mathematics

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Fuzzy Partial Metric Spaces and Fixed Point Theorems

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Abstract: Partial metrics constitute a generalization of classical metrics for which self-distance may not be zero. They were introduced by S.G. Matthews in 1994 in order to provide an adequate mathematical framework for the denotational semantics of programming languages. Since then, different works were devoted to obtaining counterparts of metric fixed-point results in the more general context of partial metrics. Nevertheless, in the literature was shown that many of these generalizations are actually obtained as a corollary of their aforementioned classical counterparts. Recently, two fuzzy versions of partial metrics have been introduced in the literature. Such notions may constitute a future framework to extend already established fuzzy metric fixed point results to the partial metric context. The goal of this paper is to retrieve the conclusion drawn in the aforementioned paper by Haghia et al. to the fuzzy partial metric context. To achieve this goal, we construct a fuzzy metric from a fuzzy partial metric. The topology, Cauchy sequences, and completeness associated with this fuzzy metric are studied, and their relationships with the same notions associated to the fuzzy partial metric are provided. Moreover, this fuzzy metric helps us to show that many fixed point results stated in fuzzy metric spaces can be extended directly to the fuzzy partial metric framework. An outstanding difference between our approach and the classical technique introduced by Haghia et al. is shown.

Keywords: fuzzy partial metric; fixed point; completeness; convergence; Cauchyness

MSC: 54A05; 54A20; 47H10

Citation: Aygün, H.; Güner, E.; Miñana, J.-J.; Valero, O. Fuzzy Partial Metric Spaces and Fixed Point Theorems. *Mathematics* **2022**, *10*, 3092. https://doi.org/10.3390/math10173092

Academic Editor: Vladimir Balan

Received: 10 August 2022
Accepted: 23 August 2022
Published: 28 August 2022

1. Introduction

The concept of metrics becomes essential in many real problems in which a kind of measurement between elements or objects is needed, for instance, when evaluating how separated two objects are or how different they are. Indeed, it seems to be natural that a tool to calculate such measurements should fit with the axioms that define a metric, which are separation (and self-distance 0), symmetry, and triangular inequality. However, the essence of the addressed problem is that it can be too restrictive to be fulfilled by the aforementioned three axioms. Therefore, different generalizations of the notion of metrics have appeared in the literature. To address this problem, it is worth mentioning the concept of a partial metric space. Such a notion constitutes a generalization of the metrics in which the self-distance 0 is not required. Partial metrics were introduced in 1994 by Matthews in [1]. Since then, different works have been conducted on the theoretical study of partial metrics, especially in the context of fixed point theory. Specifically, different authors have contributed to this topic by obtaining classical metric fixed-point results in the more general context of partial metrics. Such a topic is relevant nowadays (see, for instance, [2–7]). Nevertheless, the adaptation of a classical metric, fixed-point result to the partial-metric context does not often actually constitute a generalization but, on the contrary, a particular case. Indeed, in [8], many fixed point results established in partial metric spaces were obtained as corollaries of classical metrics' fixed points through the use of a classical metric constructed from a partial one.

Coming back to the limitations of metrics, some real problems entail uncertainty in itself when the measurement between objects has to be carried out. Fuzzy theory seems to be a proper framework to tackle such situations. In this framework, we can find the notion of fuzzy metric introduced by Kramosil and Michalek in [9]. This metric constitutes a fuzzy version of the concept of a metric, which provides a degree of nearness between two objects with respect to a parameter. Although fuzzy metrics in that sense, as well as the later modification of them given by George and Veeramani in [10], are metrizable [11], they are very interesting when purely metrical properties are considered. Indeed, such fuzzy metrics have been shown to be useful (overcoming the limitations of classical metrics) in engineering problems such as image processing [12–14], perceptual color difference [15,16], task allocation [17,18], or model estimation [19–21]. Moreover, fixed-point theory in fuzzy metrics shows significant differences when comparing with the classical counterpart (see, for instance, [22–26]). Various recent works are devoted to adapting classical fixed point theorems to the fuzzy metric context (see, for instance, [27–31]).

In spite of the aforementioned usefulness of fuzzy metrics in engineering problems, we may again notice some restrictiveness in them. Motivated by this fact, different works have been devoted to obtaining generalizations of the above-mentioned concepts of fuzzy metrics by removing or relaxing some of the axioms that define them. In particular, we can find some studies in the literature that obtain a fuzzy version of distinct generalizations of the notion of classical metrics. Hence, Gregori et al. introduced in [32] the concept of fuzzy partial metric space, both in Kramosil and Michalek's sense and in George and Veeramani's sense. Taking into account that a usual issue in fixed point theory consists of extending published fixed -point results to more general frameworks, one could expect to find coming research works aimed to prove, in the context of fuzzy partial metric spaces, those fixed-point theorems already established in (fuzzy) metrics. Then, keeping in mind the discovery by Haghia et al. in [8], we wonder whether the situation in the fuzzy setting could be similar.

The aim of this paper is to retrieve, for fuzzy partial metric spaces, as introduced in [32], the main conclusions obtained in [8]. With this aim, we construct a fuzzy set M_P on $X \times X \times [0, \infty[$ from a given fuzzy partial metric space $(X, P, *)$. Furthermore, we show that M_P is a fuzzy metric, provided that the condition **(FPKM4*)** (see Proposition 5 in Section 3) holds for the considered fuzzy partial metric. Subsequently, we define in a natural way the notions of a Cauchy sequence and completeness in the context of fuzzy partial metric spaces. Then, we show that such a completeness in a fuzzy partial metric satisfying **(FPKM4*)** implies that the fuzzy metric space $(X, M_P, *)$ is complete. Finally, we illustrate by means of a particular case that the exposed results allow one to easily extend fixed-point theorems that are already established in fuzzy metric spaces to the context of fuzzy partial metric spaces whenever condition **(FPKM4*)** is fulfilled.

The remainder of this paper is organized as follows. The next section is devoted to gathering the basics of partial metrics, fuzzy metrics, and fuzzy partial metrics, which will be useful throughout this paper. In Section 3, we present the main results provided in this study that allow us to retrieve the main conclusions drawn in [8] in a particular case. In addition, we expose a way of obtaining a fixed-point theorem in fuzzy partial metrics as a corollary of a fixed-point theorem established in fuzzy metric spaces and using our developed theory. Finally, Section 4 presents the conclusions of the results provided throughout the paper, as well as suggests future work in the context of fixed-point theory in fuzzy partial metrics spaces.

2. Preliminaries

This section is devoted to compile definitions and results that will be useful throughout the paper. It is divided in two subsections devoted to recall the basics of partial metric spaces and fuzzy partial metric spaces, respectively.

2.1. Partial Metric Spaces

In [1], Matthews introduced the notion of partial metric space as follows:

Definition 1. *A partial metric space is a pair (X, p) such that X is a (non-empty) set and p is a real-valued function on $X \times X$ satisfying, for all $x, y, z \in X$, the following conditions:*

(PM1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;

(PM2) $0 \leq p(x, x) \leq p(x, y)$;

(PM3) $p(x, y) = p(y, x)$;

(PM4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

If (X, p) is a partial metric space, then we say that p is a partial metric on X.

It is clear that a metric space (X, d) is a partial metric space that satisfies, in addition, the following condition:

$$d(x, x) = 0 \text{ for all } x \in X. \tag{1}$$

According to [1], each partial metric p on X induces a T_0 topology $\tau(p)$ on X that has as a base the family of open balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$. Moreover, in such a topology $\tau(p)$, a sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$. Furthermore, a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a partial metric space (X, p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite. On account of [33], $\{x_n\}_{n\in\mathbb{N}}$ is called a 0-Cauchy sequence whenever $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. So, as usual, a partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to a point $x \in X$; that is, $p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m)$. Furthermore, (X, p) is said to be 0-complete if each 0-Cauchy sequence in X converges to a point $x \in X$ such that $p(x, x) = 0$. Note that every complete partial metric space is 0-complete but the converse is not true, as was shown in [33].

In [1], Mathews proved that every partial metric induces, in a natural way, a metric. Indeed, given a partial metric space (X, p), then the function $d_p : X \times X \to [0, \infty)$, defined by $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ for all $x, y \in X$, is a metric on X. According to [34], every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in a partial metric space (X, p) is a Cauchy sequence in (X, d_p). Hence, each 0-Cauchy sequence in (X, p) is Cauchy in (X, d_p).

Another technique to obtain a metric from a given partial metric was established by Hitzler and Seda in [35]. Let us recall such a technique in the following.

Proposition 1. *Let (X, p) be a partial metric space; then, the function $d : X \times X \to [0, \infty)$ defined by*

$$d(x, y) = \begin{cases} 0, & x = y \\ p(x, y), & x \neq y \end{cases}, \tag{2}$$

is a metric on X such that $\tau(d_p) \subseteq \tau(d)$. Moreover, (X, d) is complete if and only if (X, p) is 0-complete.

In [8], the preceding result was crucial to show that some extensions of fixed-point results to the partial metric context can be proved as corollaries of the celebrated classical counterparts that they try to extend.

2.2. Fuzzy (Partial) Metric Spaces

In this subsection, we gather both the concepts of fuzzy metrics and fuzzy partial metrics, which we will discuss from now on, as well as some necessary results related to them that will be key for our subsequent study. To start, we recall the concept of continuous triangular norm, which is crucial to the concept of fuzzy (partial) metrics that we deal with. For an outstanding reference to triangular norms, we refer the reader to [36].

Definition 2. *A* triangular norm *(briefly, t-norm) is a binary operation $*$ on $[0,1]$ such that, for all $x, y, z \in [0,1]$, the following axioms are satisfied:*

(T1) $x * y = y * x$;

(T2) $x * (y * z) = (x * y) * z$;

(T3) $x * y \geq x * z$, *whenever* $y \geq z$;

(T4) $x * 1 = x$.

If, in addition, $*$ is continuous (with respect to the usual topology) as a function defined on $[0,1] \times [0,1]$, we will say that $*$ is a continuous t-norm.

Now we are able to present the notion of fuzzy metric space due to Kramosil and Michalek [9]. It is worth mentioning that nowadays, this concept is commonly used in the literature following its reformulation given by Grabiec in [23], which is given as follows.

Definition 3. *A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, \infty)$ satisfying, for all $x, y, z \in X$ and $s, t \in (0, \infty)$, the following conditions:*

(KM1) $M(x, y, 0) = 0$;

(KM2) $M(x, y, t) = 1$ *for all* $t > 0$ *if and only if* $x = y$;

(KM3) $M(x, y, t) = M(y, x, t)$;

(KM4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;

(KM5) *The function $M_{x,y} : (0, \infty) \to [0, 1]$ is left-continuous, where $M_{x,y}(t) = M(x, y, t)$ for each $t \in (0, \infty)$.*

Below, we can find the modification of the preceding definition given by George and Veeramani in [10].

Definition 4. *A GV-fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm, and M is a fuzzy set on $X \times X \times (0, \infty)$, satisfying, for all $x, y, z \in X$ and $s, t \in (0, \infty)$, conditions* **(KM3)**, **(KM4)** *and the following ones:*

(GV1) $M(x, y, t) > 0$;

(GV2) $M(x, y, t) = 1$ *if and only if* $x = y$;

(GV5) *The function $M_{x,y} : (0, \infty) \to [0, 1]$ is continuous.*

As usual, if $(X, M, *)$ is a (GV-)fuzzy metric space, we say that $(M, *)$, or simply M, is a (GV-)fuzzy metric on X.

It should be noted that a GV-fuzzy metric M can be regarded as a fuzzy metric defining $M(x, y, 0) = 0$ for each $x, y \in X$. Then, GV-fuzzy metric spaces can be considered a particular case of fuzzy metric.

According to [10], each GV-fuzzy metric M on X induces a T_2 topology τ_M on X that has as a base the family of open balls $\{B_M(x, r, t) : x \in X, r \in (0, 1), t \in (0, \infty)\}$, where $B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$. Moreover, in such a topology, convergent sequences are characterized as follows: a sequence $\{x_n\}_{n \in \mathbb{N}}$ in a fuzzy metric space $(X, M, *)$

converges to a point $x \in X$ if $\lim_{n\to\infty} M(x, x_n, t) = 1$ for all $t \in (0, \infty)$. Although the preceding statements were established for GV-fuzzy metrics in [10], it is well-known that they remain true in the general context of fuzzy metrics.

The notions of Cauchy sequence and complete fuzzy metric space will play a relevant role in our study. Let us recall them as follows [10].

Definition 5. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a fuzzy metric space $(X, M, *)$ is called a Cauchy sequence if $\lim_{n,m\to\infty} M(x_n, x_m, t) = 1$ for all $t \in (0, \infty)$. A fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges with respect to τ_M to a point $x \in X$.*

The last part of this subsection is devoted to recalling the fundamental of fuzzy partial metric spaces in the sense of [32]. In [32], the notion of fuzzy partial metric was introduced following the essence of the Matthews concept, both in Kramosil and Michalek's sense and in George and Veeramani's sense. In both definitions, the residuum operator associated with a continuous t-norm is essential. The next proposition determines the expression of such a residuum operator.

Proposition 2. *Let $*$ be a continuous t-norm; then, $*$-residuum operator $\rightarrow_*$ of $*$ is uniquely determined by the formula*

$$x \rightarrow_* y = \begin{cases} 1, & \text{if } x \leq y; \\ \max\{z \in [0,1] : x * z = y\}, & \text{if } x > y. \end{cases} \tag{3}$$

To find a deeper treatment of the residuum operator we refer the reader to [36] (see also [37]). Following [36], we have the following propositions, which will be useful later on the residuum operator.

Proposition 3. *Let $*$ be a continuous t-norm and denote by $\wedge$ the minimum t-norm. Then*

$$x \wedge y = x * (x \rightarrow_* y) \text{ for all } x, y \in [0,1]. \tag{4}$$

An immediate corollary of the previous propositions is the following one.

Corollary 1. *Let $*$ be a continuous t-norm. Then $x \rightarrow_* y \geq y$ for all $x, y \in [0,1]$.*

In light of the presented concepts we are ready to present the aforementioned notion of a fuzzy partial metric space introduced in [32].

Definition 6. *A fuzzy partial metric space is an ordered triple $(X, P, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm, and P is a fuzzy set on $X \times X \times [0, \infty)$, satisfying, for all $x, y, z \in X$ and $s, t \in (0, \infty)$, the following conditions:*

(FPKM0) $P(x, y, 0) = 0$;

(FPKM1) $P(x, y, t) \leq P(x, x, t)$;

(FPKM2) $P(x, y, t) = P(x, x, t) = P(y, y, t)$ *for all* $t > 0$ *if and only if* $x = y$;

(FPKM3) $P(x, y, t) = P(y, x, t)$;

(FPKM4) $P(x, x, t+s) \rightarrow_* P(x, z, t+s) \geq (P(x, x, t) \rightarrow_* P(x, y, t)) * (P(y, y, s) \rightarrow_* P(y, z, s))$;

(FPKM5) *The function $P_{x,y} : (0, \infty) \to [0,1]$ is left-continuous, where $P_{x,y}(t) = P(x, y, t)$ for each $t \in (0, \infty)$.*

According to the notion of a GV-fuzzy metric space, the following refinement of the previous concept was introduced in [32].

Definition 7. *A GV-fuzzy partial metric space is an ordered triple $(X, P, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm, and P is a fuzzy set on $X \times X \times (0, \infty)$, satisfying, for all $x, y, z \in X$ and $s, t \in (0, \infty)$, conditions* **(FPKM3)**, **(FPKM4)**, *as well as the following conditions*

(FPGV1) $0 < P(x, y, t) \leq P(x, x, t)$;

(FPGV2) $P(x, y, t) = P(x, x, t) = P(y, y, t)$ *if and only if* $x = y$;

(FPGV5) *The function* $P_{x,y} : (0, \infty) \to [0, 1]$ *is continuous.*

Again, if $(X, P, *)$ is a (GV-)fuzzy partial metric space, we say that $(P, *)$, or simply P, is a (GV-)fuzzy partial metric on X.

In [32], it was proven that, given a fuzzy partial metric $(X, P, *)$, a T_0 topology τ_P on X can be defined in such a way that the family of sets $\{B_P(x, r, t) : x \in X, r \in (0, 1), t \in (0, \infty)\}$ is a base, where, for each $x \in X$, $r \in (0, 1)$ and $t \in (0, \infty)$, $B_P(x, r, t) = \{y \in X : P'(x, y, t) > 1 - r\}$ and $P'(x, y, t) = \sup\{P(x, x, s) \to_* P(x, y, s) : s \in (0, t)\}$.

3. The Main Results

We begin this section by showing that, unlike the classic partial metric case (see Proposition 1), given a fuzzy partial metric space $(X, P, *)$, the fuzzy set $M_P : X \times X \times [0, \infty)$ defined, for each $x, y \in X$, by

$$M_P(x, y, t) = \begin{cases} 0, & if\ t = 0 \\ 1, & x = y \text{ and } t > 0\,, \\ P(x, y, t), & x \neq y \text{ and } t > 0 \end{cases} \tag{5}$$

may not be a fuzzy metric. The following example corroborates such an affirmation.

Example 1. *Denote by $\mathbb{R}$ the set of real numbers. Let $X = \mathbb{R}$, and consider the fuzzy set $P : X \times X \times [0, \infty) \to [0, 1]$ given, for each $x, y \in X$ and for each $t \in [0, \infty)$, by*

$$P(x, y, t) = \begin{cases} 0, & \text{if } t = 0 \\ e^{-t}, & \text{if } x = y \text{ and } t > 0\,. \\ \frac{1}{2}e^{-t}, & \text{if } x \neq y \text{ and } t > 0 \end{cases} \tag{6}$$

In [32], Example 3.8, $(X, P, *_p)$ was proved to be a fuzzy partial metric space, where $*_P$ denotes the product t-norm. The aforementioned mapping M_P is given, for each $x, y \in X$ and for each $t \in [0, \infty)$, by

$$M_P(x, y, t) = \begin{cases} 0, & \text{if } t = 0 \\ 1, & \text{if } x = y \text{ and } t > 0\,. \\ \frac{1}{2}e^{-t}, & \text{if } x \neq y \text{ and } t > 0 \end{cases} \tag{7}$$

Next, we will see that $(M_P, *_P)$ is not a fuzzy metric by showing that **(KM4)** *is not fulfilled.*

Take $x, y, z \in X$ such that $x \neq z$ and $y = z$. Set $s, t \in (0, \infty)$. Then, we have $M_P(x, z, t + s) = \frac{1}{2}e^{-(s+t)}$, $M_P(x, y, t) = \frac{1}{2}e^{-t}$ and $M_P(y, z, s) = 1$. This means that $M_P(x, z, t + s) < M_P(x, y, t) *_p M_P(y, z, s)$ and, hence, $(M_P, *_P)$ is not a fuzzy metric on X.

In light of the preceding example, we are interested in finding under what conditions the previous fuzzy set M_P becomes a fuzzy metric. Observe that in the preceding example, for each different $x, y \in X$, the function $P_{x,y} : (0, \infty) \to [0, 1]$ fails to be non-decreasing. Next we will see that the monotony of the function $P_{x,y}$ is crucial to show that M_P is a fuzzy metric. With this aim, we need to prove the next result, which states a condition that every fuzzy partial metric P must satisfy when the function $P_{x,y}$ is non-decreasing.

Proposition 4. *Let $(X, P, *)$ be a fuzzy partial metric space. The following assertions are equivalent:*

(1) *The function* $P_{x,y} : (0, \infty) \to [0, 1]$ *is non-decreasing for all* $x, y \in X$.
(2) *For each* $x, y, z \in X$ *and* $t, s \in (0, \infty)$, *the following condition is satisfied:*

(FPKM4*) $\quad P(x, z, t + s) \geq P(x, y, t) * (P(y, y, s) \to_* P(y, z, s))$.

Proof. $(1) \Rightarrow (2)$. Let $x, y, z \in X$ and $s, t \in (0, \infty)$. Since the function $P_{x,x}$ is non-decreasing, we have that $P(x, x, t + s) \geq P(x, x, t)$. So, by axiom **(FPKM4)**, we have the following:

$$\begin{array}{l} P(x, x, t+s) * (P(x, x, t+s) \to_* P(x, z, t+s)) \geq \\ P(x, x, t) * (P(x, x, t) \to_* P(x, y, t)) * (P(y, y, s) \to_* P(y, z, s)) \end{array} \tag{8}$$

Whence we deduce, by Proposition 3, that

$$P(x, x, t+s) \wedge P(x, z, t+s) \quad \geq \quad (P(x, x, t) \wedge P(x, y, t)) * (P(y, y, s) \to_* P(y, z, s)). \tag{9}$$

Thus, by axiom **(FPKM1)**, we conclude that

$$P(x, z, t+s) \quad \geq \quad P(x, y, t) * (P(y, y, s) \to_* P(y, z, s)), \tag{10}$$

which means that condition **(FPKM4*)** holds.

$(2) \Rightarrow (1)$. Let $(X, P, *)$ be a fuzzy partial metric space satisfying the condition **(FPKM4*)**. For the purpose of contradiction, suppose that there exists $x, y \in X$ such that $P_{x,y}$ is not non-decreasing. Therefore, we can find $t, s \in (0, \infty)$, with $s < t$, such that $P(x, y, s) > P(x, y, t)$. Then, we have from condition **(FPKM4*)** that the following is satisfied:

$$P(x, z, t) \geq P(x, y, s) * (P(y, y, t-s) \to_* P(y, z, t-s)). \tag{11}$$

Taking $z = y$ in the preceding inequality, we have that

$$\begin{array}{rcl} P(x, y, t) & \geq & P(x, y, s) * (P(y, y, t-s) \to_* P(y, y, t-s)) \\ & = & P(x, y, s) * 1 = P(x, y, s) > P(x, y, t), \end{array} \tag{12}$$

which is a contradiction. Hence, the function $P_{x,y} : (0, \infty) \to [0, 1]$ is non-decreasing for all $x, y \in X$. $\square$

From the above proposition, we deduce the following two corollaries. They give two examples of fuzzy partial metric spaces, which were provided in [32], satisfying condition **(FPKM4*)**. Such examples of fuzzy partial metric spaces are constructed from a partial metric space. This fact shows a connection between classical and fuzzy partial metrics.

Corollary 2. *Let* (X, p) *be a partial metric space and* $(X, P_e, *_P)$ *be the fuzzy partial metric space where, for all* $x, y \in X$, $P_e(x, y, 0) = 0$ *and* $P_e(x, y, t) = e^{\frac{-p(x,y)}{t}}$ *for all* $t \in (0, \infty)$. *Then,* $(X, P_e, *_P)$ *satisfies the condition* **(FPKM4*).**

Corollary 3. *Let* (X, p) *be a partial metric space and* $(X, P_d, *_H)$ *be the fuzzy partial metric space, where* $*_H$ *denotes the Hamacher t-norm and, for all* $x, y \in X$, $P_d(x, y, 0) = 0$ *and* $P_d(x, y, t) = \frac{t}{t + p(x,y)}$ *for all* $t \in (0, \infty)$. *Then,* $(X, P_d, *_H)$ *satisfies the condition* **(FPKM4*)**.

Observe that, as mentioned before, Example 1 yields an instance of fuzzy partial metric space which does not satisfies the property "$P_{x,y} : (0, \infty) \to [0, 1]$ is non-decreasing for all $x, y \in X$".

Next we show that each fuzzy partial metric induces, in a natural way, a fuzzy metric through the technique, inspired by Proposition 1, exposed at the beginning of Section 3 when such a fuzzy partial metric fulfils condition **(FPKM4*)**.

Proposition 5. *Let $(X, P, *)$ be a fuzzy partial metric space satisfying the condition* **(FPKM4*)**. *Let $M_P : X \times X \times [0, \infty)$ be the fuzzy set defined, for each $x, y \in X$ and for each $t \in [0, \infty)$, by*

$$M_P(x,y,t) = \begin{cases} 0, & \text{if } t = 0 \\ 1, & \text{if } x = y \text{ and } t > 0 \\ P(x,y,t), & \text{if } x \neq y \text{ and } t > 0 \end{cases}. \tag{13}$$

Then $(X, M_P, *)$ is a fuzzy metric space.

Proof. Let $x, y, z \in X$ and $t, s \in (0, \infty)$. We will see that M_P satisfies all the axioms of Definition 3. Observe that axioms **(KM1)** and **(KM3)** are obviously satisfied by the construction of M_P. Therefore, we focus on showing that M_P satisfies the rest of the axioms in Definition 3.

(KM2) Suppose that $M_P(x,y,t) = 1$ for each $t \in (0, \infty)$. For the purpose of contradiction, assume that $x \neq y$. It follows that $P(x,y,t) = 1$ for all $t \in (0, \infty)$, and, by axiom **(FPKM1)**, we conclude that $P(x,x,t) = 1$ and $P(y,y,t) = 1$ for all $t \in (0, \infty)$. Therefore, axiom **(FPKM2)** ensures that $x = y$, which is a contradiction. Now, if $x = y$, then, by the definition of M_P, we obtain the result that $M_P(x,y,t) = 1$ for each $t \in (0, \infty)$.

(KM4) Let $x, y, z \in X$ and $t, s \in (0, \infty)$. We only consider the case in which $x \neq z$, since otherwise the required condition is satisfied trivially. Nex,t we distinguish two cases:

Case 1. $x \neq y$ and $y \neq z$. Applying **(FPKM4*)** and Corollary 1, we have that

$$\begin{aligned} M_P(x,z,t+s) &= P(x,z,t+s) \geq P(x,y,t) * (P(y,y,s) \rightarrow_* P(y,z,s)) \\ &\geq P(x,y,t) * P(y,z,s) = M_P(x,y,t) * M_P(y,z,s). \end{aligned} \tag{14}$$

Case 2. $x = y$ or $y = z$. Notice that $x = y$ and $y = z$ cannot be satisfied at the same time, since $x \neq z$. Assume that $x = y$ and $y \neq z$. It follows, by Proposition 4, that the function $P_{y,z}$ is non-decreasing. Thus, we have that

$$\begin{aligned} M_P(x,z,t+s) &= P(x,z,t+s) \geq P(y,z,s) = M_P(y,z,s) \\ &= 1 * M_P(y,z,s) = M_P(x,y,t) * M_P(y,z,s). \end{aligned} \tag{15}$$

(KM5) The function $(M_P)_{x,y} : (0, \infty) \rightarrow [0,1]$ is left-continuous for all $x, y \in X$. Indeed, if $x \neq y$, then $(M_P)_{x,y} = P_{x,y}$, which is left-continuous by **(FPKM5)**. Moreover, if $x = y$, then $(M_P)_{x,y}(t) = 1$ for all $t \in (0, \infty)$, which is obviously (left-)continuous.

Therefore, we conclude that $(X, M_P, *)$ is a fuzzy metric space. □

We leave it to the reader to verify that, if $(X, P, *)$ is a GV-fuzzy partial metric space satisfying the condition **(FPKM4*)**, then $(X, M_P, *)$ is a GV-fuzzy metric space.

Once we have seen that M_P is a fuzzy metric on X when considering a fuzzy partial metric space $(X, P, *)$ satisfying **(FPKM4*)**, we are now interested in establishing the relationship between the topologies τ_{M_P} and τ_P induced by M_P and P, respectively. Next result shows that the topology τ_P is included in τ_{M_P}.

Theorem 1. *Let $(X, P, *)$ be a fuzzy partial metric space satisfying the condition* **(FPKM4*)** *and consider $(X, M_p, *)$ the fuzzy metric space defined in Proposition 5. Then the topology τ_{M_P} is finer than the topology τ_P, i.e., $\tau_P \subseteq \tau_{M_P}$.*

Proof. Let $(X, P, *)$ be a fuzzy partial metric space satisfying the condition **(FPKM4*)**. Observe that $A \in \tau_{M_P}$ ($A \in \tau_P$) if and only if for each $x \in A$ there exists $r \in (0,1)$ and $t \in (0, \infty)$ such that $B_{M_P}(x,r,t) \subseteq A$ ($B_P(x,r,t) \subseteq A$) . Therefore, in order to show that $\tau_P \subseteq \tau_{M_P}$, we just need to prove that $B_{M_P}(x,r,t) \subseteq B_P(x,r,t)$, for each $x \in X$, $r \in (0,1)$ and $t \in (0, \infty)$.

First of all, observe that $x \in B_P(x,r,t)$ for each $x \in X$, $r \in (0,1)$ and $t \in (0,\infty)$. Indeed, $P(x,x,s) \rightarrow_* P(x,x,s) = 1$ for each $s \in (0,t)$.

Now, fix $x \in X$, $r \in (0,1)$ and $t \in (0,\infty)$, and consider $B_{M_p}(x,r,t)$. We distinguish two cases:

Case 1. Suppose that for each $y \in X$, with $x \neq y$, we have $P(x,y,t) \leq 1 - r$. It follows that

$$B_{M_p}(x,r,t) = \{x\} \subseteq B_P(x,r,t). \tag{16}$$

Case 2. Assume that there exists $y \in X$, with $x \neq y$, such that $P(x,y,t) > 1 - r$. In this case, we can find $y \in B_{M_P}(x,r,t)$ with $y \neq x$. Next, we show that $B_{M_P}(x,r,t) \subseteq B_P(x,r,t)$. To this end, let $y \in B_{M_P}(x,r,t)$. If $y = x$, then we conclude that $y \in B_P(x,r,t)$. Therefore, suppose that $y \neq x$. The construction of M_P implies that $P(x,y,t) > 1 - r$. Then, by axiom **(FPKM5)**, we can find $s \in (0,t)$ such that $P(x,y,s) > 1 - r$. Therefore, by Corollary 1, we obtain the following

$$P(x,x,s) \rightarrow_* P(x,y,s) \quad \geq \quad P(x,y,s) > 1 - r. \tag{17}$$

Furthermore, we have that

$$\sup\{P(x,x,\alpha) \rightarrow_* P(x,y,\alpha) : \alpha \in (0,t)\} \geq P(x,x,s) \rightarrow_* P(x,y,s) > 1 - r. \tag{18}$$

Whence we deduce that $y \in B_P(x,r,t)$.

Thus, $B_{M_P}(x,r,t) \subseteq B_P(x,r,t)$, for each $x \in X$, $r \in (0,1)$ and $t \in (0,\infty)$, which implies that $\tau_P \subseteq \tau_{M_P}$. □

The next example shows that the inclusion $\tau_{M_P} \subseteq \tau_P$ is not satisfied in general.

Example 2. *Let $X = [0,\infty[$ and let (X,p) be the partial metric space, where $p(x,y) = \max\{x,y\}$ for each $x,y \in X$. Consider the fuzzy partial metric space $(X, P_d, *_H)$ introduced in Corollary 3, i.e., for each $x,y \in X$ (see [32], Proposition 3.4),*

$$P_d(x,y,0) = 0 \text{ and } P_d(x,y,t) = \frac{t}{t + p(x,y)}, \text{ for all } t \in (0,\infty). \tag{19}$$

Corollary 3 guarantees that $(X, P_d, *_H)$ satisfies property **(FPKM4*)***. Now we will show that the topologies induced by P_d and M_{P_d}, respectively, are not the same. First, observe that, for each $x,y \in X, t \in (0,\infty)$, (see [38], Example 3 (ii) and Theorem 2), we have*

$$P(x,x,t) \rightarrow_{*_H} P(x,y,t) = \frac{t}{t + p(x,y) - p(x,x)}. \tag{20}$$

Now, we are able to show that τ_{P_d} and $\tau_{M_{P_d}}$ are not the same; i.e., $\tau_{M_{P_d}} \not\subseteq \tau_{P_d}$.

On the one hand, $\{1\} \in \tau_{M_{P_d}}$ since $B_{M_{P_d}}(1,1/2,1/2) = \{1\}$. Indeed, if $y \in B_{M_{P_d}}(1,1/2, 1/2)$ with $y \neq 1$, then we have

$$\frac{1}{2} \leq M_{P_d}(1,y,1/2) = P_d(1,y,1/2) = \frac{1/2}{1/2 + \max\{1,y\}} \leq \frac{1/2}{1/2+1} = \frac{1}{3}, \tag{21}$$

which provides a contradiction.

On the other hand, $[0,x] \subseteq B_{P_d}(x,r,t)$, for each $r \in (0,1)$, $t \in (0,\infty)$ and $x \in X$. Indeed, for each $y \in [0,x]$ we have

$$P(x,x,s) \rightarrow_{*_H} P(x,y,s) = \frac{s}{s + p(y,x) - p(x,x)} = 1 \text{ for each } s \in (0,\infty). \tag{22}$$

Therefore $y \in B_{P_d}(x,r,t)$ for all $r \in (0,1)$ and $t \in (0,\infty)$.

Hence we conclude that $\{1\} \notin \tau_{P_d}$ *because otherwise there should exist* $x \in X$, $r \in (0,1)$ *and* $t \in (0,\infty)$ *such that* $[0,x] \subseteq B_{P_d}(x,r,t) \subseteq \{1\}$.

After establishing the relationship between τ_P and τ_{M_P} when the fuzzy partial metric space satisfies the property **(FPKM4*)**, we continue studying the completeness of such fuzzy partial metrics. With this aim, we will characterize convergent sequences, but beforehand, we should take into account the following remark and the subsequent lemma.

Remark 1. *Let* $(X,P,*)$ *be a fuzzy partial metric space. Then, for each* $x \in X$, $r \in (0,1)$ *and* $t \in (0,\infty)$, *we have that* $\widetilde{B_P}(x,r,t) = \{y \in X : P(x,x,t) \rightarrow_* P(x,y,t) > 1-r\}$ *is a neighbourhood of* x *in* τ_P. *Indeed, given* $t > 0$, *we have that* $P(x,x,t) \rightarrow_* P(x,y,t) \geq P(x,x,s) \rightarrow_* P(x,y,s)$, *for each* $s \in (0,t)$ *due to the fact that, by condition* **(FPKM4)**, *the function* $P(x,x,\cdot) \rightarrow_* P(x,y,\cdot)$ *is non-decreasing on* t. *Then,* $P(x,x,t) \rightarrow_* P(x,y,t) \geq \sup\{P(x,x,s) \rightarrow_* P(x,y,s) : s \in (0,t)\}$ *and so, for each* $r \in (0,1)$ *and* $t > 0$, *we have that* $x \in B_P(x,r,t) \subseteq \widetilde{B_P}(x,r,t)$.

The next lemma will be useful to characterize convergent sequences with respect to τ_P.

Lemma 1. *Let* $*$ *be a continuous t-norm and consider* $a,b,c \in [0,1]$ *with* $a \geq b$. *If* $b > a * c$, *then* $a \rightarrow_* b > c$.

Proof. Since $*$ is continuous, Proposition 2.5.2 in [37] warranties that $a * c \leq b \Leftrightarrow a \rightarrow_* b \geq c$ and, in addition, Proposition 2 gives that $a \rightarrow_* b = \max\{z \in [0,1] : a * z = b\}$. Assume that $a \rightarrow_* b = c$. Then, we have that $b = a * c < b$, which is a contradiction. Therefore, we conclude that $a \rightarrow_* b > c$. □

Below, we can find the promised characterization of convergent sequences with respect to τ_P

Theorem 2. *Let* $(X,P,*)$ *be a fuzzy partial metric space and* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence in* X. *Then,* $\{x_n\}_{n\in\mathbb{N}}$ *converges in* τ_P *to* $x \in X$ *if and only if* $\lim\limits_{n\to\infty} P(x_n,x,t) = P(x,x,t)$ *for all* $t \in (0,\infty)$.

Proof. ($\Rightarrow$) Suppose that $\{x_n\}_{n\in\mathbb{N}}$ converges in τ_P to $x \in X$. Then, for each neighbourhood U of x in τ_P, there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for each $n \geq n_0$. Now, fix $t \in (0,\infty)$ and let $\varepsilon \in (0,1)$. By Remark 1, there exists $n_0 \in \mathbb{N}$ such that $x_n \in \widetilde{B_P}(x,\varepsilon,t)$ for each $n \geq n_0$; i.e., $P(x,x,t) \rightarrow_* P(x,x_n,t) > 1-\varepsilon$ for each $n \geq n_0$. In such a case, by Proposition 2.5.2 in [37], we know that $a * c \leq b \Leftrightarrow a \rightarrow_* b \geq c$ for all $a,b,c \in [0,1]$ and, thus, we get that $P(x,x_n,t) \geq (1-\varepsilon) * P(x,x,t)$ for each $n \geq n_0$. Moreover, taking into account axiom **(FPKM1)**, we have that $P(x,x,t) \geq P(x,x_n,t)$ for each $n \in \mathbb{N}$. Therefore, we conclude that $P(x,x,t) \geq P(x,x_n,t) \geq (1-\varepsilon) * P(x,x,t)$, for each $n \geq n_0$. Therefore, $\lim\limits_{n\to\infty} P(x,x_n,t) = P(x,x,t)$ for all $t \in (0,\infty)$, since t is arbitrary.

($\Leftarrow$) Suppose that $\lim\limits_{n\to\infty} P(x,x_n,t) = P(x,x,t)$ for all $t \in (0,\infty)$. Taking into account that, for each $t \in (0,\infty)$, we have that $P(x,x,t) \geq P(x,x_n,t)$ for all $n \in \mathbb{N}$, our assumption implies the following: for all $t \in (0,\infty)$, given $\varepsilon \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $P(x,x_n,t) > (1-\varepsilon) * P(x,x,t)$ for all $n \geq n_0$. Then, Lemma 1 ensures that $P(x,x,t) \rightarrow_* P(x,x_n,t) > 1-\varepsilon$ for all $n \geq n_0$.

Fix $t \in (0,\infty)$ and let $s_0 \in (0,t)$. Given $\varepsilon \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $P(x,x,s_0) \rightarrow_* P(x,x_n,s_0) > 1-\varepsilon$ for all $n \geq n_0$. Thus, we have, for all $s \in (s_0,t)$, that $P(x,x,s) \rightarrow_* P(x,x_n,s) > 1-\varepsilon$ for all $n \geq n_0$, since the function $P(x,x,\cdot) \rightarrow_* P(x,y,\cdot)$ is non-decreasing (see Remark 1). Therefore, such an argument ensures that $\sup\{P(x,x,s) \rightarrow_* P(x,x_n,s) : s \in (0,t)\} > 1-\varepsilon$ for each $n \geq n_0$, which is equivalent to $x_n \in B_P(x,\varepsilon,t)$ for each $n \geq n_0$.

Let U be a neighbourhood of x in τ_P. Then, there exists $r \in (0,1)$ and $t > 0$ such that $B_P(x,r,t) \subseteq U$. Hence, by what has been said before, there exists $n_0 \in \mathbb{N}$ such that $x_n \in B_P(x,r,t)$ for all $n \geq n_0$. Hence, $\{x_n\}_{n\in\mathbb{N}}$ converges to x in τ_P. □

As mentioned above, we are interested in studying completeness in fuzzy partial metrics satisfying **(FPKM4*)**. In this context, we introduce the following natural definition, which is inspired by the classical definition for partial metric spaces (see Section 2.1).

Definition 8. *Let $(X, P, *)$ be a fuzzy partial metric space and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X.*

(i) *$\{x_n\}_{n\in\mathbb{N}}$ is said to be a Cauchy sequence if for all $t \in (0, \infty)$ there exists $\lim_{n,m\to\infty} P(x_n, x_m, t)$ and it is greater than 0. In the case that $\lim_{n,m\to\infty} P(x_n, x_m, t) = 1$ for all $t \in (0, \infty)$, then $\{x_n\}_{n\in\mathbb{N}}$ is said to be a 1-Cauchy sequence.*

(ii) *$(X, P, *)$ is said to be complete if each Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to a point $x \in X$ in τ_P and $\lim_{n,m\to\infty} P(x_n, x_m, t) = \lim_{n\to\infty} P(x_n, x, t) = P(x, x, t)$ for all $t \in (0, \infty)$. In the case that each 1-Cauchy sequence in X converges to a point $x \in X$ in τ_P such that $P(x, x, t) = 1$ for all $t \in (0, \infty)$, the fuzzy partial metric space is said to be 1-complete.*

Unlike the fuzzy metric case, each convergent sequence in a fuzzy partial metric space may not be a Cauchy sequence, as the following example shows.

Example 3. *Let $(X, P_d, *_H)$ be the fuzzy partial metric space of Example 2. Let us consider the sequence $\{x_n\}_{n\in\mathbb{N}} = (0, 1, 0, 1, ..., 0, 1, ...)$. Observe that*

$$\lim_{n\to\infty} P(x_n, 1, t) = \lim_{n\to\infty} \frac{t}{t + p_{max}(x_n, 1)} = \frac{t}{t+1} = P(1, 1, t) \tag{23}$$

for all $t \in (0, \infty)$. Then Theorem 2 gives that $\{x_n\}_{n\in\mathbb{N}} = \{0, 1, 0, 1, ..., 0, 1, ...\}$ converges to the point $1 \in X$ in τ_{P_d}. However, $\lim_{n,m\to\infty} P(x_n, x_m, t)$ does not exist. Indeed, $\lim_{n\to\infty} P(x_n, x_{n+2}, t) = 1$ for all $t \in (0, \infty)$ whenever n is odd, whereas $\lim_{n\to\infty} P(x_n, x_{n+2}, t) = \frac{t}{t+1}$, for all $t \in (0, \infty)$ whenever n is even. Therefore, $\{x_n\}_{n\in\mathbb{N}} = \{0, 1, 0, 1, ..., 0, 1, ...\}$ is not a Cauchy sequence in $(X, P_d, *_H)$.

Observe that each complete fuzzy partial metric space is 1-complete. Moreover, 1-completeness in a fuzzy partial metric space $(X, P, *)$ satisfying the condition that **(FPKM4*)** is equivalent to completeness of $(X, M_P, *)$, as the next theorem below shows.

Theorem 3. *Let $(X, P, *)$ be a fuzzy partial metric space satisfying the condition* **(FPKM4*)** *and consider $(X, M_p, *)$ the fuzzy metric space defined in Proposition 5. Then, a sequence $\{x_n\}_{n\in\mathbb{N}}$ non-eventually constant is Cauchy in $(X, M_P, *)$ if and only if $\{x_n\}_{n\in\mathbb{N}}$ is 1-Cauchy in $(X, P, *)$. Furthermore, $(X, P, *)$ is 1-complete if and only if $(X, M_P, *)$ complete.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a non-eventually constant sequence in X.

With the aim of showing the direct implication, suppose that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in $(X, M_P, *)$. Then, we have $\lim_{n,m\to\infty} M_P(x_n, x_m, t) = 1$ for all $t \in (0, \infty)$. Taking into account that $\{x_n\}_{n\in\mathbb{N}}$ is non-eventually constant, by the construction of M_P, we conclude that $\lim_{n,m\to\infty} P(x_n, x_m, t) = 1$ for all $t \in (0, \infty)$. Thus, $\{x_n\}_{n\in\mathbb{N}}$ is a 1-Cauchy sequence in $(X, P, *)$. The proof of the converse implication runs following similar arguments.

Now, assume that $(X, P, *)$ is a 1-complete fuzzy partial metric space and let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(X, M_P, *)$. If $\{x_n\}_{n\in\mathbb{N}}$ is eventually constant, then it is obviously convergent. Therefore, assume that $\{x_n\}_{n\in\mathbb{N}}$ is non-eventually constant. It follows that $\{x_n\}_{n\in\mathbb{N}}$ is a 1-Cauchy sequence in $(X, P, *)$. Since $(X, P, *)$ is 1-complete, then there exists a point $x \in X$ such that $\lim_{n,m\to\infty} P(x_n, x_m, t) = \lim_{n\to\infty} P(x_n, x, t) = P(x, x, t) = 1$ for all $t \in (0, \infty)$. Therefore $\lim_{n\to\infty} M_P(x_n, x, t) = 1$ for all $t \in (0, \infty)$, which implies that $(X, M_P, *)$ is a complete fuzzy metric space.

Next assume completeness in $(X, M_P, *)$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a 1-Cauchy sequence in $(X, P, *)$. If $\{x_n\}_{n\in\mathbb{N}}$ is eventually constant, then there exists $n_0 \in \mathbb{N}$ such that $x_n = x_{n_0}$ for each $n \geq n_0$. So, $1 = \lim_{n,m \to \infty} P(x_n, x_m, t) = P(x_{n_0}, x_{n_0}, t)$ for all $t \in (0, \infty)$. Thus, $\{x_n\}_{n \in \mathbb{N}}$ converges to x_{n_0} in τ_P and $P(x_{n_0}, x_{n_0}, t) = 1$ for each $t \in (0, \infty)$. The case of $\{x_n\}_{n \in \mathbb{N}}$ being non-eventually constant is proved following a similar argument to the one shown above and, in addition, taking into account that $P(x, x_n, t) \leq P(x, x, t)$ for all $x \in X$ and for all $t \in (0, \infty)$. □

Taking into account the previous theorem, we conclude that for each complete fuzzy partial metric $(X, P, *)$ satisfying **(FPKM4*)**, the fuzzy metric $(X, M_P, *)$ is complete.

The remainder of the paper is devoted to illustrating how such a conclusion allows to prove fixed-point results in fuzzy partial metrics, specifically, those results in which a contractive condition given in the context of fuzzy metrics is established for fuzzy partial metrics. In this context, we present a notion of contractivity in fuzzy metric spaces introduced in [27]. First, let us recall that a binary operation $\diamond$ on $[0, 1]$ is called a t-conorm if, for each $a, b, c \in [0, 1]$, it satisfies axioms **(T1)–(T3)** for t-norms given in Definition 2 and additionally the following one:

(S4) $a \diamond 0 = a$.

A t-conorm $\diamond$ is said to be continuous when it is continuous (with respect to the usual topology) as a function defined on $[0, 1] \times [0, 1]$. Moreover, if for each $a, b \in]0, 1[$ there exists $n \in \mathbb{N}$ such that $a \diamond \cdots^{n)} \diamond a > b$, then the t-conorm is called Archimedean (see [36] for a deeper treatment on t-conorms), where $a \diamond \cdots^{n)} \diamond a$ denotes n-power of a with respect to $\diamond$ (see [36], Remark 1.10).

Below, we can find this concept of contractivity.

Definition 9. *Let $(X, M, *)$ be a fuzzy metric space. With fixed $k \in (0, 1)$ and a continuous t-conorm $\diamond$, we will say that a mapping $T : X \to X$ is a* fuzzy k-$\diamond$-contraction *in $(X, M, *)$ if, for each $x, y \in X$ and $t \in (0, \infty)$, the following condition holds:*

$$M(T(x), T(y), t) \geq k \diamond M(x, y, t). \tag{24}$$

In [27], the following fixed point theorem was established for fuzzy k-$\diamond$-contractions in the context of fuzzy metric spaces.

Theorem 4. *Let $(X, M, *)$ be a complete fuzzy metric space and let $T : X \to X$ be a fuzzy k-$\diamond$-contraction in $(X, M, *)$. If $\diamond$ is Archimedean, then T has a unique fixed point.*

As pointed out in [32], a fuzzy metric space is a particular case of fuzzy partial metric space. In addition, we can easily extend the notion of fuzzy k-$\diamond$-contractions to the context of fuzzy partial metrics in the obvious way. Thus, one can try to generalize the preceding theorem in fuzzy partial metrics instead of fuzzy metrics. Nevertheless, it can be established for fuzzy partial metrics satisfying **(FPKM4*)** just as a mere corollary of Theorem 4, as we show below.

Theorem 5. *Let $(X, P, *)$ be a complete fuzzy partial metric space satisfying* **(FPKM4*)** *and let $T : X \to X$ be a fuzzy k-$\diamond$-contraction in $(X, P, *)$. If $\diamond$ is Archimedean, then T has a unique fixed point $x^\star$. Moreover, $P(x^\star, x^\star, t) = 1$ for all $t \in (0, \infty)$.*

Proof. Let $(X, P, *)$ be a complete fuzzy partial metric space satisfying **(FPKM4*)**. Then, by Theorem 3, the fuzzy metric space $(X, M_P, *)$ introduced in Proposition 5 is complete. Let $T : X \to X$ be a fuzzy k-$\diamond$-contraction in $(X, P, *)$ for $\diamond$ being Archimedean. Then, for all $x, y \in X$ and for all $t \in (0, \infty)$, we have that

$$P(Tx, Ty, t) \geq k \diamond P(x, y, t). \tag{25}$$

Therefore, for each $x, y \in X$, with $x \neq y$, and $t \in (0, \infty)$, we obtain that

$$M_P(Tx, Ty, t) \geq P(Tx, Ty, t) \geq k \diamond P(x, y, t) = k \diamond M_P(x, y, t). \tag{26}$$

Furthermore, if $x = y$, then $M_P(Tx, Ty, t) = 1 \geq k \diamond 1 = k \diamond M_P(x, y, t)$ for all $t \in (0, \infty)$. Therefore, T is also a fuzzy k-$\diamond$-contraction in $(X, M_P, *)$. Theorem 4 ensures the existence and uniqueness of a fixed point. Let $x^\star$ be the fixed point of T. We will show by contradiction that $P(x^\star, x^\star, t) = 1$ for all $t \in (0, \infty)$. Therefore, assume that $P(x^\star, x^\star, t) < 1$ for some fixed $t_0 \in (0, \infty)$. We know that

$$P(x^\star, x^\star, t_0) \geq k \diamond P(x^\star, x^\star, t_0) \geq \max\{k, P(x^\star, x^\star, t_0)\} \geq P(x^\star, x^\star, t_0). \tag{27}$$

Hence, we obtain the result that $k \diamond P(x^\star, x^\star, t_0) = P(x^\star, x^\star, t_0)$. The fact that $\diamond$ is Archimedean and continuously shows that there exists an additive generator (see [36], Definition 3.39) $g_\diamond : [0, 1] \to [0, \infty]$ such that

$$P(x^\star, x^\star, t_0) = k \diamond P(x^\star, x^\star, t_0) = g_\diamond^{(-1)}(g_\diamond(k) + g_\diamond(P(x^\star, x^\star, t_0))). \tag{28}$$

Since $P(x^\star, x^\star, t_0) < 1$ we deduce that $g_\diamond(k) + g_\diamond(P(x^\star, x^\star, t_0) < g_\diamond(1)$. It follows that

$$P(x^\star, x^\star, t_0) = g_\diamond^{-1}(g_\diamond(k) + g_\diamond(P(x^\star, x^\star, t_0))). \tag{29}$$

Hence

$$g_\diamond(P(x^\star, x^\star, t_0)) = g_\diamond(k) + g_\diamond(P(x^\star, x^\star, t_0)). \tag{30}$$

Hence, we have that $g_\diamond(k) = 0$ and, thus, that $k = 0$. However, $k \in (0, 1)$. Therefore $P(x^\star, x^\star, t) = 1$ for all $t \in (0, \infty)$. $\square$

4. Conclusions and Future Work

An interesting topic in fixed-point theory consists in extending a result established in a certain context to another more general context. To achieve this goal, many authors obtained generalizations of different fixed-point results, already established in metric spaces, in the context of partial metric spaces. Nevertheless, in [8], it was shown that many of the aforementioned generalizations were actually corollaries of their metric counterpart. In this paper, we have shown that the main conclusions drawn in [8] can be retrieved in the fuzzy setting when we consider the notions of fuzzy partial metric space introduced in [32]. This is achieved by means of a technique for generating a fuzzy metric from fuzzy partial metrics that extend the classical one. The relationship between the topology and completeness of both the fuzzy partial metric space and the associated fuzzy metric space have been explored. However, unlike partial metric spaces, in order to obtain the conclusions obtained in the classical case, a condition must be required on the fuzzy partial metric. Specifically, we have shown that we can extend the aforementioned technique in a direct way whenever the condition **(FPKM4*)** is fulfilled. Moreover, a kind of fixed-point results already established in fuzzy metric spaces have been extended to the more general context of fuzzy partial metric spaces whenever the condition **(FPKM4*)** is also satisfied. Therefore, the main results presented in this paper will allow us to avoid many repetitive contributions devoted to extending fixed-point results in fuzzy metric spaces to their partial counterpart.

Nonetheless, different topics remain to be explored in fixed-point theory for fuzzy partial metrics. For instance, on the one hand, one can explore contractive conditions for fuzzy partial-metric spaces that have not been studied in the context of fuzzy metrics. On the other hand, one can try to study whether versions of fixed-point results already established in fuzzy metrics remain valid in the fuzzy partial-metric framework when the condition **(FPKM4*)** is not assumed.

Author Contributions: All authors contribute equally to this work. Formal analysis, J.-J.M. and O.V.; Funding acquisition, H.A., J.-J.M. and O.V.; Investigation, E.G. and J.-J.M.; Methodology, E.G., J.-J.M. and O.V.; Project administration, J.-J.M.; Resources, J.-J.M. and O.V.; Supervision, H.A., J.-J.M. and O.V.; Validation, J.-J.M. and O.V.; Visualization, H.A., E.G., J.-J.M. and O.V.; Writing—original draft, E.G.; Writing—review & editing, J.-J.M. and O.V. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Proyecto PGC2018-095709-B-C21 financiado por MCIN/AEI/10.13039/501100011033 y FEDER "Una manera de hacer Europa" and from project BUGWRIGHT2. This last project has received funding from the European Union's Horizon 2020 research and innovation program under grant agreements No. 871260. This publication reflects only the authors' views, and the European Union is not liable for any use that may be made of the information contained therein.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

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