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Research and Pedagogies for Early Math

Douglas H. Clements *, Renee Lizcano * and Julie Sarama *

College of Education, University of Denver, Denver, CO 80208, USA
* Correspondence: douglas.clements@du.edu (D.H.C.); liza.lizcano@du.edu (R.L.); julie.sarama@du.edu (J.S.)

Abstract: The increasing interest in early childhood mathematics education for decades has increased the need for empirically supported pedagogical strategies. However, there is little agreement on how early math might best be taught. We draw from the empirical literature to paint a picture of research-based and research-validated pedagogical approaches and strategies for teaching early math. Most approaches share core characteristics, including concern for children's interests and engagement and for working on content matched to children's level of thinking. Learning trajectories are an especially useful organizing structure because they combine and integrate educational goals, development of children's thinking, and empirically supported pedagogical strategies. Therefore, they help teachers interpret what the child is doing, thinking, and constructing, and offer instructional activities that extend children's mathematical thinking. Simultaneously, teachers can see instructional strategies from the child's perspective, offering meaningful and joyful opportunities to engage in learning.

Keywords: early childhood; learning trajectories; pedagogy; teaching strategies

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There has been increasing interest in early childhood mathematics education for decades. However, there is less agreement on how early math might best be taught. Here we draw from the empirical literature to paint a picture of what research tells us about pedagogical approaches and strategies for teaching early math. These include understanding learning trajectories, formative assessment, small-group instruction, rich math discussions, strong examples and non-examples, and ensuring children from culturally and linguistically diverse backgrounds are represented in classrooms, curricula, and all educational experiences [1].

1. Importance of Early Childhood Math and the Need for High-Quality Pedagogy

Increased attention to early math has been driven by factors such as high-stakes assessment. However, a more child-centered view is the recognition that *all* children are able and interested in engaging in important mathematics. If we do not teach math, we are not teaching the "whole child". Further, burgeoning research shows math's surprising importance. First, math has a growing role in economies and cultures, but some countries, such as the US, have not improved mathematics education (http://ncee.org/pisa-2018-lessons, accessed on 1 June 2023). Differences appear as early as the primary and preschool grades [2,3]. Interest in such countries around the globe is also motivated by a particular concern for children who have not been provided opportunities to learn [4–6]. The pandemic exacerbated these equity concerns, and more for math than other domains [7]. Educators must provide comprehensive approaches targeted to educators and families, especially for marginalized populations, to improve children's experiences learning mathematics [8,9].

Second, the math that children know when they enter kindergarten is the best predictor of graduating high school [10,11], and number and arithmetic knowledge at age 7 years predicts socioeconomic status at age 42 (even controlling for all other variables) [12].

Third, math predicts and supports children's development in other domains, such as reading [13] and oral language [14]. Math plays a central role in the sciences—throughout the grades [15]. For example, the more math courses students take in high school, the higher their performance in college math, biology, chemistry, and physics. Taking more high school math courses increases achievement in the sciences as much as, or even more than, taking more science courses [16]. Math is a core component of cognition.

Even farther transfer has also been documented. For example, early math is related to executive function, and high-quality math experiences develop executive function, even more so than approaches designed to do so [17,18]. Perhaps more surprisingly, high-quality math learning contributes to students' development of social competencies, such as preschoolers' ability to share [19] and engage in high-level socio-dramatic play [20].

Although there are many benefits, a caveat is critical: These are multiple positive outcomes of *high-quality* early math experiences. What constitutes high-quality teaching?

2. Useful Evidence

To help answer this question, we summarize and synthesize studies that provide evidence useful to researchers and practitioners: studies with designs that address important pedagogical issues. We also distinguish between claims that teaching approaches are research-*based* or research-*validated*. Most educators claim that their approach to teaching is based on research. However, these can be placed on a continuum, from mentioning theories or studies on "students' thinking" vaguely to drawing explicit connections from the theories and empirical research to the design and implementation of the approaches of the curriculum [21,22]. For example, in early childhood, early applications of Piaget's theories often led to suggestions that children be taught to perform accurately on Piagetian clinical tasks. Some incorporated materials directly adapted from those tasks [23,24]. Unfortunately, these were not particularly successful. Even detailed analyses of Piagetian research failed to guide the development of programs or curricula in directly useful ways [25]. However, as Piaget himself argued, it is not his clinical tasks, but his research showing that children were *active learners who constructed knowledge* that was central to education. "To understand is to invent" [26]. Many recent pedagogies take this position.

Therefore, *research* bases for teaching make strong contributions, if interpreted with care. Complementing these approaches are research-validated; this indicates that the actual pedagogy was evaluated. Such studies may either examine efficacy or effectiveness. Although these terms can be used interchangeably, efficacy often implies smaller evaluations that provide initial confirmation of the value the curriculum adds to a "business as usual" condition, whereas effectiveness can imply larger studies into the generalizability of this value added. At one end of the spectrum, efficacy may be measured in a *superrealization* context, an ideal situation to see what the curriculum can accomplish at its best [27]. At the other end, effectiveness might be measured in multiple locations at scale, involving not just larger numbers but greater complexity in four interrelated dimensions: depth, sustainability, spread, and shift in ownership to schools and teachers [28]. These are almost always quantitative studies, although for many models, complementary qualitative methods are essential [28,29]. Further, essential phases in the research-and-development process between these ends of the spectrum feature mainly qualitative methods, including those that provide causal evidence [30]. These include a wide variety of methodologies, such as teaching experiments, design studies, and classroom case studies (provides a comprehensive list and description) [29], provides a comprehensive list and description. Constraints on space required that we present a "best evidence" synthesis in which we were selective in the topics and studies we included. Our goal was to complement other reviews on pedagogies in early math, e.g., [31–37]. To do so, we synthesized our comprehensive reviews using all studies available via searches of databases, journals, and books over decades [38–40], supplemented with a search of ERIC, PsycINFO, and Google Scholar for new studies in the last 3 years.

As stated, the second way we wish the summary to be useful is to address important pedagogical issues. That is, we focus our discussions on pedagogical approaches with evidence of success and those widely used ones without. Finally, we focus on pedagogical approaches that *matter* according to theory and research. That is, research on teaching has addressed backgrounds and characteristics of teachers, process-product studies relating teaching practices to student outcomes, backgrounds and characteristics of students, different ways students engage and process what teachers present, and so on. Although these have all contributed to knowledge, they generally have not found practically meaningful links between specific teacher actions and student learning outcomes, e.g., [41–43] and analyses from the complex array of variables across these types of studies similarly provide little guidance. What *can* guide teaching? Understanding how to provide adequate high-quality educative experiences to achieve mathematical learning goals (which may stem from standards, or from recognition of a child's interests and needs). Thus, teachers can consistently focus on creating research-based, empirically validated Sustained Learning Opportunities (SLO) [44]. Educators develop and plan fecund instructional tasks and patterns of interactions, realized by teachers and children collaboratively so they are meaningful to all. These are based on teachers' understanding of students' levels of thinking and their development across SLOs, progressing toward the educational goal [5,39]. Students' intellectual work occurs within the triarchic interaction of teachers, students, and mathematics content and activities.

Finally, as useful as these findings are, they are general teaching strategies. High-quality teaching also depends on knowledge of content, how children think and learn about that content, and how specifically to teach that content for each important topic in early math [39,45–54]. This applies to intentional teaching, and perhaps more so to child-initiated contexts such as play, so as to fully understand how to support children's creative math thinking and learning [5,39,55]. We ground our interpretations within our theory of Hierarchical Interactionalism, a synthesis of empiricist, nativist, and especially constructivist theories, that emphasizes these three knowledge domains [40,56].

3. Children's Learning with Different Approaches to Teaching

As one of the most complex human enterprises, teaching is difficult to define and study. Here, we define the teaching of math as intentional interactions among children and teachers around mathematics content using deliberately arranged environments, contexts, and tasks, all designed to promote children's learning of increasingly powerful and sophisticated math competencies and positive dispositions. Those goals—competencies and dispositions—lead to our first issue.

3.1. General Teaching Approaches for Different Goals

When not recognized, differences in these goals can lead us to believe that research is contradictory when it is not because different pedagogical approaches can be effective for different goals [41]. For example, when learning skills, or targeting instrumental understanding (rules without reasons) [57], is the primary goal, certain teaching strategies, such as whole group organization, clear directions and explanations with modeling, fast pacing, emphasis on mastery, and careful review are effective [58–62]. In contrast, goals focused on relational understanding (knowing both what to do and why) [57] include skills and also competencies such as conceptual knowledge, mathematical practices, general cognitive competencies (e.g., executive functions) *and* positive dispositions [63]. Here, effective teaching strategies include attending explicitly to concepts and *connections* among facts, skills, and the key ideas of mathematics with consistent math talk among all participants, creating a shared coherent mathematical structure [54,64], and an emphasis on children struggling with the key math ideas [41]. "Struggle" does not indicate frustration but rather trying to make sense of math and figure out how to understand or solve a problem without following prescribed procedures.

Addressing relational understanding promotes full mathematical learning and development [39,65–68] and supports skill fluency as well as focusing mainly on skills [41,69,70]. As one example of these benefits, low-SES, urban first, and second graders learned to use the standard arithmetic algorithms skillfully and to understand them conceptually, when taught conceptually, by connecting place-value blocks and written representations. Second graders and high-ability first graders performed higher than third graders receiving traditional skills-based instruction [65]. In sum, teaching for relational goals is research-validated and important to do in collaboration with marginalized families [9]. Therefore, we focus on useful studies of effective teaching for relational understanding.

3.2. Teaching for Relational Understanding: Confronting the Dichotomies

The previous descriptions of effective teaching methods do not mention some that are most debated [71]. The reason is that many dichotomies are not consistently explanatory, such as student-directed versus child-centered approaches, using "real-world" problems or not [67], or play versus intentional instruction [72]. So many discussions, even by experts, e.g., [73] tend to phrase issues as debates between two incommensurable positions. However, research indicates that both sides of these dichotomies can be used to teach most goals and that synthesis is often the most productive strategy. Often, such dichotomies are used for rhetorical reasons, of course, but in so doing can prevent a dialectical synthesis that is the better pedagogical approach.

For example, teaching only non-constrained and higher-level skills may be counter-productive given that "lower-level" knowledge may be necessary for effective learning and use of higher-level processes, perhaps especially in hierarchical content domains such as math, e.g., [39,74,75]. Further, researchers of color have argued that avoiding lower-level skills and knowledge may not serve the needs of some communities because they have not had equitable opportunities to learn them [76,77]. Therefore, effective teachers support both differentiation and meaningful syntheses of goals and skills when appropriate [78], and a focus on specific goals for a child when more appropriate.

The issue of play "versus" intentional teaching is one of the most pernicious false dichotomies in early childhood [72,79–81]. Possible teaching approaches are varied and nuanced, ranging from "free" or unstructured play to guided play to playful teacher-directed instruction.

Starting with the most unstructured approach, math arises naturally and frequently from children's free play across a range of topics [82,83] and in children as young as toddlers [84,85]. The effects on learning are less well known. More striking, children in classrooms emphasizing math were likelier to be engaged at a higher-quality level during free choice (play) time [20]. Thus, high-quality math and free play do not have to compete for time. Doing both makes each richer. However, research in multiple countries shows minimal math learning during free play [39]; without guidance, children may build experiential foundations for later math learning but not explicit math concepts. Talking to children about math in their play promotes learning [86,87]. Specifically, interactions that are a good fit with what children are playing and those that engage children's thinking and discussions about math topics promote math achievement with no harm to their play [88]. Communication as to the math content is an important issue [31].

Therefore, seek and use teachable moments in everyday play and routines [89]. Attend to all children, including very young children, who may not be seen as "doing math" [90]. However, recognize that these moments will constitute only a small portion of the math activities needed in most situations. Further, they should contribute to the SLOs that serve children's learning needs [44].

Other approaches to play help children learn math reliably. A systematic review of free play, guided play, and direct instruction found that guided play was particularly important in math, with a greater positive effect than direct instruction on early math overall and shape recognition specifically, and then free play on spatial vocabulary [91]. This is consistent with research-validated experiments showing that unguided play or playful teaching approaches are more effective than unguided play [92], especially for children with fewer previous opportunities to learn math [93]. Notably, the guided approach supports equitable education [94,95]. Further, programs based only on an "everyday" or "play" approach to math education frequently show negligible gains. In comparison, academic approaches have strong, consistent, positive effects [96] with no harm to social–emotional development [97]. High-quality guided play, see also [98,99], includes having a clear learning goal, ensuring children have a degree of choice and agency, and using their understanding of children's thinking and interests to choose strategies, such as open-ended questions, hints, prompts, and modeling [91,100].

Finally, a playful but intentional teaching approach is more effective in promoting math learning than laissez-fair approaches or teaching based only on "teachable moments" [55,89,101–103], including in free play contexts, such as the block center [88,104]. This is especially true for children with disabilities [105]. Later sections address intentional teaching.

Unsurprisingly, these issues and suggestions mirror similar findings in the debates on discovery learning, in which unguided discovery is more effective than guided discovery teaching [60,106–108] and better at developing concepts that direct instruction alone [109].

However, even direct instruction can play an important role in a multidimensional pedagogical toolkit, especially at appropriate junctures with discovery- or inquiry-based learning contexts [109,110]. As a simple example, direct instruction is necessary and efficient for Piaget's social-arbitrary knowledge, such as spelling "four", writing "4" or other mathematical symbols, conventions, or simple procedures. Physical knowledge is learned by activity on objects. In contrast, logical-mathematics knowledge is learned from *thinking about* one's actions on the objects [111]. Intentional, playful experiences and guided discovery approaches develop deep understanding and transfer needed for relational understanding in all math topics [39,112]. Strategies from the pedagogical toolkit are best deployed depending on the content, context, and children. For example, children who explore math ideas playfully before intentional instruction use a greater variety of strategies and attend to the features of problems more than those instructed first [113].

In summary, those teaching for relational understanding view children as active learners who initiate explorations of and interactions with the surrounding world and both adults and peers [26,33,114–121]. They avoid a preponderance of passive "reception" of knowledge, understanding that children construct knowledge from a wide variety of experiences [122], including direct instruction when it contributes to their learning. Such experiences support learning and development and minimize wasted time in passive experiences such as waiting [123]. Teachers support learning by using an equity lens to watch and listen to children and the way they express their ideas [76]. By observing, interacting, and being reflective, they base interactions and activities on children's thinking and learning [114,120]. In these ways, they promote joyful, engaged learning for all children [124] from birth (we recognize that space limitations did not allow addressing infants and toddlers) [33], we recognize that space limitations did not allow addressing infants and toddlers.

3.3. Intentional Teaching and the Central Role of Children's Thinking and Learning

A critical feature of teaching approaches that develop relational thinking is that they *base teaching on an understanding of children's thinking and learning* [5,125]. A research-validated approach that does so and seamlessly integrates goals, children's thinking, and the teacher is the learning trajectories (LT) construct [39,40,56]. A LT has three interrelated components: an educational goal, a developmental progression, and teaching practices and activities. To attain a specific competence in a given subject or content area (the goal), children progress through several levels of thinking (the developmental progression), aided by intentionally planned environments, interactions, experiences, and challenges (the teaching practices that create SLOs) designed to build the mental actions that enable thinking at each higher level (and the Zone of Proximal Development) [126], and the Zone of Proximal Development. Therefore, to support children's development across content areas, teachers need to understand the goals for children's learning, children's current thinking in reference to those goals, and how to design learning opportunities to move children from their current understandings toward the learning goal.

In this way, the Hierarchical Interactionalism theory posits that LTs are a particularly fecund instructional approach. That is, each LT level is a pattern of thinking including specific, mental actions-on-objects. Instructional environments or tasks present a problem;

actions and strategies to solve the problem are represented. There is reflection on whether the problem is solved, or partially solved, which leads to new understandings (mental actions and objects, organized into strategies and structures) and actions [54]. Specific learning trajectories are the main bridge that connect the "grand theory" of Hierarchic Interactionalism [40] to particular theories and educational practice.

All three components of a LT can be misunderstood. Table 1 addresses misunderstandings and myths to make the theory and its application clear.

Table 1. Myths and Understandings of Learning Trajectories.

Component	Misunderstanding/Myths	True Learning Trajectories
Goal	Narrow behavioral objective	"Big ideas and proficiencies, central and coherent, consistent with children's thinking, and generative of future learning. Math practices and investigations [33] Positive dispositions
Developmental Progression (DP)	Rigid sequence of skills in "small steps"	Broad levels of learning; patterns of thinking including concepts and structures [31,54], skills, practices, etc.
Instructional Activities	Rote-skill based or Generic	Connected to each level of the DP–concepts, skills, and problem solving. Designed to promote thinking at that level–the actions-on-objects (often right in the activity—unitizing, composing, etc.)
Learning Trajectories	Break down skills into sequences, all followed in lock step	*Building up* children from and through their natural ways of thinking (asset-based) [32].

The LT approach has been research validated in multiple studies for a wide variety of math topics [32,92,127–135]. In most, teachers used all the strategies in the previously described multidimensional pedagogical toolkit. Further, they combined brief, active, whole-group sessions, individual work (sometimes using educational technology), incidental learning throughout the day, and small-group sessions. The last was especially important due to the personal involvement and close interactions, supporting their understanding and *use* of children's thinking to differentiate instruction. Such *formative assessment* is one of the most strongly empirically supported teaching approaches [67,136,137]. Formative assessment is the ongoing understanding of children's thinking and learning to inform and adapt instruction for groups and individuals. However, formative assessment is not useful if teaching is not adapted based on that understanding [67,138]. Effective teachers ask and answer the following questions: what do children need to learn, where are children now, and how do I help them progress? [137]. Importantly, these questions align with the three components of LTs: goal, developmental progression, and linked teaching activities and strategies. This may be why LTs support and contribute to teachers' professional development and teaching prowess [139–141] and children's learning [92,127,133,142,143].

Considering the validating studies cited, it is important to note that many have involved a specific curriculum, so that the LTs may have been confounded by other differences between the compared groups. Therefore, studies that rigorously compared LT-based instruction to the same instruction without a critical aspect of LTs address their specific contribution. In most cases, these experiments validated the LT approach [144–148]. In the case of no significant difference, the LT itself may have been under-researched, patterning in one case [149].

In summary, teachers who know how to use the three components of a learning trajectory are more effective in supporting children's learning [150]. Without such knowledge, teachers of young children might offer tasks that are either too easy or too hard for children, and this mismatch may limit children's learning [39,151]. Playful, meaningful, content-rich education based on learning trajectories benefits all children. Indeed, it is especially important for children with disabilities (CWD) [152]. CWD might operate at levels different from their peers and quite different levels in one topic (say, counting) than others (such as geometry). Learning trajectories offer different ways to introduce math topics, such as arithmetic (e.g., counting, subitizing, or partitioning), so children can build on their individual strengths. Learning trajectories' levels are clusters of ideas and processes, not just skills. So, children can both learn and show competence using a variety of modalities and representations. Finally, learning trajectories can be aligned with formative assessment and the Individualized Education Program (IEP) or the Individualized Family Service Plan (IFSP) process. For all children with disabilities or math difficulties, tiered support is important and validated as effective [153,154].

The remainder of this section consists of brief reviews of specific teaching strategies for relational understanding. We start with additional research on formative assessment.

3.4. Formative Assessment

Formative assessment, the ongoing monitoring of student learning to inform instruction, was mentioned previously. Of the 10 instructional practices the National Mathematics Advisory Panel (NMAP) researched, only a few had an adequate number of rigorous studies supporting them. One of the most strongly supported was teachers' use of formative assessment [67] so that teachers can monitor the class and individuals within it. Although the youngest children in NMAP's rigorous studies were in the upper primary grades, other studies confirm that regular assessment and individualization are key to effective education in general [155] and early math education [156,157] in particular, including internationally [158]. Teachers should observe not just answers but strategies as well [159]. Second graders experiencing such individualization gained *four months more* achievement than those in regular classrooms without formative assessment [156]. Consistent use helps all children learn but helps marginalized children the most because instruction builds on what they know and can do, and thus they learn more content and gain higher-order competencies [137,160].

3.5. Group Size and Structure

Small-group work sessions can significantly increase children's scores on tests aligned with that work [161]. Small-group work is where formative assessment with LTs is particularly effective [92,162]. One relevant finding is that these groups do not have to be small (and certainly not individual): groups with two children may not be any more effective than those with five at a time [163]. Children can also transfer the knowledge they learned in small-group activities to tasks they have not been taught [164].

Learning centers can also contribute to children's learning. They are most effective when combined with other group sizes and structures and when carefully planned, introduced, and *guided* by the teacher [165]. Finally, small class size has positive effects on math achievement in grades K–3, especially when class size is 22 or fewer, children are from marginalized populations, or reduction in class sizes is well planned and implemented in consecutive grades [166,167].

3.6. Math Talk Discussions and Connections

Most educators agree that discourse aids development. However, some studies show teachers of young children use language and feedback infrequently to teach concepts [168].

Effective teachers engage their students in mathematics discussions and use open-ended questions more than less effective teachers. They ask students, "Why?" and "How do you know?" They ensure multiple opportunities for children to talk with, not just listen to, teachers and interact with peers by applying research on productive dialogues, "think-pair-share" strategies, to incorporate the best of such interactions between teachers and children and between children, e.g., [169–171]. They expect students to share strategies, explain their thinking, work together to solve problems, and listen to and understand one another. They appropriately help students summarize critical ideas at the end of each lesson. They explicitly discuss connections between the properties and relationships of mathematics and connections between ideas and applications to everyday situations throughout the day [88,92,157,162,172–179]. This is especially important for Latinx children [180] and other marginalized groups.

Effective teachers encourage math talk throughout the day and do not limit it to the common "number talk" [176]. Math talk often productively features sharing and analyzing strategies. A recent finding regarding classroom strategy diversity is that encouraging such diversity early in learning and working with children to use a smaller number of research-validated, more sophisticated strategies is more effective than other approaches [159]. This is a new theory, and the results are not causal, but the findings are consistent with several previous studies on strategy use [181,182].

3.7. Adapting Activities and Implementation of Research-Validated Approaches and Curricula

Ann Brown [183] contrasted two ways to adapt a given activity or curriculum sequence. The first, a lethal mutation, no longer captures the pedagogical essence of the intervention and can be harmful. For example, "simplifying" a game by removing a step in which children turn to their partner and ask, "Am I right?" limits peer interaction and removes an opportunity for productive disagreement [39]. In contrast, the second, a productive adaptation, positively reinterprets a curriculum, preserving this essence while tailoring the learning experience to the strengths, needs, and characteristics of particular classrooms and children [183]. For example, in a game, teachers might provide some children a number cube with only 1, 2, 3, 1, 2, 3 on the six faces and others with numerals from 5 to 10 depending on their level of thinking [39]. Such formative assessment is discussed in the next section.

This raises the question of the role of teachers in implementing research-validated approaches and curricula. In contrast to the notion that individual teachers create all aspects of the curriculum (which should not be expected) [184], systematic, scientifically based practice is more effective than private, idiosyncratic practice [185]. This does not imply using a "scripted" curriculum; rather, focusing on the shared scientific base is a more effective and efficient way to improve education. Further, such scientifically grounded shared practice is, somewhat paradoxically, more likely to generate creative contributions. Teachers may modify shared practices, which will be accessible to discussion and further research. Further and more extensively, productive adaptations and flexible curriculum planning are necessary for teachers and children in different sociocultural contexts and with different individual strengths, assets, interests, and needs [186]. From this perspective, fidelity is being true to the research guidance and the vision of the curriculum as supporting all children's development, not compliance with a rigid script. Curricula, related resources, and professional development need to highlight and support the research-validated approaches and strategies so that teachers can understand, implement, and adapt them for their classrooms and support relational understanding.

3.8. Thoughtful Examples and Non-Examples

An oft-neglected, constructive activity is providing children with examples and non-examples of a math concept so that they can discover for themselves the critical (defining) attributes of the idea [107,187–189]. For example, "Wow! That's *not* two horses. That's *three* horses!" This is especially important in geometry: varied examples (e.g., "tilted" squares and obtuse triangles) and non-examples help children understand attributes of shapes that are mathematically relevant as well as those (orientation, size) that are not [190].

Challenging non-examples of shapes can be paired with an example, such as a triangle next to a visually-similar quadrilateral with one short side [39].

A study of examples in arithmetic found that second-grade children notice structure by analyzing worked examples, and they try to make sense of them based on prior knowledge [191]. Therefore, the first worked example *contrast* is important as it confirms or challenges their prior understanding. For example, using contrasting cases, such as having students compare the problem $5 - 3 = 2$ with $3 - 5 = -2$, can help them notice important features. Another study confirms the benefits of teachers asking children to compare and contrast ways of reasoning used on problems of different problem types to evoke different strategies [50]. Children then see and understand features of each problem that made one way of reasoning easier for solving one problem type than another. Such comparisons develop better problem solving and flexible mathematical thinking [50].

3.9. "Concrete" Manipulatives for "Abstract" Ideas

Teachers often move from "concrete" (e.g., using manipulatives) to "abstract" experiences for children. Although generally research supports this sequence, there are some critical nuances [125,192]. As an example of a study validating the approach, second graders randomly assigned to be taught with manipulatives achieved and retained significantly more on a place value comprehension test than students assigned to be taught by conventional methods using algorithmic procedures and drill and practice [193]. In addition, a case study of third graders with disabilities showed a relationship between the sequence and a place value assessment, including generalization to new tasks [194]. For example, just providing connecting cubes increased the math scores of second graders [195].

However, manipulatives do not guarantee success. Students taught multiplication emphasizing understanding performed well whether they used manipulatives or symbols [196,197]. Further, the students randomly assigned to be taught with symbols scored higher on an immediate transfer test involving different factors [198]. Manipulatives do not "carry" mathematical ideas. *They may help in teaching concretely at first, but only if such concrete teaching emphasizes quantitative or spatial ideas.*

Why might concrete manipulatives help? The answer has an interesting twist. Many would say that because they are physical objects that students can grasp with their hands, this sensory characteristic makes manipulatives "real", connected with one's intuitively meaningful personal self, and therefore helpful. However, concepts cannot be "read off" manipulatives. Expert teacher John Holt said that he and his fellow teacher "were excited about the [Cuisenaire] rods because we could see strong connections between the world of rods and the world of numbers. We therefore assumed that children, looking at the rods and doing things with them, could see how the world of numbers and numerical operations worked. The trouble with this theory is that [my colleague] and I already knew how the numbers worked. We could say, 'Oh, the rods behaved just the way numbers do'. But if we hadn't known how numbers behaved, would looking at the rods enable us to find out? Maybe so, maybe not" (Holt 1982, pp. 138–139). That is, the physical objects may be manipulated without the concepts being illuminated. Concrete materials may help students build meaning, but the students must reflect on their actions with manipulatives. Said in another way, "understanding does not travel through the fingertips and up the arm". [199] (p. 47). They need teachers to reflect on their students' representations for mathematical ideas and help them develop increasingly sophisticated and mathematical representations.

Children have Sensory-Concrete knowledge when they need to use sensory material to make sense of an idea [197,200]. For example, very young children need to count objects they can see to count meaningfully [39]. Later, teachers can help them develop Integrated-Concrete knowledge that connects concrete experiences to more abstract math concepts. There is a shift in what the adjective "concrete" describes. Sensory-Concrete refers to knowledge that demands the support of concrete objects and students' knowledge of manipulating these objects. Integrated-Concrete refers to knowledge that is concrete at a higher level because it is connected to other knowledge, both physical knowledge that has been abstracted and thus distanced from concrete objects and abstract knowledge of a variety of types.

Multiple studies have shown the benefit of supporting children in progressing from Sensory-Concrete to Integrated-Concrete cognition [192,200,201]. Usually, teachers first develop children's Sensory-Concrete implicit levels of thinking, at which perceptual supports are necessary and fundamental, and reasoning may be restricted to limited cases. Then they learn explicit, verbally-enhanced generalizations and abstractions that characterize Integrated-Concrete understandings, involving internalized mental imagery and linked verbal schemes that generate operations and abstractions that are increasingly sophisticated and powerful [192].

Too often, teachers use manipulatives to "make math fun", where "manipulative math" and "real math" are seen as different enterprises [202]. Justifications for their *instructional* role are often that they are concrete and thus understandable. Research offers guidelines to make manipulatives more effective [197,200,201].

- *Model with manipulatives.* Sensory-concrete support is useful if the manipulatives help students investigate and understand mathematical structures and processes. For example, students benefited more from using (bendable) chenille sticks than pictures to make nontriangles into triangles [203]. They merely drew on top of the pictures, but they transformed the chenille sticks, engendering more learning.
- *Encourage appropriate play with manipulatives* [204]. Is it reasonable to let children play with manipulatives? Usually yes, sometimes no. Most teachers recognize that if young children have not explored a manipulative on their own (say, toy dinosaurs), getting them to address the teacher's agenda (say, counting) can be inefficient and, at worst, near impossible. Further, children can learn pre-mathematical foundations through self-directed play, especially with structured manipulatives, such as pattern blocks or building blocks [83,204]. However, these experiences are rarely mathematical without teacher guidance, and teachers of young children often fail to extend children's thinking [168]. Counterintuitively, play can sometimes be counterproductive. When a physical object is intended to serve as a symbol, playing with the object can interfere with understanding. For example, having children play with a model of a room decreased young children's success in using it as a symbol in a map search task, and eliminating any interaction increased their success [205]. Thus, the purpose and intended learning with the manipulatives must be considered carefully.
- *Ensure manipulatives serve as symbols.* Students need to interpret the manipulative as representing a mathematical idea. For example, connecting work based on *place*-value blocks with verbal representations of number and arithmetic can help build both concepts and skills successfully [204,206–208]. Further, too many attributes can distract young children [206].
- *Use drawings and symbols, moving away from manipulatives as soon as possible.* Students use manipulatives in second grade to do arithmetic, and many continue to do so even in fourth grade [204,209]. That is a failure to move along the learning trajectory.
- *Use digital manipulatives too.* They can be more manageable, "clean", flexible, and extensible than their physical counterparts [200]. Further, children who used both physical and software manipulatives showed greater sophistication in classification and logical thinking than did a control group that used physical manipulatives only [210]. Other studies support using physical and concrete manipulatives [200,211–213] as they can reveal mathematical thinking in new ways [31].

In summary, research-based manipulative use is helpful when implemented consistent with research. However, the caution is that although widely accepted notions such as "concrete to abstract" often have a good deal of truth behind them, they can also become immune from critical reflection [200].

3.10. Practice

Research shows that teaching for relational understanding also develops skills. That does not mean young children do not need practice [214]. Fortunately, research offers clear guidelines. Rather than substantial time spent on drill, repeated experiences with many contexts and different types of activities support generalization and transfer [40,215]. Moreover, distributed, spaced practice is better than massed (all in one session, repetition of the same item repeatedly) practice [216,217]. Unfortunately, such practice is hotly debated.

Contrary to those who believe practice has no role and the so-called "science of math" movement that promotes memorization through drill without caveats, practice should be used at the correct developmental juncture and to the appropriate degree [218]. Because competencies in subitizing, counting, and arithmetic combinations support math thinking and learning throughout life, short, frequent practice sessions of facts and skills *whose conceptual foundations have been well learned and understood* are recommended. Finally, a classic conceptualization describes three levels of practice: the level of drill, application, or problem solving [219]. Practice at the problem-solving level teaches all the competencies of relational understanding. Meaningful practice develops more abilities and superior skills [181].

3.11. Affect, Motivation, and Engagement

Recall that productive disposition was one of the goals of relational understanding: a consistent view of math as sensible, useful, and worthwhile and of oneself as capable and engaged [63]. Contrary to this goal, one US cultural belief is that math achievement depends mostly on native aptitude or ability. In contrast, people from other countries, such as Japan, believe that achievement comes from effort [220]. Even more disturbing, research shows that the US belief hurts children and is not *valid.* Students who believe—or are helped to understand—that they can learn if they try to work on tasks longer and achieve better throughout their school careers than students who believe that one either "has it" (or "gets it") or does not [221]. This view often leads to failure, anxiety, and "learned helplessness" [221,222]. Similarly, students who have mastery-oriented goals (i.e., students who try to learn and see the point of school to develop knowledge and skills) achieve more than students whose goals are directed toward high grades or outperforming others [67,223].

Children's math anxiety predicts future math achievement over and above cognitive math ability, especially tackling challenging problems [224]. This adverse effect may be through children's visuospatial system [225]. Surprisingly, children with high achievement and high working memory may avoid using more advanced solution strategies *due* to math anxiety [226]. Unsurprisingly, most of these mirror the pedagogical strategies discussed previously, but the point here is that these have *also* been identified as improving children's attitudes and beliefs about math.

Fortunately, most young children have positive feelings about math and are motivated to explore numbers and shapes [223]. However, after only a couple of years in typical schools, they begin to believe that "only some people have the ability to do math". Children who experience math as a sense-making activity will build positive feelings about math throughout their school careers.

Teachers can help by providing meaningful tasks that make sense to students and connect with their everyday interests and lives. The right degree of challenge and novelty can promote interest, and promoting and discussing skill improvement can promote a mastery orientation. For example, researchers have estimated that students should be successful about 70% of the time to maximize motivation [223].

A common core of characteristics of learning environments enhances students' attitudes and beliefs about mathematics [227–233].

- Use problems that have meaning for children (both practical and mathematical). (Note that even instruction that increases, for example, memorization via drill in the short run, may damage children's motivation.)

- Expect that children will invent, explain, and critique their solution strategies within a social context.
- Provide opportunities for creative invention and practice and promote inquiry [234].
- Use manipulatives [195,235].
- Use technology [235–237].
- Encourage and support children progressing toward increasingly sophisticated and abstract math methods and understandings and to understand and develop more efficient and elegant solution strategies.
- Help children see connections between various types of knowledge and topics, with the goal of having each child build a well-structured, coherent knowledge of math.
- Ensure that your expectations of and interactions with girls about math are positive and equal to that with boys [238].
- Engagement of children with math difficulties is facilitated by playing games, using number lines to represent whole numbers, using manipulatives and technology, and learning a range of content beyond the number domain [235].

3.12. Collaboration with Families

Families influence children's earliest development of math [8,39,110,235,239,240], primarily through stimulating, informal, experiences developing foundational ideas [241,242] and more intentional formal activities developing symbolic competencies [243,244]. Parents' math talk with their children about number and spatial ideas are related to children's talk about and achievement in those topics [245], with prompts to talk about math more effective than statements [246]. Further, effective teachers promote reciprocal partnerships with families and family engagement [120,239,247] because supporting parent-child math interactions can make a meaningful difference, especially with math-anxious parents [245]. For example, one study compared business as usual to a school intervention and a school plus home intervention and found significant effects only if the home intervention was added to the school intervention [248]. Similar benefits of encouraging parents to play math games at home have been validated [249]. Interestingly, in studies showing the home math environment does not relate to children's math competencies, the reason may be strong early math education in schools [250]. This is especially important for families from marginalized communities [240].

To provide culturally and linguistically relevant curricula and ensure equity, developers and teachers engage and learn from families and communities [251]. Curricula build in communication supports to promote ongoing two-way partnerships to bring the school into the home and vice versa [252,253].The following section addresses how to ensure that curricula, classrooms, and teaching strategies are responsive to and representative of children from culturally and linguistically diverse backgrounds.

4. Ensuring Pedagogy Represents Children from Culturally and Linguistically Diverse Backgrounds

Reviewing studies that have focused on culturally relevant pedagogy and culturally responsive teaching in early childhood mathematics education can support the creation and implementation of inclusive curricula and experiences for children from culturally and linguistically diverse backgrounds [1,31,36]. The keywords and phrases used to search for articles in this and the following sections included culturally relevant pedagogy, culturally responsive teaching, early childhood education, mathematics, culturally responsive mathematics education, and combinations of the terms and phrases. Ladson-Billings proposed the term Culturally Relevant Pedagogy (CRP) to describe a model of practice that "helps students to accept and affirm their cultural identity while developing critical perspectives that challenge inequities that schools (and other institutions) perpetuate" [254] (p. 469). She outlined three elements of culturally relevant pedagogy: academic achievement, cultural competence, and cultural critique, and observed "exceptional" teachers recommended by parents and teachers and provided concrete examples of these elements in action [254–256].

Teachers held high standards for children's academic work but provided tools and strategies for students who needed extra support to reach proficiency.

Building upon Ladson-Billings and others' work, Gay [257] defined Culturally Responsive Teaching (CRT) with eight descriptors: validating, comprehensive and inclusive, multidimensional, empowering, transformative, emancipatory, humanistic, and normative and ethical. Gay [257] supported the veracity of these descriptors by weaving in stories from researchers and practitioners of culturally responsive teaching as well as her own experiences in the classroom as a culturally responsive educator. A qualitative metasynthesis of math teaching strategies that support CRT and CRP reviewed 12 studies from pre-kindergarten through 12th grade [258]. Five teaching practices supported CRT and CRP: caring for learners as participants in their education, knowing and using the context of students' lives at school and home (funds of knowledge) [259], having cultural competency, setting high expectations for learners and themselves, and using high quality, interactive math instruction. Before instruction even begins, early childhood educators can engage in CRT and CRP by validating and affirming the importance of cultural diversity in their classrooms and building and sustaining positive cultural identities for all children [260]. As a new field, such practices are at least partially research-based and need additional study to establish research-validated pedagogies.

4.1. Culturally Responsive Classroom Environments

One element of practice that Gay outlined was "symboling", [257] (p. 48) or including positive visual imagery, classroom books, and other media from different racial, ethnic, and cultural backgrounds. As mentioned, teachers can begin this practice before children even enter the classroom. Once the educator knows more about the students, families, and communities they will be working with, educators can ensure the classroom reflects the learners' experiences. As an example of this element, Gay described a kindergarten teacher that asked her students' families to send in photos for a classroom photograph montage. Children were able to see photos from their families as well as from families from different backgrounds doing different activities (sometimes culturally specific and sometimes not). This same kindergarten teacher asked parents to donate two books to the classroom library at the beginning of the school year. Parents were asked to donate a book about their own ethnic group and a second book about another ethnic group. By including photos, books, and media representation, this kindergarten teacher was creating a culturally responsive classroom environment. Educators can also create culturally responsive classrooms by encouraging the use of children's home languages if students are bilingual or multilingual. In math, one kindergartner class engaged in bilingual counting. One child pointed as the other students said the numbers aloud in Spanish. After they laughed about their pronunciations, they discussed that those learning a language do not sound like those who are native speakers. Rosita, a native Spanish speaker, was asked to model, and the others followed her [257] (p. 51). Studies confirm that bilingual counting or counting in Spanish and then reporting the result in English supports students' mathematical thinking [261].

Ethnomathematics is an approach designed to change deficit views to use indigenous knowledge to benefit both indigenous and non-indigenous educational contexts [261]. Compared to approaches that tend to essentialize, considering all children in an ethnic group, this approach builds from a local culture. An Alaskan native curriculum was based on different aspects of the subsistence cycle, such as collecting or gathering foods, including locating a good place to gather berries or collect eggs, estimating distance and time (how far and how long will it take to get there), and estimating volume (how many filled buckets will be needed), sorting by attributes such as type of berry, storing (other units of measure—freezer bag full), use (recipes—measuring), and sharing/redistributing the food. An earlier experiment validated the efficacy of this approach, providing initial validation [256].

Ladson-Billings [256] highlighted an elementary teacher who used rap song lyrics as a way to teach poetry. By including media that her students were familiar with, the students were able to see their cultural styles appreciated in the classroom. Similar application of music could be used in math, e.g., patterning, [92,262] and related fields such as computational thinking [263].

In summary, early childhood educators can set the stage for academic learning by creating culturally responsive classroom environments in which diverse backgrounds, ethnicities, and cultures are acknowledged and celebrated [264–266]. Positive cultural representation must also be featured in textbooks and other curricular materials [257]. Having culturally responsive stories and characters that students identify with ethnically can inspire elementary students to engage more with mathematics and demonstrate more mathematical behaviors [267].

4.2. Funds of Knowledge

Because CRT "builds bridges of meaningfulness between home and school experiences as well as between academic abstractions and lived sociocultural realities" [257] (p. 37), it is imperative that educators learn about their students' home lives, families, and communities and incorporate those elements into the classroom. One way that educators can learn about the resources in students' households is by conducting home visits [259]. When visiting students' homes, teachers should look for funds of knowledge within the household, or the knowledge, cultural resources, and artifacts that families have developed over time and that are essential to their daily routines. Educators may look for artifacts such as family photos, books, household tools, or religious items [268]. They may ask about family members' occupations, places of travel, pets, or livestock. When observing households, the teacher will need to center himself or herself as a learner of the family's routines and experiences while withholding judgement and avoiding stereotyping.

If home visits are not possible, educators can ask students and their caregivers about their households through surveys or informal or formal interviews. Teachers should ask about family traditions, languages used at home, family values, occupations, household tools, educational activities, and any chores that children do to contribute to the household. After collecting information about the funds of knowledge that exist in the child's home, educators should reflect upon how they can incorporate these funds into activities in the classroom.

Mathematical funds of knowledge could include math that is involved in sewing, gardening, cooking, construction, and time management; however, it may be difficult for caretakers and educators to see the underlying mathematical principles in these daily activities. That is, mathematics is a particularly challenging domain in which to incorporate funds of knowledge [269]. During one home visit, a teacher noticed that a student was selling Mexican candy to neighbors and friends [259]. The teacher decided to make a theme around candy. Students compared Mexican candy and candy sold in the United States, made graphs of their favorite types of candy, had a classroom parent visit the class to assist in making candy, and then packaged and priced the candy to sell at a school event. The teacher was able to add additional lessons in mathematics to a strength that the students already possessed outside of the classroom.

Aguirre and Zavala [270] created a culturally responsive mathematical lesson analysis tool that could be used with K–8 teachers. The tool contains an element titled "Funds of Knowledge/culture/community support" that can be used to determine if/how the lesson is connecting math to relevant aspects of students' lives. A score of 1 indicates that the lesson contains culturally neutral contexts and a score of 5 indicates that the hook, activity, assessment, and closure involve strong connections to the students' funds of knowledge and community.

4.3. Critical Reflection: High Standards and Warm Demanders

To engage in CRT, educators must engage in critical reflection, confront, and eliminate any deficit thoughts they may hold toward students from culturally and linguistically diverse backgrounds and instead view differences as assets [271]. Durden and others [272] charge teachers to "engage in critical reflective practice to examine [their] own ideological stance towards the children in [their] classroom[s]" (p. 224) including examining their beliefs about bilingual education and any racial stereotypes they may hold. Early childhood educators can use tools to engage in self-study that will help them examine their diversity practices [273]. Math educators must set high expectations for themselves and adapt their instruction based on their students' needs [258] while also setting high expectations for their learners. Teachers must be "demanding, yet supportive and encouraging" [257] (p. 64).

Ladson-Billings [254] called for culturally relevant pedagogy to be included in teacher preparation programs, pointing out the "growing disparity between the racial, ethnic, and cultural characteristics of teachers and students" (p. 483). This need for culturally relevant pedagogy and critical reflection is underscored by recent research such as that by Essien and Wood [274] who found that Black children experienced a number of microaggressions including being treated as second-class citizens in early childhood education settings during the pandemic.

4.4. Learning about Math in Students' Homes

One way that early childhood educators can be culturally responsive is by conducting home visits and learning more about how mathematics is taught in students' homes. As Moll et al. [259] described, home visits are critical in helping educators acknowledge, appreciate, and incorporate strengths and values that students' families foster in their daily lives. Educators can then incorporate these values into classroom lessons, creating a bridge between home and school life and learning.

Over a third of Latina mothers of preschool to first graders supported children's math learning at home by using daily living activities such as cooking, grocery shopping, working on household finances, counting rosary beads, and sharing food with family members [275]. Such support strengthened and expanded children's understanding of how math is embedded in their own daily activities. The researchers suggested that early childhood educators and programs reinforce these positive math experiences in the home and work toward building stronger family-school partnerships. For example, home visits or surveys could reveal activities they could incorporate in the classroom. Further, they could send home weekly newsletters highlighting a math concept to foster in daily activities, such as looking for shapes when walking around the neighborhood [275].

Sonnenschien and colleagues [276] interviewed Chinese and Latina immigrant mothers of young children (preschool through first grade) on their beliefs and practices about math and their engagement in math-related activities at home. The researchers found that both groups highlighted the importance of learning math and engaged their children in learning math skills and concepts in the home. One difference that the researchers found was that Chinese mothers had systematic, long-term plans for their children's learning, while Latina mothers did not often mention using a systematic approach over time. Early childhood educators could help parents create their own plans for children's learning or send monthly suggestions of skills to work on and incorporate into their home lives.

Researchers working with Indigenous cultures in Australia found that Aboriginal Teaching Assistants helped build community connectedness and relationships between teachers and families [36]. More intensive teaching and combining direct teaching with play-based opportunities, and using familiar contexts, also enhanced young Indigenous students' learning.

In summary, to provide culturally and linguistically relevant curricula and ensure equity, teachers engage and learn from families and communities [251], building communication supports to promote ongoing two-way partnerships to bring the school into the home and vice versa [252,253].

4.5. Children, Parents, and Caregivers as Experts in the Classroom

As Gay [257] described, CRT should be empowering. Teachers can empower students by ensuring that students are "consumers and producers of knowledge" (p. 41). Students can also be positioned as experts (as in the case of Rosita counting in Spanish). This allows students to move from passive recipients of knowledge to active learners and mentors who are developing confidence and self-efficacy.

Early childhood educators can also invite parents and caregivers to the classroom to serve as experts and capitalize on funds of knowledge in the community [277]. Parents and caregivers can teach skills, crafts, lead culturally specific activities, read books, and support math instruction.

4.6. Learning about How Math Is Taught in Other Cultures and Countries

Teachers who do not engage in culturally responsive mathematics education may hold the belief that mathematics is culturally neutral [278]. Teacher education and professional development sessions should work to deconstruct teachers' views that may include "beliefs about mathematics as a culturally-neutral subject, as universal truth, as a non-reasoning system, and, as an exclusively European and Western discipline" [278] (p. 51). By learning about the differences in how mathematics is taught and learned in countries and cultures around the world, early childhood educators can understand and appreciate differences in students' thinking and provide students with multiple strategies to approach their learning of mathematics.

An alternative teaching strategy that was used by early childhood educators in India involved having the children use their fingers to count in ways different from other cultures [279]. Children begin with the fingers of one hand, then the fingers of both hands, and then extended to using the joints and finger lines of both hands for a total count of 40. If students or their families have recently immigrated, early childhood educators can learn more about how children are taught mathematics in their countries of origin. Having knowledge of cultural mathematical practices like this strategy from India could help educators and students by providing them with additional tools. Teachers with children from multiple cultures should avoid imposing specific ways to count on their fingers, but rather enrich children's representations through discussions of different ways to show numbers.

Mathematics teacher educators can strengthen their programs by incorporating the history of mathematics as well as the more contemporary contributions of people from African, Latino/a, Asian, and Native American backgrounds [280]. This will allow teachers to learn about the contributions of different racial and ethnic groups to mathematics, both throughout history and in more recent times. Early Childhood Educators can share this information with their students, helping children develop a more inclusive view of the field of mathematics.

5. Caveats

As stated previously, this article is a "best evidence" synthesis from our perspectives and research reviews. That is, of the myriad studies in early childhood math, we selected studies on pedagogies that we deemed (a) strongly, more directly research-based or research-validated and (b) useful in addressing important pedagogical issues in early math. Although our goal as scientists was to ensure a full consideration of all evidence, this review reflects our choices—an essential caveat. We look forward to constructively critical reactions, the only way to move the field forward.

Space constraints limited our coverage of many topics and issues, such as teaching with literature and other worthwhile approaches. Specific research on teaching math topics and recommendations for future research in early math pedagogy can be found in other reviews [32,33,35–37,39,40,281].

6. Final Words

Teachers matter more than other factors, and teachers in the early years matter the most [282]. So, early math teachers must use the best pedagogical strategies [39].

Teaching techniques are tools and must be used carefully, thoughtfully, and appropriately. Every strategy, from play to direct instruction, can be educative or mis-educative. "Any experience is mis-educative that has the effect of arresting or distorting the growth of further experience" [283] (p. 25). For example, mis-educative experiences resulting from inappropriate direct teaching may decrease sensitivity to the wide range of applications of math ideas or develop automatic skills but narrow the range of other experiences with the idea underlying the skill. Conversely, child-centered education that rejects the structures or sequencing of subject matter content may be so disconnected as to limit later integrative experiences. As Dewey said, "Just because traditional education was a matter of routine in which the plans and programs were handed down from the past, it does not follow that progressive education is a matter of planless improvisation" (p. 28).

Regardless of instructional approach or strategy, educators must remember that young children's ideas can be uniquely different from those of adults [31,39,284,285]. Early childhood teachers must be careful not to assume that children "see" situations, problems, or solutions as adults do. Based on their interpretations of children's thinking, teachers conjecture what the child might be able to learn or abstract from his or her experiences. Similarly, when interacting with the child, they also consider their own actions from the child's point of view. This makes early childhood teaching both demanding and rewarding. Such sensitivity, however, is necessary to fully benefit from this chapter's pedagogical suggestions, especially the core contention of the central role of children's thinking and learning, as well as the use of formative assessment, and a variety of teaching strategies at each particular phase of learning. Knowledge of developmental paths in learning trajectories can enhance teachers' understanding of children's thinking, helping teachers assess children's level of understanding and offer instructional activities at the next level and thus offer meaningful and joyful opportunities to engage in learning.

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