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Photon-Added Deformed Peremolov Coherent States and Quantum Entanglement

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Abstract: In the present article, we build the excitedcoherent states associated with deformed $su(1,1)$ algebra (DSUA), called photon-added deformed Perelomov coherent states (PA-DPCSs). The constructed coherent states are obtained by using an alterationof the Holstein–Primakoff realization (HPR) for DSUA. A general method to resolve of the problem of the unitary operator was developed for these kinds of quantum states. The Mandel parameter is considered to examine the statistical properties of PA-DPCSs. Furthermore, we offer a physical method to generate the PA-DPCSs in the framework of interaction among fields and atoms. Finally, we introduce the concept of entangled states for PA-DPCSs and examine the entanglement properties for entangled PA-DPCSs.

Keywords: photon-added states; quantum groups; lie algebra; statistical properties; atom–field interaction; quantum correlation

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1. Introduction

The investigation of coherent states (C-Ss) and their uses in several physics fields have been the subject of several works over the last four decades. These states have significant applications in quantum optics, statistical mechanics, nuclear physics, and condensed matter physics [1–3]. Schrödinger was the first to introduce the concept of coherent states for the harmonic oscillator system. His goal was to obtain quantum states that show an intimate close link between quantum and classical descriptions [4]. In quantum optics, Glauber introduced the importance of C-Ss as eigenstates of the lowering operator [5,6], and he figured out that these C-Ss have the property of minimizing the uncertainty in both position and momentum. Klauder reintroduced the identical states in [7,8]. The similarity here was that these C-Ss were the specific states of the system of the harmonic oscillator. Due to their significant properties, these C-Ss were then generalized from either a physical or mathematical standpoint to other systems. For a review of all these generalizations, see Refs. [9–11].

Perelomov [12] and Gilmore [13] independently realized a set of C-Ss related to any Lie groups. These states are referred to as $SU(1,1)$ C-Ss, which are related to the $SU(1,1)$ group. They characterize several quantum systems and have many intriguing applications in condensed matter physics, statistical mechanics, and quantum optics [9–11]. On another side, the establishment of quantum groups served as a formal characterization of deformed Lie algebras [14,15], which allowed us to build deformed C-Ss that were considered an extension of the concept of CSs. Then, generalized deformations of Glauber states are constructed as C-Ss associated with deformed oscillators. Additionally, deformed $SU(1,1)$ C-Ss were built as C-Ss associated with the DSUA [16,17]. Mathematically, Klauder [7,8] notes that the minimal set of conditions necessary to construct C-Ss includes normalization condition, continuity in labels, and the presence of a positive weighting function to deal with the resolution of unity.

The primary feature of quantum states is the quantum entanglement that is not present in classical physics. As noticed by Schr̆odinger [18], this is a characteristic feature of quantum mechanics, one that forces it to move completely away from classical lines of thought. The confusion of Einstein, Podolsky, and Rosen about quantum entanglement led them to suggest an alternative theory of quantum mechanics that was eventually disproved by a theory devised by Bell and confirmed experimentally by what we call the Bell inequality test [19–23]. The pioneer's workon understanding entanglement led to the emergence of quantum information science with the main goal of finding methods for exploiting quantum mechanical effects in nature, such as the superposition principal and entanglement, to perform tasks of information processing that would not be possible in the traditional (classical) world. Due to the brilliant work developed by Bennett et al., we now understand the role of entanglement in the development of numerous quantum protocols such as dense coding [24], teleportation [25], and quantum key distribution [26]. Using the fundamentals of quantum mechanics and entanglement as a tool has led to the development of information-processing tasks. Recently, there is also a link between quantum entanglement and other disciplines, including high-energy and condensed matter physics, where the entanglement phenomenon can be viewed as a clue about quantum time and space [27] and the signature of quantum phase transition and quantum orders [28].

Agarwal and Tara gave the first description of the PA-CSs for the harmonic oscillator [29]. These quantum states attracted a great deal of attention and provided several physical applications [30–32]. In light of their possible uses, several generalizations have been proposed [33–39]. Thus, these kinds of CSs could be useful. In the present article, we shall build the excitedcoherent states associated with DSUA. The constructed coherent states are obtained by using an alterationof HPR for DSUA. A general way for the resolution of the problem of the unitary operator will be developed for this kind of quantum state. The Mandel parameter will be considered to examine the physical properties of PA-DPCSs. Furthermore, we shall propose a way to produce the PA-DPCSs by considering the interaction among fields and atoms. Finally, we shall introduce the concept of entangled states in the context of PA-DPCSs and examine the entanglement properties for entangled PA-DPCSs.

The manuscript is outlined as follows. In Section 1, we introduce the scheme for constructing the PA-DPCSs in the framework of DSUA and examine the non-classicality of these states by evaluating the Mandel parameter. In Section 2, we propose a physical way to generate the PA-DPCSs. Section 3 is devoted to studying the bipartite entanglement of PA-DPCSs. A summary is provided in the last section.

2. PA-DPCSs and Statistical Properties

The DSUA is characterized by the commutation relations of the generators $\hat{\mathbb{K}}_z^\kappa$ and $\hat{\mathbb{K}}_\pm^\kappa$

$$[\hat{\mathbb{K}}_z^\kappa, \hat{\mathbb{K}}_\pm^\kappa] = \pm\hat{\mathbb{K}}_\pm^\kappa \quad ; \quad [\hat{\mathbb{K}}_+^\kappa, \hat{\mathbb{K}}_-^\kappa] = [2\hat{\mathbb{K}}_z^\kappa]\,, \tag{1}$$

where $\hat{\mathbb{K}}_z^\kappa$ and $\hat{\mathbb{K}}_\pm^\kappa$ satisfy the Hermiticity restrictions

$$(\hat{\mathbb{K}}_z^\kappa)^\dagger = \hat{\mathbb{K}}_z^\kappa \qquad ; \qquad (\hat{\mathbb{K}}_+^\kappa)^\dagger = \hat{\mathbb{K}}_-^\kappa \qquad ; \qquad (\hat{\mathbb{K}}_-^\kappa)^\dagger = \hat{\mathbb{K}}_+^\kappa. \tag{2}$$

The function [ ] in Equation (1) defines the deformation of DSUA. The choice of particular forms of [ ] leads to a specific deformation of the DSUA. When $[\mathcal{Y}] = \mathcal{Y}$ we obtain the undeformed SUA. In this manuscript, the following box function [40–42]

$$[\mathcal{Y}] = \frac{\kappa^{\mathcal{Y}} - 1}{\kappa - 1}, \tag{3}$$

represents the standard deformation of the SUA. Here, κ is the deformed parameter and considered to be real.

The unitary irreducible representation of the DSUA is obtained through the unitary representation [43] of the undeformed SUA as

$$\hat{\mathbb{K}}^{\kappa}_{z}|k,m\rangle = m|k,m\rangle \ ,$$
$$\hat{\mathbb{K}}^{\kappa}_{\pm}|k,m\rangle = ([m \mp k \pm 1][m \pm k])^{\frac{1}{2}}|k,m \pm 1\rangle. \tag{4}$$

Here, the vector $|k,m\rangle$ defines an orthonormal basis of the irreducible representation space for $k \in \{\frac{1}{2}, 1, \frac{3}{2}, \cdots\}$, which is the Bargman index that labels the representation and $m \in \{k, k+1, k+2, \cdots\}$.

We should write the basis vectors $|j,m\rangle$ in terms of the Fock states ($|k,m\rangle \sim |n\rangle$) in order to examine the photons statistics of PA-DPCSs. We use an alteration

$$\hat{\mathbb{K}}^{\kappa}_{+} = \hat{A}^{\kappa}_{\dagger}\sqrt{[2k+N]} \ , \ \hat{\mathbb{K}}^{\kappa}_{-} = \sqrt{[2k+N]} \ \hat{A}^{\kappa} \ , \ \hat{\mathbb{K}}^{\kappa}_{z} = N + k \ . \tag{5}$$

Here, $\hat{A}^{\kappa}$, $\hat{A}^{\kappa}_{+}$ represent deformed annihilation and creation operators that act on $|n\rangle$ as

$$\hat{A}^{\kappa}_{+}|n\rangle = \sqrt{[n+1]} \ |n+1\rangle \quad , \quad \hat{A}^{\kappa}|n\rangle = \sqrt{[n]} \ |n-1\rangle. \tag{6}$$

According to Klauder's work [7,8], the following minimum requirements must be met in order to obtain coherent states:

(a) Normalization condition

$$\langle\zeta|\zeta\rangle = 1, \tag{7}$$

where ζ represents the amplitude of the coherent state.

(b) Continuity property

$$\Big||\zeta\rangle - |\zeta^{'}\rangle\Big| \longrightarrow 0 \text{ when } |\zeta - \zeta^{'}|^{2} \longrightarrow 0. \tag{8}$$

(c) Resolution of the unity operator

$$\int\int d\mu(\zeta)|\zeta\rangle\langle\zeta| = I, \tag{9}$$

where $d\mu(\zeta)$ represents the measure in the label space.

As we will show in this manuscript, the last requirement is unquestionably the most significant and restricted one. It is challenging to determine if a resolution relation exists, and for a wide class of coherent states, this problem has not yet been solved. Here, we give a general scheme for constructing PA-DPCSs by discussing the minimum requirements needed to build Klauder's coherent states.

The PCSs related to DSUA are defined by the formula

$$|\zeta,k\rangle = \mathcal{N}(|\zeta|^{2})\mathbf{E}_{\kappa}(\zeta\hat{\mathbb{K}}^{\kappa}_{+})|k,k\rangle, \tag{10}$$

where $\mathbf{E}_{\kappa}(x)$ represents a deformation of the ordinary exponential function

$$\mathbf{E}_{\kappa}(x) = \sum_{n=0}^{\infty}\frac{x^{n}}{[n]!} \quad \text{with} \quad [n]! = [1]\ldots[n-1][n]. \tag{11}$$

The condition of normalization requires that

$$\mathcal{N}(|\zeta|^{2}) = \left(\sqrt{(1-|\zeta|^{2})^{(-2k)}}\right)^{-1}, \tag{12}$$

where

$$(a+b)^{(m)} := \sum_{n=0}^{m}\begin{bmatrix} m \\ n \end{bmatrix} a^{m-n}b^{n} \ , \tag{13}$$

with the deformed binomial formula

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{[m]!}{[m]![m-n]!} \qquad \text{for} \qquad m \geq n\,. \tag{14}$$

The formula defined by Equation (13) is already considered by mathematicians [17,44]. The DPCS can be given by the formula

$$|\zeta,k\rangle = \left(\sqrt{(1-|\zeta|^2)^{-(2k)}}\right)^{-1} \sum_{m=k}^{\infty} \sqrt{\begin{bmatrix} m+k-1 \\ m-k \end{bmatrix}} \zeta^{m-k} |k,m\rangle. \tag{15}$$

Using the HPR, the DPCSs can be re-written as a function of $|n\rangle$ as

$$|\zeta,k\rangle = \left(\sqrt{(1-|\zeta|^2)^{-(2k)}}\right)^{-1} \sum_{n=0}^{\infty} \sqrt{\begin{bmatrix} 2k+n-1 \\ n \end{bmatrix}} \zeta^{n} |n\rangle. \tag{16}$$

The PA-DPCSs $|\xi,k,\boldsymbol{m}\rangle$ can be built by repeated application of the operator $\hat{\mathbb{K}}_+^\kappa$ to DPCSs $|\xi,k\rangle$

$$|\zeta,k,\boldsymbol{m}\rangle = N(|\zeta|^2)(\hat{\mathbb{K}}_+^\kappa)^{\boldsymbol{m}}|\zeta,k\rangle \tag{17}$$

where $\boldsymbol{m}$ is a nonegative integer that defines the number of added photons. By using Equation (4), we obtain

$$(\hat{\mathbb{K}}_+^\kappa)^{\boldsymbol{m}}|n\rangle = \left[\frac{[n+\boldsymbol{m}]![n+2k+\boldsymbol{m}-1]!}{[n]![n+2k-1]!}\right]^{\frac{1}{2}} |n+\boldsymbol{m}\rangle. \tag{18}$$

Substituting (18) into (17), we find that

$$|\zeta,k,\boldsymbol{m}\rangle = N(|\zeta|^2) \sum_{n=0}^{\infty} \left[\frac{[n+\boldsymbol{m}]![n+2k+\boldsymbol{m}-1]!}{([n]!)^2[2k-1]!}\right]^{\frac{1}{2}} \zeta^n |n+\boldsymbol{m}\rangle, \tag{19}$$

where the normalization function

$$N(|\zeta|^2) = \left(\sum_{n=0}^{\infty} \left[\frac{[n+\boldsymbol{m}]![n+2k+\boldsymbol{m}-1]!}{([n]!)^2[2k-1]!}\right] |\zeta|^{2n}\right)^{-\frac{1}{2}}. \tag{20}$$

In the $\boldsymbol{m} \to 0$ limit, we can recover the DPCSs $|\xi,k\rangle$.

As of right now, it appears that the selected states are essentially unrestricted by the requirements (a) and (b). We investigate the condition on the measure $d\mu = \mathbb{W}d^2\zeta$ in (9), where for $d^2\zeta = d(Re[\zeta])d(Im[\zeta])$ the following equations should be satisfied

$$\int\int_{\mathcal{C}} d^2\zeta |\zeta,k,\boldsymbol{m}\rangle\langle\zeta,k,\boldsymbol{m}|\mathbb{W}\left(|\zeta|^2\right) = \sum_n |n\rangle\langle n|. \tag{21}$$

Concurrent with our construction of the PA-DPCSs in Equation (19), we introduce the polar decomposition $\zeta = re^{i\theta}$ to obtain

$$\int_0^{+\infty} x^n N^2(x)\mathbb{W}(x)dx = \frac{1}{\pi}\left(\left[\frac{[n+\boldsymbol{m}]![n+2k+\boldsymbol{m}-1]!}{([n]!)^2[2k-1]!}\right]\right)^{-1}, \tag{22}$$

Instead of solving Equation (22) for $W(x)$, with $x = r^2$, we examine the existence of solution to integrable equations given by

$$\int_0^{+\infty} x^n \widehat{\mathbb{W}}(x)dx = \xi(n) \tag{23}$$

where

$$\widehat{\mathbb{W}}(x) = N^2(x)\mathbb{W}(x),\ \xi(n) = \frac{1}{\pi}\left(\left[\frac{[n+\boldsymbol{m}]![n+2k+\boldsymbol{m}-1]!}{([n]!)^2[2k-1]!}\right]\right)^{-1}.$$

The well-known Stieltjes moment problem is represented by Equation (23).

To solve the integral equation, we consider similar approach as used in [45] by applying the concept of Fourier transforms. We multiply Equation (23) by $((iy)^n/n!)$ and perform a sum over the integer n

$$\int_0^{+\infty} \widehat{\mathbb{W}}(x)\, e^{ixy}\, dx = \sum_{n=0}^{\infty}\left(\frac{(iy)^n}{n!}\right)\xi(n) = \overline{\mathbb{W}}(y). \tag{24}$$

It is clear that the series on the right-hand side must converge. This is the case when the parameters of the standard box function are real. Thus, in this case we can obtain the inverse Fourier transform of $\overline{\mathbb{W}}(y)$

$$\widehat{\mathbb{W}}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{\mathbb{W}}(y)\, e^{-ixy}\, dy. \tag{25}$$

The function $\mathbb{W}(x)$ permitting the resolution of the identity operator is given by

$$\mathbb{W}(x) = \frac{N^{-2}(x)}{2\pi}\int_{-\infty}^{+\infty} \overline{\mathbb{W}}(y)\, e^{-ixy}\, dy. \tag{26}$$

This fulfills the Klauder requirements that must be met by the PA-DPCSs presented in this work in order for them to qualify as coherent states.

For examining what is going on and obtaining a handle on the physical nature of the PA-DPCSs, the HPR of the DSUA should be used to compare the obtained states with Glauber's coherent states. To do this, the Mandel parameter, $\mathbb{M}_{\mathbb{P}}$, is used as a measure for deciding whether the probability distribution is sub-Poissonian, Poissonian, or super-Poissonian. The parameter $\mathbb{M}_{\mathbb{P}}$ is defined by the formula [46]

$$\mathbb{M}_{\mathbb{P}} = \frac{\langle(\Delta\hat{\mathbb{N}})^2\rangle - \langle\hat{\mathbb{N}}\rangle}{\langle\hat{\mathbb{N}}\rangle}. \tag{27}$$

Here, $\langle\hat{\mathbb{N}}\rangle$ represents the average number of photons in the PA-DPCS $|z,k,\boldsymbol{m}\rangle$ and $\langle(\Delta\hat{\mathbb{N}})^2\rangle = \langle\hat{\mathbb{N}}^2\rangle - \langle\hat{\mathbb{N}}\rangle^2$ defines the standard deviation such that

$$\langle N\rangle = N^2(|\zeta|^2)\sum_{n=\boldsymbol{m}}^{\infty}\left[\frac{[n][n]![n+2k-1]!}{([n-\boldsymbol{m}]!)^2[2k-1]!}\right]|\zeta|^{2(n-\boldsymbol{m})}, \tag{28}$$

and

$$\langle N^2\rangle = N^2(|\zeta|^2)\sum_{n=\boldsymbol{m}}^{\infty}\left[\frac{[n]^2[n]![n+2k-1]!}{([n-\boldsymbol{m}]!)^2[2k-1]!}\right]|\zeta|^{2(n-\boldsymbol{m})}. \tag{29}$$

The probability distribution is said to be sub-Poissonian if $-1 \le \mathbb{M}_{\mathbb{P}} < 0$, Poissonian if $\mathbb{M}_{\mathbb{P}} = 0$, and super-Poissonian if $\mathbb{M}_{\mathbb{P}} > 0$.

In Figures 1 and 2, we display the Mandel $\mathbb{M}_{\mathbb{P}}$ parameter as a function of $|\zeta|$ considering different values of the parameters $\boldsymbol{m}$ and κ with $k = 5$ and $k = 10$. From the figures, it can be seen that an increase in the value of $\boldsymbol{m}$ leads to a decrease in the value of the parameter $\mathbb{M}_{\mathbb{P}}$. We can observe that the Mandel parameter $\mathbb{M}_{\mathbb{P}}$ can obtain positive and negative values when $\kappa \to 1$ indicates super-Poissonian and sub-Poissonian distributions of photons that depend on the values of $|\zeta|$. When κ gets away from 1, the parameter $\mathbb{M}_{\mathbb{P}}$ has a negative value, indicating a sub-Poissonian distribution of photons for different values of $\boldsymbol{m}$. Furthermore, the increase of k leads to a decrease in the value of $\mathbb{M}_{\mathbb{P}}$ with respect to the C-Ss parameters.

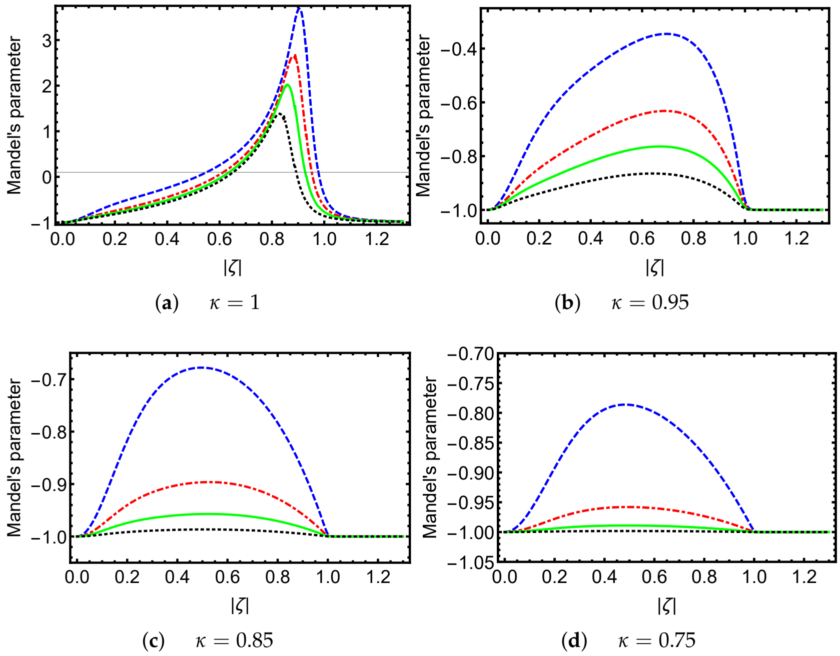


Figure 1. $\mathbb{M}_{\mathbb{P}}$ of the PA-DPCSs as function of $|\zeta|$ for different values of $\boldsymbol{m}$ and κ with the Bargman index $k = 5$. Labels (**a**–**d**) are considered for $\kappa = 1$, $\kappa = 0.95$, $\kappa = 0.85$, and $\kappa = 0.75$, respectively. Blue (dashed curve): PA-DPCSs for $\boldsymbol{m} = 1$; red (dashed-dotted curve): PA-DPCSs for $\boldsymbol{m} = 3$; green (solid curve): PA-DPCSs for $\boldsymbol{m} = 5$; black (dotted curve): PA-DPCSs for $\boldsymbol{m} = 8$. When $\boldsymbol{m}$ increases and κ gets away from 1, the parameter $\mathbb{M}_{\mathbb{P}}$ has a negative value and sub-Poissonian distribution, which results in an enhancement in the non-classicality for the PA-DPCSs.

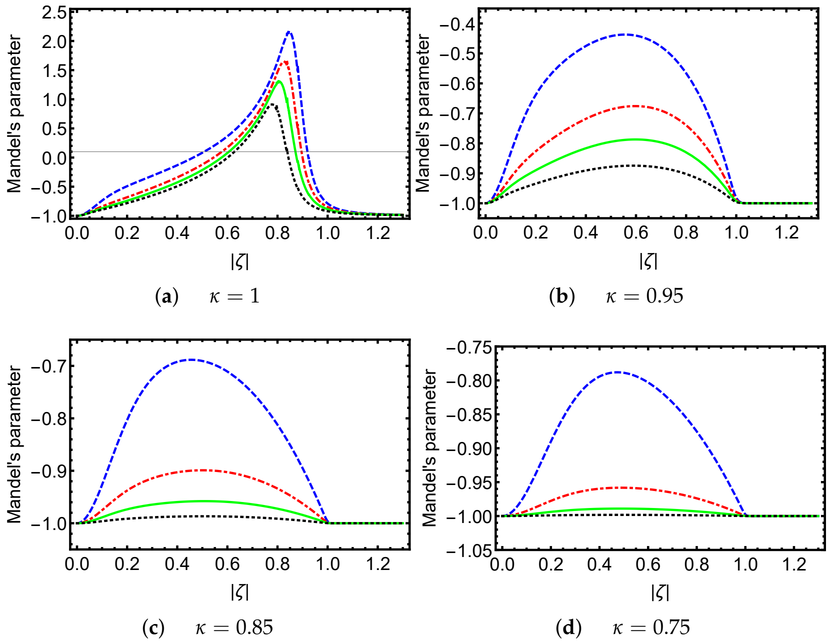


Figure 2. $\mathbb{M}_{\mathbb{P}}$ of the PA-DPCSs as function of $|\zeta|$ for different values of $\boldsymbol{m}$ and κ with the Bargman index $k = 10$. Labels (**a**–**d**) are considered for $\kappa = 1$, $\kappa = 0.95$, $\kappa = 0.85$, and $\kappa = 0.75$, respectively. Blue (dashed curve): PA-DPCSs for $\boldsymbol{m} = 1$; red (dashed-dotted curve): PA-DPCSs for $\boldsymbol{m} = 3$; green (solid curve): PA-DPCSs for $\boldsymbol{m} = 5$; black (dotted curve): PA-DPCSs for $\boldsymbol{m} = 8$. Generally, the increase of k leads to a decreased $\mathbb{M}_{\mathbb{P}}$ and then results in an enhancement of the non-classicality for the PA-DPCSs.

3. Generation of the PA-DPCS

For physically generating the PA-DPCS in (19), we consider a slab of excited atoms with two levels through a cavity. Let $|\Psi(0)\rangle = |\zeta, k, \boldsymbol{m}\rangle|e\rangle$ represent the atom–field state at the instant $t = 0$, where the state $|e\rangle$ describes the excited state of an atom. The configuration of the Hamiltonian of an interaction is

$$\mathbb{H} = \hbar\gamma\left(\sigma_+\hat{\mathbb{K}}_-^\kappa + \hat{\mathbb{K}}_+^\kappa\sigma_-\right). \tag{30}$$

Here, $\sigma_+ = |e\rangle\langle g|$ and $\sigma_- = |g\rangle\langle e|$ are the standard atomic two-level transition operators acting on the excited state $|e\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and ground state $|g\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as follows: $\sigma_+|g\rangle = |e\rangle$ and $\sigma_-|e\rangle = |g\rangle$.

The state $|\Psi(0)\rangle$ at subsequent times is

$$|\Psi(t)\rangle = \exp\left[-i\eta\left(\sigma_+\hat{\mathbb{K}}_-^\kappa + \hat{\mathbb{K}}_+^\kappa\sigma_-\right)\right]|\Psi(0)\rangle, \tag{31}$$

where we have set $\eta = \hbar\gamma$, which with γ denotes the coupling constant. For $\eta \ll 1$, we obtain

$$|\Psi(t)\rangle \cong \left(1 - i\eta\left(\sigma_+\hat{\mathbb{K}}_-^\kappa + \hat{\mathbb{K}}_+^\kappa\sigma_-\right)\right)|\zeta, k, \boldsymbol{m}\rangle|e\rangle. \tag{32}$$

The atom–field state can be given as

$$|\Psi(t)\rangle = |\zeta, k, \boldsymbol{m}\rangle|e\rangle - i\eta\hat{\mathbb{K}}_+^\kappa|\zeta, k, \boldsymbol{m}\rangle|g\rangle. \tag{33}$$

Then, the field state can be transferred to $\hat{\mathbb{K}}_+^\kappa|\zeta, k, \boldsymbol{m}\rangle$, with the atom in the ground state $|g\rangle$, which corresponds to the PA-DPCS $|\zeta, k, 1\rangle$ and leads to generating the state $|\zeta, k, 1\rangle$. Based on the above method, we can produce the PA-DPCSs with different values of $\boldsymbol{m}$ via the Hamiltonian operator

$$\mathbb{H}_{\boldsymbol{m}} = \hbar\lambda\left(\sigma_+\left(\hat{\mathbb{K}}_-^\kappa\right)^{\boldsymbol{m}} + \left(\hat{\mathbb{K}}_+^\kappa\right)^{\boldsymbol{m}}\sigma_-\right). \tag{34}$$

4. Entangled PA-DPCSs

In this section, we introduce sufficient conditions allowing us to test whether the bipartite PA-PSCSs are entangled, and we analyze the amount of entanglement of the entangled PA-PSCSs in terms of k, $\boldsymbol{m}$, κ, and $|\zeta|$.

Two inequalities have been presented recently for the detection of entanglement [47]. Let $\mathbb{H} = \mathbb{H}_{\mathbb{A}} \otimes \mathbb{H}_{\mathbb{B}}$ be a ket space for a bipartite system $\mathbb{AB}$. Here, $\mathbb{H}_{\mathbb{A}}$ represents the ket space for system $\mathbb{A}$ and $\mathbb{H}_{\mathbb{B}}$ is for system $\mathbb{B}$. Let $\hat{\mathbb{A}}$ and $\hat{\mathbb{B}}$ be two operators acting on the ket spaces $\mathbb{H}_{\mathbb{A}}$ and $\mathbb{H}_{\mathbb{B}}$, respectively. The quantum state of system $\mathbb{AB}$ is entangled if

$$\begin{aligned} |\langle\hat{\mathbb{A}}\hat{\mathbb{B}}^\dagger\rangle|^2 &> \langle\hat{\mathbb{A}}^\dagger\hat{\mathbb{A}}\hat{\mathbb{B}}^\dagger\hat{\mathbb{B}}\rangle \\ |\langle\hat{\mathbb{A}}\hat{\mathbb{B}}\rangle|^2 &> \langle\hat{\mathbb{A}}^\dagger\hat{\mathbb{A}}\rangle\langle\hat{\mathbb{B}}^\dagger\hat{\mathbb{B}}\rangle. \end{aligned} \tag{35}$$

Let us now exploit the implications of these inequalities for the bipartite PA-DPCSs, where each subsystem is described by a PA-DPCS as

$$|\phi\rangle = \mathbb{C}|\zeta_1, k_1, \boldsymbol{m}\rangle \otimes |\zeta_2, k_2, \boldsymbol{m}\rangle + \mathbb{D}|\zeta_1', k_1, \boldsymbol{m}\rangle \otimes |\zeta_2', k_2, \boldsymbol{m}\rangle. \tag{36}$$

In terms of $|n_i\rangle$ (i = 1, 2), we have

$$|\phi\rangle = \sum_{n_1=\boldsymbol{m}}^{\infty}\sum_{n_2=\boldsymbol{m}}^{\infty} \mathcal{B}_{n_1,n_2}^{k_1,k_2}|n_1\rangle \otimes |n_2\rangle, \tag{37}$$

where

$$\mathcal{B}^{k_1,k_2}_{n_1,n_2} = \left[\mathbb{CN}(|\zeta_1|^2)N(|\zeta_2|^2)\zeta_1^{n_1-m}\zeta_2^{n_2-m} + \mathbb{DN}(|\zeta_1'|^2)N(|\zeta_2'|^2)\zeta_1'^{\,n_1-m}\zeta_2'^{\,n_2-m}\right]\left(\frac{[n_1]![n_2]![n_1+2k_1-1]![n_2+2k_2-1]!}{([n_1-m]![n_2-m]!)^2[2k_1-1]![2k_2-1]!}\right)^{\frac{1}{2}}. \quad (38)$$

Let $\hat{\mathbb{K}}^{\kappa}_{1-}$ and $\hat{\mathbb{K}}^{\kappa}_{2-}$ be the lowering operators of the $su_{\kappa_1}(1,1)$ and $su_{\kappa_2}(1,1)$ quantum algebras, respectively, with raising operators $\hat{\mathbb{K}}^{\kappa}_{1+}$ and $\hat{\mathbb{K}}^{\kappa}_{2+}$. In this case, the conditions for entanglement are given

$$|\langle\hat{\mathbb{K}}^{\kappa}_{1-}\hat{\mathbb{K}}^{\kappa}_{2+}\rangle|^2 \;>\; \langle\hat{\mathbb{K}}^{\kappa}_{1+}\hat{\mathbb{K}}^{\kappa}_{1-}\hat{\mathbb{K}}^{\kappa}_{2+}\hat{\mathbb{K}}^{\kappa}_{2-}\rangle \quad (39)$$

$$|\langle\hat{\mathbb{K}}^{\kappa}_{1-}\hat{\mathbb{K}}^{\kappa}_{2-}\rangle|^2 \;>\; \langle\hat{\mathbb{K}}^{\kappa}_{1+}\hat{\mathbb{K}}^{\kappa}_{1-}\rangle\langle\hat{\mathbb{K}}^{\kappa}_{2+}\hat{\mathbb{K}}^{\kappa}_{2-}\rangle. \quad (40)$$

Using (37), we obtain that

$$\langle\phi|\hat{\mathbb{K}}^{\kappa}_{1-}\hat{\mathbb{K}}^{\kappa}_{2-}|\phi\rangle = \sum_{n_1=m}^{\infty}\sum_{n_2=m}^{\infty}\mathcal{B}^{*k_1,k_2}_{n_1,n_2}\mathcal{B}^{k_1,k_2}_{n_1+1,n_2+1}([n_1+1][n_2+1][2k_1+n_1][2k_2+n_2])^{\frac{1}{2}} \quad (41)$$

$$\langle\phi|\hat{K}^{\kappa}_{1^+}\hat{K}^{\kappa}_{1-}|\phi\rangle = \sum_{n_1=m}^{\infty}\sum_{n_2=m}^{\infty}|\mathcal{B}^{k_1,k_2}_{n_1,n_2}|^2[n_1][2k_1+n_1-1] \quad (42)$$

and

$$\langle\phi|\hat{K}^{\kappa}_{2^+}\hat{K}^{\kappa}_{2-}|\phi\rangle = \sum_{n_1=m}^{\infty}\sum_{n_2=m}^{\infty}|\mathcal{B}^{k_1,k_2}_{n_1,n_2}|^2[n_2][2k_2+n_2-1]. \quad (43)$$

The entanglement condition is

$$|\langle\phi|\hat{K}^{\kappa}_{1-}\hat{K}^{\kappa}_{2-}|\phi\rangle|^2 > \langle\phi|\hat{K}^{\kappa}_{1+}\hat{K}^{\kappa}_{1-}|\phi\rangle\langle\phi|\hat{K}^{\kappa}_{2+}\hat{K}^{\kappa}_{2-}|\phi\rangle. \quad (44)$$

We note that the quantum state (36) is separable if one of the following conditions is verified: $\mathbb{C}=0$, $\mathbb{D}=0$, $|\zeta_1,k_1,m\rangle = \pm e^{i\theta_1}|\zeta_1',k_1,m\rangle$, or $|\zeta_2,k_2,m\rangle = \pm e^{i\theta_2}|\zeta_2',k_2,m\rangle$.

Let us examine the degree of quantum entanglement for entangled PA-DPCSs introduced as

$$\begin{aligned}|\Phi\rangle \;&=\; \mathcal{N}_\theta\Big(|\zeta_1,k_1,m\rangle\otimes|-\zeta_2,k_2,m\rangle \\ &+\; e^{i\theta}|-\zeta_1,k_1,m\rangle\otimes|\zeta_2,k_2,m\rangle\Big).\end{aligned} \quad (45)$$

The ket states $|\zeta_1,k_1,m\rangle$ and $|-\zeta_1,k_1,m\rangle$ described the first subsystem, and $|\zeta_2,k_2,m\rangle$ and $|-\zeta_2,k_2,m\rangle$ described the second subsystem, spanning two dimensional subspaces considered to be linearly independent.The normalization factor $\mathcal{N}_\theta$ should satisfy

$$\mathcal{N}_\theta = [2+2\cos\theta\mu_1\mu_2]^{-\frac{1}{2}}. \quad (46)$$

where

$$\mu_1 = \langle\zeta_1,k_1,m|-\zeta_1,k_1,m\rangle, \quad \mu_2 = \langle-\zeta_2,k_2,m|\zeta_2,k_2,m\rangle. \quad (47)$$

The state (45) describes a bipartite system in the context of $su_\kappa(1,1)$ algebra.

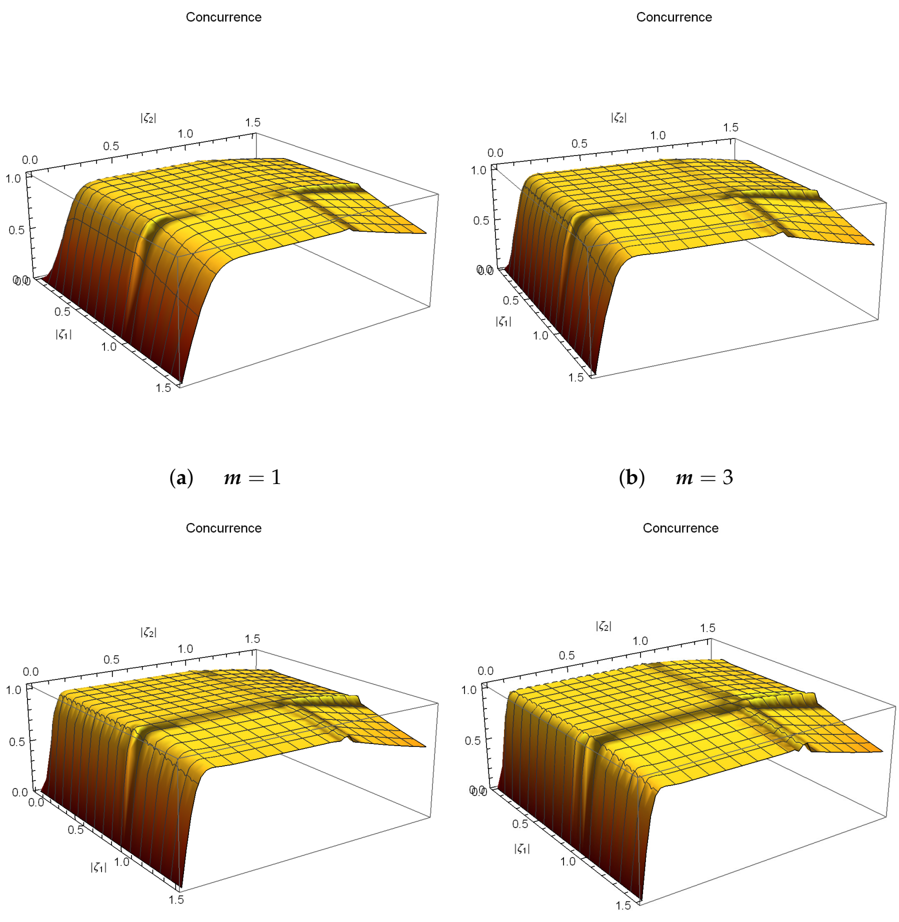
Various measures of bipartite quantum states have been considered in the literature. In the present manuscript, the concurrence is used to examine the degree of entanglement of PA-DPCSs. It is defined for state $|\Phi\rangle$ by Formula [48]

$$C(|\Phi\rangle) = |\langle\Phi|\tilde{\Phi}\rangle|, \quad (48)$$

where the tilde represents the operation $|\tilde{\Phi}\rangle = \sigma_y\otimes\sigma_y|\Phi^*\rangle$. $|\Phi^*\rangle$ denotes the complex conjugate of $|\Phi\rangle$ and σ_y is the pauli y-operator $\left[i\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right]$. The concurrence C ranges from the value zero for a separable state to the value one for a maximally entangled state ($C=1$). The two conditions for which the states (45) become maximally entangled are

(1) $\mu_1 \neq 0$ and $\mu_2 \neq 0$, $C = 1$ as $\mu_1 = \mu_2$ and $\theta = \pi$; (2) $\mu_1 = \mu_2 = 0$, $C = 1$ for different values of θ.

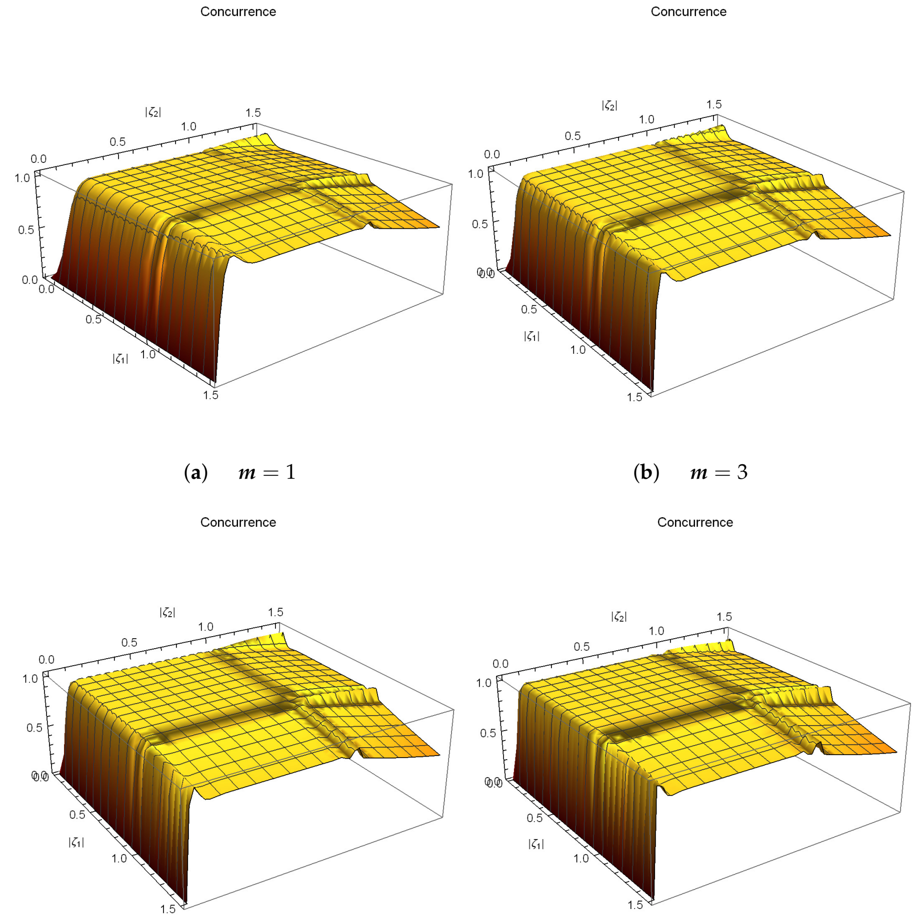
In Figures 3 and 4, we plot the concurrence of the entangled PA-DPCSs as a function of $|\zeta_1|$ and $|\zeta_2|$ for $k = 5$ and $k = 10$, respectively, considering various values of $\boldsymbol{m}$ with $\kappa = 1$. Labels (a), (b), (c), and (d) are considered for $\boldsymbol{m} = 1$, $\boldsymbol{m} = 3$, $\boldsymbol{m} = 5$, and $\boldsymbol{m} = 8$, respectively. From the figures, it can be seen that function C depends on the parameters $\boldsymbol{m}$ according to the change $|\zeta_1|$ and $|\zeta_2|$. Interestingly, large values of function C are obtained for small values of the amplitude ζ_1 and ζ_2 for which $C = 1$. On the other hand, the measure of entanglement C is not largely affected by the increase of the Bargman parameters k_1 and k_2. In Figures 5 and 6, we plot the concurrence of the entangled PA-DPCSs in terms of $|\zeta_1|$ and $|\zeta_2|$ for $k = 5$ and $k = 10$, respectively, in the case of $\kappa = 0.85$. From the figure, we note that the degree of the quantum entanglement of entangled PA-DPCSs can be manipulated through a convenable selection of parameter κ. We find that the degree of entanglement will be significant as the parameter κ gets near to one with respect to the physical parameters.



(**a**) $\boldsymbol{m} = 1$ (**b**) $\boldsymbol{m} = 3$

(**c**) $\boldsymbol{m} = 5$ (**d**) $\boldsymbol{m} = 8$

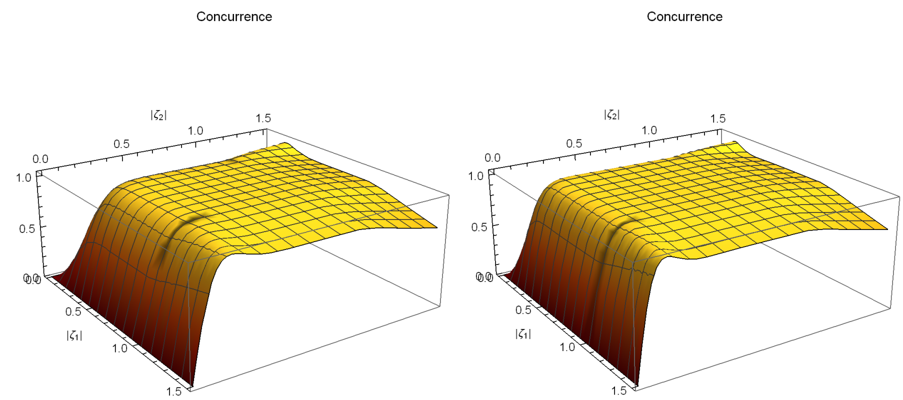
Figure 3. The variation in concurrence, defined by Equation (48), in terms of $|\zeta_1|$ and $|\zeta_2|$ for different values of $\boldsymbol{m}$ with $k_1 = k_2 = 5$ and $\kappa = 1$. Labels (**a**–**d**) are considered for $\boldsymbol{m} = 1$, $\boldsymbol{m} = 3$, $\boldsymbol{m} = 5$, and $\boldsymbol{m} = 8$, respectively. Generally, the degree of entanglement of entangled PA-DPCSs depends on the selection of the physical parameters.



(**a**) $m = 1$ (**b**) $m = 3$

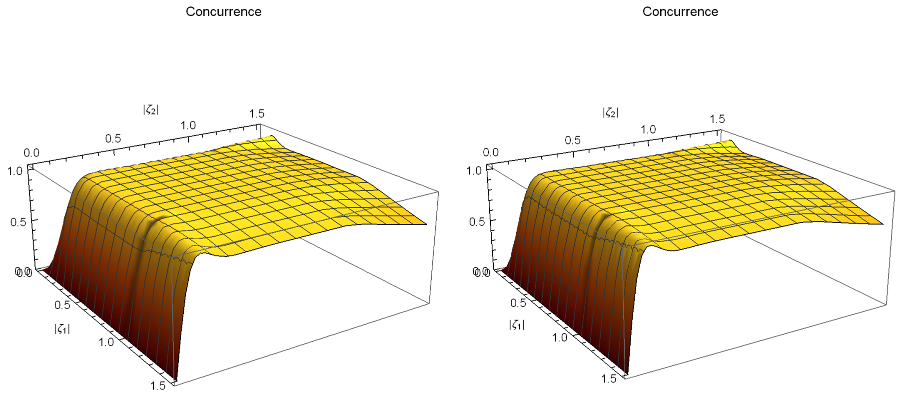
(**c**) $m = 5$ (**d**) $m = 8$

Figure 4. The variation in concurrence, defined by Equation (48), in terms of $|\zeta_1|$ and $|\zeta_2|$ for different values of m with $k_1 = k_2 = 10$ and $\kappa = 1$. Labels (**a**–**d**) are considered for $m = 1$, $m = 3$, $m = 5$, and $m = 8$, respectively. Generally, the degree of entanglement of entangled PA-DPCSs depends on the selection of the physical parameters.



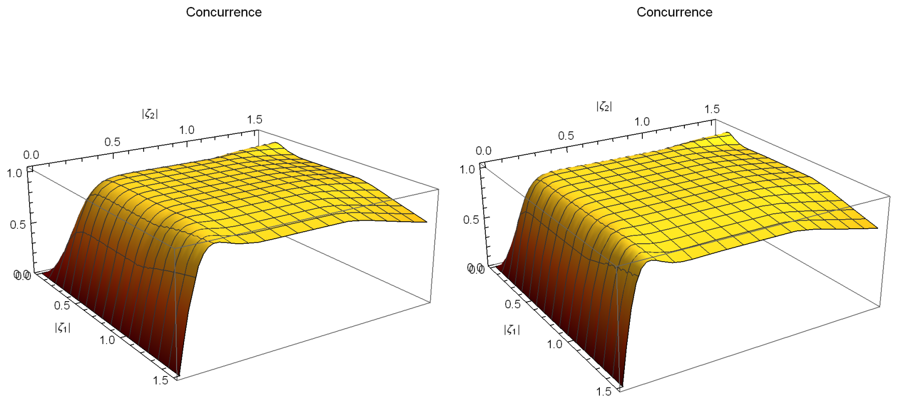
(**a**) $m = 1$ (**b**) $m = 3$

Figure 5. *Cont*.

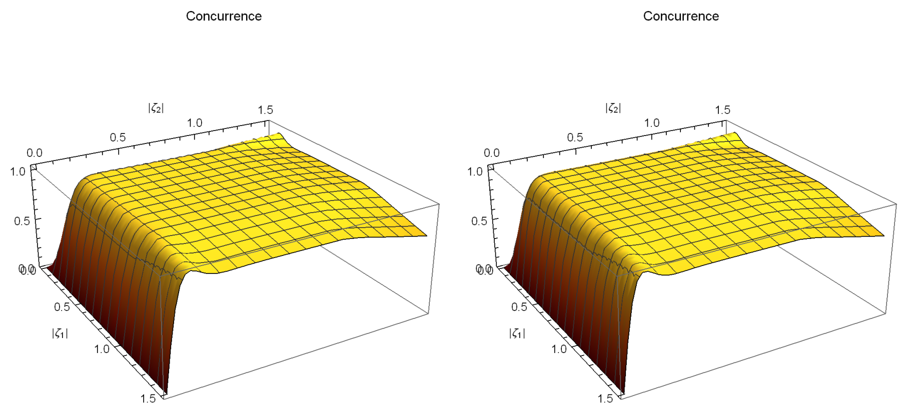


(**c**) $m = 5$ (**d**) $m = 8$

Figure 5. The variation in concurrence, defined by Equation (48), in terms of $|\zeta_1|$ and $|\zeta_2|$ for different values of $\boldsymbol{m}$ with $k_1 = k_2 = 5$ and $\kappa = 0.85$. Labels (**a**–**d**) are considered for $\boldsymbol{m} = 1$, $\boldsymbol{m} = 3$, $\boldsymbol{m} = 5$, and $\boldsymbol{m} = 8$, respectively. Generally, the degree of entanglement of entangled PA-DPCSs depends on the selection of the physical parameters.



(**a**) $m = 1$ (**b**) $m = 3$



(**c**) $m = 5$ (**d**) $m = 8$

Figure 6. The variation in concurrence, defined by Equation (48), in terms of $|\zeta_1|$ and $|\zeta_2|$ for different values of $\boldsymbol{m}$ with $k_1 = k_2 = 10$ and $\kappa = 0.85$. Labels (**a**–**d**) are considered for $\boldsymbol{m} = 1$, $\boldsymbol{m} = 3$, $\boldsymbol{m} = 5$, and $\boldsymbol{m} = 8$, respectively. Generally, the degree of entanglement of entangled PA-DPCSs depends on the selection of the physical parameters.

5. Conclusions

In summary, we have built the photon-added coherent states associated with DSUA. The constructed coherent states are obtained by using an alterationof the HPR for DSUA. We have developed a general method for the resolution of the problem of the unitary operator for these kinds of quantum states. We considered the Mandel parameter for examining the photon statistics of PA-DPCSs. We have shown that this parameter can obtain positive and negative values, exhibiting super- and sub-Poissonian distribution; depending on the values of state parameters and proper selection of the parameters, there can be an enhancement in the non-classicality of the PA-DPCSs.Furthermore, we have proposed a physical way to generate the PA-DPCSs in the framework of interaction among fields and atoms. Finally, we have introduced the concept of entangled states in the context of PA-DPCSs and examined entanglement properties for entangled PA-DPCSs with respect to the physical parameters. The obtained results can stimulate and propose further applications and studies for PA-DPCSs.

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