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# On the Extremality of Harmonic Beltrami Coefficients

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**Abstract:** We prove a general theorem, which provides a broad collection of univalent functions with equal Grunsky and Teichmüller norms and thereby the Fredholm eigenvalues and the reflection coefficients of associated quasicircles. It concerns an important problem to establish the exact or approximate values of basic quasiinvariant functionals of Jordan curves, which is crucial in applications and in the numerical aspect of quasiconformal analysis.

**Keywords:** univalent function; Grunsky operator; quasiconformal extension; harmonic Beltrami coefficients; quasireflections; universal Teichmüller space

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## 1. Preliminaries

### *1.1. The Teichmüller and Grunsky Norms of Univalent Functions*

Consider the class $S_Q$ of univalent functions $f(z) = a_1 + a_z z^2 + \dots$ in the unit disk $\mathbb{D} = \{|z| < 1\}$ admitting quasiconformal extensions across the boundary unit circle $\mathbb{S}^1$, hence to the whole Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. To have compactness in the topology of locally uniform convergence on $\mathbb{C}$, one must add the third normalization condition, for example, $f(\infty) = \infty$.

The Beltrami coefficients of extensions are supported in the complementary disk

$$\mathbb{D}^* = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} = \{z \in \widehat{\mathbb{C}} : \ |z| > 1\}$$

and run over the unit ball

$$\mathrm{Belt}(\mathbb{D}^*)_1 = \{\mu \in L_\infty(\mathbb{C}) : \ \mu(z)|D = 0, \ \ \|\mu\|_\infty < 1\}.$$

Each $\mu \in \mathrm{Belt}(\mathbb{D}^*)_1$ determines a unique homeomorphic solution to the Beltrami equation $\bar{\partial} w = \mu \partial w$ on $\mathbb{C}$ (quasiconformal automorphism of $\widehat{\mathbb{C}}$) normalized by $w^\mu(0) = 0$, $(w^\mu)'(0) = 1$, $w^\mu(\infty) = \infty$, whose restriction to $\mathbb{D}$ belongs to $S_Q(\infty)$.

The **Schwarzian derivatives** of these functions

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2, \quad f(z) = w^\mu(z)|\mathbb{D},$$

belong to the complex Banach space $\mathbf{B} = \mathbf{B}(\mathbb{D})$ of hyperbolically bounded holomorphic functions in the disk $\mathbb{D}$ with norm

$$\|\varphi\|_{\mathbf{B}} = \sup_{\mathbb{D}} (1 - |z|^2)^2 |\varphi(z)|$$

and run over a bounded domain in $\mathbf{B}$ modeling the **universal Teichmüller space T**. The space $\mathbf{B}$ is dual to the Bergman space $A_1(\mathbb{D})$, a subspace of $L_1(\mathbb{D})$ formed by integrable holomorphic functions (quadratic differentials $\varphi(z)dz^2$) on $\mathbb{D}$. On Teichmüller space theory and its deep applications to various fields of Mathematics see, e.g., [1–5].

One defines for any $f \in S_Q(\infty)$ its **Grunsky coefficients** $\alpha_{mn}$ from the expansion

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n=1}^{\infty} \alpha_{mn} z^m \zeta^n, \quad (z, \zeta) \in \mathbb{D}^2, \tag{1}$$

where the principal branch of the logarithmic function is chosen. These coefficients satisfy the inequality

$$\Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\, \alpha_{mn}(f) x_m x_n \Big| \le 1 \tag{2}$$

for any sequence $\mathbf{x} = (x_n)$ from the unit sphere $S(l^2)$ of the Hilbert space $l^2$ with norm $\|\mathbf{x}\| = \big(\sum\limits_1^{\infty} |x_n|^2\big)^{1/2}$; conversely, the inequality (2) also is sufficient for univalence of a locally univalent function in $\mathbb{D}$ (cf. [6]).

The minimum $k(f)$ of dilatations $k(w^\mu) = \|\mu\|_\infty$ among all quasiconformal extensions $w^\mu(z)$ of $f$ onto the whole plane $\widehat{\mathbb{C}}$ (forming the equivalence class of $f$) is called the **Teichmüller norm** of this function. Hence,

$$k(f) = \tanh d_{\mathbf{T}}(\mathbf{0}, S_f),$$

where $d_{\mathbf{T}}$ denotes the Teichmüller-Kobayashi metric on $\mathbf{T}$. This quantity dominates the **Grunsky norm**

$$\varkappa(f) = \sup \Big\{ \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\, \alpha_{mn} x_m x_n \Big| : \mathbf{x} = (x_n) \in S(l^2) \Big\}$$

by $\varkappa(f) \le k(f)$. These norms coincide only when any extremal Beltrami coefficient $\mu_0$ for $f$ (i.e., with $\|\mu_0\|_\infty = k(f)$) satisfies

$$\|\mu_0\|_\infty = \sup \Big\{ \Big| \iint_{\mathbb{D}^*} \mu(z)\psi(z) dxdy \Big| : \psi \in A_1^2(\mathbb{D}^*),\ \|\psi\|_{A_1} = 1 \Big\} = \varkappa(f) \quad (z = x + iy).$$

Here $A_1(\mathbb{D}^*)$ denotes the subspace in $L_1(\mathbb{D}^*)$ formed by integrable holomorphic functions (quadratic differentials) on $\mathbb{D}^*$ (hence, $\psi(z) = c_4 z^{-4} + c_5 z^{-5} + \dots$), so $\psi(z) = O(z^{-4})$ as $z \to \infty$, and $A_1^2(\mathbb{D}^*)$ is its subset consisting of $\psi$ with zeros even order in $\mathbb{D}^*$, i.e., of the squares of holomorphic functions (see, e.g., [7–9]). Note that every $\psi \in A_1^2(\mathbb{D}^*)$ has the form

$$\psi(z) = \frac{1}{\pi} \sum_{m+n=4}^{\infty} \sqrt{mn}\, x_m x_n z^{-(m+n)}$$

and $\|\psi\|_{A_1(\mathbb{D}^*)} = \|\mathbf{x}\|_{l^2} = 1$, $\mathbf{x} = (x_n)$.

The method of Grunsky inequalities was generalized in several directions, even to bordered Riemann surfaces $X$ with a finite number of boundary components. In particular, for any unbounded simply connected domain, $D^* \ni \infty$, the expansion (1) assumes the form

$$-\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n=1}^{\infty} \frac{\beta_{mn}}{\sqrt{mn}\, \chi(z)^m\, \chi(\zeta)^n},$$

where $\chi$ denotes a conformal map of $D^*$ onto the disk $\mathbb{D}^*$ so that $\chi(\infty) = \infty$, $\chi'(\infty) > 0$ (cf. [10]).

Accordingly, the generalized Grunsky norm is defined by

$$\varkappa_{D^*}(f) = \sup \Big\{ \Big| \sum_{m,n=1}^{\infty} \beta_{mn}\, x_m x_n \Big| : \mathbf{x} = (x_n) \in S(l^2) \Big\}.$$

### 1.2. Extremality and Substantial Points

In the general case, $\mu_0 \in \mathrm{Belt}(\mathbb{D}^*)_1$ is *extremal in its class if and only if*

$$\|\mu_0\|_\infty = \sup \Big\{ \Big| \iint_{D^*} \mu(z)\psi(z)dxdy \Big| : \ \psi \in A_1(\mathbb{D}^*), \ \|\psi\|_{A_1} = 1 \Big\}.$$

Moreover, if $\varkappa(f) = k(f)$ and the equivalence class of $f$ is a **Strebel point** of $\mathbf{T}$, which means that this class contains the Teichmüller extremal extension $f^{k|\psi_0|/\psi_0}$ with $\psi_0 \in A_1(\mathbb{D})$, then necessarily $\psi_0 = \omega^2 \in A_1^2$ (cf. [7,11–13]). The Strebel points are dense in any Teichmüller space (see [3]).

Assume that $\mu \in \mathrm{Belt}(\mathbb{D}^*)_1$ is extremal in its class but not of Teichmüller type. A point $z_0 \in \mathbb{S}^1$ is called **substantial** (or essential) for $\mu_0$ if for any $\varepsilon > 0$ there exists a neighborhood $U_0$ of $z_0$ such that

$$\sup_{\mathbb{D}^* \setminus U_0} |\mu_0(z)| < \|\mu_0\|_\infty - \varepsilon;$$

the maximal dilatation $k(w^{\mu_0}) = \|\mu_0\|_\infty$ is attained on $\mathbb{D}^*$ only by approaching this point.

In addition, there exists a sequence $\{\psi_n\} \subset A_1(|D^*)$ such that $\psi_m(z) \to 0$ locally uniformly on $\mathbb{D}^*$ but $\|\psi_n\| = 1$ for any $n$, and

$$\lim_{n\to\infty} \iint_{\mathbb{D}^*} \mu_0(z)\psi_n(z)dxdy = \|\mu_0\|_\infty.$$

Such sequences are called degenerated.

The image of a substantial point is a common point of two quasiconformal arcs, which can be of spiral type.

### 1.3. Fredholm Eigenvalues and Quasireflections

The Teichmüller and Grunsky norms are intrinsically connected with quasiconformal reflections, Fredholm eigenvalues and other quasiinvariants of quasiconformal curves. We outline briefly the main notions; the details see, e.g., in [1,14–16].

The **quasiconformal reflections** (or quasireflections) are the orientation reversing quasiconformal homeomorphisms of the sphere $\widehat{\mathbb{C}}$ which preserve point-wise some (oriented) quasicircle $L \subset \widehat{\mathbb{C}}$ and interchange its interior and exterior domains. One defines for $L$ its **reflection coefficient**

$$q_L = \inf k(f) = \inf \|\partial_z f / \partial_{\bar{z}} f\|_\infty,$$

taking the infimum over all quasireflections across $L$. Due to [1,14], the dilatation

$$Q_L = (1 + q_L)/(1 - q_L) \geq 1$$

is equal to the quantity $Q_L = (1 + k_L)^2/(1 - k_L)^2$, where $k_L$ is the minimal dilatation among all orientation preserving quasiconformal automorphisms $f_*$ of $\widehat{\mathbb{C}}$ carrying the unit circle onto $L$, and $k(f_*) = \|\partial_{\bar{z}} f_* / \partial_z f_*\|_\infty$.

The **Fredholm eigenvalues** $\rho_n$ of an oriented smooth closed Jordan curve $L \subset \widehat{\mathbb{C}}$ are the eigenvalues of its double-layer potential. These values are crucial in many applications.

The least positive eigenvalue $\rho_L = \rho_1$ plays is naturally connected with conformal and quasiconformal maps and can be defined for any oriented closed Jordan curve $L$ by

$$\frac{1}{\rho_L} = \sup \frac{|\mathcal{D}_G(u) - \mathcal{D}_{G^*}(u)|}{\mathcal{D}_G(u) + \mathcal{D}_{G^*}(u)},$$

where $G$ and $G^*$ are, respectively, the interior and exterior of $L$; $\mathcal{D}$ denotes the Dirichlet integral, and the supremum is taken over all functions $u$ continuous on $\widehat{\mathbb{C}}$ and harmonic on $G \cup G^*$.

A rough upper bound for $\rho_L$ is given by Ahlfors' inequality

$$\frac{1}{\rho_L} \le q_L,$$

where $q_L$ denotes the minimal dilatation of quasireflections across $L$ [17].

One of the basic tools in quantitative estimating the Freholm eigenvalues $\rho_L$ of quasicircles is given by the Kühnau-Schiffer theorem [9,18], which states that *the value $\rho_L$ is reciprocal to the Grunsky norm $\varkappa(f)$ of the Riemann mapping function of the exterior domain of $L$.*

For all functions $f \in S_Q$ with $k(f) = \varkappa(f)$, we have the exact explicit values

$$q_{f(\mathbb{S}^1)} = 1/\rho_{f(\mathbb{S}^1)} = \varkappa(f).$$

*1.4. Harmonic Beltrami Coefficients*

By the Ahlfors-Weill theorem strengthening the classical Nehari's result on univalence in terms of the Schwarzians, every $\varphi \in \mathbf{B}$ with $\|\varphi\|_{\mathbf{B}} < 2$ is the Schwarzian derivative of a univalent function $f$ in $D^*$, and this function has quasiconformal extension onto the disk $\mathbb{D}^*$ with Beltrami coefficient

$$\nu_\varphi(z) = -\frac{1}{2}(|z|^2 - 1)^2 \varphi(1/\bar{z})(1/\bar{z}^4), \quad z \in \mathbb{D}^*; \tag{3}$$

see [2,3,19]. The Beltrami coefficients of such form are called harmonic.

## 2. A Global Theorem for the Disk

The aim of this paper is to prove the following theorem which provides a broad collection of univalent functions with equal Grunsky and Teichmüller norms and thereby the Fredholm eigenvalues $\rho_L$ and the reflection coefficients of associated quasicircles. It concerns the important problem to establish the exact or approximate values of basic qiasiinvariant functionals of Jordan curves, which is crucial in both applications and in numerical aspect of quasiconformal analysis.

This problem has different aspects: analytic, geometric, potential. It still remains open and far from its complete solving. The known results in this direction are presented, for example, in [14,16,20].

**Theorem 1.** *Suppose that a function $f(z) \in S_Q$ is $C^3$ smooth on some open arc $\gamma_0 \subset \mathbb{S}^1 \setminus \{z_1, \dots, z_n\}$, and let its Schwarzian $S_f$ satisfy: $\|S_f\|_{\mathbf{B}} < 2$, and there exists a point $z_0 \in \mathbb{S}^1$ at which the function $|\nu_{S_f}(z)|$ given by (3) attains its maximum on $\mathbb{D}$ (hence, $|\nu_{S_f}(z_0)| = \frac{1}{2}\|S_f\|_{\mathbf{B}}$). Then*

$$k(f) = \varkappa(f) = \|\nu_{S_f}\|_\infty. \tag{4}$$

It will be seen from the proof of the theorem that $z_0$ must be a substantial point for the extremal extension of $f$ onto $\mathbb{D}^*$.

We mention two important consequences.

**Corollary 1.** *Every harmonic Beltrami coefficient $r\nu_{S_f}$ with $0 < r \le 1$ and $S_f$ satisfying that assumptions of Theorem 1 is extremal in its equivalence class (and also obeys (4) for the corresponding maps $f_r$). Hence, this class does not contain the extremal coefficient $\mu$ of Teichmüller type.*

**Corollary 2.** *For any function $f \in S_Q$ satisfying the assumptions of Theorem 1, we have the exact explicit values*

$$q_{f(\mathbb{S}^1)} = 1/\rho_{f(\mathbb{S}^1)} = \frac{1}{2}\|(|z|^2 - 1)^2 \varphi(1/\bar{z})/\bar{z}^4\|_\infty.$$

Special results of such type related to phenomenon of extremality of harmonic coefficients have been established in [21]. These results involve the Schwarz-Christoffel integral representation of conformal map onto polygons and provide an important consequence for the problem of starlikness of Teichmüller spaces.

The classical examples (Strebel's chimney domain [22] and horizontal stretching of a strip by Belinskii, presented in [23]) yield that the functions with a finite number of substantial points exist for any quasiconformal dilatation $k$. The assumption $\|S_f\|_{\mathbf{B}} < 2$ is crucial for the proof of Theorem 1.

**Proof of Theorem 1.** To establish that the Beltrami coefficient $\nu_{S_f}$ is extremal in its equivalence class, we show that

$$\sup_{\|\psi\|_{A_1(\mathbb{D}^*)}=1} \Big| \iint_{\mathbb{D}^*} \nu_{S_f}(z)\psi(z)dxdy \Big| = \sup_{\|\psi\|_{A_1^2}=1} \Big| \iint_{\mathbb{D}^*} \nu_{S_f}(z)\psi(z)dxdy \Big|, \tag{5}$$

where

$$A_1(\mathbb{D}^*) = \{f \in L_1(\mathbb{D}^*) : \ f \text{ is holomorphic in } \mathbb{D}^*\},$$
$$A_1^2 = \{f \in A_1(\mathbb{D}^*) : \ f = \omega^2, \ \omega \text{ is holomorphic in } \mathbb{D}^*\}.$$

□

Such sets of holomorphic functions on the planar regions (more generally, holomorphic quadratic differentials on the Riemann surfaces) are intrinsically connected with extremal quasiconformal maps. Note that the space $\mathbf{B}$ is dual to $A_1(\mathbb{D})$ and that elements of $A_1^2$ naturally arising in quasiconforaml theory of the Grunsky coefficients are the squares of abelian differentials on $\mathbb{D}$.

To prove (5), we choose a substantial point $z_{j_0}$ and map conformally the half-strip

$$\Pi_+ = \{\zeta = \xi + i\eta : \ \xi > 0, \ 0 < \eta < 1\}$$

onto $\mathbb{D}^*$ by the function $g = \sigma \circ g_0$, where $g(\zeta) = -\cosh \pi\zeta$ (so $g(\Pi_+)$ is the upper half-plane $U$ and $g(\infty) = \infty$) and $\sigma$ is the additional Moebius map of $U$ onto $\mathbb{D}^*$ chosen so that the images of the points $\zeta = 0, i, \infty$ under $g$ are, respectively, the endpoints of the arc $g_0$ and $z_{j_0}$. Thereby, the coefficient $\nu_{S_f}$ is pulled back to Beltrami coefficient

$$\nu_*(\zeta) := g^* \nu_{S_f}(\zeta) = (\nu_{S_f} \circ g)(\zeta)\, \overline{g'(\zeta)}/g'(\zeta)$$

on $\Pi_+$. Noting that the map $g_0$ carries the horizontal lines

$$l_\eta = \{\zeta = \xi + i\eta : \ -\infty < \xi < \infty\}, \quad 0 < \eta < 2\pi,$$

into hyperbolae $\Gamma$ with foci $\pm 1$ in $U$ moving then to the curves $\sigma(G) \subset D^*$, one derives from the assumption that $z_{j_0}$ (and hence, its inverse image $\zeta_0 = g^{-1}(z_{j_0})$ on $\partial\Pi_+$) is a substantial point for $f$,

$$\lim_{\xi \to \infty} |\nu_*(\xi + i\eta)| = \|\nu_*\|_\infty = \|\nu_{S_f}\|_\infty.$$

In view of the smoothness of $g$ on $\partial\Pi_+ \setminus \{0, i, \infty\}$, there exists the limit function

$$\nu_*(\zeta_0) = \lim_{\zeta \to \zeta_0 \in \partial\Pi_+} \nu_*(\zeta),$$

at least for all $\zeta \in \partial\Pi_+$ different from the points $0, i, \infty, g^{-1}(z_1), \ldots, g^{-1}(z_n)$.

The smoothness of $f$ on the arc $\gamma_0$ implies that $S_f$ is bounded on $\gamma_0$, and therefore,

$$\nu_*(i\eta) = 0. \tag{6}$$

Let us take the sequence

$$\omega_m(\zeta) = \frac{1}{m} e^{-\zeta/m}, \quad \zeta \in \Pi_+ \ (m = 1, 2, \ldots);$$

all these $\omega_m$ belong to $A_1^2(\Pi_+)$ and $\omega_m(\zeta) \to 0$ uniformly on $\Pi_+ \cap \{|\zeta| < M\}$ for any $M < \infty$. Furthermore, $\|\omega_m\|_{A_1(\Pi_+)} = 1$ (moreover, $|\iint_{\Pi_+} \omega_m d\xi d\eta| = 1 - O(1/m)$), which shows that $\{\omega_m\}$ is a degenerating sequence for the affine horizontal stretching of $\Pi_+$.

We prove that this sequence is degenerating also for $\nu_*$. Indeed,

$$\iint\limits_{\Pi_+} \nu_*(\zeta)\omega_m(\zeta) d\xi d\eta = \int\limits_0^1 e^{-i\eta/m} d\eta \left( \frac{1}{m} \int\limits_0^\infty \nu_*(\xi + i\eta) e^{-\xi/m} d\xi \right), \tag{7}$$

and integrating the inner integral by parts, one obtains applying (6),

$$\int\limits_0^\infty \frac{\partial \nu_*(\xi + i\eta)}{\partial \xi} e^{-\xi/m} d\xi = \frac{1}{m} \int\limits_0^\infty \nu_*(\xi + i\eta) e^{-\xi/m} d\xi.$$

Now applying Abels's theorem for Laplace transform in $\xi$, one obtains that nontangential limit

$$\lim_{s \to 0} \int\limits_0^\infty \frac{\partial \nu_*(\xi + i\eta)}{\partial \xi} e^{-s\xi} d\xi = \int\limits_0^\infty \frac{\partial \nu_*(\xi + i\eta)}{\partial \xi} d\xi = \nu_*(\infty) - \nu_*(i\eta).$$

Hence

$$\lim_{m \to \infty} \frac{1}{m} \int\limits_0^\infty \nu_*(\xi + i\eta) e^{-\xi/m} d\xi = \nu_*(\infty).$$

By Lebesgue's theorem on dominated convergence, the iterated integral in (7) is estimated as follows:

$$\lim_{m \to \infty} \Big| \iint\limits_{\Pi_+} \nu_*(\zeta)\omega_m(\zeta) d\xi d\eta \Big| = \Big| \int\limits_0^1 d\eta \ \lim_{m \to \infty} \frac{1}{m} \int\limits_0^\infty \nu_*(\xi + i\eta) e^{-\xi/m} d\xi \Big| = 1. \tag{8}$$

Using the inverse conformal map $\zeta = g_0^{-1}(z)$: $U^* \to \Pi_+$, i.e., a suitable branch of $\frac{1}{\pi} cosh^{-1}(-z)$, one obtains the sequence $\{\psi_m = (\omega_m \circ g^{-1})(g')^{-2}\} \subset A_1^2$, which is a degenerating sequence for the initial Beltrami coefficient $\nu_{S_f}$ on $\mathbb{D}^*$, and by (8),

$$\lim_{m \to \infty} \Big| \iint\limits_{\mathbb{D}^*} \nu_{S_f}(z)\psi_m(z) dx dy \Big| = \lim_{m \to \infty} \Big| \iint_{\Pi_+} \nu_*(\zeta)\omega_m(\zeta) d\xi d\eta \Big| = |\nu_*(\infty)|. \tag{9}$$

In view of the assumption $|\nu_{S_f}(z_0)| = \frac{1}{2}\|S_f\|_{\mathbf{B}}$, all terms in (9) are equal to $\|\nu_*\|_\infty$, which proves (5).

Then, by the criterion of extremality, mentioned in Section 1.1, the equality (5) implies that the Beltrami coefficient $\nu_{S_f}$ must be extremal in the class of maps with the boundary values $f|\mathbb{S}^1$.

Moreover, since the limit (maximal value) in (9) is attained on the functions from $A_1^2$ (abelian differentials), the disk of Beltrami coefficients $\{t\nu_{S_f}: \ |t| < 1\}$ determines in the space $\mathbf{T}$ a holomorphic disk which is simultaneously geodesic in the Teichmüller, Kobayashi and Carathéodory metrics (see, e.g., [13]). This completes the proof of the theorem.

## 3. Open Questions and Concluding Remarks

### *3.1. General Quasicircles*

It remains an **open question** on the extent in which Theorem 1 can be generalized to arbitrary quasiconformal domains (quasidisks) $D \subset \widehat{\mathbb{C}}$, even in a weaker form.

Denote the complementary (unbounded) domain by $D^*$ and assume that the common boundary curve $L$ for $D$ and $D^*$ contains a $C^{1+}$-smooth arc $L_0$.

Denote by $\lambda_D(z)|dz|$ the hyperbolic metric on $D$ of Gaussian curvature $-4$ and consider the complex Banach space $\mathbf{B}(D)$ of holomorphic functions on $D$ with finite norm

$$\|\varphi\|_{\mathbf{B}(D)} = \sup_D \lambda_D(z)^{-2}|\varphi(z)|.$$

Let $\mathrm{Belt}(D)_1$ be the unit ball of Beltrami coefficients $\mu$ on $\mathbb{C}$ supported in $D$.

The are two natural generalizations of the Ahlfors-Weill extension. The firs one is based on applying the Douady–Earle **conformally natural extension** of quasisymmetric homeomorphisms of the unit circle and the related Earle–Nag reflection. The description of these subjects and obtained results are presented, for example, in [3] (Ch. 14). This approach implies the existence of a number $\varepsilon_0 > 0$ such that for any univalent function $f(z)$ in domain $D^*$ with the expansion $f(z) = z + b_0 + b_1 z^{-1} + \dots$ near $z = \infty$ and $\|S_f\|_{\mathbf{B}(D^*)} < \varepsilon_0$ having at most a finite number of substantial boundary points, we have the equalities

$$k(f) = \varkappa_{D^*}(f) = \|s(\varphi)\|_\infty,$$

where $\varphi = S_f$,

$$s(\varphi)(z) = \frac{(z - j(z))^2 \varphi(j(z)) j_{\bar{z}}(z)}{2 + \varphi(j(z))(z - j(z))^2 j_z(z)} \in \mathrm{Belt}(D)_1, \tag{10}$$

and $j(z)$ means the Earle–Nag reflection across $L$. This reflection is Lipschitz continuous, so the needed equality (6) is preserved. However, the coefficient (10) is not harmonic.

The second approach is given by the Bers extension theorem [24] which yields that for some $\varepsilon > 0$, there exists an anti-holomorphic homeomorphism $\tau$ (with $\tau(\mathbf{0}) = \mathbf{0}$) of the ball $V_\varepsilon = \{\varphi \in \mathbf{B}(D^*) : \|\varphi\| < \varepsilon\}$ into $\mathbf{B}(D)$ such that every $\varphi$ in $V_\varepsilon$ is the Schwarzian derivative of some univalent function $f$ which is the restriction to $D^*$ of a quasiconformal automorphism $\widehat{f}$ of Riemann sphere $\widehat{\mathbb{C}}$. This $\widehat{f}$ can be chosen in such a way that its Beltrami coefficient is harmonic on $D$, i.e., of the form

$$\mu_{\widehat{f}}(z) = \lambda_D^{-2}(z)\overline{\psi(z)}, \quad \psi = \tau(\varphi).$$

However, this homeomorphism $\tau$ does not insure that the equality (6) remains valid.

### *3.2. Connection with Fredholm Eigenvalues*

Another **open question** is how the quantity $\varkappa_{D^*}(f)$ relates to Fredholm eigenvalues of curves $f(L)$.

### *3.3. Concluding Remarks*

As was mentioned above, Theorem 1 provides exactly the Fredholm eigenvalues and the reflection coefficients for a new broad collection of quasiconformal curves, and the proof of this theorem yields an analytic algorithm for establishing these intrinsic quasiinvariants of curves.

On the other hand, the harmonic Beltrami coefficients are intrinsically connected with the Kodaira–Spencer deformation theory of complex structures, so Theorem 1 and its corollaries bridge in some measure this theory with extremal quasiconformal maps, whose role in geometric function theory is fundamental, and also open direction of research.

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