\mathrm{C}_1 x^2 + x^3}{\sqrt{n}} + [\mathrm{C}_2 + \mathrm{C}_3 x] \frac{\ln n}{\sqrt{n}} \Big). \tag{36}$$

hold for all $0 \le x = o(\sqrt{n})$ *as* $n \to \infty$.

Proof. Condition (34) is also satisfied by a sub-Gaussian random variable, provided that there exists a positive constant $C_1 > 0$,

$$\begin{aligned} \mathrm{P}(\mathrm{Z}_1 + \mathrm{Y}_1 - m \ge x) &= \mathrm{P}(\mathrm{Z}_1 - m \ge x - \mathrm{Y}_1) \\ &\le \mathrm{C}_1 \exp\Big(- \frac{[x - \mathrm{Y}_1]^2}{\mathrm{C}_1} \Big) \\ &< \mathrm{C}_1 \exp(-\frac{x^2}{\mathrm{C}_1}), \mathrm{Y}_1 > x \end{aligned} \tag{37}$$

Indeed, it is evident that, for all $p \ge 2$,

$$\begin{aligned} \mathrm{E}|(\mathrm{Z}_1 + \mathrm{Y}_1) - m|^p &\le \int_0^\infty p x^{p-1} \mathrm{P}(\mathrm{Z}_1 + \mathrm{Y}_1 - m \ge x) dx + m^p \cdot \mathrm{P}(\mathrm{Z}_1 + \mathrm{Y}_1 - m < 0) \\ &\le \int_0^\infty p x^{p-1} \mathrm{C}_1 \exp(-\frac{x^2}{\mathrm{C}_1}) dx + m^p \\ &= \mathrm{C}_1 (\sqrt{\frac{\mathrm{C}_1}{2}})^{p-1} \int_0^\infty p y^{p-1} \exp(-\frac{y^2}{2}) dy + m^p \\ &= \mathrm{C}_1 (\sqrt{\frac{\mathrm{C}_1}{2}})^{p-1} p!! + m^p \end{aligned} \tag{38}$$

According to Remark 2.1 in [18],

$$\mathrm{E}|\mathrm{Z}_1 + \mathrm{Y}_1 - m|^p \le p!(p-1)^{-\frac{l}{2}} \mathrm{C}_2 (\sqrt{\frac{\mathrm{C}_2}{2}})^{p-1} (p-1)^{\frac{l}{2}} e^{\frac{p+1}{2}} \tag{39}$$

which implies (34) holds for sufficiently large c and $p \ge 3$.

On the other hand, for the case of $p = 2$, condition (34) obviously holds. It is clear from the definition of $H_{n_0,n}$ that it can be expressed as follows:

$$\begin{aligned} H_{n_0,n} &= \frac{1}{\mu\sqrt{n}} \sum_{k=n_0}^{n_0+n-1} \Big(\sqrt{X_k}[\frac{X_{k+1}}{X_k} - m] - \frac{m}{\sqrt{X_k}} \Big) \\ &= \frac{1}{\mu\sqrt{n}} \sum_{k=n_0}^{n_0+n-1} \Big(\sum_{i=1}^{X_k} \frac{{Z_i}^k - m}{\sqrt{X_k}} + \frac{Y_k - m}{\sqrt{X_k}} \Big) \end{aligned} \tag{40}$$

According to the formula for the total probability, we have

$$\begin{aligned} P(H_{n_0,n} \geq x) &= \sum_{j=1}^{\infty} P(Y_n = j) P\Big(\sum_{k=n_0}^{n_0+n-1} \sum_{i=1}^{X_k} \frac{{Z_i}^n - m}{\mu\sqrt{nX_k}} \geq x - \frac{j-m}{\mu\sqrt{nX_k}} \Big) \\ &= \sum_{q=1}^{\infty} P(X_k = q) \sum_{j=1}^{\infty} P(Y_n = j) P\Big(\sum_{k=n_0}^{n_0+n-1} \sum_{i=1}^{X_k} \frac{{Z_i}^n - m}{\mu\sqrt{nq}} \geq x - \frac{j-m}{\mu\sqrt{nq}} \Big) \\ &=: \sum_{q=1}^{\infty} P(X_k = q) \sum_{j=1}^{\infty} P(Y_n = j) \acute{F}_k(x - \frac{j-m}{\mu\sqrt{nq}}) \end{aligned} \tag{41}$$

By Theorem 2.1 in [18], we have

$$F_k(x - \frac{j-m}{\mu\sqrt{nq}}) < C_1\Big(1 - \phi(x - \frac{j-m}{\mu\sqrt{nq}})\Big) \exp\Big(\frac{[x - \frac{j-m}{\mu\sqrt{nq}}]^3}{\sqrt{n}} + [1 + x - \frac{j-m}{\mu\sqrt{nq}}] \frac{\ln n}{\sqrt{n}} \Big), \tag{42}$$

$$F_k(x - \frac{j-m}{\mu\sqrt{nq}}) > C_2\Big(1 - \phi(x - \frac{j-m}{\mu\sqrt{nq}})\Big) \exp\Big(\frac{[x - \frac{j-m}{\mu\sqrt{nq}}]^3}{\sqrt{n}} + [1 + x - \frac{j-m}{\mu\sqrt{nq}}] \frac{\ln n}{\sqrt{n}} \Big), \tag{43}$$

By (12), we have

$$\begin{aligned} \frac{1}{\sqrt{2\pi}(1 + x - \frac{j-m}{\mu\sqrt{nq}})} \exp\Big(\frac{-[x - \frac{j-m}{\mu\sqrt{nq}}]^2}{2} \Big) &\leq 1 - \phi(x - \frac{j-m}{\mu\sqrt{nq}}) \\ &\leq \frac{1}{\sqrt{\pi}(1 + x - \frac{j-m}{\mu\sqrt{nq}})} \exp\Big(\frac{-[x - \frac{j-m}{\mu\sqrt{nq}}]^2}{2} \Big) \end{aligned} \tag{44}$$

Combining (12) and (44), we have

$$\begin{aligned} \frac{1+x}{\sqrt{2}(1 + x - t_2)} \exp\big(xt_2 - \frac{{t_2}^2}{2}\big) &< \frac{1 - \phi(x - \frac{j-m}{\mu\sqrt{nq}})}{1 - \phi(x)} \\ &< \frac{\sqrt{2}(1+x)}{1 + x - t_2} \exp(xt_2 - \frac{{t_2}^2}{2}), t_2 = \frac{j-m}{\mu\sqrt{nq}} \end{aligned} \tag{45}$$

Based on (42) and (45), we have

$$\begin{aligned}
&\mathrm{P}\Big(\frac{1}{\mu\sqrt{n\mathrm{X}_k}}\sum_{i=1}^{\mathrm{X}_k}[\mathrm{Z}_i{}^n-m]>x-\frac{j-m}{\mu\sqrt{n\mathrm{X}_k}}\Big)\\
&<C_4\Big(1-\phi(x)\Big)\exp\Big(\frac{[x-t_2]^3}{\sqrt{n}}+[1+x-t_2]\frac{\ln n}{\sqrt{n}}+xt_2-\frac{t_2{}^2}{2}\Big)\\
&<C_4\Big(1-\phi(x)\Big)\exp\Big(\frac{x^3}{\sqrt{n}}+[1+x]\frac{\ln n}{\sqrt{n}}+x\frac{j-m}{\mu\sqrt{nk}}-\frac{[j-m]^2}{2\mu^2nk}\Big)\\
&<C_4\Big(1-\phi(x)\Big)\exp\Big(\frac{C_1x^2+x^3}{\sqrt{n}}+[C_2+C_3x]\frac{\ln n}{\sqrt{n}}\Big)
\end{aligned} \tag{46}$$

and

$$\mathrm{P}\Big(\frac{1}{\mu\sqrt{n\mathrm{X}_k}}\sum_{i=1}^{\mathrm{X}_n}[\mathrm{Z}_i{}^n-m]\Big)>C_5\Big(1-\phi(x)\Big)\exp\Big(-[\frac{C_1x^2+x^3}{\sqrt{n}}+\{C_2+C_3x\}\frac{\ln n}{\sqrt{n}}]\Big) \tag{47}$$

Finally, for all $k\geq 1$, we have

$$\Big|\ln\frac{\mathrm{P}(\mathrm{H}_{n_0,n}\geq x)}{1-\phi(x)}\Big|=O\Big(\frac{C_1x^2+x^3}{\sqrt{n}}+[C_2+C_3x]\frac{\ln n}{\sqrt{n}}\Big).$$

The proof is complete. □

According to Theorem 3, we can further obtain the following conclusions, and we give the relevant results in the form of remarks.

Remark 2. *Let*

$$\mathrm{E}|\mathrm{Z}_1+\mathrm{Y}_1-m|^p\leq\frac{1}{2}p!(p-1)^{-\frac{l}{2}}C^{p-2}E(Z_1+Y_1-m)^2, p\geq 2.$$

hold for some positive constant, c. Then,

$$\frac{\mathrm{P}(\mathrm{H}_{n_0,n}\geq x)}{1-\phi(x)}=1+O\Big(\frac{C_1x^2+x^3}{\sqrt{n}}+[C_2+C_3x]\frac{\ln n}{\sqrt{n}}\Big) \tag{48}$$

and

$$\frac{\mathrm{P}(\mathrm{H}_{n_0,n}\leq -x)}{1-\phi(x)}=1+O\Big(\frac{C_1x^2+x^3}{\sqrt{n}}+[C_2+C_3x]\frac{\ln n}{\sqrt{n}}\Big) \tag{49}$$

hold for all $0\leq x\leq n^{\frac{1}{6}}$ *and* $n\geq 3$ *.*

In particular, we can also naturally conclude the following result.

Remark 3. *Let*

$$\mathrm{E}|\mathrm{Z}_1+\mathrm{Y}_1-m|^p\leq\frac{1}{2}p!(p-1)^{-\frac{l}{2}}C^{p-2}E(Z_1+Y_1-m)^2, p\geq 2.$$

hold for some positive constant, c. Then,

$$\frac{\mathrm{P}(\mathrm{H}_{n_0,n}\geq x)}{1-\phi(x)}=1+o(1) \tag{50}$$

and

$$\frac{\mathrm{P}(\mathrm{H}_{n_0,n} \leq -x)}{1-\phi(x)} = 1 + o(1) \tag{51}$$

hold for all $0 \leq x = o(n^{\frac{1}{6}})$ *as* $n \to \infty$.

4. Applications

Moderate deviation plays a crucial role in statistics. In this section, we will present an empirical analysis that applies our proposed methodology to statistically significant domains. This will not only demonstrate the practical relevance of the work but also provide a concrete example of its applicability in solving real-world problems.

Cramér moderate deviations can be utilized in constructing confidence intervals for m. Specifically, we can employ the findings from Theorem 3 to establish the following confidence intervals.

Proposition 1. *Let*

$$\mathrm{E}|\mathrm{Z}_1 + \mathrm{Y}_1 - m|^p \leq \frac{1}{2}p!(p-1)^{-\frac{l}{2}}C^{p-2}E(\mathrm{Z}_1 + \mathrm{Y}_1 - m)^2, p \geq 2.$$

hold for some positive constant c and $|\ln c_n| = o(n^{\frac{1}{3}})$. *If* $c_n \in (0,1)$, *then for sufficiently large n,* $[a_n, b_n]$ *is a* $1 - c_n$ *confidence interval for m, where*

$$a_n = \hat{m}_n - \frac{\mu n^{\frac{1}{2}} \phi^{-1}(1 - \frac{c_n}{2})}{\sum_{i=1}^n \sqrt{\mathrm{X}_i}}$$

and

$$a_n = \hat{m}_n + \frac{\mu n^{\frac{1}{2}} \phi^{-1}(1 - \frac{c_n}{2})}{\sum_{i=1}^n \sqrt{\mathrm{X}_i}},$$

where ϕ^{-1} *is the inverse function of* ϕ.

Proof. It follows from Theorem 3 that we can derive the confidence interval for m under the assumptions. □

5. Discussions

For Galton–Watson processes, the generational evolution of population size and the branching structure are the typical topics of discussion. Analyzing the Cramér moderate deviations in these systems is an important area of interest, since it can provide insight into the behavior of extreme population growth variations and small probability events.

Different from previous research, in this study, we introduced immigration into the classical Galton–Watson process. The inclusion of immigration introduces additional complexity to the population dynamics, potentially impacting the growth rate and offspring distribution. Such changes can lead to differences in the occurrence and magnitude of moderate deviations, influencing the overall stability and variability of the population sizes.

Furthermore, exploring the Cramér moderate deviations of the GWIP provides valuable insights into the role of external factors, such as immigration, in shaping the population dynamics. By examining moderate deviations in the branching processes with immigration, researchers can obtain a deeper understanding of how different mechanisms influence the long-term behavior and resilience of populations.

In conclusion, the investigation of Cramér moderate deviations for the GWIP provides an interesting perspective on the interplay between intrinsic reproduction dynamics and external influences. This analysis not only enriches our theoretical understanding of population processes but also has practical implications for modeling and predicting population behaviors in various biological and ecological systems.

Author Contributions: Conceptualization, J.W. and C.P.; writing—original draft preparation, C.P.; writing—review and editing, J.W.; funding acquisition, J.W. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Sciences Foundation of China (No. 11901392).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The authors thank the reviewers for their professional review work, constructive comments and valuable suggestions for our manuscript.

Conflicts of Interest: The authors declare no conflicts of interest.

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