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Stability of the Borell–Brascamp–Lieb Inequality for Multiple Power Concave Functions

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Abstract: In this paper, we prove the stability of the Brunn–Minkowski inequality for multiple convex bodies in terms of the concept of relative asymmetry. Using these stability results and the relationship of the compact support of functions, we establish the stability of the Borell–Brascamp–Lieb inequality for multiple power concave functions via relative asymmetry.

Keywords: Borell–Brascamp–Lieb inequality; Brunn–Minkowski inequality; stability

MSC: 52A20; 26A51

1. Introduction

We start by recalling the classical Brunn–Minkowski inequality. Let $\mathcal{K}^n$ denote the class of convex bodies (compact convex subsets with interior points) in an n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathcal{K}_o^n$ denote the subset of convex bodies containing the origin in their interiors in $\mathcal{K}^n$. Let $K_1, K_2 \in \mathcal{K}^n$ and $\lambda \in (0,1)$. The classical Brunn–Minkowski inequality states that

$$|K_\lambda|^{\frac{1}{n}} \geq (1-\lambda)|K_1|^{\frac{1}{n}} + \lambda|K_2|^{\frac{1}{n}}, \tag{1}$$

where $|\cdot|$ denotes the Lebesgue measure (i.e., the n-dimensional volume), and

$$K_\lambda = (1-\lambda)K_1 + \lambda K_2 = \{(1-\lambda)x + \lambda y : x \in K_1, y \in K_2\}$$

denotes the Minkowski convex combination of K_1 and K_2, which is still a convex body. The set sum "+" is said to be the Minkowski sum. Moreover, the equality in (1) holds if and only if K_1 and K_2 are homothetic, i.e., $K_1 = sK_2 + x$, for some $s > 0$ and $x \in \mathbb{R}^n$. The Brunn–Minkowski inequality is one of the fundamental results in the theory of convex bodies, and several other important inequalities, e.g., the isoperimetric inequality, can be deduced from it; see [1–6], for example.

Stability results of an inequality answer the following question: is the inequality that we consider sensitive to small perturbations of the maximizers (or minimizers) of the inequality? In other words, if a function almost reaches the equality in a given inequality, is it possible for this function to be close to the maximizers (or minimizers) of the inequality? For example, the classical isoperimetric inequality in two-dimensional Euclidean space states that for any convex body K in the plane, one has

$$P(K)^2 \geq 4\pi A(K),$$

where $P(K)$ and $A(K)$ denote the perimeter and area of K. The equality holds if and only if $K = B_2^n(r)$ for $r > 0$. The stability question of this isoperimetric inequality asks that when a convex body K makes the isoperimetric deficit $P(K)^2 - 4\pi A(K) \leq \varepsilon$ for small $\varepsilon > 0$, how is the body K close to the Euclidean ball $B_2^n(r)$ for $r > 0$? Stability results for the classical

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isoperimetric inequality for convex bodies were proven by Bonnesen (see [7], e.g.). If $\varepsilon > 0$ is small such that $P(K)^2 - 4\pi A(K) \le \varepsilon$, then

$$\delta(K, B_2^n) \le \frac{1}{4\sqrt{\pi}}\varepsilon^{\frac{1}{2}},$$

where $\delta(K, B_2^n)$ denotes the Hausdorff distance between K and ball B_2^n. The result says that if ε converges to 0 (i.e., the equality almost attained), then K is close to a Euclidean ball with speed $\varepsilon^{\frac{1}{2}}$ in the Hausdorff distance. Moreover, the stability strengthens the isoperimetric inequality. Letting $\varepsilon = P(K)^2 - 4\pi A(K)$, we have

$$P(K)^2 - 4\pi A(K) \ge 16\pi\delta(K, B_2^n)^2,$$

which also is referred to the stability of the isoperimetric inequality.

Since the convex bodies K_1 and K_2 are homothetic when the equality holds in (1), it is natural to ask the following stability question: if the equality almost holds in inequality (1), are K_1 and K_2 close to each other up to being homothetic?

To answer this question, one needs to define what "close" means. Two natural ways to measure how "close" two convex bodies are were deduced from the Hausdorff distance and from the volume. Using the Hausdorff distance between the convex body K_1 and K_2, Minkowski himself established the first stability result of the Brunn–Minkowski inequality (see Groemer [7]), and Diskant [8] and Groemer [9] offered a stability version which is close to optimal. However, using the "homothetic" distance deduced from the volume of the symmetric difference is a more natural way to compare the distances, and was used by Figalli, Maggi, and Pratelli [10,11] to establish the optimal result. To state this result, we recall the relative asymmetry of two sets K_1 and K_2, which is defined by

$$\mathcal{A}(K_1, K_2) := \inf_{x\in\mathbb{R}^n}\left\{\frac{|K_1\Delta(x+\lambda K_2)|}{|K_1|}, \lambda = \left(\frac{|K_1|}{|K_2|}\right)^{\frac{1}{n}}\right\}. \tag{2}$$

In addition, let $\Lambda = \max\left\{\frac{\lambda}{1-\lambda}, \frac{1-\lambda}{\lambda}\right\}$ and $\sigma(K_1, K_2) := \max\left\{\frac{K_1}{K_2}, \frac{K_2}{K_1}\right\}$. Figalli, Maggi, and Pratelli [11] showed the following quantitative form of the Brunn–Minkowski inequality.

Theorem 1 (Figalli, Maggi, Pratelli [11])**.** *Let $K_1, K_2 \in \mathcal{K}_o^n, 0 < \lambda < 1$. Then,*

$$|K_\lambda| \ge \mathcal{M}_{\frac{1}{n}}(|K_1|, |K_2|, \lambda)\left(1 + \frac{1}{\Lambda\sigma(K_1, K_2)^{\frac{1}{n}}}\left(\frac{\mathcal{A}(K_1, K_2)}{C(n)}\right)^2\right),$$

where $\mathcal{M}_p(a, b, \lambda)$ denotes the (λ-weighted) p-mean of a, b (see Section 2 for more details) and $C(n)$ is a constant depending on n with polynomial growth. In particular,

$$C(n) = \frac{362n^7}{(2 - 2^{\frac{n}{n-1}})^{3/2}}.$$

Note that the exponent 2 in $\mathcal{A}(K_1, K_2)^2$ is optimal, see Figalli, Maggi, Pratelli [11]. The constant $C(n)$ was improved to cn^7 by Segal [12], and to $cn^{5.5}$ by Kolesnikov and Milman ([13] Theorem 12.12). The best bound found for $C(n)$ up to now is $cn^{5+o(1)}$, which is obtained by combining the general estimate of Kolesnikov and Milman [13] with the bound $n^{o(1)}$ ($o(1)$ represents an infinitesimal of a higher order than the constants) on the Cheeger constant of a convex body in an isotropic position that follows from Chen's work [14].

There are many ways to generalize inequalities, such as considering different spaces, different quantities of geometry (or functions), and different scales, see [5,6,15,16] for examples. Among them, both [15,16] utilize the Hölder inequality, which is closely related to our study. In this paper, we consider the Brunn–Minkowski inequality for multiple convex bodies and the Borell–Brascamp–Lieb inequality for multiple functions.

For the Brunn–Minkowski inequality for multiple convex bodies, suppose $K_i \in \mathcal{K}^n$, $\lambda_i \in (0,1)$ and $\sum\limits_{i=1}^{m} \lambda_i = 1$,

$$|K_\lambda|^{\frac{1}{n}} \geq \lambda_1 |K_1|^{\frac{1}{n}} + \cdots + \lambda_m |K_m|^{\frac{1}{n}}, \tag{3}$$

where $K_\lambda = \lambda_1 K_1 + \cdots + \lambda_m K_m = \{\lambda_1 x_1 + \cdots + \lambda_m x_m : x_1 \in K_1, \cdots, x_m \in K_m\}$. Obviously, if $n = 2$, inequality (3) turns to the classical Brunn–Minkowski inequality.

The first aim of this paper is to show the stability of the Brunn–Minkowski inequality for multiple convex bodies in terms of the concept of relative asymmetry.

Let $\Gamma = \{K_1, \cdots, K_m\}$ be the set of bounded convex sets. Its relative asymmetry is defined by

$$\mathcal{A}(\Gamma) := \inf_{\substack{x_{ij} \in \mathbb{R}^n \\ i \neq j = 1, \cdots, m}} \left\{ \frac{|K_i \Delta (\alpha_{i,j} K_j + x_{ij})|}{|K_i|} : \alpha_{i,j} = \left(\frac{|K_i|}{|K_j|} \right)^{1/n} \right\}. \tag{4}$$

In the case $m = 2$, it turns to the classical relative asymmetry (2). Note that it is essentially the minimum of $\mathcal{A}(K_i, K_j)$, that is,

$$\mathcal{A}(\Gamma) = \inf_{i \neq j = 1, \cdots, m} \{\mathcal{A}(K_i, K_j)\}.$$

In Section 3, we prove the following stability of inequality (3), which is a generalization of Theorem 1.

Theorem 2. *Let* $\Gamma = \{K_1, \cdots, K_m\}, \lambda = (\lambda_1, \cdots, \lambda_m)$, $K_i \in \mathcal{K}_o^n, \lambda_i \in (0,1), i = 1, \cdots, m$, $\sum\limits_{i=1}^{m} \lambda_i = 1$, *and* $K_\lambda = \sum\limits_{i=1}^{m} \lambda_i K_i$. *Then,*

$$|K_\lambda| \geq \mathcal{M}_{\frac{1}{n}}(|K_1|, \cdots, |K_m|, \lambda) \left(1 + \frac{\mathcal{A}(\Gamma)^2}{\Lambda \sigma(\Gamma)^{\frac{1}{n}} C_1(n)^2} \right), \tag{5}$$

where $C_1(n) = 2C(n)$, $\sigma(\Gamma) = \max\limits_{i \neq j = 1, \cdots, m} \{|K_i|/|K_j| : K_i, K_j \in \Gamma\}$ *and* $\Lambda = \max\limits_{i \neq j = 1, \cdots, n} \{\lambda_i / \lambda_j\}$.

Next, we state the Borell–Brascamp–Lieb inequality for multiple functions. Throughout this paper, $u_i \in L^1(\mathbb{R}^n), (n \geq 1)$ is a real non-negative bounded function with compact support Ω_i, $i = 1, \cdots, m$. To avoid triviality, we assume that

$$I_i = \int_{\mathbb{R}^n} u_i \mathrm{d}x > 0, \text{ for } i = 1, \cdots, m.$$

The Borell–Brascamp–Lieb inequality for multiple functions (BBL(m) inequality below) claims that

Theorem 3 (BBL(m) inequality)**.** *Let* $\lambda = (\lambda_1, \cdots, \lambda_m)$, $\lambda_i \in (0,1), i = 1, \cdots, m$, $\sum\limits_{i=1}^{m} \lambda_i = 1$, $-\frac{1}{n} \leq p \leq +\infty$, *and* $0 \leq h \in L^1(\mathbb{R}^n)$ *and assume that*

$$h(\lambda_1 x_1 + \cdots + \lambda_m x_m) \geq \mathcal{M}_p(u_1(x_1), \cdots, u_m(x_m), \lambda),$$

for every $x_i \in \Omega_i, i = 1, \cdots, m$. *Then,*

$$\int_{\Omega_\lambda} h(x) \mathrm{d}x \geq \mathcal{M}_{\frac{p}{np+1}}(I_1, \cdots, I_m, \lambda),$$

where $\Omega_\lambda = \lambda_1 \Omega_1 + \cdots + \lambda_m \Omega_m$.

The proof of Theorem 3 is very similar to the Borell–Brascamp–Lieb inequality, see [17] (Section 3) for an example. It suffices to utilize the concept of the quantity $\mathcal{M}_p(a_1, \cdots, a_m, \lambda)$ and Lemma 1. To avoid redundancy, we omit the proof.

Note that the number $p/(pn+1)$ can be interpreted in extremal cases; it is equal to $-\infty$ when $p = -1/n$, and to $1/n$ when $p = +\infty$. In the case of $m = 2$, the BBL(m) inequality is the classical Borell–Brascamp–Lieb inequality (BBL inequality below). The BBL inequality was first proven for $p > 0$ (in a slightly different form) by Henstock and Macbeath [18] and by Dinghas [19], and was generalized by Brascamp and Lieb [20] and by Borell [21]. In the case of $p = 0$, the inequality is known as the Prékopa–Leindler inequality, which was originally established by Prékopa [22] and Leindler [23], and later rediscovered by Brascamp and Lieb in [24].

A non-negative function u is called p-concave for some $p \in [-\infty, +\infty]$ if

$$u((1-\lambda)x + \lambda y) \geq \mathcal{M}_p(u(x), u(y), \lambda) \quad \text{for every } x, y \in \mathbb{R}^n \text{ and } \lambda \in (0,1).$$

Recently, Ghilli and Salani [17] studied the stability of the BBL inequality for power concave functions with compact support. It is shown in [17] (Theorem 4.1) that in the same assumption of Theorem 3, $p > 0$ and u_1, u_2 are p-concave functions and for some (small enough) $\varepsilon > 0$,

$$\int_{\Omega_\lambda} h(x)\mathrm{d}x \leq \mathcal{M}_{\frac{p}{np+1}}(I_1, I_2, \lambda) + \varepsilon,$$

It holds that

$$|\Omega_\lambda| \leq \mathcal{M}_{\frac{1}{n}}(|\Omega_1|, |\Omega_2|, \lambda)\left[1 + \eta\varepsilon^{\frac{p}{p+1}}\right],$$

where $\eta \leq 2(n + \mathcal{M}_{\frac{p}{np+1}}(I_1, I_2, \lambda)^{-1})$.

Another contribution of this paper is to establish the stability of the Borell–Brascamp–Lieb inequality for multiple power concave functions via the concept of relative asymmetry.

Theorem 4. *Assume that $\lambda = (\lambda_1, \cdots, \lambda_m)$ with $\lambda_i \in (0,1)$ and $\sum_{i=1}^{m} \lambda_i = 1$, $p > 0$, and u_i are non-negative bounded and p-concave functions in $\mathbb{R}^n$ with convex compact supports Ω_i, respectively. If, for some (small enough) $\varepsilon > 0$, it holds that*

$$\int_{\Omega_\lambda} h(x)\mathrm{d}x \leq \mathcal{M}_{\frac{p}{np+1}}(I_1, \cdots, I_m, \lambda) + \varepsilon, \tag{6}$$

then

$$|\Omega_\lambda| \leq \left[1 + 2\left(n + \mathcal{M}_{\frac{p}{np+1}}(I_1, \cdots, I_m, \lambda)^{-1}\right)\varepsilon^{\frac{p}{p+1}}\right]\mathcal{M}_{\frac{1}{n}}(|\Omega_1|, \cdots, |\Omega_m|, \lambda). \tag{7}$$

By using the stability result between the compact supports of the involved function obtained in Theorem 4, we obtain a quantitative version of the BBL(m) inequality. With the adjective "quantitative", we mean to estimate the distance of support sets Ω_i of the functions u_i precisely in terms of the relative asymmetry of Ω_i. That is, when $\mathcal{A}(\Gamma)$ is small enough, we have a strengthened BBL(m) inequality by means of the relative asymmetry of their support sets Ω_i.

Theorem 5. *In the same assumptions and notation of Theorem 4, $\Gamma = \{\Omega_1, \cdots, \Omega_m\}$. If $\mathcal{A}(\Gamma)$ is small enough, it holds that*

$$\int_{\Omega_\lambda} h(x)\mathrm{d}x \geq \mathcal{M}_{\frac{p}{np+1}}(I_1, \cdots, I_m, \lambda) + \beta\mathcal{A}(\Gamma)^{\frac{2(p+1)}{p}}, \tag{8}$$

where

$$\beta = \left(8C(n)^2\Lambda\sigma(\Gamma)^{\frac{1}{n}}\left(n + \mathcal{M}_{\frac{p}{np+1}}(I_1, \cdots, I_m, \lambda)^{-1}\right)\right)^{-\frac{p+1}{p}}.$$

Note that this theorem strengthens the BBL(m) inequality, since the right side of (8) is greater than or equal to $\mathcal{M}_{\frac{p}{np+1}}(I_1, \cdots, I_m, \lambda)$.

The rest of this paper is as follows. In Section 2, we recall some basic results in convex geometry. The main results of this paper will be proven in Section 3, including Theorems 2, 4 and 5. In Section 4, we construct some examples as applications of our results. Finally, the conclusion of this paper is given in Section 5.

2. Preliminaries

2.1. Means of Non-Negative Numbers

For $p \in [-\infty, +\infty]$, $\lambda = (\lambda_1, \cdots, \lambda_m)$ with $\lambda_i \in (0,1)$ and $\sum\limits_{i=1}^{m} \lambda_i = 1$, the quantity $\mathcal{M}_p(a_1, \cdots, a_m, \lambda)$ represents the (λ-weighted) p-mean of non-negative numbers $a_i, i = 1, \cdots, m$, which is defined by $\mathcal{M}_p(a_1, \cdots, a_m, \lambda) = 0$ for $a_1, \cdots, a_m \geq 0$ and $a_1 \cdots a_m = 0$, and for $a_1, \cdots, a_m > 0$, we set

$$\mathcal{M}_p(a_1, \cdots, a_m, \lambda) = \begin{cases} \max\{a_1, \cdots, a_m\}, & \text{if } p = +\infty, \\ \left[\lambda_1 a_1^p + \cdots + \lambda_m a_m^p\right]^{\frac{1}{p}}, & \text{if } 0 \neq p \in \mathbb{R}, \\ a_1^{\lambda_1} \cdots a_m^{\lambda_m}, & \text{if } p = 0, \\ \min\{a_1, \cdots, a_m\}, & \text{if } p = -\infty. \end{cases}$$

We next recall some useful facts, see [25,26] for more details. Note that for all p and λ, $\mathcal{M}_p(a_1, \cdots, a_m, \lambda)$ is non-decreasing with respect to $a_i, i = 1, \cdots, m$. Using Jensen's inequality, $\mathcal{M}_p(a_1, \cdots, a_m, \lambda)$ is non-decreasing with respect to p, that is,

$$\mathcal{M}_p(a_1, \cdots, a_m, \lambda) \leq \mathcal{M}_q(a_1, \cdots, a_m, \lambda), \quad \text{if } \ p \leq q.$$

We also have the following technical lemma, which can be found in [25].

Lemma 1. *Let $\lambda_i \in (0,1)$, $\sum\limits_{i=1}^{m} \lambda_i = 1$ and $a_1, \cdots, a_m, b_1, \cdots, b_m$ be non-negative numbers. If $p + q > 0$, then*

$$\mathcal{M}_p(a_1, \cdots, a_m, \lambda)\mathcal{M}_q(b_1, \cdots, b_m, \lambda) \geq \mathcal{M}_s(a_1 b_1, \cdots, a_m b_m, \lambda), \tag{9}$$

where $s = \frac{pq}{p+q}$. Moreover, the result holds trivially with $s = 0$ if $p = q = 0$.

2.2. Convex Body

Throughout this article, K (possibly with subscripts) denotes a convex body (convex, compact set with non-empty interior) in $\mathbb{R}^n$. We denote by $\mathcal{K}^n$ the class of convex bodies in $\mathbb{R}^n$. Let $\mathcal{K}_o^n$ denote the subset of convex bodies containing the origin in their interiors in $\mathcal{K}^n$. For $K \in \mathcal{K}_o^n$, we define the weight function in direction ν:

$$\|\nu\|_* := \sup\{x \cdot \nu : x \in K\}, \quad \nu \in S^{n-1}.$$

Let E be an open subset of $\mathbb{R}^n$, with a smooth or polyhedral boundary ∂E oriented by its outer unit normal vector ν_E. The anisotropic perimeter of E is defined as

$$P_K(E) := \int_{\partial E} \|\nu_E(x)\|_* \mathrm{d}\mathcal{H}^{n-1}(x), \tag{10}$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$. If $K_1, K_2 \in \mathcal{K}_o^n$, E is a set of finite perimeter, then

$$P_{K_1}(E) + P_{K_2}(E) = P_{K_1+K_2}(E). \tag{11}$$

From the definition of $P_K(E)$, we have $P_K(K) = n|K|$. For more properties about anisotropic perimeters, we refer to [11]. Now, we recall the anisotropic isoperimetric inequality, that is,

$$P_K(E) \geq n|K|^{1/n}|E|^{1/n'}, \quad \text{if} \quad |E| < \infty, \tag{12}$$

where $n' = \frac{n}{n-1}$.

Figalli, Maggi, and Pratelli [11] have provided a quantitative form of (12) in terms of the asymmetry index, shown in the following.

Lemma 2. *Let E be a set of finite perimeters with $|E| < \infty$; then,*

$$P_K(E) \geq n|K|^{1/n}|E|^{1/n'}\left(1 + \left(\frac{\mathcal{A}(E)}{C(n)}\right)^2\right), \tag{13}$$

where $\mathcal{A}(E)$ represents the asymmetry index of E:

$$\mathcal{A}(E) := \inf_{x\in\mathbb{R}^n}\left\{\frac{|E\Delta(x+\lambda K)|}{|E|}, \lambda = \left(\frac{|E|}{|K|}\right)^{\frac{1}{n}}\right\}.$$

Note that the asymmetry index of E is the relative asymmetry between E and K, that is, $\mathcal{A}(E) = \mathcal{A}(E, K)$. Note that the triangle inequality holds for the relative asymmetry. For $K_1, K_2, L \in \mathcal{K}_o^n$,

$$\mathcal{A}(K_1, K_2) \leq \mathcal{A}(K_1, L) + \mathcal{A}(L, K_2). \tag{14}$$

This can be proven easily by the triangle inequality of the symmetric difference and the property of scaling and translation invariance of the relative asymmetry.

2.3. Power Concave Function and (p, λ)-Convolution of Non-Negative Functions

Let Ω be a convex set in $\mathbb{R}^n$ and $p \in [-\infty, +\infty]$. We say that a non-negative function u defined in Ω is p-concave if

$$u((1-\lambda)x + \lambda y) \geq \mathcal{M}_p(u(x), u(y), \lambda),$$

for every $x, y \in \Omega$ and $\lambda \in (0, 1)$. In short, u is p-concave if it has convex support Ω and

(1) u^p is concave in Ω for $p > 0$;
(2) $\log u$ is concave in Ω for $p = 0$;
(3) u^p is convex in Ω for $p < 0$;
(4) u is quasi-concave, i.e., all its superlevel sets are convex, for $p = -\infty$;
(5) u is a non-negative constant in Ω for $p = +\infty$.

Let $p \in [-\infty, +\infty]$ and u_i be non-negative functions with compact convex support $\Omega_i, i = 1, \cdots, m$. The (p, λ)-convolution of u_i is the function defined as

$$u_{p,\lambda}(x) = \sup\{\mathcal{M}_p(u_1(x_1), \cdots, u_n(x_m), \lambda) : x = \lambda_1 x_1 + \cdots + \lambda_m x_m, x_i \in \bar{\Omega}_i\}.$$

From above the definition of $u_{p,\lambda}$ and the monotonicity of the p-mean with respect to p, we obtain

$$u_{p,\lambda} \leq u_{q,\lambda} \quad \text{for } -\infty \leq p \leq q \leq +\infty.$$

Obviously, the support of $u_{p,\lambda}$ is $\Omega_\lambda = \lambda_1\Omega_1 + \cdots + \lambda_m\Omega_m$ and the continuity of $u_i, i = 1, \cdots, m$ yields the continuity of $u_{p,\lambda}$. For more details on supremal convolution of convex/concave functions, see [26–29].

3. Proofs of Theorems 2, 4 and 5

Motivated by a similar method, to prove Theorem 1, we now use Lemma 2 to prove Theorem 2.

Proof of Theorem 2. According to the definitions of $\mathcal{A}, \sigma$, they are all translation invariant. Thus, we may assume that $K_i, i = 1, \cdots, m$ contain the origin. By Lemma 2, we have

$$P_{\lambda_1 K_1}(K_\lambda) \geq n|\lambda_1 K_1|^{\frac{1}{n}}|K_\lambda|^{\frac{n-1}{n}}\left(1 + \left(\frac{\mathcal{A}(K_\lambda, \lambda_1 K_1)}{C(n)}\right)^2\right),$$

$$\vdots$$

$$P_{\lambda_m K_m}(K_\lambda) \geq n|\lambda_m K_m|^{\frac{1}{n}}|K_\lambda|^{\frac{n-1}{n}}\left(1 + \left(\frac{\mathcal{A}(K_\lambda, \lambda_m K_m)}{C(n)}\right)^2\right).$$

Adding up the above inequalities, thanks to (10) and the fact that

$$\sum_{i=1}^{m} P_{\lambda_i K_i}(K_\lambda) = P_{\lambda_1 K_1 + \cdots + \lambda_m K_m}(K_\lambda),$$

$$P_{\lambda_1 K_1 + \cdots + \lambda_m K_m}(K_\lambda) = n|K_\lambda|,$$

we obtain

$$\begin{aligned}\frac{|K_\lambda|^{\frac{1}{n}}}{\sum\limits_{i=1}^{m}|\lambda_i K_i|^{\frac{1}{n}}} - 1 &\geq \frac{|\lambda_1 K_1|^{\frac{1}{n}}}{\sum\limits_{i=1}^{m}|\lambda_i K_i|^{\frac{1}{n}}}\left(\frac{\mathcal{A}(K_\lambda, \lambda_1 K_1)}{C(n)}\right)^2 + \cdots + \frac{|\lambda_m K_m|^{\frac{1}{n}}}{\sum\limits_{i=1}^{m}|\lambda_i K_i|^{\frac{1}{n}}}\left(\frac{\mathcal{A}(K_\lambda, \lambda_m K_m)}{C(n)}\right)^2 \\ &\geq \frac{\sum\limits_{i=1}^{m}(\mathcal{A}(K_\lambda, K_i))^2}{mC(n)^2 \Lambda \sigma(\Gamma)^{\frac{1}{n}}},\end{aligned}$$

where the last inequality follows from the definitions of Λ and $\sigma(\Gamma)$. Using

$$m\sum_{i=1}^{m} a_i^2 \geq \left(\sum_{i=1}^{m} a_i\right)^2, \quad \forall a_i > 0, i = 1, \cdots, m,$$

and the fact that

$$\mathcal{A}(K_\lambda, K_i) + \mathcal{A}(K_\lambda, K_j) \geq \mathcal{A}(K_i, K_j) \geq \mathcal{A}(\Gamma), i \neq j,$$

it follows that

$$\begin{aligned}\frac{|K_\lambda|^{\frac{1}{n}}}{\sum\limits_{i=1}^{m}|\lambda_i K_i|^{\frac{1}{n}}} - 1 &\geq \frac{\left[\sum\limits_{i=1}^{m}\mathcal{A}(K_\lambda, K_i)\right]^2}{m^2 C(n)^2 \Lambda \sigma(\Gamma)^{\frac{1}{n}}} \\ &\geq \frac{((m/2)\mathcal{A}(\Gamma))^2}{m^2 C(n)^2 \Lambda \sigma(\Gamma)^{\frac{1}{n}}} = \frac{\mathcal{A}(\Gamma)^2}{4C(n)^2 \Lambda \sigma(\Gamma)^{\frac{1}{n}}}.\end{aligned}$$

That is,

$$|K_\lambda| \geq \mathcal{M}_{\frac{1}{n}}(|K_1|, \cdots, |K_m|, \lambda)\left(1 + \frac{\mathcal{A}(\Gamma)^2}{C_1(n)^2 \Lambda \sigma(\Gamma)^{\frac{1}{n}}}\right).$$

□

We next prove Theorem 4.

Proof of Theorem 4. For simplicity, we recall some notation and concepts. For the given bounded functions $u_i, i = 1, \cdots, m$, their distribution functions are given by

$$\mu_i(s) = |\{u_i \geq s\}|,\ s \in [0, L_i],$$

where $L_i = \max\limits_{\Omega_i} u_i < +\infty$. And, for $\lambda_i \in (0,1)$ and $p > 0$, it follows that

$$L_\lambda = \max_{\Omega_\lambda} u_{p,\lambda} = \mathcal{M}_p(L_1, \cdots, L_m, \lambda),$$

and

$$\mu_\lambda(s) = \left|\{u_{p,\lambda} \geq s\}\right|,$$

for $s \in [0, L_\lambda]$. From the p-concavity of u_i and $u_{p,\lambda}$, we obtain that the distribution functions μ_i and μ_λ are continuous. The Brunn–Minkowski inequality yields

$$|\Omega_\lambda| \geq \mathcal{M}_{\frac{1}{n}}(|\Omega_1|, \cdots, |\Omega_m|, \lambda).$$

In the case of equality, the final result holds trivially. Then, we may assume that for some $\tau > 0$,

$$|\Omega_\lambda| = \mathcal{M}_{\frac{1}{n}}(|\Omega_1|, \cdots, |\Omega_m|, \lambda) + \tau. \tag{15}$$

Now, we want to find a function to estimate τ depending on ε, that is, to search for a function f such that $\lim\limits_{\varepsilon \to 0} f(\varepsilon) = 0$ and $\tau < f(\varepsilon)$. The definition of $u_{p,\lambda}$ yields

$$\left\{u_{p,\lambda} \geq \mathcal{M}_p(s_1, \cdots, s_m, \lambda)\right\} \supseteq \lambda_1\{u_1 \geq s_1\} + \cdots + \lambda_m\{u_m \geq s_m\},$$

for $s_i \in [0, L_i], i = 1, \cdots, m$. Thus, the Brunn–Minkowski inequality implies

$$\mu_\lambda\big(\mathcal{M}_p(s_1, \cdots, s_m, \lambda)\big) \geq \mathcal{M}_{\frac{1}{n}}(\mu_1(s_1), \cdots, \mu_m(s_m), \lambda). \tag{16}$$

For $i = 1, \cdots, m$, we set functions $s_i : [0,1] \to [0, L_i]$ such that for $t \in [0,1]$

$$\int_0^{s_i(t)} \mu_i(s)\mathrm{d}s = tI_i, \tag{17}$$

where $I_i = \int_{\mathbb{R}^n} u_i(x)\mathrm{d}x$. Note that s_i is strictly increasing, and it is differentiable almost everywhere. Then, by differentiating (17), we have

$$\mu_i(s_i(t)){s_i}'(t) = I_i \quad \text{for a.e.} \quad t \in [0,1]. \tag{18}$$

We now define the map $s_\lambda : [0,1] \to [0, L_\lambda]$ as

$$s_\lambda = \mathcal{M}_p(s_1(t), \cdots, s_m(t), \lambda), \quad t \in [0,1]. \tag{19}$$

The continuity of s_i implies the continuity of the function s_λ; thus, it follows that

$$s_\lambda'(t) = \left(\sum_{i=1}^m \lambda_i s_i'(t) s_i(t)^{p-1}\right) s_\lambda(t)^{1-p}, \quad t \in [0,1].$$

Applying Lemma 1 with $p = \frac{1}{n}$ and $q = 1$, we have, for $s_i(t) \in [0, L_i]$,

$$\begin{aligned}
&\mathcal{M}_{\frac{1}{n}}(\mu_1(s_1(t)), \cdots, \mu_m(s_m(t)), \lambda)\mathcal{M}_1\Big({s_1}'(t)s_1(t)^{p-1}, \cdots, s_m'(t)s_m(t)^{p-1}, \lambda\Big) \\
\geq &\mathcal{M}_{\frac{1}{n+1}}\Big(\mu_1(s_1(t))s_1(t)^{p-1}s_1'(t), \cdots, \mu_m(s_m(t))s_m(t)^{p-1}s_m'(t), \lambda\Big).
\end{aligned}$$

Then, the identity

$$s_\lambda(t)^{1-p} = \mathcal{M}_p(s_1(t), \cdots, s_m(t), \lambda)^{1-p} = \mathcal{M}_{\frac{p}{1-p}}\Big(s_1(t)^{1-p}, \cdots, s_m(t)^{1-p}, \lambda\Big), \tag{20}$$

together with Lemma 1 and (18), implies that

$$\begin{aligned}
&\mathcal{M}_{\frac{1}{n}}(\mu_1(s_1(t)), \cdots, \mu_m(s_m(t)), \lambda)s'_\lambda(t) \\
=&\mathcal{M}_{\frac{1}{n}}(\mu_1(s_1(t)), \cdots, \mu_m(s_m(t)), \lambda)\mathcal{M}_1\Big(s'_1(t)s_1(t)^{p-1}, \cdots, s'_m(t)s_m(t)^{p-1}, \lambda\Big)s_\lambda(t)^{1-p} \\
\geq&\mathcal{M}_{\frac{1}{n+1}}(\mu_1(s_1(t))s_1(t)^{p-1}s'_1(t), \cdots, \mu_m(s_m(t))s_m(t)^{p-1}s'_m(t), \lambda)\mathcal{M}_{\frac{p}{1-p}}(s_1(t)^{1-p}, \cdots, s_m(t)^{1-p}, \lambda) \\
\geq&\mathcal{M}_{\frac{p}{np+1}}(I_1, \cdots, I_m, \lambda).
\end{aligned} \tag{21}$$

Next, for any given $\delta > 0$, set

$$F_\delta = \Big\{t \in [0,1] : \mu_\lambda(s_\lambda(t)) > \mathcal{M}_{\frac{1}{n}}(\mu_1(s_1(t)), \cdots, \mu_m(s_m(t)), \lambda) + \delta\Big\},$$

and

$$\Gamma_\delta = \{s_\lambda(t) : t \in F_\delta\}.$$

Then, we have

$$\begin{aligned}
I_\lambda &= \int_0^{L_\lambda} \mu_\lambda(s)\mathrm{d}s = \int_0^1 \mu_\lambda(s_\lambda(t))s_\lambda(t)'\mathrm{d}t \\
&= \int_{F_\delta} \mu_\lambda(s_\lambda(t))s_\lambda(t)'\mathrm{d}t + \int_{[0,1]\setminus F_\delta} \mu_\lambda(s_\lambda(t))s_\lambda(t)'\mathrm{d}t \\
&\geq \int_{F_\delta} \Big(\mathcal{M}_{\frac{1}{n}}(\mu_1, \cdots, \mu_m, \lambda) + \delta\Big)s_\lambda(t)'\mathrm{d}t + \int_{[0,1]\setminus F_\delta} \mu_\lambda(s_\lambda(t))s_\lambda(t)'\mathrm{d}t \\
&\geq \int_0^1 \mathcal{M}_{\frac{1}{n}}(\mu_1, \cdots, \mu_m, \lambda)s'_\lambda(t)\mathrm{d}t + \delta|\Gamma_\delta|,
\end{aligned} \tag{22}$$

where the last inequality follows from (16). From (21), it follows that

$$\int_0^1 \mathcal{M}_{\frac{1}{n}}(\mu_1, \cdots, \mu_m, \lambda)s'_\lambda(t)\mathrm{d}t \geq \mathcal{M}_{\frac{p}{np+1}}(I_1, \cdots, I_m, \lambda). \tag{23}$$

Combining (22), (23) and the condition (5), we have

$$\mathcal{M}_{\frac{p}{np+1}}(I_1, \cdots, I_m, \lambda) + \varepsilon \geq I_\lambda \geq \mathcal{M}_{\frac{p}{np+1}}(I_1, \cdots, I_m, \lambda) + \delta|\Gamma_\delta|,$$

which implies

$$|\Gamma_\delta| \leq \frac{\varepsilon}{\delta}.$$

For some $\alpha \in (0,1)$, we take $\delta = \varepsilon^\alpha L_\lambda^{-1}$. Then,

$$L_\lambda^{-1}|\Gamma_{\varepsilon^\alpha L_\lambda^{-1}}| \leq \varepsilon^{1-\alpha}. \tag{24}$$

Letting ε be small enough, (24) implies that the set $[0,1]\setminus\Gamma_{\varepsilon^\alpha L_\lambda^{-1}}$ is big enough. Then, there exists $\bar{t} \notin F_{\varepsilon^\alpha L_\lambda^{-1}}$ such that $s_\lambda(\bar{t}) \leq \eta$ for any small η. Therefore, there exists $\bar{t} \in [0,1]$ such that

$$s_\lambda(\bar{t}) \leq \varepsilon^{1-\alpha}L_\lambda, \tag{25}$$

and

$$\mu_\lambda(s_\lambda(\bar{t})) \leq \mathcal{M}_{\frac{1}{n}}(\mu_1(s_1(\bar{t})), \cdots, \mu_m(s_m(\bar{t})), \lambda) + \varepsilon^\alpha L_\lambda^{-1}. \tag{26}$$

Suppose $\xi_i \in (0,1)$ and $\sum\limits_{i=1}^m \xi_i = 1$. Let

$$\xi_m = \left(\frac{s_\lambda(\bar{t})}{L_\lambda}\right)^p. \tag{27}$$

From (25), we have

$$\xi_m \leq \varepsilon^{(1-\alpha)p}.$$

For $i = 1, \cdots, m$, by the assumption that u_i are p-concave, we have that $u_{p,\lambda}$ is p-concave. Set $l_1 = \cdots = l_{m-1} = 0, l_m = L_\lambda$. Then,

$$\{z : u_{p,\lambda}(z) \geq \mathcal{M}_p(l_1, \cdots, l_{m-1}, L_\lambda, \xi)\} \supseteq \sum_{i=1}^{m-1} \xi_i\{x_i : u_{p,\lambda}(x_i) \geq 0\} + \xi_m\{x_m : u_{p,\lambda}(x_m) \geq L_\lambda\}.$$

Since $s_\lambda(\bar{t}) = \mathcal{M}_p(0, \cdots, 0, L_\lambda, \xi)$ from (27), the above inclusion can be written as

$$\{u_{p,\lambda}(z) \geq s_\lambda(\bar{t})\} \supseteq \sum_{i=1}^{m-1} \xi_i \Omega_\lambda + \xi_m\{u_{p,\lambda} \geq L_\lambda\}.$$

By the Brunn–Minkowski inequality, we get

$$\left|\{u_{p,\lambda} \geq s_\lambda(\bar{t})\}\right| \geq \left(\sum_{i=1}^{m-1} \xi_i |\Omega_\lambda|^{\frac{1}{n}} + \xi_m \left|\{u_{p,\lambda} \geq L_\lambda\}\right|^{\frac{1}{n}}\right)^n. \tag{28}$$

Since the set $\{u_{p,\lambda} \geq L_\lambda\} = \sum_{i=1}^{m} \{u_i \geq L_i\}$ is a single point set under the assumption that the involved functions are strictly p-concave, we then have that $\{u_{p,\lambda} \geq L_\lambda\}$ has zero measure. Then, using (26) and the fact $\mu_i(s_i(\bar{t})) \leq |\Omega_i|$, and then using (28) and (15), we have

$$\begin{aligned}
&\mathcal{M}_{\frac{1}{n}}(|\Omega_1|, \cdots, |\Omega_m|, \lambda) + \varepsilon^\alpha L_\lambda^{-1} \\
\geq &\mu_\lambda(s_\lambda(\bar{t})) = \left|\{u_{p,\lambda} \geq s_\lambda(\bar{t})\}\right| \\
\geq &\left(\sum_{i=1}^{m-1} \xi_i |\Omega_\lambda|^{\frac{1}{n}}\right)^n \\
\geq &(1-\xi_m)^n \mathcal{M}_{\frac{1}{n}}(|\Omega_1|, \cdots, |\Omega_m|, \lambda) + (1-\xi_m)^n \tau.
\end{aligned}$$

Since $(1-\xi_m)^n \geq 1 - n\xi_m \geq \frac{1}{2}$ for $0 \leq \xi_m \leq \frac{1}{2n}$, we get

$$\begin{aligned}
\tau &\leq \frac{\left(\varepsilon^\alpha L_\lambda^{-1} + [1-(1-\xi_m)^n]\mathcal{M}_{\frac{1}{n}}(|\Omega_1|, \cdots, |\Omega_m|, \lambda)\right)}{(1-\xi_m)^n} \\
&\leq 2\left(\varepsilon^\alpha L_\lambda^{-1} + n\xi_m \mathcal{M}_{\frac{1}{n}}(|\Omega_1|, \cdots, |\Omega_m|, \lambda)\right).
\end{aligned}$$

Taking $\alpha = \frac{p}{p+1}$ and a small enough ε (precisely, $\varepsilon \leq \left(\frac{1}{2n}\right)^{\frac{p+1}{p}}$) and combining $\xi_m \leq \varepsilon^{(1-\alpha)p}$, we have

$$|\Omega_\lambda| \leq \mathcal{M}_{\frac{1}{n}}(|\Omega_1|, \cdots |\Omega_m|, \lambda) + 2\left(L_\lambda^{-1} + n\mathcal{M}_{\frac{1}{n}}(|\Omega_1|, \cdots |\Omega_m|, \lambda)\right)\varepsilon^{\frac{p}{p+1}}. \tag{29}$$

Since clearly $I_i \leq L_i|\Omega_i|$ for $i = 1, \cdots, m, \lambda$, we get

$$L_\lambda = \mathcal{M}_p(L_1, \cdots, L_m, \lambda) \geq \mathcal{M}_p\left(\frac{I_1}{|\Omega_1|}, \cdots, \frac{I_m}{|\Omega_m|}, \lambda\right),$$

and recalling Lemma 1,

$$L_\lambda \geq \frac{\mathcal{M}_{\frac{p}{np+1}}(I_1, \cdots, I_m, \lambda)}{\mathcal{M}_{\frac{1}{n}}(|\Omega_1|, \cdots, |\Omega_m|, \lambda)}.$$

Finally, combining the above inequality with (29) we obtain

$$|\Omega_\lambda| \leq \left[1 + 2\left(n + \mathcal{M}_{\frac{p}{np+1}}(I_1, \cdots, I_m, \lambda)^{-1}\right)\varepsilon^{\frac{p}{p+1}}\right]\mathcal{M}_{\frac{1}{n}}(|\Omega_1|, \cdots, |\Omega_m|, \lambda),$$

and the proof is complete. □

Now, by virtue of Theorem 2 and Theorem 4 obtained above, we are in the position to prove Theorem 5.

Proof of Theorem 5. We argue by contradiction. Suppose that

$$\int_{\Omega_\lambda} h(x)\mathrm{d}x < \mathcal{M}_{\frac{p}{np+1}}(I_1, \cdots, I_m, \lambda) + \beta\mathcal{A}(\Gamma)^{\frac{2(p+1)}{p}},$$

where β is defined in Theorem 5. Then, we apply Theorem 4 and we have

$$|\Omega_\lambda| \leq \left[1 + \frac{\mathcal{A}(\Gamma)^2}{\Lambda\sigma(\Gamma)^{\frac{1}{n}}C_1(n)^2}\right]\mathcal{M}_{\frac{1}{n}}(|\Omega_1|, \cdots |\Omega_m|, \lambda).$$

Then, according to Theorem 2, we can easily get a contradiction and finish the proof. □

4. Examples of Theorems 2, 4 and 5

Now, we give some specific examples to further reveal the application and significance of our results. We consider three convex bodies in $\mathbb{R}^2$, one of which is a ball $K_1 = B_2^2(r)$ for $r = \pi^{-1/2}$, one is a square K_2 with sides of 1, and the other K_3 is a rectangle with sides of lengths ε and $\frac{1}{\varepsilon}$ which has a big difference in shape from the above two bodies. Moreover, the area of them is one, i.e., $|K_1| = |K_2| = |K_3| = 1$.

Example 1 (Example of Theorem 2)**.** *Let* $\Gamma = \{K_1, K_2, K_3\}$ *with* $K_1 = \{x \in \mathbb{R}^2 : |x| \leq \pi^{-1/2}\}$, $K_2 = [-1/2, 1/2]^2$ *and* $K_3 = [-1/(2\varepsilon), 1/(2\varepsilon)] \times [-\varepsilon/2, \varepsilon/2]$, $\lambda = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ *and* $K_\lambda = \frac{1}{3}K_1 + \frac{1}{3}K_2 + \frac{1}{3}K_3$. *It follows from Theorem 2 that*

$$|K_\lambda| \geq \left(1 + \frac{C_2}{C_1(2)^2}\right),$$

where $C_1(2) = 2C(2)$ *with* $C(2)$ *defined in Theorem 1, and*

$$C_2 = 1 - 4\sqrt{\frac{1}{\pi} - \frac{1}{4}} + \frac{6}{\pi}\arcsin\sqrt{1 - \frac{\pi}{4}} - \frac{2}{\pi}\arcsin\sqrt{\frac{\pi}{4}}.$$

Proof. Using the assumption and the definitions of Λ and $\sigma(\Gamma)$, it is easy to obtain $\Lambda = \sigma(\Gamma) = 1$. Let us calculate $\mathcal{A}(\Gamma)$. Thanks to the fact that $\mathcal{A}(\Gamma) = \inf_{i \neq j = 1,2,3}\{\mathcal{A}(K_i, K_j)\}$ and the definition of K_3, which has a great difference in shape from K_1 and K_2, we only need to deduce the relative asymmetry between K_1 and K_2. By (4) and calculus of integrals, we get

$$C_2 := \mathcal{A}(\Gamma) = \mathcal{A}(K_1, K_2) = 1 - 4\sqrt{\frac{1}{\pi} - \frac{1}{4}} + \frac{6}{\pi}\arcsin\sqrt{1 - \frac{\pi}{4}} - \frac{2}{\pi}\arcsin\sqrt{\frac{\pi}{4}}.$$

□

Example 2 (Example of Theorems 4)**.** *Let* $K_1 = \{x \in \mathbb{R}^2 : |x| \leq \pi^{-1/2}\}$, $K_2 = [-1/2, 1/2]^2$ *and* $K_3 = [-1/(2\varepsilon), 1/(2\varepsilon)] \times [-\varepsilon/2, \varepsilon/2]$, *and assume that* $\lambda = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $p > 0$, *and* $u_i \in L^1(\mathbb{R}^2), i = 1, 2, 3$ *are non-negative bounded and p-concave functions in* $\mathbb{R}^2$ *with convex com-*

pact supports K_i, *respectively. Assume that* $\int_{\mathbb{R}^2} u_i(x)\mathrm{d}x = 1$ *and for* $\varepsilon > 0$ $\int_{K_\lambda} h(x)\mathrm{d}x \le 1+\varepsilon$. *From Theorem 4, we get*

$$|K_\lambda| \le 1 + 6\varepsilon^{\frac{p}{p+1}}.$$

Example 3 (Example of Theorems 5)**.** *Let* $\Gamma = \{K_1, K_2, K_3\}$ *with* $K_1 = \{x \in \mathbb{R}^2 : |x| \le \pi^{-1/2}\}$, $K_2 = [-1/2, 1/2]^2$ *and* $K_3 = [-1/(2\varepsilon), 1/(2\varepsilon)] \times [-\varepsilon/2, \varepsilon/2]$, *and assume that* $\lambda = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $p > 0$ *and* $u_i \in L^1(\mathbb{R}^2), i = 1,2,3$ *are non-negative bounded and p-concave functions in* $\mathbb{R}^2$ *with convex compact supports* K_i, *respectively. Assume that* $\int_{\mathbb{R}^2} u_i(x)\mathrm{d}x = 1$, *and for* $\varepsilon > 0$ $\int_{K_\lambda} h(x)\mathrm{d}x \le 1+\varepsilon$. *Using Theorem 5, it holds that*

$$\int_{K_\lambda} h(x)\mathrm{d}x \ge 1 + \beta C_2^{\frac{2(p+1)}{p}},$$

where $\beta = \left(24C(2)^2\right)^{-\frac{p+1}{p}}$, C_2 *is obtained in Example 1 and* $C(2)$ *is the value of* $C(n)$ *which is defined in Theorem 1 when* $n = 2$.

5. Conclusions

This paper has deepened the understanding of the stability of the Brunn–Minkowski inequality for multiple convex bodies by incorporating the concept of relative asymmetry. Additionally, applying the established stability estimations of the Brunn–Minkowski inequality and the property of compact support, we also established the stability of the Borell–Brascamp–Lieb inequality for multiple power concave functions. Furthermore, some examples are also given in Section 4 as applications of the main results.

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References

1. Fang, N.; Hu, J.; Zhao, L. From the Brunn-Minkowski inequality to a class of generalized Poincaré-type inequalities for torsional rigidity. *J. Geom. Anal.* **2024**, *34*, 114. [CrossRef]
2. Li, W.; Zhu, B. The affine Convex Lorentz-Sobolev inequalities. *J. Geom. Anal.* **2024**, *34*, 30. [CrossRef]
3. Ma, L.; Zeng, C.; Wang, Y. The log-Minkowski inequality of curvature entropy. *Proc. Amer. Math. Soc.* **2023**, *151*, 3587–3600.
4. Shan, J.; Xu, W.; Yin, L. L_p Blaschke-Santaló and Petty projection inequalities in Gaussian space. *Arch. Math.* **2024**, *122*, 331–342. [CrossRef]
5. Wu, D.; Bu, Z.-H. The measure-comparison problem for polar (p,μ)-centroid bodies. *Adv. Appl. Math.* **2022**, *137*, 102332. [CrossRef]
6. Wu, D.; Guo, P.; Bu, Z.-H. Sylvester-Busemann type functionals with respect to Orlicz centroid bodies. *J. Geom. Anal.* **2023**, *33*, 197. [CrossRef]
7. Groemer, H. Stability of geometric inequalities. In *Handbook of Convex Geometry*; Gruber, P.M., Wills, J.M., Eds.; Elsevier: Amsterdam, The Netherlands, 1993; pp. 125–150.
8. Diskant, V.I. Stability of the solution of a Minkowski equation. *Sib. Math. J.* **1974**, *14*, 466–473.
9. Groemer, H. On the Brunn-Minkowski theorem. *Geom. Dedicata* **1988**, *27*, 357–371. [CrossRef]
10. Figalli, A.; Maggi, F.; Pratelli, A. A refined Brunn-Minkowski inequality for convex sets. *Ann. Inst. Henri Poincaré* **2009**, *26*, 2511–2519. [CrossRef]
11. Figalli, A.; Maggi, F.; Pratelli, A. A mass transportation approach to quantitative isoperimetric inequality. *Invent. Math.* **2010**, *182*, 167–211. [CrossRef]
12. Segal, A. Remark on stability of Brunn-Minkowski and isoperimetric inequalities for convex bodies. In *Geometric Aspects of Functional Analysis, Lecture Notes in Math*; Springer: Berlin/Heidelberg, Germany, 2012; Volume 2050, pp. 381–391.
13. Kolesnikov, A.V.; Milman, E. Local L^p-Brunn-Minkowski inequalities for $p < 1$. *Mem. Am. Math. Soc.* **2022**, *277*, 78.

14. Chen, Y. An almost constant lower bound of the isoperimetric coefficient in the KLS conjecture. *Geom. Funct. Anal.* **2021**, *31*, 34–61. [CrossRef]
15. Saker, S.H.; Tunç, C.; Mahmoud, R.R. New Carlson-Bellman and Hardy-Littlewood dynamic inequalities. *Math. Inequal. Appl.* **2018**, *21*, 967–983. [CrossRef]
16. Agarwal, R.P.; Mahmoud, R.R.; Saker, S.H.; Tunç, C. New generalizations of Németh-Mohapatra type inequalities on time scales. *Acta Math. Hungar.* **2017**, *152*, 383–403. [CrossRef]
17. Ghilli, D.; Salani, P. Quantitative Borell-Brascamp-Lieb inequalities for power concave functions. *J. Convex Anal.* **2017**, *24*, 857–888.
18. Henstock, R.; Macbeath, A.M. On the measure of sum sets. I. The theorems of Brunn, Minkowski and Lusternik. *Proc. Lond. Math. Soc.* **1953**, *3*, 182–194. [CrossRef]
19. Dinghas, A. Über eine Klasse superadditiver Mengenfunktionale von Brunn-Minkowski-Lusternikschem Typus. *Math. Z.* **1957**, *68*, 111–125. [CrossRef]
20. Brascamp, H.J.; Lieb, E.H. On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *J. Funct. Anal.* **1976**, *22*, 366–389. [CrossRef]
21. Borell, C. Convex set functions in d-space. *Period. Math. Hung.* **1975**, *6*, 111–136. [CrossRef]
22. Prékopa, A. Logarithmic concave measures with application to stochastic programming. *Acta Sci. Math.* **1971**, *32*, 301–316.
23. Leindler, L. On a certain converse of Hölder's inequality. II. *Acta Sci. Math.* **1972**, *33*, 217–223.
24. Brascamp, H.J.; Lieb, E.H. Some inequalities for Gaussian measures and the long-range order of one-dimensional plasma. In *Inequalities*; Lieb, E.H., Ed.; Springer: Berlin/Heidelberg, Germany, 2002; pp. 403–416.
25. Hardy, G.; Littlewood, J.E.; Pólya, G. *Inequalities*, 2nd ed.; Cambridge University Press: Cambridge, UK, 1989.
26. Wu, D.; Qin, M.; Bu, Z.-H. Quantitative stability of Brunn-Minkowski inequalities for the p-torsional rigidity. 2024, *submitted* .
27. Bu, Z.-H.; Qin, M.; Wu, D. The functional geometric inequality for the (p, α)-torsional rigidity and its extension in Orlicz spaces. 2024, *submitted*.
28. Rockafellar, R.T. *Convex Analysis*; Princeton Mathematical Series 28; Princeton University Press: Princeton, NJ, USA, 1970.
29. Strömberg, T. The operation of infimal convolution. *Diss. Math.* **1996**, *352*, 1–58.