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Applications of Double ARA Integral Transform

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Abstract: This paper describes our construction of a new double transform, which we call the double ARA transform (DARAT). Our novel double-integral transform can be used to solve partial differential equations and other problems. We discuss some fundamental characteristics of our approach, including existence, linearity, and several findings relating to partial derivatives and the double convolution theorem. DARAT can be used to precisely solve a variety of partial differential equations, including the heat equation, wave equation, telegraph equation, Klein–Gordon equation, and others, all of which are crucial for physical applications. Herein, we use DARAT to solve model integral equations to obtain exact solutions. We conclude that our novel method is easier to use than comparable transforms.

Keywords: ARA transform; double ARA transform; partial differential equations

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1. Introduction

Integral transformations are considered to be the most efficient method for solving partial differential equations (PDE). PDEs can mathematically describe a wide range of phenomena in mathematical physics and several other scientific domains, which makes them valueable [1–13]. Using integral transforms, these equations can also be changed to identify precise PDE solutions. The straightforward power of transform techniques has inspired ongoing research aimed at understanding and enhancing them.

Many integral transforms have been developed and put into practice to solve partial and integral differential equations. These transforms enable us to obtain the exact solutions of target equations without the need for linearization or discretization. They are used to convert partial differential equations into ordinary equations when using a single transform and into algebraic equations when using a double integral transform. Examples include the Laplace transform [14], novel transform [15], M-transform [16], Sumudu transform [17], Elzaki transform [18], natural transform [19], Kamal transform [20], Aboodh transform [21], and ARA transform [22,23], but there are many others [24–27].

Double transformations are considered to be extremely successful in handling PDEs compared to other numerical approaches, as they have been extensively used to solve PDEs with unknown functions of two variables. [28–30]. Extensions of double transforms have been developed in the relevant literature, such as the double Laplace transform, double Shehu transform [29], double Kamal transform [30], double Sumudu transform [31–36], double Elzaki transform [37], double Laplace–Sumudu transform [38] and ARA–Sumudu transform [39–42]. All the double transforms cited above can be considered as special cases of the general double transform described by Meddahi et al. [43], but there is value in the study of special variants of double transforms for comparison and consideration, as the endeavor reveals the unique properties of each one and facilitates understanding of their respective best applications [44].

Saadeh and others recently introduced the ARA transform [23], a novel integral transform. In this paper, we describe our creation of a new double ARA transform that takes advantage of this powerful transform approach. It has some novel properties, including the ability to generate numerous transforms by altering the value of the index n, and it can overcome the singularity point at zero, as introduced in [23]. This new strategy is what we call the double ARA transform (DARAT).

This paper describes and proves the main properties and theorems of DARAT. Moreover, we compute the values of DARAT for some elementary functions. We establish some new relations of DARAT with partial derivatives and the double convolution property. Some of these results are implemented to solve PDEs.

We also show the novelty of DARAT in terms of its advantages over the ARA transform, including its simplicity and versatility in application. Moreover, unlike other transforms, when we apply DARAT on constants, the same constants emerge in the results: the transformed constants are constants without any variables, which reduces the computational load when we use the transform to solve equations.

Many researchers have studied solutions of partial differential equations, and one of the most important techniques that has been developed is the double transform. Its strength lies in its ability to produce exact, rather than approximate, solutions [31–36]. In this study, we implement DARAT to solve PDEs of the form

$$Aq_{xx}(x,\ t) + Bq_{xt}(x,\ t) + Cq_{tt}(x,\ t) + Dq_x(x,\ t) + Eq_t(x,\ t) + Fq(x,\ t) = r(x,t),$$

under the following conditions: $q(x,\ t)$ is the unknown function, $r(x,t)$ is a given source function, and $A,\ B,\ C,\ D,\ E,\ F,$ and M are constants.

We have established a general formula for the solution of the above problem, and we will later use it to solve some examples.

The structure of this paper is as follows: Section 2 introduces fundamental concepts and properties of the ARA transformation, Section 3 presents the new double transform DARAT and some properties and theorems, Section 4 describes the method for using DARAT in solving PDEs, and Section 5 presents some problems and their solutions using DARAT.

2. Basic Definitions and Theorems of the ARA Transform

In this section, we introduce the definition and the basic properties of the ARA transform [23].

Definition 1. *The ARA integral transform of order n of a continuous function $q(t)$ on the interval $(0,\ \infty)$ is defined as*

$$\mathcal{G}_n[q(t)](s) = Q(n,s) = s\int_0^\infty t^{n-1}e^{-st}q(t)dt,\ s>0. \tag{1}$$

The ARA integral transform of order one, denoted by $\mathcal{G}_1[q(t)]$, is defined as

$$\mathcal{G}_1[q(t)](s) = Q(s) = s\int_0^\infty e^{-st}q(t)dt, s>0. \tag{2}$$

For simplicity, let us denote $\mathcal{G}_1[q(t)]$ by $\mathcal{G}[q(t)]$. Our study focuses on ARA transform of order one, only.

Lemma 1. *The inverse ARA integral transform of order one of a piecewise continuous function $q(t)$ on the interval $(0,\ \infty)$ is defined as*

$$\begin{aligned}\mathcal{G}^{-1}[Q(s)] &= \tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tfrac{e^{st}}{s}\left[s\int_0^\infty e^{-st}q(t)dt\right]ds \\ &= \tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tfrac{e^{st}}{s}[Q(s)]ds = q(t),\ t>0.\end{aligned} \tag{3}$$

Theorem 1. *If $q(t)$ is a piecewise continuous integrable function in every finite interval $0\le t\le \alpha,\ \alpha>0$, and $q(t)$ is of exponential order, that is, if it satisfies*

$$|q(t)| \le Re^{\alpha t},\ \ t\in[0,\alpha], \tag{4}$$

where R is a positive real constant independent of t, then the ARA transform $\mathcal{G}$ exists for all $s > \alpha$.

Proof. Using the definition of the ARA transform, we obtain

$$|\mathcal{G}[q(t)| = |Q(s)| = |s\int_0^\infty e^{-st}q(t)dt|.$$

Using the property of improper integral, we obtain

$$\begin{aligned}|Q(s)| = |s\textstyle\int_0^\infty e^{-st}q(t)dt| \le s|\int_0^\infty e^{-st}q(t)dt|\\ \le s\textstyle\int_0^\infty e^{-st}|q(t)|dt \le s\int_0^\infty e^{-st}Re^{\alpha t}dt\\ = sR\textstyle\int_0^\infty e^{-(s-\alpha)t}dt = \frac{sR}{s-\alpha}.\end{aligned}$$

Thus, the improper integral converges for all $s > \alpha$, and $\mathcal{G}[q(t)]$ exists. □

In the following arguments, we state some basic properties of the ARA transform of order one.

Assume that $Q(s) = \mathcal{G}[q(t)]$ and $P(s) = \mathcal{G}[p(t)]$ and $a, b \in \mathcal{R}$. Then, we have

$$\mathcal{G}[a\ q(t) + b\ p(t)] = a\ \mathcal{G}_n[q(t)] + b\ \mathcal{G}_n[p(t)].$$

$$\mathcal{G}^{-1}[a\ Q(n,s) + b\ P(n,s)] = a\ \mathcal{G}_n{}^{-1}[Q(n,s)] + b\ \mathcal{G}_n{}^{-1}[P(n,s)].$$

$$\mathcal{G}[t^\alpha] = \frac{\Gamma(\alpha+n)}{s^{\alpha+n-1}}, \quad \alpha > 0.$$

$$\mathcal{G}\left[e^{at}\right] = \frac{s\Gamma(n)}{(s-a)^n}\ , \quad a \in \mathcal{R}.$$

$$\mathcal{G}[sinat] = \frac{as}{s^2+a^2}\ , \quad a \in \mathcal{R}.$$

$$\mathcal{G}\left[q^{(n)}(t)\right] = s^n Q(s) - \sum_{k=1}^n s^{n-k}q^{(k-1)}(0).$$

3. Double ARA Transform (DARAT)

This section introduces DARAT, a novel double ARA transform. We provide the fundamental properties and characteristics including the existence conditions, linearity, and the inverse of our proposed double transform. Moreover, some important properties and results are provided and used to compute DARAT for some elementary functions. The double convolution theorem and the derivatives properties of the new transform are also presented and illustrated.

Definition 2. *Let $q(x,t)$ be a continuous function of two positive variables x and t. Then, DARAT of $q(x,t)$ is defined as*

$$\mathcal{G}_x\mathcal{G}_t[q(x,t)] = Q(v,s) = vs\int_0^\infty\int_0^\infty e^{-(vx+st)}q(x,t)dxdt, \quad v,s > 0\ , \tag{5}$$

provided that the integral exists.

Clearly, DARAT is a linear integral transformation, as shown below.

$$\mathcal{G}_x\mathcal{G}_t[A\ q(x,t) + B\ p(x,t)] = vs\int_0^\infty\int_0^\infty e^{-(vx+st)}[A\ q(x,t) + B\ p(x,t)]dxdt$$

$$= Avs\int_0^\infty\int_0^\infty e^{-(vx+st)}[q(x,t)]dxdt + Bvs\int_0^\infty\int_0^\infty e^{-(vx+st)}[p(x,t)]dxdt$$

$$= A\ \mathcal{G}_x\mathcal{G}_t[\ q(x,t)] + B\ \mathcal{G}_x\mathcal{G}_t[\ p(x,t)].$$

where A and B are constants and $\mathcal{G}_x\mathcal{G}_t[q(x,t)]$, $\mathcal{G}_x\mathcal{G}_t[\,p(x,t)]$ exist.

The inverse DARAT is given by

$$\begin{aligned} \mathcal{G}_x^{-1}\mathcal{G}_t^{-1}[Q(v,s)] &= \mathcal{G}_x^{-1}\left[\mathcal{G}_t^{-1}[Q(v,s)]\right] \\ &= \left(\tfrac{1}{2\pi i}\right)\int_{c-i\infty}^{c+i\infty}\tfrac{e^{vx}}{v}dv\left(\tfrac{1}{2\pi i}\right)\int_{r-i\infty}^{r+i\infty}\tfrac{e^{st}}{s}Q(v,s)ds = q(x,t). \end{aligned} \tag{6}$$

In the following, we present some properties of DARAT.

Property 1. *Let* $q(x,t) = w(x)u(t), x > 0, t > 0$. *Then*

$$\mathcal{G}_x\mathcal{G}_t[q(x,t)] = \mathcal{G}_x[w(x)]\mathcal{G}_t[u(t)].$$

Proof of Property 1.

$$\begin{aligned} \mathcal{G}_x\mathcal{G}_t[q(x,t)] = \mathcal{G}_x\mathcal{G}_t[w(x)u(t)] &= vs\int_0^\infty\int_0^\infty e^{-(vx+st)}[w(x)u(t)]dxdt \\ &= v\int_0^\infty w(x)e^{-vx}dx\cdot s\int_0^\infty u(t)e^{-st}dt \\ &= \mathcal{G}_x[w(x)]\mathcal{G}_t[u(t)]. \end{aligned}$$

□

3.1. DARAT for Some Elementary Functions

In this section, we apply DARAT for some basic functions.

i. Let $q(x,t) = 1, x > 0, t > 0$. Then,

$$\mathcal{G}_x\mathcal{G}_t[1] = vs\int_0^\infty\int_0^\infty e^{-(vx+st)}dxdt = v\int_0^\infty e^{-vx}dx\cdot s\int_0^\infty e^{-st}dt = \mathcal{G}_x[1]\mathcal{G}_t[1] = 1,$$

where $Re(v) > 0$ and $Re(s) > 0$.

ii. Let $q(x,t) = x^\alpha t^\beta$, $x > 0$, $t > 0$ and α,β be constants. Then,

$$\mathcal{G}_x\mathcal{G}_t\left[x^\alpha t^\beta\right] = vs\int_0^\infty\int_0^\infty e^{-(vx+st)}\left[x^\alpha t^\beta\right]dxdt = v\int_0^\infty e^{-vx}[x^\alpha]dx\cdot s\int_0^\infty e^{-st}\left[t^\beta\right]dt = \mathcal{G}_x[x^\alpha]\mathcal{G}_t\left[t^\beta\right].$$

From the properties of ARA transform, we obtain

$$\mathcal{G}_x\mathcal{G}_t\left[x^\alpha t^\beta\right] = \mathcal{G}_x[x^\alpha]\mathcal{G}_t\left[t^\beta\right] = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{v^\alpha\, s^\beta}, Re(\alpha) > -1 \text{ and } Re(\beta) > -1.$$

iii. Let $q(x,t) = e^{\alpha x+\beta t}, x > 0, t > 0$ and α,β be constants. Then,

$$\begin{aligned} \mathcal{G}_x\mathcal{G}_t\left[e^{\alpha x+\beta t}\right] &= vs\int_0^\infty\int_0^\infty e^{-(vx+st)}\left[e^{\alpha x+\beta t}\right]dxdt \\ &= v\int_0^\infty e^{-vx}[e^{\alpha x}]dx\cdot s\int_0^\infty e^{-st}\left[e^{\beta t}\right]dt = \mathcal{G}_x[e^{\alpha x}]\mathcal{G}_t\left[e^{\beta t}\right]. \end{aligned}$$

From the properties of ARA transform, we obtain

$$\mathcal{G}_x\mathcal{G}_t\left[e^{\alpha x+\beta t}\right] = \frac{v\,s}{(v-\alpha)(s-\beta)}\ .$$

Similarly,

$$\mathcal{G}_x\mathcal{G}_t\left[e^{i(\alpha x+\beta t)}\right] = \frac{v\,s}{(v-i\alpha)(s-i\beta)}.$$

Using some properties of complex analysis, we find

$$\mathcal{G}_x\mathcal{G}_t\left[e^{i(\alpha x+\beta t)}\right] = \frac{vs(sv-\alpha\beta)+ivs(v\beta+s\alpha)}{(v^2+\alpha^2)(s^2+\beta^2)}.$$

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i}\ ,\ \cos x = \frac{e^{ix}+e^{-ix}}{2}\ \ \sinh x = \frac{e^{x}-e^{-x}}{2}\ ,\ \cosh x = \frac{e^{x}+e^{-x}}{2}.$$

Following, we find DARAT of the following functions as

$$\mathcal{G}_x\mathcal{G}_t[\sin(\alpha x+\beta t)] = \frac{vs(v\beta+s\alpha)}{(v^2+\alpha^2)(s^2+\beta^2)}.$$

$$\mathcal{G}_x\mathcal{G}_t[\cos(\alpha x+\beta t)] = \frac{vs(sv-\alpha\beta)}{(v^2+\alpha^2)(s^2+\beta^2)}.$$

$$\mathcal{G}_x\mathcal{G}_t[\sinh(\alpha x+\beta t)] = \frac{vs(v\beta+s\alpha)}{(v^2-\alpha^2)(s^2-\beta^2)}.$$

$$\mathcal{G}_x\mathcal{G}_t[\cosh(\alpha x+\beta t)] = \frac{vs(sv+\alpha\beta)}{(v^2-\alpha^2)(s^2-\beta^2)}.$$

iv. Let $q(x,t) = J_0\left(c\sqrt{xy}\,\right)$. Then,

$$\begin{aligned}\mathcal{G}_x\mathcal{G}_t\left[J_0\left(c\sqrt{xt}\,\right)\right] &= vs\int_0^\infty\int_0^\infty e^{-vx-st}\,J_0\left(c\sqrt{xt}\,\right)dxdt\\ &= s\int_0^\infty e^{-st}dt\ v\int_0^\infty e^{-v\,x}J_0\left(c\sqrt{xy}\,\right)dx\\ &= \mathcal{G}_t\left[e^{-\frac{c^2t}{4s}}\right] = \frac{4vs}{4vs+c^2}\ ,\end{aligned}$$

where J_0 is the modified Bessel function of order zero.

3.2. Existence Conditions for DARAT

Let $q(x,t)$ be a function of exponential orders α and β as $x\to\infty$ and $t\to\infty$. If there exists a positive N such that $\forall x > X$ and $t > T$, we have

$$|q(x,t)| \le Ne^{\alpha x+\beta t}.$$

We can write $q(x,t) = O\left(e^{\alpha x+\beta t}\right)$ as $x\to\infty$ and $t\to\infty$, $v>\alpha$ and $s>\beta$.

Theorem 2. *Let $q(x,t)$ be a continuous function on the region $[0,X)\times[0,T)$ of exponential orders α and β . Then, $\mathcal{G}_x\mathcal{G}_t[q(x,t)]$ exists for v and s provided that $Re(v)>\alpha$ and $Re(s)>\beta$.*

Proof. Using the definition of DARAT, we find

$$\begin{aligned}|Q(v,s)| &= \left|vs\int_0^\infty\int_0^\infty e^{-(vx+st)}[q(x,t)]dxdt\right| \le vs\int_0^\infty\int_0^\infty e^{-(vx+st)}|q(x,t)|dxdt\\ &\le Nv\int_0^\infty e^{-(v-\alpha)x}dx\cdot s\int_0^\infty e^{-(s-\beta)t}dt\\ &= \frac{Nvs}{(v-\alpha)(s-\beta)},\ Re(v)>\alpha \text{ and } Re(s)>\beta.\end{aligned}$$

Thus, $\mathcal{G}_x\mathcal{G}_t[q(x,t)]$ exists for v and s provided $Re(v)>\alpha$ and $Re(s)>\beta$. □

3.3. Some Theorems of DARAT

Theorem 3. *(Shifting Property). Let $q(x,t)$ be a continuous function and $\mathcal{G}_x\mathcal{G}_t[q(x,t)] = Q(v,s)$. Then,*

$$\mathcal{G}_x\mathcal{G}_t\left[e^{\alpha x+\beta t}q(x,t)\right] = \frac{vs}{(v-\alpha)(s-\beta)}Q(v-\alpha,s-\beta). \tag{7}$$

Proof. From the definition of DARAT, we find

$$\mathcal{G}_x\mathcal{G}_t\left[e^{\alpha x+\beta t}q(x,t)\right] = \frac{vs}{(v-\alpha)(s-\beta)}Q(v-\alpha,s-\beta).$$

$$\mathcal{G}_x\mathcal{G}_t\left[e^{\alpha x+\beta t}q(x,t)\right] = vs\int_0^\infty\int_0^\infty e^{-(v-\alpha)x-(s-\beta)t}[q(x,t)]dxdt$$

$$= \frac{vs}{(v-\alpha)(s-\beta)}(v-\alpha)(s-\beta)\int_0^\infty\int_0^\infty e^{-(v-\alpha)x}e^{-(s-\beta)t}[q(x,t)]dxdt$$

$$= \frac{v}{(v-\alpha)}\frac{s}{(s-\beta)}Q(v-\alpha,s-\beta).$$

□

Theorem 4. *(Periodic Function). Let $\mathcal{G}_x\mathcal{G}_t[q(x,t)]$ exist, where $q(x,t)$ describes a periodic function of periods α and β such that*

$$q(x+\alpha,t+\beta) = q(x,t)\ ,\ \ \forall x,y.$$

Then,

$$\mathcal{G}_t[q(x,t)] = \frac{1}{\left(1-e^{-(v\alpha+s\beta)}\right)}\left(vs\int_0^\alpha\int_0^\beta e^{-(vx+st)}(q(x,t))dxdt\right) \tag{8}$$

Proof. Using the definition of DARAT, we obtain

$$\mathcal{G}_x\mathcal{G}_t[q(x,t)] = vs\int_0^\infty\int_0^\infty e^{-(vx+st)}[q(x,t)]dxdt. \tag{9}$$

Using the property of improper integral, Equation (9) can be written as

$$\mathcal{G}_x\mathcal{G}_t[q(x,t)] = vs\int_0^\alpha\int_0^\beta e^{-(vx+st)}(q(x,t))dxdt + vs\int_\alpha^\infty\int_\beta^\infty e^{-(vx+st)}(q(x,t))dxdt. \tag{10}$$

Putting $x = \alpha+\rho$ and $t = \beta+\tau$ on the second integral in Equation (10), we obtain

$$Q(v,s) = vs\int_0^\alpha\int_0^\beta e^{-(vx+st)}(q(x,t))dxdt + vs\int_0^\infty\int_0^\infty e^{-(v(\alpha+\rho)+s(\beta+\tau))}(q(\alpha+\rho,\beta+\tau))d\rho d\tau. \tag{11}$$

Using the periodicity of the function $q(x,t)$, Equation (11) can be written as

$$Q(v,s) = vs\int_0^\alpha\int_0^\beta e^{-(vx+st)}(q(x,t))dxdt + e^{-(v\alpha+s\beta)}vs\int_0^\infty\int_0^\infty e^{-(v\rho+s\tau)}(q(\rho,\tau))d\rho d\tau. \tag{12}$$

Using the definition of DARAT, we obtain

$$Q(v,s) = vs\int_0^\alpha\int_0^\beta e^{-(vx+st)}(q(x,t))dxdt + e^{-(v\alpha+s\beta)}Q(v,s). \tag{13}$$

Thus, Equation (12) can be simplified as

$$Q(v,s) = \frac{1}{\left(1-e^{-(v\alpha+s\beta)}\right)}\left(vs\int_0^\alpha\int_0^\beta e^{-(vx+st)}(q(x,t))dxdt\right).$$

□

Theorem 5. *(Heaviside Function). Let $\mathcal{G}_x\mathcal{G}_t[q(x,t)]$ exist and $\mathcal{G}_x\mathcal{G}_t[q(x,t)] = Q(v,s)$. Then,*

$$\mathcal{G}_x\mathcal{G}_t[q(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)] = e^{-v\delta-s\varepsilon}Q(v,s). \tag{14}$$

where $H(x-\delta,t-\varepsilon)$ is the Heaviside unit step function defined as

$$H(x-\delta,t-\varepsilon)=\begin{cases}1\,, & x>\delta,t>\varepsilon\\ 0\,, & Otherwise.\end{cases}$$

Proof. Using the definition of DARAT, we find

$$\begin{aligned}\mathcal{G}_x\mathcal{G}_t[q(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)] &= vs\int_0^\infty\int_0^\infty e^{-(vx+st)}(q(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon))dxdt\\ &= vs\int_0^\infty\int_0^\infty e^{-(vx+st)}(q(x-\delta,t-\varepsilon))dxdt.\end{aligned} \tag{15}$$

Putting $x-\delta=\rho$ and $t-\varepsilon=\tau$ in Equation (15), we obtain

$$\mathcal{G}_x\mathcal{G}_t[u(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)] = vs\int_0^\infty\int_0^\infty e^{-v(\delta+\rho)-s(\varepsilon+\tau)}(q(\rho,\tau))d\rho d\tau. \tag{16}$$

Thus, Equation (16) can be simplified as

$$\mathcal{G}_x\mathcal{G}_t[q(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)] = e^{-v\delta-s\varepsilon}\left(vs\int_0^\infty\int_0^\infty e^{-v\rho-s\tau}(q(\rho,\tau))d\rho d\tau\right) = e^{-v\delta-s\varepsilon}Q(v,s). \tag{17}$$

□

Theorem 6. *(Convolution Theorem). Let $\mathcal{G}_x\mathcal{G}_t[q(x,t)]$ and $\mathcal{G}_x\mathcal{G}_t[p(x,t)]$ exist and $\mathcal{G}_x\mathcal{G}_t[q(x,t)] = Q(v,s)$, $\mathcal{G}_x\mathcal{G}_t[p(x,t)] = P(v,s)$. Then,*

$$\mathcal{G}_x\mathcal{G}_t[q(x,t)**p(x,t)] = \frac{1}{v\,s}Q(v,s)P(v,s), \tag{18}$$

where $q(x,t)**p(x,t) = \int_0^x\int_0^t q(x-\rho,t-\tau)p(\rho,\tau)d\rho d\tau$ and the symbol $**$ denotes the double convolution with respect to x and t.

Proof. Using the definition of DARAT, we find

$$\begin{aligned}\mathcal{G}_x\mathcal{G}_t[q(x,t)**p(x,t)] &= vs\int_0^\infty\int_0^\infty e^{-(vx+st)}[q(x,t)**p(x,t)]dxdt\\ &= vs\int_0^\infty\int_0^\infty e^{-(vx+st)}\left(\int_0^x\int_0^t q(x-\rho,t-\tau)p(\rho,\tau)d\rho d\tau\right)dxdt.\end{aligned} \tag{19}$$

Using the Heaviside unit step function, Equation (19) can be written as

$$\mathcal{G}_x\mathcal{G}_t[(q**p)(x,t)] = vs\int_0^\infty\int_0^\infty e^{-(vx+st)}\left(\int_0^\infty\int_0^\infty q(x-\rho,t-\tau)H(x-\rho,t-\tau)p(\rho,\tau)d\rho d\tau\right)dxdt. \tag{20}$$

Thus, Equation (20) can be written as

$$\begin{aligned}\mathcal{G}_x\mathcal{G}_t[u**w(x,t)] &= v\int_0^\infty\int_0^\infty p(\rho,\tau)d\rho d\tau\left(s\int_0^\infty\int_0^\infty e^{-v(x+\rho)-s(t+\tau)}q(x-\rho,t-\tau)H(x-\rho,t-\tau)\right)dxdt\\ &= \int_0^\infty\int_0^\infty p(\rho,\tau)d\rho d\tau(e^{-v\rho-s\tau}Q(v,s))\\ &= Q(v,s)\int_0^\infty\int_0^\infty e^{-v\rho-s\tau}p(\rho,\tau)d\rho d\tau = \frac{1}{vs}Q(v,s)P(v,s).\end{aligned}$$

□

Theorem 7. *(Derivatives Properties). Let $q(x,t)$ be a continuous function and $\mathcal{G}_x\mathcal{G}_t[q(x,t)] = Q(v,s)$. Then, we obtain the following derivatives properties:*

$$\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial q(x,t)}{\partial t}\right] = sQ(v,s) - s\,\mathcal{G}[q(x,0)].$$
$$\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial q(x,t)}{\partial t}\right] = sQ(v,s) - s\,\mathcal{G}[q(x,0)].$$
$$\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial q(x,t)}{\partial x}\right] = vQ(v,s) - v\,\mathcal{G}[q(0,t)].$$
$$\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial^2 q(x,t)}{\partial t^2}\right] = s^2Q(v,s) - s^2\mathcal{G}[q(x,0)] - s\,\mathcal{G}\left[\frac{\partial q(x,0)}{\partial t}\right].$$
$$\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial^2 q(x,t)}{\partial x^2}\right] = v^2Q(v,s) - v^2\mathcal{G}[q(0,t)] - v\,\mathcal{G}\left[\frac{\partial q(0,t)}{\partial x}\right].$$
$$\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial^2 q(x,t)}{\partial x\partial t}\right] = vsQ(v,s) - vs\mathcal{G}[q(x,0)] - vs\mathcal{G}[q(0,t)] + vsq(0,0).$$

Proof.

$$\begin{aligned}\text{Proof of part a)}:\ \mathcal{G}_x\mathcal{G}_t\left[\frac{\partial q(x,t)}{\partial t}\right] &= vs\int_0^\infty\int_0^\infty e^{-(st+vx)}\left[\frac{\partial q(x,t)}{\partial t}\right]dxdt \\ &= v\int_0^\infty e^{-vx}dx\cdot s\int_0^\infty e^{-st}\left(\frac{\partial q(x,t)}{\partial t}\right)dt.\end{aligned} \tag{21}$$

Using integration by part for the second integration, we obtain

$$s\int_0^\infty e^{-st}\left(\frac{\partial q(x,t)}{\partial t}\right)dt = s\left(-q(x,0) + s\int_0^\infty e^{-st}q(x,t)dt\right).$$

Substituting the above value in (21), we obtain the required result as

$$\therefore\ \mathcal{G}_x\mathcal{G}_t\left[\frac{\partial q(x,t)}{\partial t}\right] = sQ(v,s) - s\mathcal{G}[q(x,0)].$$

$$\begin{aligned}\text{Proof of part c)}:\quad \mathcal{G}_x\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] &= vs\int_0^\infty\int_0^\infty e^{-(st+vx)}\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right]dxdt \\ &= v\int_0^\infty e^{-vx}dx\cdot s\int_0^\infty e^{-st}\left(\frac{\partial^2 u(x,t)}{\partial t^2}\right)dt.\end{aligned}$$

Using integration by part again for the second integration, we obtain

$$s\int_0^\infty e^{-st}\left(\frac{\partial^2 u(x,t)}{\partial t^2}\right)dt = s\left(-\frac{\partial u(x,0)}{\partial t} + s\int_0^\infty e^{-st}\left(\frac{\partial u(x,t)}{\partial t}\right)dt\right).$$

Using the result in part a, we have

$$\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] = s^2Q(v,s) - s^2\mathcal{G}[u(x,0)] - s\mathcal{G}\left[\frac{\partial u(x,0)}{\partial x}\right]. \tag{22}$$

The proof of the remaining relations can be obtained by similar arguments. □

3.4. Comparisons of DARAT and Other Transforms

This section presents some comparisons between DARAT and the double Laplace transform, double ARA–Sumud transform, and double Laplace–Sumudu transform for some functions.

- The double Laplace transform of a function $q(x,t)$ is given by $L_xL_t[q(x,t)] = Q(v,s) = \int_0^\infty\int_0^\infty e^{-(vx+st)}q(x,t)dxdt,\ v,s>0$.
- The double ARA–Sumudu transform of a function $q(x,t)$ is given by $\mathcal{G}_xS_t[q(x,t)] = Q(v,s) = \frac{v}{s}\int_0^\infty\int_0^\infty e^{-(vx+t/s)}q(x,t)dxdt,\ v,s>0$.
- The double Laplace–ARA transform of a function $q(x,t)$ is given by $L_x\mathcal{G}_t[q(x,t)] = Q(v,s) = s\int_0^\infty\int_0^\infty e^{-(vx+st)}q(x,t)dxdt,\ v,s>0$.

In the following table, Table 1, we present the values of the previous double transforms for some functions. As these results show, we note that the values from DARAT are simpler than those obtained by other transforms, suggesting DARAT's potential for simplifying analogous computations.

Table 1. Comparisons between DARAT and other transforms.

$q(x,t)$	$\mathcal{G}_x\mathcal{G}_t[q(x,t)]$	$L_xL_t[q(x,t)]$	$\mathcal{G}_xS_t[q(x,t)]$	$L_x\mathcal{G}_t[q(x,t)]$
1	1	$\frac{1}{v\ s}$	1	$\frac{1}{v\ s}$
$x^\alpha t^\beta$	$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{v^\alpha\ s^\beta}$	$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{v^{\alpha+1}\ s^{\beta+1}}$	$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\ s^\beta}{v^\alpha}$	$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{v^{\alpha+1}\ s^\beta}$
$e^{\alpha x+\beta t}$	$\frac{v\ s}{(v-\alpha)(s-\beta)}$	$\frac{v\ s}{(v-\alpha)(s-\beta)}$	$\frac{v}{(v-\alpha)(1-\beta s)}$	$\frac{s}{(v-\alpha)(s-\beta)}$
$\sin(\alpha x+\beta t)$	$\frac{vs(v\beta+s\alpha)}{(v^2+\alpha^2)(s^2+\beta^2)}$	$\frac{(v\beta+s\alpha)}{(v^2+\alpha^2)(s^2+\beta^2)}$	$\frac{\alpha v(1+vs)}{(v^2+\alpha^2)(1+\beta^2 s^2)}$	$\frac{s(v\beta+s\alpha)}{(v^2+\alpha^2)(s^2+\beta^2)}$

4. Method of Double ARA Transform

To illustrate the method of using DARAT for solving PDEs, let us consider the following PDE with two independent variables x and t :

$$Aq_{xx}(x,\ t)+Bq_{xt}(x,\ t)+Cq_{tt}(x,\ t)+Dq_x(x,\ t)+Eq_t(x,\ t)+Fq(x,\ t)=r(x,t). \tag{23}$$

Assume initial conditions of

$$q(x,\ 0)=h_1(x),\ q_t(x,\ 0)=h_2(x) \tag{24}$$

and boundary conditions of

$$q(0,\ t)=k_1(t),\ q_x(0,\ t)=k_2(t). \tag{25}$$

Let us further assume that $q(0,\ 0)=M$, where $q(x,\ t)$ is the unknown function, $r(x,t)$ is a given function, and $A,B,C,D,E,F,$ and M are constants.

The main idea of this method is to operate DARAT to Equation (23) and the ARA transform to the initial and boundary conditions in Equations (24) and (25) as follows.

Operating the ARA transform to the prescribed conditions yields

$$\begin{gathered}\mathcal{G}[q(x,\ 0)]=\mathcal{G}[h_1(x)]=H_1(v)=H_1,\ \mathcal{G}[q_t(x,\ 0)]=\mathcal{G}[h_2(x))]=H_2(v)=H_2.\\ \mathcal{G}[q(0,\ t)]=\mathcal{G}[k_1(t)]=K_1(s)=K_1,\ \mathcal{G}[u_x(0,\ t)]=\mathcal{G}[k_2(t)]=K_2(s)=K_2.\end{gathered}$$

Importantly, $R(v,s)=R=\mathcal{G}_x\mathcal{G}_t[r(x,t)]$.

Next, by operating DARAT to both sides of Equation (23), we obtain

$$\begin{gathered}Aq_{xx}(x,\ t)+Bq_{xt}(x,\ t)+Cq_{tt}(x,\ t)+Dq_x(x,\ t)+Eq_t(x,\ t)+Fq(x,\ t)=r(x,t),\\ \mathcal{G}_x\mathcal{G}_t[Aq_{xx}(x,\ t)+Bq_{xt}(x,\ t)+Cq_{tt}(x,\ t)+Dq_x(x,\ t)+Eq_t(x,\ t)+Fq(x,\ t)]=\mathcal{G}_x\mathcal{G}_t[r(x,t)].\end{gathered}$$

The derivative properties of DARAT and the above conditions yield

$$\begin{gathered}A\left[v^2Q(v,\ s)-v^2G_1-vG_2\right]+B[vsQ(v,\ s)-svF_1-vsG_1+vsM]+C\left[s^2Q(v,\ s)-s^2F_1-sF_2\right]+\\ +D[vQ(v,\ s)-vG_1]+E[sQ(v,\ s)-sF_1]+F[Q(v,\ s)]=R(v,\ s).\end{gathered} \tag{26}$$

Equation (26) can be simplified as follows:

$$Q(v,\ s)=\frac{\left(Av^2+Bvs+Dv\right)G_1+AvG_2-BvsM+\left(Bvs+Cs^2+Es\right)F_1+CsF_2+R}{Av^2+Bvs+Cs^2+Dv+Es+F} \tag{27}$$

Operating with the inverse DARAT on both sides of Equation (27), we obtain

$$q(x,\ t)=\mathcal{G}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{\left(Av^2+Bvs+Dv\right)G_1+AvG_2-BvsM+\left(Bvs+Cs^2+Es\right)F_1+CsF_2+R}{Av^2+Bvs+Cs^2+Dv+Es+F}\right] \tag{28}$$

where $q(x,\ t)$ represents the term arising from the known function $r(x,t)$ and all conditions.

5. Applications of the Double ARA Transform

Many physical phenomena can be modeled by a set of governing equations, several of them being partial differential equations. One may encounter PDEs in many branches of sciences, including but not limited to the following:

- Quantum mechanics
- Particle physics
- Astrophysics
- Chemistry
- Biology
- Environmental science

Nonetheless, solving the partial differential equations that emerge in these fields is another challenge altogether. The current state of mathematics is unable to generate closed solution, and more advances are yet to come. Still, many numerical techniques have been developed for solving PDEs. In this section, we introduce solutions for some familiar PDEs, such as the wave equation, heat equation, telegraph equation, and others. All the following figures for the selected examples were obtained using Mathematica 13 software.

5.1. Example 1

Let us consider the homogeneous wave equation,

$$q_{xx}(x,t) - q_{tt}(x,t) = 0, \text{ where } x \text{ and } t \geq 0, \tag{29}$$

with the initial conditions $q(x,0) = \sin x, q_t(x,0) = 2$, and the boundary conditions $q(0,t) = 2t, q_x(0,t) = \cos t$.

Applying the ARA transform to the initial conditions and boundary conditions yields

$$F_1 = \frac{v}{v^2+1}\ ,\ F_2 = 2\ ,\ G_1 = \frac{2}{s}\ ,\ G_2 = \frac{s^2}{s^2+1}.$$

By substituting the values of the functions F_1 , F_2, G_1 , G_2 and $A = 1, C = -1, B = D = E = F = R = M = 0$ into the obtained formula in Equation (28), we obtain

$$Q(v,\ s) = \frac{2}{s} + \frac{vs^2}{(s^2+1)(v^2+1)}. \tag{30}$$

Applying the inverse DARAT to Equation (29), the solution of Equation (30) is

$$u(x,\ t) = \mathcal{G}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{2}{s} + \frac{vs^2}{(s^2+1)(v^2+1)}\right] = 2t + \cos t \sin x.$$

5.2. Example 2

Let us consider the homogeneous Laplace equation

$$q_{xx}(x,t) + q_{tt}(x,t) = 0,\ x \text{ and } t > 0, \tag{31}$$

with the initial conditions $q(x,0) = 0,\ q_t(x,0) = \cos x,$ and the boundary conditions $q(0,t) = sinht,\ q_x(0,t) = 0$.

Upon applying the ARA transform to the initial and boundary conditions, we obtain

$$F_1 = 0\ ,\ F_2 = \frac{v^2}{v^2+1}\ ,\ G_1 = \frac{s}{s^2-1}\ ,\ G_2 = 0.$$

By substituting the values of the functions F_1 , F_2, G_1 , G_2 and $A = 1, C = 1, B = D = E = F = M = R = 0$ in the general formula in Equation (28), we obtain

$$Q(v,\ s) = \frac{v^2 s}{(s^2-1)(v^2+1)}. \tag{32}$$

Then, by applying the inverse DARAT to Equation (32), the solution of Equation (31) is

$$q(x,\ t) = \mathcal{G}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{v^2 s}{(s^2-1)(v^2+1)}\right] = \cos x \sinh t.$$

5.3. Example 3

Let us consider the homogeneous telegraph equation,

$$q_{xx}(x,t) = q_{tt}(x,t) + 4q_t(x,t) + 4q(x,t),\ x,t \geq 0, \tag{33}$$

with the initial conditions $q(x,0) = 1 + e^{2x}$, $q_t(x,0) = -2$. and the boundary conditions $q(0,t) = 1 + e^{-2t}$, $q_x(0,t) = 2$.

Applying the ARA transform to the initial and boundary conditions, we obtain

$$F_1 = 1 + \frac{v}{v-2}\ ,\ \ F_2 = -2\ ,\ \ G_1 = 1 + \frac{s}{s+2}\ ,\ \ G_2 = 2.$$

By substituting the values of the functions F_1 , F_2, G_1 , G_2 and $A = 1, C = -1, E = F = -4, B = D = R = M = 0$ in the general formula in Equation (28), we obtain

$$Q(v,\ s) = \frac{v}{v-2} + \frac{s}{(s+2)}. \tag{34}$$

Then, by applying the inverse DARAT to Equation (34), the solution of Equation (33) is

$$q(x,\ t) = \mathcal{G}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{v}{v-2} + \frac{s}{(s+2)}\right] = e^{2x} + e^{-2t}.$$

5.4. Example 4

Let us consider the nonhomogeneous heat equation,

$$q_{xx}(x,t) - q_t(x,t) - 3q(x,t) = -3,\ x \text{ and } t \geq 0, \tag{35}$$

with the initial conditions $q(x,0) = 1 + \sin x$ and the boundary conditions $q(0,t) = 1,\ q_x(0,t) = e^{-4t}$.

Applying the ARA transform to the initial and boundary conditions, we obtain

$$F_1 = 1 + \frac{v}{v^2+1}\ ,\ \ G_1 = 1\ ,\ \ G_2 = \frac{s}{s+4}.$$

By substituting the values of the functions F_1, G_1, G_2 and $A = 1, E = -1, F = -3, B = C = D = M = 0,\ \ R = 3$ in the general formula in Equation (28), we obtain

$$Q(v,\ s) = 1 + \frac{vs}{(s+4)(v^2+1)}. \tag{36}$$

Then, by applying the inverse DARAT to Equation (36), the solution of Equation (35) is

$$q(x,\ t) = \mathcal{G}_x^{-1}\mathcal{G}_t^{-1}\left[1 + \frac{vs}{(s+4)(v^2+1)}\right] = 1 + e^{-4t}\sin x.$$

5.5. Example 5

Let us consider the Klein–Gordon equation,

$$q_{tt}(x,t) - q(x,t) = q_{xx}(x,t) - \cos x \cos t, \tag{37}$$

with the initial conditions $q(x,0) = \cos x,\ \ q_t(x,0) = 0$, and the boundary conditions $q(0,t) = \cos t\ ,\ \ q_x(0,t) = 0$.

Applying the ARA transform to the initial and boundary conditions, we obtain

$$F_1 = \frac{v^2}{v^2+1}\ ,\ F_2 = 0\ ,\ \ G_1 = \frac{s^2}{s^2+1}\ ,\ \ G_2 = 0.$$

By substituting the values of the functions $F_1,\ F_2\ , G_1,$ and G_2 and $A = 1, C = -1$, $F = 1, B = D = E = M = 0$, $R = \frac{v^2 s^2}{(s^2+1)(v^2+1)}$ in the general formula in Equation (28), we obtain

$$Q(v,\ s) = \frac{v^2 s^2}{(s^2+1)(v^2+1)}. \tag{38}$$

Then, by applying the inverse DARAT to Equation (38), the solution of Equation (37) is

$$q(x,\ t) = \mathcal{G}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{v^2 s^2}{(s^2+1)(v^2+1)}\right] = \cos x \cos t.$$

5.6. Example 6

Let us consider the integral equation

$$q(x,\mathrm{t}) = 1 - \lambda \int_0^x \int_0^t g(r,u) dr\, du, \tag{39}$$

where a and λ are two constants and $q(x,t)$ is the unknown function.

Applying DARAT to Equation (39) and using the convolution theorem, we obtain

$$Q(v,s) = \mathcal{G}_x\mathcal{G}_t[1] - \lambda\mathcal{G}_x\mathcal{G}_t[1{*}{*}q(x,t)] = 1 - \lambda\frac{1}{vs}Q(s,u). \tag{40}$$

Simplifying Equation (40) yields

$$Q(v,s) = \frac{vs}{vs+\lambda}. \tag{41}$$

Then, running the inverse DARAT on Equation (41), we obtain the solution of Equation (39) as

$$q(x,t) = \mathcal{G}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{vs}{vs+\lambda}\right] = J_0\Big(2\sqrt{\lambda x t}\Big).$$

6. Conclusions

This work introduced DARAT, a novel transform technique. We described several of its properties and theorems regarding linearity, existence, partial derivatives, and the double convolution theorem. The expanded results were used to create a new formula for resolving various types of PDEs, and we also used the obtained results of the convolution theorem to solve some integral equations. We described a few numerical examples and used this new double transform to obtain precise results. In the future, DARAT applications may be created and used to resolve coupled differential equations and systems and PDEs with variable coefficients. As an additional benefit, we emphasize that this novel double transform can be used in conjunction with an iterative numerical approach to solve nonlinear PDEs.

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