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Characterization of Pseudo-Differential Operators Associated with the Coupled Fractional Fourier Transform

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Abstract: The main aim of this article is to derive certain continuity and boundedness properties of the coupled fractional Fourier transform on Schwartz-like spaces. We extend the domain of the coupled fractional Fourier transform to the space of tempered distributions and then study the mapping properties of pseudo-differential operators associated with the coupled fractional Fourier transform on a Schwartz-like space. We conclude the article by applying some of the results to obtain an analytical solution of a generalized heat equation.

Keywords: Fourier transform; fractional Fourier transform; coupled fractional Fourier transform; Schwartz space; pseudo-differential operator

MSC: 33C10; 42B10; 42A38; 35S05; 46F12; 47G30

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1. Introduction

In 1980, Namias [1] formulated the fractional Fourier transform as a path to find out the solutions of certain differential equations which occasionally appear in quantum mechanics. Later on, his results were polished by McBride and Kerr [2], who developed an operational calculus for the fractional Fourier transform.

Due to numerous applications in the area of image processing, signal analysis and optics, fractional Fourier transform has received more attention in the last several years. This transform plays an important role for solving various problems in quantum physics [1,3], signal processing and optics [4–9]. The fractional Fourier transform, which is a generalization of usual Fourier transform, has been studied in several areas of mathematical analysis, for instances wavelets [10,11], pseudo-differential operators [12] and generalized functions [13–15].

The well-known Fourier transform, denoted by $\mathcal{F}$ of a function f, is defined as

$$\mathcal{F}[f](y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{i\langle x,y\rangle} dx, \tag{1}$$

so that its inverse is given by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathcal{F}[f](y) e^{-i\langle x,y\rangle} dy, \tag{2}$$

provided the integrals exist.

We recall the one-dimensional fractional Fourier transform [5,14,16,17] of a function $f(x) \in L^1(\mathbb{R})$ with angle α,

$$\mathcal{F}_\alpha[f](y) = \int_{\mathbb{R}} f(x) \mathcal{K}_\alpha(x,y) dx, \tag{3}$$

where $\mathcal{K}_\alpha(x,y)$ is given by

$$\mathcal{K}_\alpha(x,y) = \begin{cases} \sqrt{\frac{1-i\cot\alpha}{2\pi}} e^{-\frac{i}{2}(x^2+y^2)\cot\alpha + ixy\csc\alpha}, & \alpha \neq n\pi, \\ \frac{1}{\sqrt{2\pi}} e^{ixy}, & \alpha = \frac{\pi}{2}, \\ \delta(x-y), & \alpha = 2n\pi, \\ \delta(x+y), & \alpha = (2n-1)\pi, n \in \mathbb{Z}. \end{cases} \tag{4}$$

Exploiting the tensor product of n copies of the one-dimensional fractional Fourier transform each of order $\alpha_d, d = 1, 2, \ldots, n$ [16], the fractional Fourier transform has been extended to higher-dimensional transform.

We assume that $\boldsymbol{\alpha} = (\alpha_1, \alpha_2), \mathbf{x} = (x, \xi), \mathbf{y} = (y, \zeta), \mathcal{K}_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{y}) = \mathcal{K}_{\alpha_1}(x, \xi) \cdot \mathcal{K}_{\alpha_2}(y, \zeta) = \mathcal{K}^{\alpha_1,\alpha_2}(x, y, \xi, \zeta)$, where $\mathcal{K}_{\alpha_1}(x, \xi)$ and $\mathcal{K}_{\alpha_2}(y, \zeta)$ defined in (4).

For $f(x,y) \in L^1(\mathbb{R}^2)$, two-dimensional fractional Fourier transform is defined as

$$\begin{aligned} \mathcal{F}_{\boldsymbol{\alpha}}(f)(\xi,\zeta) = \mathfrak{F}_{\alpha_1,\alpha_2}(f)(\xi,\zeta) &= \int_{\mathbb{R}^2} \mathcal{K}_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y}) f(x,y)\,dxdy \\ &= \int_{\mathbb{R}^2} \mathcal{K}_{\alpha_1}(x,\xi)\mathcal{K}_{\alpha_2}(y,\zeta) f(x,y)\,dxdy \\ &= \int_{\mathbb{R}^2} \mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta) f(x,y)\,dxdy. \end{aligned} \tag{5}$$

The corresponding inversion formula of (5) is given by

$$f(x,y) = \int_{\mathbb{R}^2} \overline{\mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta)}\, \mathfrak{F}_{\alpha_1,\alpha_2}(f)(\xi,\zeta)\,d\xi d\zeta. \tag{6}$$

It is easy to observe that for $\alpha_1 = \alpha_2 = \frac{\pi}{2}$, the two-dimensional fractional Fourier transform $\mathfrak{F}_{\alpha_1,\alpha_2}$ becomes a classical two-dimensional Fourier transform.

Bhosale [18] discussed the fractional Fourier transform on compact support distribution. Pathak [15,19] and Prasad [12,20] studied the important properties of fractional Fourier transform and pseudo-differential operator on certain function spaces like Schwartz and Sobolev spaces.

Proposition 1. *Let $\mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta)$ be the kernel of the two-dimensional fractional Fourier transform. Then, for all $\phi(x,y) \in \mathcal{S}(\mathbb{R}^2)$, we have*

(i) $\mathcal{D}^r_{x,y}\mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta) = \{i(\xi\csc\alpha_1 + \zeta\csc\alpha_2)\}^r \mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta)$,

(ii) $\int_{\mathbb{R}^2} \phi(x,y)\mathcal{D}^r_{x,y}\mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta)\,dxdy = \int_{\mathbb{R}^2} \mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta)(\mathcal{D}'_{x,y})^r\phi(x,y)\,dxdy$,

(iii) $\mathfrak{F}_{\alpha_1,\alpha_2}\{(\mathcal{D}'_{x,y})^r\phi(x,y)\}(\xi,\zeta) = \{i(\xi\csc\alpha_1 + \zeta\csc\alpha_2)\}^r(\mathfrak{F}_{\alpha_1,\alpha_2}\phi(x,y))(\xi,\zeta)$,

for all $r \in \mathbb{N}$, where $\mathcal{D}_{x,y} = [\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + i(x\cot\alpha_1 + y\cot\alpha_2)]$ and $\mathcal{D}'_{x,y} = -[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - i(x\cot\alpha_1 + y\cot\alpha_2)]$.

Proof. (i) For the case $r = 1$, we see that

$$\frac{\partial}{\partial x}\mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta) = i(\xi\csc\alpha_1 - x\cot\alpha_1)\mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta),$$

and

$$\frac{\partial}{\partial y}\mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta) = i(\zeta\csc\alpha_2 - y\cot\alpha_2)\mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta).$$

Thus

$$\begin{aligned} \mathcal{D}_{x,y}\mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta) &= \left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + i(x\cot\alpha_1 + y\cot\alpha_2)\right]\mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta) \\ &= \{i(\xi\csc\alpha_1 + \zeta\csc\alpha_2)\}\mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta). \end{aligned}$$

The result can be generalized easily for any natural number.
(ii) Integrating by parts, we have

$$\begin{aligned}&\int_{\mathbb{R}^2}\phi(x,y)\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+i(x\cot\alpha_1+y\cot\alpha_2)\right]\mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta)dxdy\\&=-\int_{\mathbb{R}^2}\mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta)\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}-i(x\cot\alpha_1+y\cot\alpha_2)\right]\phi(x,y)dxdy.\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^2}\phi(x,y)\mathcal{D}_{x,y}\mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta)dxdy=\int_{\mathbb{R}^2}\mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta)\mathcal{D}'_{x,y}\phi(x,y)dxdy.$$

The result can be generalized easily for any natural number.
(iii) Using (5) and Proposition 1(i),(ii), we have

$$\begin{aligned}&\mathfrak{F}_{\alpha_1,\alpha_2}\{(\mathcal{D}'_{x,y})^r\phi(x,y)\}(\xi,\zeta)\\&=\int_{\mathbb{R}^2}\mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta)(\mathcal{D}'_{x,y})^r\phi(x,y)dxdy\\&=\int_{\mathbb{R}^2}\phi(x,y)\mathcal{D}^r_{x,y}\mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta)dxdy\\&=\int_{\mathbb{R}^2}\phi(x,y)\{i(\xi\csc\alpha_1+\zeta\csc\alpha_2)\}^r\mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta)dxdy\\&=\{i(\xi\csc\alpha_1+\zeta\csc\alpha_2)\}^r\int_{\mathbb{R}^2}\phi(x,y)\mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta)dxdy\\&=\{i(\xi\csc\alpha_1+\zeta\csc\alpha_2)\}^r(\mathfrak{F}_{\alpha_1,\alpha_2}\phi(x,y))(\xi,\zeta).\end{aligned}$$

This completes the proof. □

In recent articles [14,17,21], Zayed developed a new two-dimensional (coupled) fractional Fourier transform $\mathcal{F}_{\alpha,\beta}$ that is not a tensor product of two copies of one-dimensional transform but is a transform which depends on two angles α,β that are coupled so that the transform parameters are $\gamma=\frac{\alpha+\beta}{2}$ and $\eta=\frac{\alpha-\beta}{2}$.

Choose $\alpha,\beta\in\mathbb{R}$ in such a way that $\alpha+\beta\neq 2n\pi, n\in\mathbb{Z}$. The coupled fractional Fourier transform of a function $f(x,y)\in L^1(\mathbb{R}^2)$ is defined by [21,22]

$$\mathcal{F}_{\alpha,\beta}(f)(\xi,\delta)=\int_{\mathbb{R}^2}f(x,y)\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)dxdy, \tag{7}$$

where $\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)=de^{[-a(x^2+y^2+\xi^2+\delta^2)+b(\xi x+\delta y)+c(\delta x-\xi y)]}, a=a(\gamma)=\frac{i\cot\gamma}{2}, b=b(\gamma,\eta)=\frac{i\cos\eta}{\sin\gamma}, c=c(\gamma,\eta)=\frac{i\sin\eta}{\sin\gamma}, d=d(\gamma)=\frac{ie^{-i\gamma}}{2\pi\sin\gamma}, \gamma=\frac{\alpha+\beta}{2}$, and $\eta=\frac{\alpha-\beta}{2}$.
The corresponding inversion formula of (7) is given by

$$f(x,y)=\int_{\mathbb{R}^2}\mathcal{F}_{\alpha,\beta}(f)(\xi,\delta)\mathcal{K}_{-\alpha,-\beta}(\xi,\delta,x,y)d\xi d\delta. \tag{8}$$

It can be easily observed that for $\alpha=\beta$ and $\alpha=\beta=\frac{\pi}{2}$, the coupled fractional Fourier transform $\mathcal{F}_{\alpha,\beta}$ becomes the tensor product of two one-dimensional fractional Fourier transforms and the classical two-dimensional Fourier transform, respectively. Throughout this manuscript, we assume that $\alpha,\beta\in\mathbb{R}$ with $\alpha+\beta\neq 2n\pi, n\in\mathbb{Z}$.

Our main objective of this present article is to investigate the continuity and boundedness properties of the coupled fractional Fourier transform and pseudo-differential operator related to it on Schwartz spaces.

2. Coupled Fractional Fourier Transform

In this section, we derive some properties of the kernel $\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)$ of the coupled fractional Fourier transform that will be used later to extend the transform and its associated pseudo-differential operator to certain Schwartz-like spaces. Let us first recall the definition of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ for two variables.

Definition 1. *The space $\mathcal{S}(\mathbb{R}^2)$ is the collection of all complex valued infinitely differentiable functions $\phi(x,y) \in \mathbb{R}^2$ for every choice of $\beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{N}_0$ which for*

$$\Gamma^{\gamma_1,\gamma_2}_{\beta_1,\beta_2}(\phi) = \sup_{(x,y)\in\mathbb{R}^2} |x^{\beta_1} y^{\beta_2} \frac{\partial^{\gamma_1}}{\partial x^{\gamma_1}} \frac{\partial^{\gamma_2}}{\partial y^{\gamma_2}} \phi(x,y)| < \infty. \tag{9}$$

The Schwartz space $\mathcal{S}(\mathbb{R}^2)$ is equipped with the topology generated by the semi-norms $\{\Gamma^{\gamma_1,\gamma_2}_{\beta_1,\beta_2}\}$. The space $\mathcal{S}(\mathbb{R}^2)$ becomes a Fréchet space. The dual of $\mathcal{S}(\mathbb{R}^2)$ is denoted by $\mathcal{S}'(\mathbb{R}^2)$.

If f is a locally integrable and polynomial growth function on $\mathbb{R}^2$, then f generates a distribution in $\mathcal{S}'(\mathbb{R}^2)$ as follows:

$$\langle f, \phi \rangle = \int_{\mathbb{R}^2} f(x,y)\phi(x,y)dxdy, \text{ for all } \phi \in \mathcal{S}(\mathbb{R}^2). \tag{10}$$

The elements of $\mathcal{S}'(\mathbb{R}^2)$ are known as tempered distributions.

Lemma 1. *A function $\psi \in \mathcal{S}(\mathbb{R}^2)$ if it is a member of C^∞ and it satisfies*

$$\rho^{\gamma_1,\gamma_2}_m(\psi) = \sup_{(x,y)\in\mathbb{R}^2} \left| (1+x^2+y^2)^{\frac{m}{2}} \frac{\partial^{\gamma_1}}{\partial x^{\gamma_1}} \frac{\partial^{\gamma_2}}{\partial y^{\gamma_2}} \psi(x,y) \right| < \infty, \tag{11}$$

for all $m, \gamma_1, \gamma_2 \in \mathbb{N}_0$.

Proof. It is easy to observe that for all $\beta_1 + \beta_2 \le m$,

$$|x^{\beta_1} y^{\beta_2}| \le [(x^2+y^2)^{1/2}]^{(\beta_1+\beta_2)} \le [1+x^2+y^2]^{\frac{\beta_1+\beta_2}{2}} \le [1+x^2+y^2]^{\frac{m}{2}}.$$

Hence, if ψ satisfies (11), then it satisfies (9).

Next, we assume that ψ satisfies (9) for all $\beta_1, \beta_2 \in \mathbb{N}_0$. We observe that for any $k \in \mathbb{N}_0$,

$$(x^2+y^2)^k = \sum_{\beta_1+\beta_2=k} p(\beta_1,\beta_2,k)(x^2)^{\beta_1}(y^2)^{\beta_2} = \sum_{\beta_1+\beta_2=k} p(\beta_1,\beta_2,k)(x^{\beta_1})^2(y^{\beta_2})^2,$$

where $p(\beta_1,\beta_2,k) = \frac{k!}{\beta_1!\beta_2!}$ are constant coefficients. Therefore, for $m \in \mathbb{N}_0$,

$$\begin{aligned}(1+x^2+y^2)^m &= \sum_{k=0}^{m} \binom{m}{k} (x^2+y^2)^k = \sum_{k=0}^{m} \sum_{\beta_1+\beta_2=k} \binom{m}{k} p(\beta_1,\beta_2,k)(x^{\beta_1})^2(y^{\beta_2})^2 \\ &\le \left(\sum_{k=0}^{m} \sum_{\beta_1+\beta_2=k} \left[\binom{m}{k} p(\beta_1,\beta_2,k) \right]^{\frac{1}{2}} x^{\beta_1} y^{\beta_2} \right)^2.\end{aligned}$$

So that

$$(1+x^2+y^2)^{\frac{m}{2}} \le \sum_{k=0}^{m} \sum_{\beta_1+\beta_2=k} \left[\binom{m}{k} p(\beta_1,\beta_2,k) \right]^{\frac{1}{2}} x^{\beta_1} y^{\beta_2}.$$

Consequently, $\rho^{\gamma_1,\gamma_2}_m(\psi) \le \sum_{k=0}^{m} \sum_{\beta_1+\beta_2=k} \left[\binom{m}{k} p(\beta_1,\beta_2,k) \right]^{\frac{1}{2}} \Gamma^{\gamma_1,\gamma_2}_{\beta_1,\beta_2}(\psi)$. $\square$

Definition 2. *The test function space $\mathcal{S}_{\gamma}(\mathbb{R}^2)$: This space contains of all those C^{∞} complex valued functions $\varphi(x,y) \in \mathbb{R}^2$, which satisfies*

$$ {}^{r,\gamma}_{\beta_1,\beta_2}\varphi(x,y) = \sup_{(x,y)\in\mathbb{R}^2} |x^{\beta_1} y^{\beta_2} \Delta^r_{x,y}\varphi(x,y)| < \infty, \textit{ for all } \beta_1, \beta_2, r \in \mathbb{N}_0, \tag{12} $$

where $\Delta_{x,y} = [\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + i(x+y)\cot\gamma]$.

Proposition 2. *Let $\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)$ be the kernel of the coupled fractional Fourier transform and $\Delta_{x,y} = [\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + i(x+y)\cot\gamma]$. Then, for all $r \in \mathbb{N}_0$*

$$ \Delta^r_{x,y}\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta) = [(b-c)\xi + (b+c)\delta]^r \mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta), \tag{13} $$

where a, b, c are defined as earlier.

Proof. Here, $\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta) = de^{-a(x^2+y^2+\xi^2+\delta^2)+b(\xi x+\delta y)+c(\delta x-\xi y)}$.
We have

$$ \frac{\partial}{\partial x}\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta) = (-ix\cot\gamma + b\xi + c\delta)\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta). \tag{14} $$

Similarly, we have,

$$ \frac{\partial}{\partial y}\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta) = (-iy\cot\gamma + b\delta - c\xi)\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta). \tag{15} $$

Now adding (14) and (15), we obtain

$$ \begin{aligned} &\frac{\partial}{\partial x}\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta) + \frac{\partial}{\partial y}\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta) \\ = \;& [-ix\cot\gamma + b\xi + c\delta - iy\cot\gamma + b\delta - c\xi]\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta). \end{aligned} $$

Hence, $\Delta_{x,y}\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta) = [(b-c)\xi + (b+c)\delta]\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)$.
Continuing in this way, we have

$$ \Delta^r_{x,y}\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta) = [(b-c)\xi + (b+c)\delta]^r \mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta). \tag{16} $$

Which completes the proof. □

Remark 1. *Let $\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)$ be the kernel of the coupled fractional Fourier transform and $\Delta_{\xi,\delta}$ defined as above. Then, for all $r \in \mathbb{N}_0$*

$$ \Delta^r_{\xi,\delta}\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta) = [(b+c)x + (b-c)y]^r \mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta), \tag{17} $$

where a, b, c are defined as earlier.

Proposition 3. *Let $\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)$ be the kernel of the coupled fractional Fourier transform and $\Delta'_{x,y} = -[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - i(x+y)\cot\gamma]$. Then, for all $\phi(x,y) \in \mathcal{S}(\mathbb{R}^2)$ and $r \in \mathbb{N}_0$, we have*

(i) $\int_{\mathbb{R}^2} \Delta^r_{x,y}\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)\phi(x,y)dxdy = \int_{\mathbb{R}^2} \mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)(\Delta'_{x,y})^r\phi(x,y)dxdy,$

(ii) $\mathcal{F}_{\alpha,\beta}((\Delta'_{x,y})^r\phi(x,y))(\xi,\delta) = [(b-c)\xi + (b+c)\delta]^r \mathcal{F}_{\alpha,\beta}(\phi(x,y)(\xi,\delta),$

(iii) $\Delta^r_{\xi,\delta}\mathcal{F}_{\alpha,\beta}(\phi(x,y))(\xi,\delta) = \mathcal{F}_{\alpha,\beta}\big([(b+c)x + (b-c)y]^r\phi(x,y)\big)(\xi,\delta),$

(iv) The operator $\mathcal{F}_{\alpha,\beta}$ is a linear and continuous mapping from $\mathcal{S}(\mathbb{R}^2)$ to $\mathcal{S}_{\gamma}(\mathbb{R}^2)$.

Proof. (i) First of all, we will prove for $r = 1$,

$$\int_{\mathbb{R}^2} [\Delta_{x,y}\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)]\phi(x,y)dxdy = \int_{\mathbb{R}^2} \mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)[\Delta'_{x,y}\phi(x,y)]dxdy.$$

Integrating by parts, we have

$$\begin{aligned}&\int_{\mathbb{R}^2}\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + i(x+y)\cot\gamma\right]\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)\phi(x,y)dxdy\\&= -\int_{\mathbb{R}^2}\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - i(x+y)\cot\gamma\right]\phi(x,y)dxdy.\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^2} [\Delta_{x,y}\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)]\phi(x,y)dxdy = \int_{\mathbb{R}^2} \mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)[\Delta'_{x,y}\phi(x,y)]dxdy.$$

Hence, in general, we have

$$\int_{\mathbb{R}^2} \Delta^r_{x,y}\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)\phi(x,y)dxdy = \int_{\mathbb{R}^2} \mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)(\Delta'_{x,y})^r\phi(x,y)dxdy.$$

(ii) Exploiting (7), Proposition 2 and Proposition 3(i), we have

$$\begin{aligned}&(\mathcal{F}_{\alpha,\beta}((\Delta'_{x,y})^r\phi(x,y)))(\xi,\delta)\\&= \int_{\mathbb{R}^2}\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)(\Delta'_{x,y})^r\phi(x,y)dxdy\\&= \int_{\mathbb{R}^2}\Delta^r_{x,y}\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)\phi(x,y)dxdy\\&= \int_{\mathbb{R}^2}[(b-c)\xi + (b+c)\delta]^r\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)\phi(x,y)dxdy\\&= [(b-c)\xi + (b+c)\delta]^r\int_{\mathbb{R}^2}\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)\phi(x,y)dxdy\\&= [(b-c)\xi + (b+c)\delta]^r\mathcal{F}_{\alpha,\beta}(\phi(x,y)(\xi,\delta).\end{aligned}$$

(iii) In viewing Proposition 2, we obtain

$$\begin{aligned}&\Delta^r_{\xi,\delta}(\mathcal{F}_{\alpha,\beta}(\phi(x,y)))(\xi,\delta)\\&= \int_{\mathbb{R}^2}\Delta^r_{\xi,\delta}\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)\phi(x,y)dxdy\\&= \int_{\mathbb{R}^2}[(b+c)x + (b-c)y]^r\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)\phi(x,y)dxdy\\&= \int_{\mathbb{R}^2}\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)\big([(b+c)x + (b-c)y]^r\phi(x,y)\big)dxdy\\&= \mathcal{F}_{\alpha,\beta}\big([(b+c)x + (b-c)y]^r\phi(x,y)\big)(\xi,\delta).\end{aligned}$$

(iv) The linearity of $\mathcal{F}_{\alpha,\beta}$ is obvious.
Assume that r, s and t be any three positive integers. Then, by Proposition 3(iii), for any sequence of functions $\{\phi_n\}_{n\in\mathbb{N}} \in \mathcal{S}(\mathbb{R}^2)$, we have

$$\begin{aligned}&\sup_{(\xi,\delta)\in\mathbb{R}^2}\left|\xi^r\delta^s\Delta^t_{\xi,\delta}[\mathcal{F}_{\alpha,\beta}(\phi_n(x,y)](\xi,\delta)\right|\\&= \sup_{(\xi,\delta)\in\mathbb{R}^2}\left|\xi^r\delta^s\mathcal{F}_{\alpha,\beta}\big([(b+c)x + (b-c)y]^t(\phi_n(x,y)\big)(\xi,\delta)\right|.\end{aligned}$$

Since $\phi_n \in \mathcal{S}(\mathbb{R}^2)$, $[(b+c)x + (b-c)y]^t\phi_n(x,y) \in \mathcal{S}(\mathbb{R}^2)$.

$$\text{So, } \mathcal{F}_{\alpha,\beta}\big([(b+c)x+(b-c)y]^t(\phi_n(x,y)\big) \in \mathcal{S}(\mathbb{R}^2).$$

Hence, ${}^{t,\gamma}_{r,s}\mathcal{F}_{\alpha,\beta}(\phi_n) = \sup\limits_{(\xi,\delta)\in\mathbb{R}^2} \left|\xi^r\delta^s\Delta^t_{\xi,\delta}\big(\mathcal{F}_{\alpha,\beta}\phi_n(x,y)\big)(\xi,\delta)\right| \to 0,$
if $\phi_n \to 0$ in $\mathcal{S}(\mathbb{R}^2)$.
which shows the continuity of $\mathcal{F}_{\alpha,\beta}$. □

3. Coupled Fractional Fourier Transform of Tempered Distributions

The coupled fractional Fourier transform was orignally defined on $L^p(\mathbb{R}^2), 1 \le p \le 2$. In this section, we extend the domain of the coupled fractional Fourier transform to the space of tempered distributions using the adjoint method. In order to do that, we need to examine the action of the coupled fractional Fourier transform on the Schwartz space of functions, $\mathcal{S}(\mathbb{R}^2)$.

Theorem 1. *The coupled fractional Fourier transform defined in* (7) *is a continuous linear mapping from* $\mathcal{S}(\mathbb{R}^2)$ *onto itself.*

Proof. In viewing the notations [22] $E(x,y) = e^{a(x^2+y^2)}, E^{-1}(x,y) = e^{-a(x^2+y^2)}, \tilde{b} = -ib, \tilde{c} = -ic,$ for all $(x,y) \in \mathbb{R}^2$, the coupled fractional Fourier transform can be rewritten as

$$\mathcal{F}_{\alpha,\beta}(\phi)(\xi,\delta) = 2\pi d E^{-1}(\xi,\delta)\Phi_{\alpha,\beta}(\xi,\delta), \tag{18}$$

where $\Phi_{\alpha,\beta}(\xi,\delta) = \mathcal{F}(\phi E^{-1})[-(\tilde{b}\xi+\tilde{c}\delta), -(\tilde{b}\delta-\tilde{c}\xi)] \in \mathcal{S}(\mathbb{R}^2)$, where $\mathcal{F}$ denotes the Fourier transform.
Now, for all $\phi \in \mathcal{S}(\mathbb{R}^2)$, we have for any $s_2 \in \mathbb{N}_0$,

$$\begin{aligned}
&\frac{\partial^{s_2}}{\partial\delta^{s_2}}\mathcal{F}_{\alpha,\beta}(\phi)(\xi,\delta)\\
&= 2\pi d\frac{\partial^{s_2}}{\partial\delta^{s_2}}[E^{-1}(\xi,\delta)\Phi_{\alpha,\beta}(\xi,\delta)]\\
&= 2\pi d\sum_{s_2'=0}^{s_2}\binom{s_2}{s_2'}\frac{\partial^{s_2'}}{\partial\delta^{s_2'}}\{E^{-1}(\xi,\delta)\}\frac{\partial^{s_2-s_2'}}{\partial\delta^{s_2-s_2'}}\Phi_{\alpha,\beta}(\xi,\delta)\\
&= 2\pi d\sum_{s_2'=0}^{s_2}\binom{s_2}{s_2'}E^{-1}(\xi,\delta)\Big(\sum_{s_2''=0}^{s_2'}m_{s_2''}(\cot\gamma)\delta^{s_2''}\Big)\frac{\partial^{s_2-s_2'}}{\partial\delta^{s_2-s_2'}}\Phi_{\alpha,\beta}(\xi,\delta),
\end{aligned}$$

where $m_{s_2''}$ are constants.

So,

$$\begin{aligned}
&\frac{\partial^{s_1}}{\partial\xi^{s_1}}\frac{\partial^{s_2}}{\partial\delta^{s_2}}(\mathcal{F}_{\alpha,\beta}(\phi)(\xi,\delta)\\
&= 2\pi d\sum_{s_2'=0}^{s_2}\binom{s_2}{s_2'}\Big(\sum_{s_2''=0}^{s_2'}m_{s_2''}(\cot\gamma)\delta^{s_2''}\Big)\frac{\partial^{s_1}}{\partial\xi^{s_1}}\Big[E^{-1}(\xi,\delta)\frac{\partial^{s_2-s_2'}}{\partial\delta^{s_2-s_2'}}\Phi_{\alpha,\beta}(\xi,\delta)\Big]\\
&= 2\pi d\sum_{s_2'=0}^{s_2}\binom{s_2}{s_2'}\Big(\sum_{s_2''=0}^{s_2'}m_{s_2''}(\cot\gamma)\delta^{s_2''}\Big)\sum_{s_1'=0}^{s_1}\binom{s_1}{s_1'}\frac{\partial^{s_1'}}{\partial\xi^{s_1'}}\{E^{-1}(\xi,\delta)\}\\
&\times\frac{\partial^{s_1-s_1'}}{\partial\xi^{s_1-s_1'}}\frac{\partial^{s_2-s_2'}}{\partial\delta^{s_2-s_2'}}\{\Phi_{\alpha,\beta}(\xi,\delta)\}
\end{aligned}$$

$$= 2\pi d \sum_{s_2'=0}^{s_2} \binom{s_2}{s_2'} \Big(\sum_{s_2''=0}^{s_2'} m_{s_2''} (\cot\gamma) \delta^{s_2''} \Big) \sum_{s_1'=0}^{s_1} \binom{s_1}{s_1'} E^{-1}(\xi,\delta) \Big(\sum_{s_1''=0}^{s_1'} n_{s_1''} (\cot\gamma) \xi^{s_1''} \Big)$$
$$\times \frac{\partial^{s_1-s_1'}}{\partial \xi^{s_1-s_1'}} \frac{\partial^{s_2-s_2'}}{\partial \delta^{s_2-s_2'}} \{\Phi_{\alpha,\beta}(\xi,\delta)\},$$

where $n_{s_1''}$ are constants.

Therefore,

$$\left| \xi^{t_1} \delta^{t_2} \frac{\partial^{s_1}}{\partial \xi^{s_1}} \frac{\partial^{s_2}}{\partial \delta^{s_2}} \mathcal{F}_{\alpha,\beta}(\phi)(\xi,\delta) \right|$$
$$= \Big| 2\pi d(\cot^2\gamma) \sum_{s_2'=0}^{s_2} \binom{s_2}{s_2'} \Big(\sum_{s_2''=0}^{s_2'} m_{s_2''} \delta^{s_2''+t_2} \Big) \sum_{s_1'=0}^{s_1} \binom{s_1}{s_1'} E^{-1}(\xi,\delta)$$
$$\times \Big(\sum_{s_1''=0}^{s_1'} n_{s_1''} \xi^{s_1''+t_1} \Big) \frac{\partial^{s_1-s_1'}}{\partial \xi^{s_1-s_1'}} \frac{\partial^{s_2-s_2'}}{\partial \delta^{s_2-s_2'}} \{\Phi_{\alpha,\beta}(\xi,\delta)\} \Big|$$

$$\leq 2\pi |d| |\cot^2\gamma| \sum_{s_2'=0}^{s_2} \binom{s_2}{s_2'} \sum_{s_2''=0}^{s_2'} |m_{s_2''}| \sum_{s_1'=0}^{s_1} \binom{s_1}{s_1'} |n_{s_1''}|$$
$$\times \Big| \xi^{s_1''+t_1} \delta^{s_2''+t_2} \frac{\partial^{s_1-s_1'}}{\partial \xi^{s_1-s_1'}} \frac{\partial^{s_2-s_2'}}{\partial \delta^{s_2-s_2'}} \{\Phi_{\alpha,\beta}(\xi,\delta)\} \Big|.$$

Hence,

$$\begin{aligned} \Gamma_{t_1,t_2}^{s_1,s_2}(\mathcal{F}_{\alpha,\beta}(\phi)(\xi,\delta) &= \sup_{(\xi,\delta)\in\mathbb{R}^2} \left| \xi^{t_1} \delta^{t_2} \frac{\partial^{s_1}}{\partial \xi^{s_1}} \frac{\partial^{s_2}}{\partial \delta^{s_2}} (\mathcal{F}_{\alpha,\beta}(\phi)(\xi,\delta) \right| \\ &\leq 2\pi |d| |\cot^2\gamma| \sum_{s_2'=0}^{s_2} \binom{s_2}{s_2'} \sum_{s_2''=0}^{s_2'} |m''_{s_2}| \sum_{s_1'=0}^{s_1} \binom{s_1}{s_1'} |n''_{s_1}| \\ &\quad \times \sup_{(\xi,\delta)\in\mathbb{R}^2} \Big| \xi^{s_1''+t_1} \delta^{s_2''+t_2} \frac{\partial^{s_1-s_1'}}{\partial \xi^{s_1-s_1'}} \frac{\partial^{s_2-s_2'}}{\partial \delta^{s_2-s_2'}} \{\Phi_{\alpha,\beta}(\xi,\delta)\} \Big| \qquad (19) \\ &< \infty, \end{aligned}$$

because $\Phi_{\alpha,\beta}(\xi,\delta) \in \mathcal{S}(\mathbb{R}^2)$. Thus $\mathcal{F}_{\alpha,\beta}(\phi)(\xi,\delta) \in \mathcal{S}(\mathbb{R}^2)$. Also from (7) and (8), we see that for all $\phi \in \mathcal{S}(\mathbb{R}^2)$,

$$(\mathcal{F}_{\alpha,\beta}\mathcal{F}_{\alpha,\beta}^{-1})(\phi) = \phi = (\mathcal{F}_{\alpha,\beta}^{-1}\mathcal{F}_{\alpha,\beta})(\phi).$$

It follows that $\mathcal{F}_{\alpha,\beta}$ is a one–one mapping from $\mathcal{S}(\mathbb{R}^2)$ onto itself. Clearly, $\mathcal{F}_{\alpha,\beta}$ is a linear map. To show that it is continuous, assume that there exists a sequence $\{\phi_n\} \to 0$ in $\mathcal{S}(\mathbb{R}^2)$, then from (19), $\{\mathcal{F}_{\alpha,\beta}\phi_n\} \to 0$ in $\mathcal{S}(\mathbb{R}^2)$; therefore, the continuity of the coupled fractional Fourier transform follows. □

Definition 3. *The generalized coupled fractional Fourier transform $\mathcal{F}_{\alpha,\beta}f$ of $f \in \mathcal{S}'(\mathbb{R}^2)$ is defined by*

$$\langle \mathcal{F}_{\alpha,\beta}f, \phi \rangle = \langle f, \mathcal{F}_{\alpha,\beta}\phi \rangle, \text{ where } \phi \in \mathcal{S}(\mathbb{R}^2). \qquad (20)$$

In a similar way, we can define the inverse of generalized coupled fractional Fourier transform $\mathcal{F}^{-1}_{\alpha,\beta}f$ of $f \in \mathcal{S}'(\mathbb{R}^2)$ as follows:

$$\langle \mathcal{F}^{-1}_{\alpha,\beta}f, \phi\rangle = \langle f, \mathcal{F}^{-1}_{\alpha,\beta}\phi\rangle,\ \phi \in \mathcal{S}(\mathbb{R}^2). \tag{21}$$

Theorem 2. *The generalized coupled fractional Fourier transform $\mathcal{F}_{\alpha,\beta}$ is a continuous linear map of $\mathcal{S}'(\mathbb{R}^2)$ onto itself.*

Proof. It is easy to observe that $\mathcal{F}_{\alpha,\beta}$ is linear on $\mathcal{S}'(\mathbb{R}^2)$.

By the previous Theorem 1, we observe that $\mathcal{F}_{\alpha,\beta}(\phi) \in \mathcal{S}(\mathbb{R}^2)$ for all $\phi \in \mathcal{S}(\mathbb{R}^2)$. The right-hand side of (20) is well defined. Also, if $\{\phi_n\} \to 0$ in $\mathcal{S}(\mathbb{R}^2)$, then by the continuity of coupled fractional Fourier transform $\{\mathcal{F}_{\alpha,\beta}(\phi_n)\} \to 0$, the right-hand side of (20) converges to zero, which in turn implies that $\{\langle \mathcal{F}_{\alpha,\beta}f, \phi_n\rangle\} \to 0$ as $n \to \infty$. Thus, $\mathcal{F}_{\alpha,\beta}$ is continuous on $\mathcal{S}'(\mathbb{R}^2)$. □

Remark 2. *The inverse of generalized coupled fractional Fourier transform $\mathcal{F}^{-1}_{\alpha,\beta}$ is a continuous linear mapping from $\mathcal{S}'(\mathbb{R}^2)$ onto itself.*

Example 1. *Let $\kappa(x-p), \kappa(y-q)$ denote the Dirac Delta functions. Then,*

$$(i)\ \mathcal{F}_{\alpha,\beta}[\kappa(x-p)\kappa(y-q)] = \mathcal{K}_{\alpha,\beta}(u,v,p,q), u,v,p,q \in \mathbb{R},$$
$$(ii)\ \mathcal{F}_{\alpha,\beta}[\kappa(x)\kappa(y)](\xi,\eta) = de^{-a(u^2+v^2)}.$$

Proof. (i) Let $\phi \in \mathcal{S}(\mathbb{R}^2)$. Then, we see that

$$\begin{aligned}\langle \mathcal{F}_{\alpha,\beta}[\kappa(x-p)\kappa(y-q)], \phi\rangle &= \langle [\kappa(x-p)\kappa(y-q)], \mathcal{F}_{\alpha,\beta}\phi\rangle = (\mathcal{F}_{\alpha,\beta}\phi)(p,q)\\ &= \int_{\mathbb{R}^2} \mathcal{K}_{\alpha,\beta}(u,v,p,q)\phi(u,v)dudv\\ &= \langle \mathcal{K}_{\alpha,\beta}(u,v,p,q), \phi(u,v)\rangle.\end{aligned}$$

(ii) It is easy to prove (ii). □

4. Pseudo-Differential Operators

Pseudo-differential operators involving Fourier and fractional Fourier transforms have been extensively studied [15,20,23–26]. The goal of this section is to extend the notion of pseudo-differential operators to the coupled fractional Fourier transform and study their continuity and boundedness on modified Schwartz-type spaces.

Pseudo-differential operator associated with $\mathfrak{F}_{\alpha_1,\alpha_2}$: A linear partial differential operator $A(x,y,\mathcal{D}'_{x,y})$ on $\mathbb{R}^2$ is given by

$$A(x,y,\mathcal{D}'_{x,y}) = \sum_{r=0}^{m} a_r(x,y)(\mathcal{D}'_{x,y})^r, \tag{22}$$

where the coefficients $a_r(x,y)$ are functions defined on $\mathbb{R}^2$ and $\mathcal{D}'_{x,y}$ is as defined above. If we replace $(\mathcal{D}'_{x,y})^r$ in (22) by monomial $\{i(\xi \csc\alpha_1 + \zeta \csc\alpha_2)\}^r$ in $\mathbb{R}^2$, then we obtain the so-called symbol

$$A(x,y,\xi,\zeta) = \sum_{r=0}^{m} a_r(x,y)\{i(\xi \csc\alpha_1 + \zeta \csc\alpha_2)\}^r. \tag{23}$$

In order to obtain another representation of the operator $A(x,y,\mathcal{D}'_{x,y})$, let us take any function $\phi \in \mathcal{S}(\mathbb{R}^2)$; then, using (5), (6) and Proposition 1(iii), we have

$$
\begin{aligned}
(A(x,y,\mathcal{D}'_{x,y})\phi)(x,y) &= \sum_{r=0}^{m} a_r(x,y)\mathfrak{F}^{-1}_{\alpha_1,\alpha_2}\mathfrak{F}_{\alpha_1,\alpha_2}(\mathcal{D}'_{x,y})^r\phi(x,y) \\
&= \sum_{r=0}^{m} a_r(x,y)\mathfrak{F}^{-1}_{\alpha_1,\alpha_2}\{i(\xi\csc\alpha_1+\zeta\csc\alpha_2)\}^r(\mathfrak{F}_{\alpha_1,\alpha_2}\phi(x,y))(\xi,\zeta) \\
&= \int_{\mathbb{R}^2} \overline{\mathcal{K}^{\alpha_1,\alpha_2}(x,y,\xi,\zeta)}A(x,y,\xi,\zeta)(\mathfrak{F}_{\alpha_1,\alpha_2}\phi)(\xi,\zeta)d\xi d\zeta.
\end{aligned}
$$

So, we have represented the partial differential operator $A(x,y,\mathcal{D}'_{x,y})$ by means of two-dimensional fractional Fourier transform. If we replace the symbol $A(x,y,\xi,\zeta)$ by a more general symbol $a(x,y,\xi,\zeta)$ which is no longer polynomial in ξ,ζ, we obtain operators more general than a partial differential operator. The operators so obtained are called pseudo-differential operators.

Pseudo-differential perator associated with $\mathcal{F}_{\alpha,\beta}$: Following the similar procedure, a linear partial differential operator $G(x,y,\Delta'_{x,y})$ of order m on $\mathbb{R}^2$ is given by

$$G(x,y,\Delta'_{x,y}) = \sum_{r=0}^{m} g_r(x,y)(\Delta'_{x,y})^r, \tag{24}$$

If we replace $(\Delta'_{x,y})^r$ in (24) by monomial $[(b-c)\xi+(b+c)\delta]^r$ in $\mathbb{R}^2$, then we obtain the so-called symbol

$$G(x,y,\xi,\delta) = \sum_{r=0}^{m} g_r(x,y)[(b-c)\xi+(b+c)\delta]^r. \tag{25}$$

In order to obtain another representation of the operator $G(x,y,\Delta'_{x,y})$, let us take any function $\phi \in \mathcal{S}(\mathbb{R}^2)$; then, by (7), (8) and Proposition 3(ii), we have

$$
\begin{aligned}
(G(x,y,\Delta'_{x,y})\phi)(x,y) &= \sum_{r=0}^{m} g_r(x,y)\mathcal{F}^{-1}_{\alpha,\beta}\mathcal{F}_{\alpha,\beta}(\Delta'_{x,y})^r\phi(x,y) \\
&= \int_{\mathbb{R}^2} \mathcal{K}_{-\alpha,-\beta}(\xi,\delta,x,y)G(x,y,\xi,\delta)(\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)d\xi d\delta,
\end{aligned}
$$

where $\mathcal{K}_{-\alpha,-\beta}(\xi,\delta,x,y)$ is as (8).

So, we have expressed the partial differential operator $G(x,y,\Delta'_{x,y})$ by means of the coupled fractional Fourier transform. If we replace the symbol $G(x,y,\xi,\delta)$ by the more general symbol $g(x,y,\xi,\delta)$, that satisfies a certain growth condition, which is no longer polynomial in ξ,δ, so we obtain operators more general than partial differential operators. The operators so obtained are called pseudo-differential operators associated with coupled fractional Fourier transform. We see that this operator is more generalized than the operator defined by means of two-dimensional fractional Fourier transform $\mathfrak{F}_{\alpha_1,\alpha_2}$.

Definition 4. *Let $m_1,m_2 \in \mathbb{R}$. Then, we define symbol class TS^{m_1,m_2} to be the set of all functions $g(x,y,\xi,\delta) \in C^\infty(\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R})$ such that for any natural numbers s_1,s_2,s_3,s_4, there exists a non-negative constant C_{s_1,s_2,s_3,s_4} depending on s_1,s_2,s_3,s_4 only, such that*

$$
\begin{aligned}
&\left|\frac{\partial^{s_1}}{\partial x^{s_1}}\frac{\partial^{s_2}}{\partial y^{s_2}}\frac{\partial^{s_3}}{\partial \xi^{s_3}}\frac{\partial^{s_4}}{\partial \delta^{s_4}}g(x,y,\xi,\delta)\right| \\
&\leq C_{s_1,s_2,s_3,s_4}(1+|\xi|)^{m_1-s_3}(1+|\delta|)^{m_2-s_4},
\end{aligned} \tag{26}
$$

for all $x,y,\xi,\delta \in \mathbb{R}$.

Definition 5. *Let g be a symbol satisfying (26). Then, the pseudo-differential operator $G_{g,\alpha,\beta}$ is defined by*

$$(G_{g,\alpha,\beta}\phi)(x,y) = \int_{\mathbb{R}^2} \mathcal{K}_{-\alpha,-\beta}(\xi,\delta,x,y)g(x,y,\xi,\delta)(\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)d\xi d\delta. \tag{27}$$

For the sake of the study of continuity of the pseudo-differential operator $G_{g,\alpha,\beta}$, we need to redefine the Schwartz-type space as follows:

Definition 6. *An infinitely differentiable complex valued function $\phi(x,y)$ is a member of $\mathcal{S}_{b,c}(\mathbb{R}^2)$ if for every choice of $\beta_1,\beta_2,\gamma_1,\gamma_2 \in \mathbb{N}_0$, it satisfies*

$$\Upsilon^{\gamma_1,\gamma_2}_{\beta_1,\beta_2}(\phi) = \sup_{(x,y)\in A_{b,c}(\mathbb{R}^2)} \left| x^{\gamma_1} k_{\gamma_2,\beta_1}(y) \frac{\partial^{\beta_1}}{\partial x^{\beta_1}} \frac{\partial^{\beta_2}}{\partial y^{\beta_2}} \phi(x,y) \right| < \infty, \tag{28}$$

where $A_{b,c}(\mathbb{R}^2) = \{(x,y) \in \mathbb{R}^2 : |bx-cy| \geq |bx|$ and $|by+cx| \geq |by|\}$ and

$$k_{\gamma_2,\beta_1}(y) = \begin{cases} y^{\gamma_2}, \text{if } |y| > 1 \\ y^{\gamma_2+\beta_1}, \text{if } |y| \leq 1. \end{cases}$$

We shall make use of the following Lemma 2 to prove Theorem 3.

Lemma 2. *Let $\beta_1^{'},\gamma_2,s_2$ be natural numbers; then, we have*

$$\begin{aligned}
&\frac{\partial^{\beta_1^{'}}}{\partial x^{\beta_1^{'}}}\left[(-by-cx)^{-(\gamma_2+s_2)}e^{a(\xi^2+\delta^2+x^2+y^2)-b(\xi x+\delta y)-c(\delta x-\xi y)}\right] \\
&= \sum_{\beta_1^{''}=0}^{\beta_1^{'}} \binom{\beta_1^{'}}{\beta_1^{''}} \sum_{\beta_1^{'''}=0}^{\beta_1^{''}} \binom{\beta_1^{''}}{\beta_1^{'''}} e^{a(\xi^2+\delta^2+x^2+y^2)-b(\xi x+\delta y)-c(\delta x-\xi y)} \sum_{s_1=0}^{\beta_1^{'''}} a_{s_1}(\cot\gamma)x^{s_1} \\
&\quad \times(-b\xi-c\delta)^{\beta_1^{''}-\beta_1^{'''}} c^{\beta_1^{'}-\beta_1^{''}} \frac{(\gamma_2+s_2+\beta_1^{'}-\beta_1^{''}-1)!}{(\gamma_2+s_2-1)!}(-by-cx)^{-(\gamma_2+s_2+\beta_1^{'}-\beta_1^{''})}.
\end{aligned}$$

Proof. Using the Leibnitz formula, we obtain

$$\begin{aligned}
&\frac{\partial^{\beta_1^{'}}}{\partial x^{\beta_1^{'}}}\left[(-by-cx)^{-(\gamma_2+s_2)}e^{a(\xi^2+\delta^2+x^2+y^2)-b(\xi x+\delta y)-c(\delta x-\xi y)}\right] \\
&= \sum_{\beta_1^{''}=0}^{\beta_1^{'}} \binom{\beta_1^{'}}{\beta_1^{''}} \frac{\partial^{\beta_1^{''}}}{\partial x^{\beta_1^{''}}}\left[e^{a(\xi^2+\delta^2+x^2+y^2)-b(\xi x+\delta y)-c(\delta x-\xi y)}\right] \\
&\quad \times \frac{\partial^{\beta_1^{'}-\beta_1^{''}}}{\partial x^{\beta_1^{'}-\beta_1^{''}}}\left((-by-cx)^{-(\gamma_2+s_2)}\right) \\
&= \sum_{\beta_1^{''}=0}^{\beta_1^{'}} \binom{\beta_1^{'}}{\beta_1^{''}} \sum_{\beta_1^{'''}=0}^{\beta_1^{''}} \binom{\beta_1^{''}}{\beta_1^{'''}} \frac{\partial^{\beta_1^{'''}}}{\partial x^{\beta_1^{'''}}}\left[e^{a(\xi^2+\delta^2+x^2+y^2)}\right] \frac{\partial^{\beta_1^{''}-\beta_1^{'''}}}{\partial x^{\beta_1^{''}-\beta_1^{'''}}}\left[e^{-b(\xi x+\delta y)-c(\delta x-\xi y)}\right] \\
&\quad \times \frac{\partial^{\beta_1^{'}-\beta_1^{''}}}{\partial x^{\beta_1^{'}-\beta_1^{''}}}\left((-by-cx)^{-(\gamma_2+s_2)}\right) \\
&= \sum_{\beta_1^{''}=0}^{\beta_1^{'}} \binom{\beta_1^{'}}{\beta_1^{''}} \sum_{\beta_1^{'''}=0}^{\beta_1^{''}} \binom{\beta_1^{''}}{\beta_1^{'''}} e^{a(\xi^2+\delta^2+x^2+y^2)-b(\xi x+\delta y)-c(\delta x-\xi y)} \sum_{s_1=0}^{\beta_1^{'''}} a_{s_1}(\cot\gamma)x^{s_1} \\
&\quad \times(-b\xi-c\delta)^{\beta_1^{''}-\beta_1^{'''}} c^{\beta_1^{'}-\beta_1^{''}} \frac{(\gamma_2+s_2+\beta_1^{'}-\beta_1^{''}-1)!}{(\gamma_2+s_2-1)!}(-by-cx)^{-(\gamma_2+s_2+\beta_1^{'}-\beta_1^{''})}.
\end{aligned}$$

This completes the proof. □

Lemma 3. *Let g be a symbol belonging to TS^{m_1,m_2} and $G_{g,\alpha,\beta}$ be the pseudo-differential operator defined in (27). Then, for any $\beta_2 \in \mathbb{N}$, we have*

$$
\begin{aligned}
&\frac{\partial^{\beta_2}}{\partial y^{\beta_2}}(G_{g,\alpha,\beta}\phi)(x,y)\\
=\;& d(-\gamma)(-1)^{(\gamma_2+s_2)}\int_{\mathbb{R}^2}\sum_{\beta_2'=0}^{\beta_2}\binom{\beta_2}{\beta_2'}\sum_{\beta_2''=0}^{\beta_2'}\binom{\beta_2'}{\beta_2''}\sum_{s_2=0}^{\beta_2''}a_{s_2}(\cot\gamma)y^{s_2}(-by-cx)^{-(\gamma_2+s_2)}\\
&\times e^{-b(\xi x+\delta y)-c(\delta x-\xi y)}\sum_{t_2=0}^{\gamma_2+s_2}\binom{\gamma_2+s_2}{t_2}\sum_{t_2'=0}^{t_2}\binom{t_2}{t_2'}e^{a(\xi^2+\delta^2+x^2+y^2)}\sum_{t_2''=0}^{t_2'}a_{t_2''}(\cot\gamma)\delta^{t_2''}\\
&\times\frac{\partial^{t_2-t_2'}}{\partial\delta^{t_2-t_2'}}\frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}}\{g(x,y,\xi,\delta)\}\frac{\partial^{\gamma_2+s_2-t_2}}{\partial\delta^{\gamma_2+s_2-t_2}}\left[(c\xi-b\delta)^{(\beta_2'-\beta_2'')}(\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)\right]d\xi d\delta,
\end{aligned}
$$

where $a_{s_2}, a_{t_2''}$ are constants.

Proof. Using (27), we have

$$
\begin{aligned}
&\frac{\partial^{\beta_2}}{\partial y^{\beta_2}}(G_{g,\alpha,\beta}\phi)(x,y)\\
=\;&\int_{\mathbb{R}^2}\frac{\partial^{\beta_2}}{\partial y^{\beta_2}}[\mathcal{K}_{-\alpha,-\beta}(\xi,\delta,x,y)g(x,y,\xi,\delta)](\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)d\xi d\delta\\
=\;& d(-\gamma)\int_{\mathbb{R}^2}\sum_{\beta_2'=0}^{\beta_2}\binom{\beta_2}{\beta_2'}\frac{\partial^{\beta_2'}}{\partial y^{\beta_2'}}[\mathcal{K}_{-\alpha,-\beta}(\xi,\delta,x,y)]\frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}}[g(x,y,\xi,\delta)]\\
&\times(\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)d\xi d\delta\\
=\;& d(-\gamma)\int_{\mathbb{R}^2}\sum_{\beta_2'=0}^{\beta_2}\binom{\beta_2}{\beta_2'}\frac{\partial^{\beta_2'}}{\partial y^{\beta_2'}}[e^{a(\xi^2+\delta^2+x^2+y^2)-b(\xi x+\delta y)-c(\delta x-\xi y)}]\\
&\times\frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}}[g(x,y,\xi,\delta)](\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)d\xi d\delta\\
=\;& d(-\gamma)\int_{\mathbb{R}^2}\sum_{\beta_2'=0}^{\beta_2}\binom{\beta_2}{\beta_2'}\sum_{\beta_2''=0}^{\beta_2'}\binom{\beta_2'}{\beta_2''}\frac{\partial^{\beta_2''}}{\partial y^{\beta_2''}}[e^{a(\xi^2+\delta^2+x^2+y^2)}]\frac{\partial^{\beta_2'-\beta_2''}}{\partial y^{\beta_2'-\beta_2''}}[e^{-b(\xi x+\delta y)}\\
&\times e^{-c(\delta x-\xi y)}]\frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}}[g(x,y,\xi,\delta)](\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)d\xi d\delta\\
=\;& d(-\gamma)\int_{\mathbb{R}^2}\sum_{\beta_2'=0}^{\beta_2}\binom{\beta_2}{\beta_2'}\sum_{\beta_2''=0}^{\beta_2'}\binom{\beta_2'}{\beta_2''}e^{a(\xi^2+\delta^2+x^2+y^2)}\sum_{s_2=0}^{\beta_2''}a_{s_2}(\cot\gamma)y^{s_2}(c\xi-b\delta)^{(\beta_2'-\beta_2'')}\\
&\times e^{-b(\xi x+\delta y)-c(\delta x-\xi y)}\frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}}[g(x,y,\xi,\delta)](\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)d\xi d\delta,
\end{aligned}
$$

where a_{s_2} are constants.

It is easy to see that $\frac{\partial^{\gamma_2+s_2}}{\partial\delta^{\gamma_2+s_2}}[e^{-b(\xi x+\delta y)-c(\delta x-\xi y)}] = (-by-cx)^{\gamma_2+s_2}e^{-b(\xi x+\delta y)-c(\delta x-\xi y)}$. Then, the above estimate becomes

$$
\begin{aligned}
= \;& d(-\gamma)\int_{\mathbb{R}^2}\sum_{\beta_2'=0}^{\beta_2}\binom{\beta_2}{\beta_2'}\sum_{\beta_2''=0}^{\beta_2'}\binom{\beta_2'}{\beta_2''}\sum_{s_2=0}^{\beta_2''}a_{s_2}(\cot\gamma)y^{s_2}(c\xi-b\delta)^{(\beta_2'-\beta_2'')}e^{a(\xi^2+\delta^2+x^2+y^2)}\\
&\times(-by-cx)^{-(\gamma_2+s_2)}\frac{\partial^{\gamma_2+s_2}}{\partial\delta^{\gamma_2+s_2}}[e^{-b(\xi x+\delta y)-c(\delta x-\xi y)}]\frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}}[g(x,y,\xi,\delta)](\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)d\xi d\delta\\
= \;& d(-\gamma)(-1)^{(\gamma_2+s_2)}\int_{\mathbb{R}^2}\sum_{\beta_2'=0}^{\beta_2}\binom{\beta_2}{\beta_2'}\sum_{\beta_2''=0}^{\beta_2'}\binom{\beta_2'}{\beta_2''}\sum_{s_2=0}^{\beta_2''}a_{s_2}(\cot\gamma)y^{s_2}(-by-cx)^{-(\gamma_2+s_2)}\\
&\times e^{-b(\xi x+\delta y)-c(\delta x-\xi y)}\frac{\partial^{\gamma_2+s_2}}{\partial\delta^{\gamma_2+s_2}}\Big[(c\xi-b\delta)^{(\beta_2'-\beta_2'')}e^{a(\xi^2+\delta^2+x^2+y^2)}\\
&\times\frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}}\{g(x,y,\xi,\delta)\}(\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)\Big]d\xi d\delta\\
= \;& d(-\gamma)(-1)^{(\gamma_2+s_2)}\int_{\mathbb{R}^2}\sum_{\beta_2'=0}^{\beta_2}\binom{\beta_2}{\beta_2'}\sum_{\beta_2''=0}^{\beta_2'}\binom{\beta_2'}{\beta_2''}\sum_{s_2=0}^{\beta_2''}a_{s_2}(\cot\gamma)y^{s_2}(-by-cx)^{-(\gamma_2+s_2)}\\
&\times e^{-b(\xi x+\delta y)-c(\delta x-\xi y)}\sum_{t_2=0}^{\gamma_2+s_2}\binom{\gamma_2+s_2}{t_2}\frac{\partial^{t_2}}{\partial\delta^{t_2}}\left[e^{a(\xi^2+\delta^2+x^2+y^2)}\frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}}g(x,y,\xi,\delta)\right]\\
&\times\frac{\partial^{\gamma_2+s_2-t_2}}{\partial\delta^{\gamma_2+s_2-t_2}}\left[(c\xi-b\delta)^{(\beta_2'-\beta_2'')}(\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)\right]d\xi d\delta\\
= \;& d(-\gamma)(-1)^{(\gamma_2+s_2)}\int_{\mathbb{R}^2}\sum_{\beta_2'=0}^{\beta_2}\binom{\beta_2}{\beta_2'}\sum_{\beta_2''=0}^{\beta_2'}\binom{\beta_2'}{\beta_2''}\sum_{s_2=0}^{\beta_2''}a_{s_2}(\cot\gamma)y^{s_2}(-by-cx)^{-(\gamma_2+s_2)}\\
&\times e^{-b(\xi x+\delta y)-c(\delta x-\xi y)}\sum_{t_2=0}^{\gamma_2+s_2}\binom{\gamma_2+s_2}{t_2}\sum_{t_2'=0}^{t_2}\binom{t_2}{t_2'}\frac{\partial^{t_2'}}{\partial\delta^{t_2'}}\{e^{a(\xi^2+\delta^2+x^2+y^2)}\}\\
&\times\frac{\partial^{t_2-t_2'}}{\partial\delta^{t_2-t_2'}}\frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}}\{g(x,y,\xi,\delta)\}\frac{\partial^{\gamma_2+s_2-t_2}}{\partial\delta^{\gamma_2+s_2-t_2}}\left[(c\xi-b\delta)^{(\beta_2'-\beta_2'')}(\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)\right]d\xi d\delta\\
= \;& d(-\gamma)(-1)^{(\gamma_2+s_2)}\int_{\mathbb{R}^2}\sum_{\beta_2'=0}^{\beta_2}\binom{\beta_2}{\beta_2'}\sum_{\beta_2''=0}^{\beta_2'}\binom{\beta_2'}{\beta_2''}\sum_{s_2=0}^{\beta_2''}a_{s_2}(\cot\gamma)y^{s_2}(-by-cx)^{-(\gamma_2+s_2)}\\
&\times e^{-b(\xi x+\delta y)-c(\delta x-\xi y)}\sum_{t_2=0}^{\gamma_2+s_2}\binom{\gamma_2+s_2}{t_2}\sum_{t_2'=0}^{t_2}\binom{t_2}{t_2'}e^{a(\xi^2+\delta^2+x^2+y^2)}\sum_{t_2''=0}^{t_2'}a_{t_2''}(\cot\gamma)\delta^{t_2''}\\
&\times\frac{\partial^{t_2-t_2'}}{\partial\delta^{t_2-t_2'}}\frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}}\{g(x,y,\xi,\delta)\}\frac{\partial^{\gamma_2+s_2-t_2}}{\partial\delta^{\gamma_2+s_2-t_2}}\left[(c\xi-b\delta)^{(\beta_2'-\beta_2'')}(\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)\right]d\xi d\delta,
\end{aligned}
$$

where $a_{t_2''}$ are constants.

This completes the proof. □

Lemma 4. *Let g be a symbol belonging to TS^{m_1,m_2} and $G_{g,\alpha,\beta}$ be the pseudo-differential operator defined in (27). Then, for $\beta_1,\beta_2\in\mathbb{N}$, we have*

$$
\begin{aligned}
& \frac{\partial^{\beta_1}}{\partial x^{\beta_1}} \frac{\partial^{\beta_2}}{\partial y^{\beta_2}} (G_{g,\alpha,\beta}\phi)(x,y) \\
= \; & d(-\gamma)(-1)^{(\gamma_1+s_1+\gamma_2+s_2)} \cot^3\gamma \int_{\mathbb{R}^2} \sum_{\beta_2'=0}^{\beta_2} \binom{\beta_2}{\beta_2'} \sum_{\beta_2''=0}^{\beta_2'} \binom{\beta_2'}{\beta_2''} \sum_{s_2=0}^{\beta_2''} a_{s_2} y^{s_2} \\
& \times \sum_{t_2=0}^{\gamma_2+s_2} \binom{\gamma_2+s_2}{t_2} \sum_{t_2'=0}^{t_2} \binom{t_2}{t_2'} \sum_{t_2''=0}^{t_2'} a_{t_2''} \delta^{t_2''} \sum_{\beta_1'=0}^{\beta_1} \binom{\beta_1}{\beta_1'} \sum_{\beta_1''=0}^{\beta_1'} \binom{\beta_1'}{\beta_1''} \\
& \times \sum_{\beta_1'''=0}^{\beta_1''} \binom{\beta_1''}{\beta_1'''} \sum_{s_1=0}^{\beta_1'''} a_{s_1} x^{s_1} (-bx+cy)^{-(\gamma_1+s_1)} (-by-cx)^{-(\gamma_2+s_2+\beta_1'-\beta_1'')} \\
& \times c^{\beta_1'-\beta_1''} \frac{(\gamma_2+s_2+\beta_1'-\beta_1''-1)!}{(\gamma_2+s_2-1)!} e^{-b(\xi x+\delta y)-c(\delta x-\xi y)} \frac{\partial^{\gamma_1+s_1}}{\partial \xi^{\gamma_1+s_1}} \Big[\\
& \times \frac{\partial^{\gamma_2+s_2-t_2}}{\partial \delta^{\gamma_2+s_2-t_2}} \{(c\xi-b\delta)^{(\beta_2'-\beta_2'')} (\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)\} e^{a(\xi^2+\delta^2+x^2+y^2)} (-b\xi-c\delta)^{\beta_1''-\beta_1'''} \\
& \times \frac{\partial^{\beta_1-\beta_1'}}{\partial x^{\beta_1-\beta_1'}} \frac{\partial^{t_2-t_2'}}{\partial \delta^{t_2-t_2'}} \frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}} \{g(x,y,\xi,\delta)\} \Big] d\xi d\delta.
\end{aligned}
$$

Proof. Exploiting (27) and Lemma 3, we have

$$
\begin{aligned}
& \frac{\partial^{\beta_1}}{\partial x^{\beta_1}} \frac{\partial^{\beta_2}}{\partial y^{\beta_2}} (G_{g,\alpha,\beta}\phi)(x,y) \\
= \; & d(-\gamma)(-1)^{(\gamma_2+s_2)} \cot^2\gamma \int_{\mathbb{R}^2} \sum_{\beta_2'=0}^{\beta_2} \binom{\beta_2}{\beta_2'} \sum_{\beta_2''=0}^{\beta_2'} \binom{\beta_2'}{\beta_2''} \sum_{s_2=0}^{\beta_2''} a_{s_2} y^{s_2} \sum_{t_2=0}^{\gamma_2+s_2} \binom{\gamma_2+s_2}{t_2} \\
& \times \sum_{t_2'=0}^{t_2} \binom{t_2}{t_2'} \Big(\sum_{t_2''=0}^{t_2'} a_{t_2''} \delta^{t_2''} \Big) \frac{\partial^{\gamma_2+s_2-t_2}}{\partial \delta^{\gamma_2+s_2-t_2}} \Big[(c\xi-b\delta)^{(\beta_2'-\beta_2'')} (\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta) \Big] \\
& \times \frac{\partial^{\beta_1}}{\partial x^{\beta_1}} \Big[(-by-cx)^{-(\gamma_2+s_2)} e^{a(\xi^2+\delta^2+x^2+y^2)-b(\xi x+\delta y)-c(\delta x-\xi y)} \\
& \times \frac{\partial^{t_2-t_2'}}{\partial \delta^{t_2-t_2'}} \frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}} \{g(x,y,\xi,\delta)\} \Big] d\xi d\delta \\
= \; & d(-\gamma)(-1)^{(\gamma_2+s_2)} \cot^2\gamma \int_{\mathbb{R}^2} \sum_{\beta_2'=0}^{\beta_2} \binom{\beta_2}{\beta_2'} \sum_{\beta_2''=0}^{\beta_2'} \binom{\beta_2'}{\beta_2''} \sum_{s_2=0}^{\beta_2''} a_{s_2} y^{s_2} \sum_{t_2=0}^{\gamma_2+s_2} \binom{\gamma_2+s_2}{t_2} \\
& \times \sum_{t_2'=0}^{t_2} \binom{t_2}{t_2'} \sum_{t_2''=0}^{t_2'} a_{t_2''} \delta^{t_2''} \frac{\partial^{\gamma_2+s_2-t_2}}{\partial \delta^{\gamma_2+s_2-t_2}} \Big[(c\xi-b\delta)^{(\beta_2'-\beta_2'')} (\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta) \Big] \\
& \times \sum_{\beta_1'=0}^{\beta_1} \binom{\beta_1}{\beta_1'} \frac{\partial^{\beta_1'}}{\partial x^{\beta_1'}} \Big[(-by-cx)^{-(\gamma_2+s_2)} e^{a(\xi^2+\delta^2+x^2+y^2)-b(\xi x+\delta y)-c(\delta x-\xi y)} \Big] \\
& \times \frac{\partial^{\beta_1-\beta_1'}}{\partial x^{\beta_1-\beta_1'}} \frac{\partial^{t_2-t_2'}}{\partial \delta^{t_2-t_2'}} \frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}} \{g(x,y,\xi,\delta)\} d\xi d\delta.
\end{aligned}
$$

Now, using Lemma 2, the above estimate becomes

$$\begin{aligned}
&\frac{\partial^{\beta_1}}{\partial x^{\beta_1}}\frac{\partial^{\beta_2}}{\partial y^{\beta_2}}(G_{g,\alpha,\beta}\phi)(x,y)\\
=\;& d(-\gamma)(-1)^{(\gamma_2+s_2)}\cot^3\gamma\int_{\mathbb{R}^2}\sum_{\beta_2'=0}^{\beta_2}\binom{\beta_2}{\beta_2'}\sum_{\beta_2''=0}^{\beta_2'}\binom{\beta_2'}{\beta_2''}\sum_{s_2=0}^{\beta_2''}a_{s_2}y^{s_2}\sum_{t_2=0}^{\gamma_2+s_2}\binom{\gamma_2+s_2}{t_2}\\
&\times\sum_{t_2'=0}^{t_2}\binom{t_2}{t_2'}\sum_{t_2''=0}^{t_2'}a_{t_2''}\delta^{t_2''}\frac{\partial^{\gamma_2+s_2-t_2}}{\partial\delta^{\gamma_2+s_2-t_2}}\left[(c\xi-b\delta)^{(\beta_2'-\beta_2'')}(\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)\right]\\
&\times\sum_{\beta_1'=0}^{\beta_1}\binom{\beta_1}{\beta_1'}\sum_{\beta_1''=0}^{\beta_1'}\binom{\beta_1'}{\beta_1''}\sum_{\beta_1'''=0}^{\beta_1''}\binom{\beta_1''}{\beta_1'''}e^{a(\xi^2+\delta^2+x^2+y^2)}\sum_{s_1=0}^{\beta_1'''}a_{s_1}x^{s_1}e^{-b(\xi x+\delta y)}\\
&\times e^{-c(\delta x-\xi y)}(-b\xi-c\delta)^{\beta_1''-\beta_1'''}c^{\beta_1'-\beta_1''}\frac{(\gamma_2+s_2+\beta_1'-\beta_1''-1)!}{(\gamma_2+s_2-1)!}\\
&\times(-by-cx)^{-(\gamma_2+s_2+\beta_1'-\beta_1'')}\frac{\partial^{\beta_1-\beta_1'}}{\partial x^{\beta_1-\beta_1'}}\frac{\partial^{t_2-t_2'}}{\partial\delta^{t_2-t_2'}}\frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}}\{g(x,y,\xi,\delta)\}d\xi d\delta,
\end{aligned}$$

where a_{s_1} is constant.

Since $\frac{\partial^{\gamma_1+s_1}}{\partial\xi^{\gamma_1+s_1}}\{e^{-b(\xi x+\delta y)-c(\delta x-\xi y)}\} = (-bx+cy)^{\gamma_1+s_1}e^{-b(\xi x+\delta y)-c(\delta x-\xi y)}$, then the above expression can be rewritten as

$$\begin{aligned}
=\;& d(-\gamma)(-1)^{(\gamma_2+s_2)}\cot^3\gamma\int_{\mathbb{R}^2}\sum_{\beta_2'=0}^{\beta_2}\binom{\beta_2}{\beta_2'}\sum_{\beta_2''=0}^{\beta_2'}\binom{\beta_2'}{\beta_2''}\sum_{s_2=0}^{\beta_2''}a_{s_2}y^{s_2}\sum_{t_2=0}^{\gamma_2+s_2}\binom{\gamma_2+s_2}{t_2}\\
&\times\sum_{t_2'=0}^{t_2}\binom{t_2}{t_2'}\sum_{t_2''=0}^{t_2'}a_{t_2''}\delta^{t_2''}\frac{\partial^{\gamma_2+s_2-t_2}}{\partial\delta^{\gamma_2+s_2-t_2}}\left[(c\xi-b\delta)^{(\beta_2'-\beta_2'')}(\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)\right]\\
&\times\sum_{\beta_1'=0}^{\beta_1}\binom{\beta_1}{\beta_1'}\sum_{\beta_1''=0}^{\beta_1'}\binom{\beta_1'}{\beta_1''}\sum_{\beta_1'''=0}^{\beta_1''}\binom{\beta_1''}{\beta_1'''}\sum_{s_1=0}^{\beta_1'''}a_{s_1}x^{s_1}e^{a(\xi^2+\delta^2+x^2+y^2)}c^{\beta_1'-\beta_1''}\\
&\times(-bx+cy)^{-(\gamma_1+s_1)}\frac{\partial^{\gamma_1+s_1}}{\partial\xi^{\gamma_1+s_1}}\left[e^{-b(\xi x+\delta y)-c(\delta x-\xi y)}\right]\frac{(\gamma_2+s_2+\beta_1'-\beta_1''-1)!}{(\gamma_2+s_2-1)!}\\
&\times(-b\xi-c\delta)^{\beta_1''-\beta_1'''}(-by-cx)^{-(\gamma_2+s_2+\beta_1'-\beta_1'')}\\
&\times\frac{\partial^{\beta_1-\beta_1'}}{\partial x^{\beta_1-\beta_1'}}\frac{\partial^{t_2-t_2'}}{\partial\delta^{t_2-t_2'}}\frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}}\{g(x,y,\xi,\delta)\}d\xi d\delta\\
=\;& d(-\gamma)(-1)^{(\gamma_1+s_1+\gamma_2+s_2)}\cot^3\gamma\int_{\mathbb{R}^2}\sum_{\beta_2'=0}^{\beta_2}\binom{\beta_2}{\beta_2'}\sum_{\beta_2''=0}^{\beta_2'}\binom{\beta_2'}{\beta_2''}\sum_{s_2=0}^{\beta_2''}a_{s_2}y^{s_2}\\
&\times\sum_{t_2=0}^{\gamma_2+s_2}\binom{\gamma_2+s_2}{t_2}\sum_{t_2'=0}^{t_2}\binom{t_2}{t_2'}\sum_{t_2''=0}^{t_2'}a_{t_2''}\delta^{t_2''}\sum_{\beta_1'=0}^{\beta_1}\binom{\beta_1}{\beta_1'}\sum_{\beta_1''=0}^{\beta_1'}\binom{\beta_1'}{\beta_1''}\\
&\times\sum_{\beta_1'''=0}^{\beta_1''}\binom{\beta_1''}{\beta_1'''}\sum_{s_1=0}^{\beta_1'''}a_{s_1}x^{s_1}(-bx+cy)^{-(\gamma_1+s_1)}(-by-cx)^{-(\gamma_2+s_2+\beta_1'-\beta_1'')}\\
&\times c^{\beta_1'-\beta_1''}\frac{(\gamma_2+s_2+\beta_1'-\beta_1''-1)!}{(\gamma_2+s_2-1)!}e^{-b(\xi x+\delta y)-c(\delta x-\xi y)}\frac{\partial^{\gamma_1+s_1}}{\partial\xi^{\gamma_1+s_1}}\Big[
\end{aligned}$$

$$\times \frac{\partial^{\gamma_2+s_2-t_2}}{\partial \delta^{\gamma_2+s_2-t_2}}\{(c\xi - b\delta)^{(\beta_2'-\beta_2'')}(\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)\}e^{a(\xi^2+\delta^2+x^2+y^2)}(-b\xi - c\delta)^{\beta_1''-\beta_1'''}$$
$$\times \frac{\partial^{\beta_1-\beta_1'}}{\partial x^{\beta_1-\beta_1'}}\frac{\partial^{t_2-t_2'}}{\partial \delta^{t_2-t_2'}}\frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}}\{g(x,y,\xi,\delta)\}\Big]d\xi d\delta. \tag{29}$$

This completes the proof. □

Theorem 3. *Let g be a symbol belonging to TS^{m_1,m_2}, with $m_1, m_2 < -1$. Then, the pseudo-differential operator $G_{g,\alpha,\beta}$ defined in (27) is a continuous linear mapping from $\mathcal{S}_{b,c}(\mathbb{R}^2)$ into itself.*

Proof. Let $\phi \in S(\mathbb{R}^2)$. Then, for any four non-negative integers $\beta_1, \beta_2, \gamma_1, \gamma_2$, we need to verify that

$$\sup_{(x,y)\in A_{b,c}(\mathbb{R}^2)} |x^{\gamma_1}k_{\gamma_2,\beta_1}(y)\frac{\partial^{\beta_1}}{\partial x^{\beta_1}}\frac{\partial^{\beta_2}}{\partial y^{\beta_2}}(G_{g,\alpha,\beta}\phi)(x,y)| < \infty.$$

To verify the above inequality, note that from (29)

$$\frac{\partial^{\gamma_1+s_1}}{\partial \xi^{\gamma_1+s_1}}\Big[\frac{\partial^{\gamma_2+s_2-t_2}}{\partial \delta^{\gamma_2+s_2-t_2}}\{(c\xi - b\delta)^{(\beta_2'-\beta_2'')}(\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)\}e^{a(\xi^2+\delta^2+x^2+y^2)}$$
$$\times (-b\xi - c\delta)^{\beta_1''-\beta_1'''}\frac{\partial^{\beta_1-\beta_1'}}{\partial x^{\beta_1-\beta_1'}}\frac{\partial^{t_2-t_2'}}{\partial \delta^{t_2-t_2'}}\frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}}g(x,y,\xi,\delta)\Big]$$
$$= \sum_{t_1=0}^{\gamma_1+s_1}\binom{\gamma_1+s_1}{t_1}\frac{\partial^{t_1}}{\partial \xi^{t_1}}\Big[e^{a(\xi^2+\delta^2+x^2+y^2)}(-b\xi - c\delta)^{\beta_1''-\beta_1'''}\frac{\partial^{\beta_1-\beta_1'}}{\partial x^{\beta_1-\beta_1'}}\frac{\partial^{t_2-t_2'}}{\partial \delta^{t_2-t_2'}}$$
$$\times \frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}}g(x,y,\xi,\delta)\Big]\frac{\partial^{\gamma_1+s_1-t_1}}{\partial \xi^{\gamma_1+s_1-t_1}}\frac{\partial^{\gamma_2+s_2-t_2}}{\partial \delta^{\gamma_2+s_2-t_2}}\{(c\xi - b\delta)^{(\beta_2'-\beta_2'')}(\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)\}$$
$$= \sum_{t_1=0}^{\gamma_1+s_1}\binom{\gamma_1+s_1}{t_1}\sum_{t_1'=0}^{t_1}\binom{t_1}{t_1'}\frac{\partial^{t_1'}}{\partial \xi^{t_1'}}\{e^{a(\xi^2+\delta^2+x^2+y^2)}(-b\xi - c\delta)^{\beta_1''-\beta_1'''}\}$$
$$\times \frac{\partial^{t_1-t_1'}}{\partial \xi^{t_1-t_1'}}\frac{\partial^{\beta_1-\beta_1'}}{\partial x^{\beta_1-\beta_1'}}\frac{\partial^{t_2-t_2'}}{\partial \delta^{t_2-t_2'}}\frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}}\{g(x,y,\xi,\delta)\}$$
$$\times \frac{\partial^{\gamma_1+s_1-t_1}}{\partial \xi^{\gamma_1+s_1-t_1}}\frac{\partial^{\gamma_2+s_2-t_2}}{\partial \delta^{\gamma_2+s_2-t_2}}\{(c\xi - b\delta)^{(\beta_2'-\beta_2'')}(\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)\}$$
$$= \sum_{t_1=0}^{\gamma_1+s_1}\binom{\gamma_1+s_1}{t_1}\sum_{t_1'=0}^{t_1}\binom{t_1}{t_1'}\sum_{t_1''=0}^{t_1'}\binom{t_1'}{t_1''}\frac{\partial^{t_1'}}{\partial \xi^{t_1'}}\{e^{a(\xi^2+\delta^2+x^2+y^2)}\}$$
$$\times \frac{\partial^{t_1'-t_1''}}{\partial \xi^{t_1'-t_1''}}\{(-b\xi - c\delta)^{\beta_1''-\beta_1'''}\}\frac{\partial^{t_1-t_1'}}{\partial \xi^{t_1-t_1'}}\frac{\partial^{\beta_1-\beta_1'}}{\partial x^{\beta_1-\beta_1'}}\frac{\partial^{t_2-t_2'}}{\partial \delta^{t_2-t_2'}}\frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}}\{g(x,y,\xi,\delta)\}$$
$$\times \frac{\partial^{\gamma_1+s_1-t_1}}{\partial \xi^{\gamma_1+s_1-t_1}}\frac{\partial^{\gamma_2+s_2-t_2}}{\partial \delta^{\gamma_2+s_2-t_2}}\{(c\xi - b\delta)^{(\beta_2'-\beta_2'')}(\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)\}$$

$$= \sum_{t_1=0}^{\gamma_1+s_1} \binom{\gamma_1+s_1}{t_1} \sum_{t_1'=0}^{t_1} \binom{t_1}{t_1'} \sum_{t_1''=0}^{t_1'} \binom{t_1'}{t_1''} e^{a(\xi^2+\delta^2+x^2+y^2)} \sum_{t_1'''=0}^{t_1''} a_{t_1'''}(\cot\gamma)\xi^{t_1'''}$$
$$\times (-b)^{t_1'-t_1''} \frac{(\beta_1''-\beta_1''')!}{(\beta_1''-\beta_1'''-t_1'+t_1'')!}(-b\xi-c\delta)^{\beta_1''-\beta_1'''-t_1'+t_1''}$$
$$\times \frac{\partial^{t_1-t_1'}}{\partial\xi^{t_1-t_1'}} \frac{\partial^{\beta_1-\beta_1'}}{\partial x^{\beta_1-\beta_1'}} \frac{\partial^{t_2-t_2'}}{\partial\delta^{t_2-t_2'}} \frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}} \{g(x,y,\xi,\delta)\}$$
$$\times \frac{\partial^{\gamma_1+s_1-t_1}}{\partial\xi^{\gamma_1+s_1-t_1}} \frac{\partial^{\gamma_2+s_2-t_2}}{\partial\delta^{\gamma_2+s_2-t_2}} \{(c\xi-b\delta)^{(\beta_2'-\beta_2'')}(\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)\}, \tag{30}$$

where $a_{t_1'''}$ are constants.
Thus, from (30) and (29), we have

$$\frac{\partial^{\beta_1}}{\partial x^{\beta_1}} \frac{\partial^{\beta_2}}{\partial y^{\beta_2}} (G_{g,\alpha,\beta}\phi)(x,y)$$
$$= d(-\gamma)(-1)^{(\gamma_1+s_1+\gamma_2+s_2)} \cot^4\gamma \int_{\mathbb{R}^2} \sum_{\beta_2'=0}^{\beta_2} \binom{\beta_2}{\beta_2'} \sum_{\beta_2''=0}^{\beta_2'} \binom{\beta_2'}{\beta_2''} \sum_{s_2=0}^{\beta_2''} a_{s_2} y^{s_2}$$
$$\times \sum_{t_2=0}^{\gamma_2+s_2} \binom{\gamma_2+s_2}{t_2} \sum_{t_2'=0}^{t_2} \binom{t_2}{t_2'} \sum_{t_2''=0}^{t_2'} a_{t_2''} \delta^{t_2''} \sum_{\beta_1'=0}^{\beta_1} \binom{\beta_1}{\beta_1'} \sum_{\beta_1''=0}^{\beta_1'} \binom{\beta_1'}{\beta_1''}$$
$$\times \sum_{\beta_1'''=0}^{\beta_1''} \binom{\beta_1''}{\beta_1'''} \sum_{s_1=0}^{\beta_1'''} a_{s_1} x^{s_1} (-bx+cy)^{-(\gamma_1+s_1)} (-by-cx)^{-(\gamma_2+s_2+\beta_1'-\beta_1'')}$$
$$\times c^{\beta_1'-\beta_1''} \frac{(\gamma_2+s_2+\beta_1'-\beta_1''-1)!}{\gamma_2+s_2-1)!} e^{-b(\xi x+\delta y)-c(\delta x-\xi y)} \sum_{t_1=0}^{\gamma_1+s_1} \binom{\gamma_1+s_1}{t_1}$$
$$\times \sum_{t_1'=0}^{t_1} \binom{t_1}{t_1'} \sum_{t_1''=0}^{t_1'} \binom{t_1'}{t_1''} e^{a(\xi^2+\delta^2+x^2+y^2)} \sum_{t_1'''=0}^{t_1''} a_{t_1'''} \xi^{t_1'''}$$
$$\times (-b)^{t_1'-t_1''} \frac{(\beta_1''-\beta_1''')!}{(\beta_1''-\beta_1'''-t_1'+t_1'')!}(-b\xi-c\delta)^{\beta_1''-\beta_1'''-t_1'+t_1''}$$
$$\times \frac{\partial^{t_1-t_1'}}{\partial\xi^{t_1-t_1'}} \frac{\partial^{\beta_1-\beta_1'}}{\partial x^{\beta_1-\beta_1'}} \frac{\partial^{t_2-t_2'}}{\partial\delta^{t_2-t_2'}} \frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}} \{g(x,y,\xi,\delta)\}$$
$$\times \frac{\partial^{\gamma_1+s_1-t_1}}{\partial\xi^{\gamma_1+s_1-t_1}} \frac{\partial^{\gamma_2+s_2-t_2}}{\partial\delta^{\gamma_2+s_2-t_2}} \{(c\xi-b\delta)^{(\beta_2'-\beta_2'')}(\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)\} d\xi d\delta.$$

Therefore,

$$\left| x^{\gamma_1} k_{\gamma_2,\beta_1}(y) \frac{\partial^{\beta_1}}{\partial x^{\beta_1}} \frac{\partial^{\beta_2}}{\partial y^{\beta_2}} (G_{g,\alpha,\beta}\phi)(x,y) \right|$$
$$\leq |d(-\gamma)\cot^4\gamma| \sum_{\beta_2'=0}^{\beta_2} \binom{\beta_2}{\beta_2'} \sum_{\beta_2''=0}^{\beta_2'} \binom{\beta_2'}{\beta_2''} \sum_{s_2=0}^{\beta_2''} |a_{s_2}| \sum_{t_2=0}^{\gamma_2+s_2} \binom{\gamma_2+s_2}{t_2} \sum_{t_2'=0}^{t_2} \binom{t_2}{t_2'}$$

$$\times \sum_{t_2''=0}^{t_2'} |a_{t_2''}| \sum_{\beta_1'=0}^{\beta_1} \binom{\beta_1}{\beta_1'} \sum_{\beta_1''=0}^{\beta_1'} \binom{\beta_1'}{\beta_1''} \sum_{\beta_1'''=0}^{\beta_1''} \binom{\beta_1''}{\beta_1'''} \sum_{s_1=0}^{\beta_1'''} |a_{s_1}|$$
$$\times |c|^{\beta_1'-\beta_1''} \frac{(\gamma_2+s_2+\beta_1'-\beta_1''-1)!}{(\gamma_2+s_2-1)!} \sum_{t_1=0}^{\gamma_1+s_1} \binom{\gamma_1+s_1}{t_1} \sum_{t_1'=0}^{t_1} \binom{t_1}{t_1'} \sum_{t_1''=0}^{t_1'} \binom{t_1'}{t_1''}$$
$$\times \sum_{t_1'''=0}^{t_1''} |a_{t_1'''}||b|^{t_1'-t_1''} \frac{(\beta_1''-\beta_1''')!}{(\beta_1''-\beta_1'''-t_1'+t_1'')!} |x^{s_1+\gamma_1}(bx-cy)^{-(\gamma_1+s_1)} y^{s_2+\gamma_2+B}$$
$$\times (by+cx)^{-(\gamma_2+s_2+\beta_1'-\beta_1'')}| \int_{\mathbb{R}^2} \frac{\partial^{t_1-t_1'}}{\partial \xi^{t_1-t_1'}} \frac{\partial^{\beta_1-\beta_1'}}{\partial x^{\beta_1-\beta_1'}} \frac{\partial^{t_2-t_2'}}{\partial \delta^{t_2-t_2'}} \frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}} \{g(x,y,\xi,\delta)\}$$
$$\times \frac{\partial^{\gamma_1+s_1-t_1}}{\partial \delta^{\gamma_1+s_1-t_1}} \frac{\partial^{\gamma_2+s_2-t_2}}{\partial \delta^{\gamma_2+s_2-t_2}} \{(c\xi-b\delta)^{(\beta_2'-\beta_2'')} (\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)\} (b\xi+c\delta)^{\beta_1''-\beta_1'''-t_1'+t_1''}$$
$$\times \xi^{t_1'''} \delta^{t_2''} d\xi d\delta.$$

Since $|bx-cy| \geq |bx|$ and $|by+cx| \geq |by|$, then the above inequality becomes

$$\begin{aligned}
\leq \quad & |d(-\gamma)\cot^4\gamma| \sum_{\beta_2'=0}^{\beta_2} \binom{\beta_2}{\beta_2'} \sum_{\beta_2''=0}^{\beta_2'} \binom{\beta_2'}{\beta_2''} \sum_{s_2=0}^{\beta_2''} |a_{s_2}| \sum_{t_2=0}^{\gamma_2+s_2} \binom{\gamma_2+s_2}{t_2} \sum_{t_2'=0}^{t_2} \binom{t_2}{t_2'} \\
& \times \sum_{t_2''=0}^{t_2'} |a_{t_2''}| \sum_{\beta_1'=0}^{\beta_1} \binom{\beta_1}{\beta_1'} \sum_{\beta_1''=0}^{\beta_1'} \binom{\beta_1'}{\beta_1''} \sum_{\beta_1'''=0}^{\beta_1''} \binom{\beta_1''}{\beta_1'''} \sum_{s_1=0}^{\beta_1'''} |a_{s_1}| \\
& \times |c|^{\beta_1'-\beta_1''} \frac{(\gamma_2+s_2+\beta_1'-\beta_1''-1)!}{(\gamma_2+s_2-1)!} \sum_{t_1=0}^{\gamma_1+s_1} \binom{\gamma_1+s_1}{t_1} \sum_{t_1'=0}^{t_1} \binom{t_1}{t_1'} \sum_{t_1''=0}^{t_1'} \binom{t_1'}{t_1''} \\
& \times \sum_{t_1'''=0}^{t_1''} |a_{t_1'''}||b|^{t_1'-t_1''-\gamma_1-s_1} \frac{(\beta_1''-\beta_1''')!}{(\beta_1''-\beta_1'''-t_1'+t_1'')!} |y|^{s_2+\gamma_2+B} \\
& \times |by|^{-(\gamma_2+s_2+\beta_1'-\beta_1'')} \int_{\mathbb{R}^2} \frac{\partial^{t_1-t_1'}}{\partial \xi^{t_1-t_1'}} \frac{\partial^{\beta_1-\beta_1'}}{\partial x^{\beta_1-\beta_1'}} \frac{\partial^{t_2-t_2'}}{\partial \delta^{t_2-t_2'}} \frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}} \Big[\\
& \times g(x,y,\xi,\delta)\Big] \frac{\partial^{\gamma_1+s_1-t_1}}{\partial \xi^{\gamma_1+s_1-t_1}} \frac{\partial^{\gamma_2+s_2-t_2}}{\partial \delta^{\gamma_2+s_2-t_2}} \{(c\xi-b\delta)^{(\beta_2'-\beta_2'')} (\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)\} \\
& \times (b\xi+c\delta)^{\beta_1''-\beta_1'''-t_1'+t_1''} \xi^{t_1'''} \delta^{t_2''} d\xi d\delta \\
\leq \quad & |d(-\gamma)\cot^4\gamma| \sum_{\beta_2'=0}^{\beta_2} \binom{\beta_2}{\beta_2'} \sum_{\beta_2''=0}^{\beta_2'} \binom{\beta_2'}{\beta_2''} \sum_{s_2=0}^{\beta_2''} |a_{s_2}| \sum_{t_2=0}^{\gamma_2+s_2} \binom{\gamma_2+s_2}{t_2} \sum_{t_2'=0}^{t_2} \binom{t_2}{t_2'} \\
& \times \sum_{t_2''=0}^{t_2'} |a_{t_2''}| \sum_{\beta_1'=0}^{\beta_1} \binom{\beta_1}{\beta_1'} \sum_{\beta_1''=0}^{\beta_1'} \binom{\beta_1'}{\beta_1''} \sum_{\beta_1'''=0}^{\beta_1''} \binom{\beta_1''}{\beta_1'''} \sum_{s_1=0}^{\beta_1'''} |a_{s_1}||c|^{\beta_1'-\beta_1''}
\end{aligned}$$

$$\times \frac{(\gamma_2+s_2+\beta_1'-\beta_1''-1)!}{(\gamma_2+s_2-1)!} \sum_{t_1=0}^{\gamma_1+s_1} \binom{\gamma_1+s_1}{t_1} \sum_{t_1'=0}^{t_1} \binom{t_1}{t_1'} \sum_{t_1''=0}^{t_1'} \binom{t_1'}{t_1''} \sum_{t_1'''=0}^{t_1''}$$
$$\times |a_{t_1'''}||b|^{t_1'-t_1''-\gamma_1-s_1-\gamma_2-s_2-\beta_1'+\beta_1''} \frac{(\beta_1''-\beta_1''')!}{(\beta_1''-\beta_1'''-t_1'+t_1'')!}$$
$$\times \int_{\mathbb{R}^2} \frac{\partial^{t_1-t_1'}}{\partial\xi^{t_1-t_1'}} \frac{\partial^{\beta_1-\beta_1'}}{\partial x^{\beta_1-\beta_1'}} \frac{\partial^{t_2-t_2'}}{\partial\delta^{t_2-t_2'}} \frac{\partial^{\beta_2-\beta_2'}}{\partial y^{\beta_2-\beta_2'}} \{g(x,y,\xi,\delta)\} \frac{\partial^{\gamma_1+s_1-t_1}}{\partial\xi^{\gamma_1+s_1-t_1}} \frac{\partial^{\gamma_2+s_2-t_2}}{\partial\delta^{\gamma_2+s_2-t_2}}$$
$$\times \{(c\xi-b\delta)^{(\beta_2'-\beta_2'')}(\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)\}(b\xi+c\delta)^{\beta_1''-\beta_1'''-t_1'+t_1''} \xi^{t_1'''} \delta^{t_2''} d\xi d\delta.$$

Now exploiting (26), we have

$$|x^{\gamma_1} k_{\gamma_2,\beta_1}(y) \frac{\partial^{\beta_1}}{\partial x^{\beta_1}} \frac{\partial^{\beta_2}}{\partial y^{\beta_2}} (G_{g,\alpha,\beta}\phi)(x,y)|$$
$$\leq |d(-\gamma)\cot^4\gamma| \sum_{\beta_2'=0}^{\beta_2} \binom{\beta_2}{\beta_2'} \sum_{\beta_2''=0}^{\beta_2'} \binom{\beta_2'}{\beta_2''} \sum_{s_2=0}^{\beta_2''} |a_{s_2}| \sum_{t_2=0}^{\gamma_2+s_2} \binom{\gamma_2+s_2}{t_2} \sum_{t_2'=0}^{t_2} \binom{t_2}{t_2'}$$
$$\times \sum_{t_2''=0}^{t_2'} |a_{t_2''}| \sum_{\beta_1'=0}^{\beta_1} \binom{\beta_1}{\beta_1'} \sum_{\beta_1''=0}^{\beta_1'} \binom{\beta_1'}{\beta_1''} \sum_{\beta_1'''=0}^{\beta_1''} \binom{\beta_1''}{\beta_1'''} \sum_{s_1=0}^{\beta_1'''} |a_{s_1}||c|^{\beta_1'-\beta_1''}$$
$$\times \frac{(\gamma_2+s_2+\beta_1'-\beta_1''-1)!}{(\gamma_2+s_2-1)!} \sum_{t_1=0}^{\gamma_1+s_1} \binom{\gamma_1+s_1}{t_1} \sum_{t_1'=0}^{t_1} \binom{t_1}{t_1'} \sum_{t_1''=0}^{t_1'} \binom{t_1'}{t_1''} \sum_{t_1'''=0}^{t_1''}$$
$$\times |a_{t_1'''}||b|^{t_1'-t_1''-\gamma_1-s_1-\gamma_2-s_2-\beta_1'+\beta_1''} \frac{(\beta_1''-\beta_1''')!}{(\beta_1''-\beta_1'''-t_1'+t_1'')!}$$
$$\times \int_{\mathbb{R}^2} C_{\beta_1-\beta_1',\beta_2-\beta_2',t_1-t_1',t_2-t_2'} (1+|\xi|)^{m_1-t_1+t_1'} (1+|\delta|)^{m_2-t_2+t_2'}$$
$$\times \frac{\partial^{\gamma_1+s_1-t_1}}{\partial\xi^{\gamma_1+s_1-t_1}} \frac{\partial^{\gamma_2+s_2-t_2}}{\partial\delta^{\gamma_2+s_2-t_2}} \{(c\xi-b\delta)^{(\beta_2'-\beta_2'')}(\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta)\}$$
$$\times (b\xi+c\delta)^{\beta_1''-\beta_1'''-t_1'+t_1''} \xi^{t_1'''} \delta^{t_2''} d\xi d\delta.$$

Since $(c\xi-b\delta)^{(\beta_2'-\beta_2'')}(\mathcal{F}_{\alpha,\beta}\phi)(\xi,\delta) \in S(\mathbb{R}^2)$, the last integral is convergent. Hence,

$$\sup_{(x,y)\in A_{b,c}(\mathbb{R}^2)} |x^{\gamma_1} k_{\gamma_2,\beta_1}(y) \frac{\partial^{\beta_1}}{\partial x^{\beta_1}} \frac{\partial^{\beta_2}}{\partial y^{\beta_2}} (G_{g,\alpha,\beta}\phi)(x,y)| < \infty.$$

This completes the proof. □

5. Application of the Coupled Fractional Fourier Transform to a Generalized Heat Equation

Example 2. *Using the coupled fractional Fourier transform, we investigate the solution of the generalized heat equation*

$$\frac{\partial}{\partial t}\{\phi(x,y,t)\} = (\Delta_{x,y}')^2 \phi(x,y,t); \quad -\infty < x,y < \infty, 0 < t < \infty, \tag{31}$$
$$\phi(x,y,0) = f(x,y).$$

Taking the coupled fractional Fourier transform from both sides of (31), we have

$$\int_{\mathbb{R}^2} \mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta) \frac{\partial}{\partial t}\{\phi(x,y,t)\} dxdy = (\mathcal{F}_{\alpha,\beta}((\Delta_{x,y}')^2 \phi(x,y,t)))(\xi,\delta,t).$$

So that,

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^2}\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)\phi(x,y,t)dxdy = \{(b-c)\xi+(b+c)\delta\}^2(\mathcal{F}_{\alpha,\beta}(\phi(x,y,t))(\xi,\delta,t).$$

Therefore,

$$\frac{\partial}{\partial t}\{\mathcal{F}_{\alpha,\beta}(\phi(x,y,t))\}(\xi,\delta,t) = \{(b-c)\xi+(b+c)\delta\}^2(\mathcal{F}_{\alpha,\beta}(\phi(x,y,t))(\xi,\delta,t).$$

Which gives

$$\mathcal{F}_{\alpha,\beta}(\phi)(\xi,\delta,t) = C(\xi,\delta)\exp[\{(b-c)\xi+(b+c)\delta\}^2 t]. \tag{32}$$

So, $(\mathcal{F}_{\alpha,\beta}(\phi))(\xi,\delta,0) = C(\xi,\delta)$. But

$$\begin{aligned}\mathcal{F}_{\alpha,\beta}(\phi))(\xi,\delta,0) &= \int_{\mathbb{R}^2}\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)\phi(x,y,0)\,dxdy\\ &= \int_{\mathbb{R}^2}\mathcal{K}_{\alpha,\beta}(x,y,\xi,\delta)f(x,y)\,dxdy\\ &= \mathcal{F}_{\alpha,\beta}(f))(\xi,\delta).\end{aligned}$$

So, $C(\xi,\delta) = \mathcal{F}_{\alpha,\beta}(f))(\xi,\delta)$.
From (32), we have

$$\mathcal{F}_{\alpha,\beta}(\phi)(\xi,\delta,t) = \exp[\{(b-c)\xi+(b+c)\delta\}^2 t]\mathcal{F}_{\alpha,\beta}(f)(\xi,\delta). \tag{33}$$

Now applying (8) of both sides of (33), we have

$$\phi(x,y,t) = \mathcal{F}^{-1}_{\alpha,\beta}\Big(\exp[\{(b-c)\xi+(b+c)\delta\}^2 t]\mathcal{F}_{\alpha,\beta}(f)(\xi,\delta)\Big)(x,y).$$

6. Conclusions

In this study, we extended the coupled fractional Fourier transform to a Schwartz-like space and exploit the adjoint method of the said transform to a space of tempered distributions. We derived certain fruitful properties of the kernel of the coupled fractional Fourier transform. Pseudo-differential operators involving coupled fractional Fourier transform is introduced. Moreover, it is shown that the pseudo-differential operators associated with coupled fractional Fourier transform create a continuous mapping on a suitably designed Schwartz-like space. This article concluded with an application of the coupled fractional Fourier transform to solve a generalized heat equation.

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