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# Event-Triggered Extended Kalman Filtering Analysis for Networked Systems

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**Abstract:** In this paper, the filtering problem of nonlinear networked systems with event-triggered data transmission tasks is studied. To reduce the transmission of excessive measurement data in the bandwidth-limited network, a data transmission mechanism with event trigger is introduced to analyze the error behavior of the extended Kalman filter. We prove that the real estimation error and error covariance matrices can be determined by restricting the initial conditions appropriately. Finally, the effectiveness of the filtering algorithm is verified by simulation.

**Keywords:** extended Kalman filter; event-triggered; nonlinear systems; estimation error analysis

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## 1. Introduction

In recent years, microelectronics technology, computing technology and wireless communication technology have developed rapidly. Various integrated microsensors work together to monitor, perceive and collect information. This wireless sensor network (WSN) has a very broad application prospect [1–5].

Kalman filter (KF) uses the minimum mean square error (MMSE) criterion to estimate the state of the system, which is an optimal dynamic estimation algorithm for linear systems [6,7]. The algorithm of KF is expressed in the form of recursion and has been studied thoroughly by scholars. However, the nonlinear phenomenon is very common in practical applications [8,9]. Therefore, extended Kalman filter (EKF) and untraced Kalman filter (UKF) have been derived for nonlinear systems [10]. For one-step prediction equations, UKF uses untracked transformation to handle the nonlinear transfer of mean and covariance. EKF uses Taylor decomposition to linearize the model and then uses Gauss hypothesis to solve the problem of difficult probability calculation. The application of EKF further improves the estimation performance of KF for nonlinear systems. The computational complexity of EKF is less than that of UKF and their performances are compared in [11,12]. EKF is widely used in control [13], optimization [14], observation [15], adaptive filtering [16], estimation [17] and neural network [18].

EKF is the most direct method to solve a nonlinear state estimation problem which has lower computational complexity and has been successfully applied in many nonlinear systems. On the other hand, most of the literature working on nonlinear systems uses the traditional time-triggered sampling method which is easy to implement. However, such a periodic transmission scheduling will lead to unnecessary transmission and waste of resources. References [19,20] designed a new event-triggered mechanism based on the event-triggered control (ETC) strategy and studied the input-to-state stability of nonlinear systems. This reduces the burden of system communication and controller updates while maintaining performance requirements. Therefore, it is of great significance to adopt appropriate sampling methods [21]. In this case, the event-triggered policies are more suitable [22–25]. Reference [22] studies the estimation of event triggering in linear systems. References [23,24] study event-triggered robust estimation problems for hidden Markov models and linear systems, respectively. Reference [25] presents a MMSE estimator with an innovation-based event trigger. Based on the event trigger in [25,26] studies the UKF problem of nonlinear networked system. However, only the boundedness of the estimation error covariance matrix is analyzed. In addition, to the best of our knowledge, little attention has been paid to the problem of event-triggered filtering of nonlinear systems. Therefore, this paper aims to close this gap. The Kalman filter has good results in terms of convergence when applied to linear systems. However, for nonlinear filtering, it is difficult to discuss the boundedness of the estimation error and it is even more difficult to add event-trigger conditions. For example, in [25], the proof of the boundedness of the estimation error is not given for a linear system with event-triggered condition; however, we give the sufficient conditions for it.

In this paper, the event-triggered MMSE estimator for a linear system proposed in [25] is extended to EKF for nonlinear systems and both boundedness conditions of the real estimation error and estimation error covariance matrix are analyzed. Data are only transmitted when certain conditions are met, effectively reducing the quantity of executions. However, such a transmission scheme leads to a complicated state estimation analysis. The linearization of the event-triggered filtering algorithm in nonlinear EKF systems will introduce step error and lead to the decrease in filtering accuracy. Therefore, it is difficult to conduct error analysis and design a stable estimator. At the same time, when the initial state error is relatively large or the system model is nonlinear, the accuracy of filtering will be severely affected or even diverged. These are the problems to be solved in this paper.

Motivated by the above analysis, this paper has proposed an EKF-based nonlinear filtering algorithm for nonlinear networked systems under event-triggered data transmission. The main contributions are embodied in the following three aspects: (1) An event-triggered data transmission scheme is introduced to lower the excessive measurement transmission. (2) An EKF type filtering algorithm is designed under event triggering for nonlinear system. (3) Sufficient initial conditions are provided to ensure the boundedness of the real estimation error and the error covariance matrices. The organization of this paper is as follow. Problem setup is presented in Section 2. Event-triggered EKF is proposed in Section 3. Boundness of the real estimation error and the error covariance matrices are discussed in Section 4. Section 5 and Section 6 present the numerical example and conclusion, respectively.

Notations: The symbols are standard in this paper. $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ represent $n$-dimensional vector space and set of the $m \times n$ matrices, respectively. $\mathrm{tr}(\cdot)$ stands for the trace of a matrix. $X'$ and $X^{-1}$ represent the transpose and inverse of the matrix $X$, respectively. $X > 0$ ($X \geq 0$) means $X$ is a positive definite (positive semi-definite) matrix. $\mathbb{E}[\cdot]$ denotes the expectation of a random variable. $\mathbb{E}[x|y]$ stands for the expectation value of $x$ with $y$ as the condition. $\mathrm{diag}(\cdot)$ denotes a diagonal matrix. $\|\cdot\|$ and $\|\cdot\|_\infty$ stand for the Euclidean norm and infinity-norm, respectively.

## 2. System and Problem Description

Consider the system with nonlinear discrete-time represented by:

$$x_{n+1} = f(x_n) + \omega_n, \tag{1}$$
$$y_n = h(x_n) + v_n, \tag{2}$$

where $x_n \in \mathbb{R}^l$ and $y_n \in \mathbb{R}^m$ are the state vector and the sensor measurement, respectively. The process noise $\omega_n \in \mathbb{R}^l$ and the measurement noise $v_n \in \mathbb{R}^m$ are uncorrelated zero-mean white noises with the covariance matrices $Q_n \geq 0$ and $R_n > 0$. Assume that at each $x$, the nonlinear function $f(x)$ and the measurement function $h(x)$ are continuous and differentiable.

In Figure 1, the smart sensor consists of the sensor and event-trigger scheduler. Assume that the wireless network channel is ideal without packet loss and delay. The smart sensor transmission $y_n$ is first sent to the event-triggered scheduler at each time $n$. According to the value of $\gamma_n$ which is calculated based on the event-trigger conditions, the sensor determines whether to transfer $y_n$ to the remote nonlinear estimator.

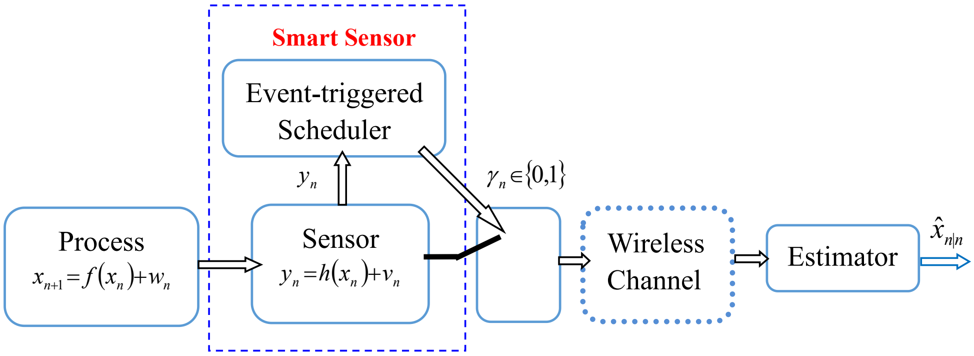


**Figure 1.** Networked system communication diagram.

We use the event-triggered communication strategy in [25] to reduce excessive measurement traffic. There exists a unitary matrix $\Gamma_n \in \mathbb{R}^{m\times m}$ satisfying:

$$\Gamma_n'(C_n P_{n|n-1} C_n' + R_n)\Gamma_n = \Lambda_n, \tag{3}$$

where $\Lambda_n = \mathrm{diag}(\lambda_n^1, ..., \lambda_n^m) \in \mathbb{R}^{m\times m}$ and the elements $\lambda_n^1, ..., \lambda_n^m$ on the diagonal are eigenvalues of $C_n P_{n|n-1} C_n' + R_n$. $P_{n|n-1}$ is the prediction error covariance and $C_n$ is a matrix related to the system, which will be defined later. If we define $S_n \triangleq \Gamma_n \Lambda_n^{-1/2} \in \mathbb{R}^{m\times m}$, the smart sensor calculates the matrix $S_n$ at each time. Define $z_n \triangleq y_n - \hat{y}_{n|n-1}$ and $\epsilon_n \triangleq S_n' z_n$, where $\hat{y}_{n|n-1}$ is the measurement step prediction. The event-trigger mechanism is as follows:

$$\gamma_n = \begin{cases} 0, & \|\epsilon_n\|_\infty \le \delta, \\ 1, & \text{others}, \end{cases} \tag{4}$$

where $\delta \ge 0$ is a threshold which can be fixed to achieve a desired compromise. The desired tradeoff can be achieved by adjusting the threshold appropriately. We let the set of available information $\mathcal{I}_n$ at instant $n$ as:

$$\mathcal{I}_n \triangleq \{\gamma_0 y_0, \cdots, \gamma_n y_n\} \cup \{\gamma_0, \cdots, \gamma_n\}. \tag{5}$$

Define the a priori and a posterior MMSE estimates $\hat{x}_{n|n-1} = \mathbb{E}[x_n|\mathcal{I}_{n-1}]$ and $\hat{x}_{n|n} = \mathbb{E}[x_n|\mathcal{I}_n]$. Let prediction error $e_{n+1|n} = x_{n+1} - \hat{x}_{n+1|n}$ and estimation error $e_{n+1|n+1} = x_{n+1} - \hat{x}_{n+1|n+1}$. $P_{n|n-1} = \mathbb{E}[e_{n|n-1}e_{n|n-1}'|\mathcal{I}_{n-1}]$ and $P_{n|n} = \mathbb{E}[e_{n|n}e_{n|n}'|\mathcal{I}_n]$ represent the corresponding error covariance matrix. The Taylor expansions of the functions $f(x)$ and $h(x)$ are expressed as

$$f(x_n) = f(\hat{x}_{n|n}) + \frac{\partial f}{\partial \hat{x}_{n|n}} e_{n|n} + \phi(x_n, \hat{x}_{n|n}), \tag{6}$$

$$h(x_n) = h(\hat{x}_{n|n-1}) + \frac{\partial h}{\partial \hat{x}_{n|n-1}} e_{n|n-1} + \psi(x_n, \hat{x}_{n|n-1}), \tag{7}$$

where the functions $\phi$, $\psi$ represent the remainder. We define $\frac{\partial f}{\partial \hat{x}_{n|n}} = A_n$ and $\frac{\partial h}{\partial \hat{x}_{n|n-1}} = C_n$. In the calculation of Section 3, we approximate $\hat{x}_{n|n}$ by $f(\hat{x}_{n|n}) + \frac{\partial f}{\partial \hat{x}_{n|n}} e_{n|n}$ and approximate $h(x_n)$ by $h(x_n) = h(\hat{x}_{n|n-1}) + \frac{\partial h}{\partial \hat{x}_{n|n-1}} e_{n|n-1}$, we have $\hat{y}_{n|n-1} = \mathbb{E}[h(\hat{x}_{n|n-1}) + C_n(x_n - \hat{x}_{n|n-1}) + v_n] = h(\hat{x}_{n|n-1})$.

## 3. Design of Event-Triggered EKF

Considering the event-triggered transformation, an EKF-type nonlinear filter is proposed:

$$\hat{x}_{n+1|n} = f(\hat{x}_{n|n}), \tag{8}$$

$$\hat{x}_{n+1|n+1} = \hat{x}_{n+1|n} + \gamma_{n+1} K_{n+1} z_{n+1}. \tag{9}$$

Note that when $\gamma_{n+1} = 0$, the state estimate $\hat{x}_{n+1|n+1}$ is equal to $\hat{x}_{n+1|n}$. $K_{n+1}$ is the state gain matrix.

**Theorem 1.** *Consider the event-triggered data transmission mechanism (4) under the nonlinear systems (1) and (2). The error covariance matrices $P_{n+1|n}$, $P_{n+1|n+1}$ and the state gain matrix $K_{n+1}$ are given as:*

$$P_{n+1|n} = A_n P_{n|n} A_n' + Q_n, \tag{10}$$

$$K_{n+1} = P_{n+1|n} C_{n+1}' (C_{n+1} P_{n+1|n} C_{n+1}' + R_{n+1})^{-1}, \tag{11}$$

$$P_{n+1|n+1} = P_{n+1|n} - [(1 - \gamma_{n+1})\beta(\delta) + \gamma_{n+1}] K_{n+1} C_{n+1} P_{n+1|n}, \tag{12}$$

*where* $\beta(\delta) = \frac{2}{\sqrt{2\pi}} \delta e^{-\frac{\delta^2}{2}} [1 - 2Q(\delta)]^{-1}$, $Q(\delta) \triangleq \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x$.

**Proof of Theorem 1.** We can obtain $P_{n+1|n} = \mathbb{E}[e_{n+1|n} e_{n+1|n}' | \mathcal{I}_n] = A_n P_{n|n} A_n' + Q_n$ by substitution, which proves (10). We have $z_{n+1} = C_{n+1} e_{n+1|n} + v_{n+1}$ and

$$\mathbb{E}[e_{n+1|n} z_{n+1}' | \mathcal{I}_n] = \mathbb{E}[e_{n+1|n} (C_{n+1} e_{n+1|n} + v_{n+1})'] = P_{n+1|n} C_{n+1}',$$
$$\mathbb{E}[(C_{n+1} e_{n+1|n} + v_{n+1})(C_{n+1} e_{n+1|n} + v_{n+1})' | \mathcal{I}_n] = C_{n+1} P_{n+1|n} C_{n+1}' + R_{n+1}.$$

Now, from Lemmas 3.3 and 3.4 in [25], one obtains

$$\mathbb{E}[(e_{n+1|n} - K_{n+1} z_{n+1})(e_{n+1|n} - K_{n+1} z_{n+1})' | \hat{\mathcal{I}}_{n+1}] = P_{n+1|n} - K_{n+1} C_{n+1} P_{n+1|n}, \tag{13}$$

$$\mathbb{E}[(e_{n+1|n} - K_{n+1} z_{n+1}) z_{n+1}' K_{n+1}' | \hat{\mathcal{I}}_{n+1}] = 0, \tag{14}$$

$$\mathbb{E}[z_{n+1} z_{n+1}' | \hat{\mathcal{I}}_{n+1}] = [I_l - \beta(\delta)](C_{n+1} P_{n+1|n} C_{n+1}' + R_{n+1}), \tag{15}$$

where $\hat{\mathcal{I}}_{n+1} = \mathcal{I}_n \cup \{\gamma_{n+1} = 0\}$. Next, two cases are considered.

(1) $\gamma_{n+1} = 1$: Smart sensor sends measurement information to remote estimator in this case. The estimation error is written as $x_{n+1} - \hat{x}_{n+1|n} - K_{n+1} z_{n+1}$ and $\mathcal{I}_{n+1} = \mathcal{I}_n \cup \{\gamma_{n+1} = 1\}$. Then the estimation error covariance is calculated as follows

$$\begin{aligned} P_{n+1|n+1} &= \mathbb{E}[e_{n+1|n+1} e_{n+1|n+1}' | \mathcal{I}_{n+1}] \\ &= \mathbb{E}[((I_l - K_{n+1} C_{n+1}) e_{n+1|n} - K_{n+1} v_{n+1})((I_l - K_{n+1} C_{n+1}) e_{n+1|n} \\ &\quad - K_{n+1} v_{n+1})' | \mathcal{I}_{n+1}] \\ &= (I_l - K_{n+1} C_{n+1}) P_{n+1|n} (I_l - K_{n+1} C_{n+1})' + K_{n+1} R_{n+1} K_{n+1}'. \end{aligned} \tag{16}$$

Take the partial derivative of $P_{n+1|n+1}$ relative to $K_{n+1}$ and let $\dfrac{\partial P_{n+1|n+1}}{\partial K_{n+1}} = 0$. Then, the filter gain is derived as

$$K_{n+1} = P_{n+1|n} C_{n+1}' (C_{n+1} P_{n+1|n} C_{n+1}' + R_{n+1})^{-1}. \tag{17}$$

Substituting (17) into (16) , $P_{n+1|n+1}$ is determined as

$$P_{n+1|n+1} = P_{n+1|n} - K_{n+1}C_{n+1}P_{n+1|n}. \tag{18}$$

(2) $\gamma_{n+1} = 0$: $\hat{x}_{n+1|n+1}$ is replaced by $\hat{x}_{n+1|n}$ due to (9). Then, we let $e_{n+1|n+1} = e_{n+1|n} - K_{n+1}z_{n+1} + K_{n+1}z_{n+1}$, and we can obtain:

$$K_{n+1}\mathbb{E}[z_{n+1}z'_{n+1}|\hat{\mathcal{I}}_{n+1}]K'_{n+1} = [1 - \beta(\delta)]K_{n+1}C_{n+1}P_{n+1|n}. \tag{19}$$

Thus, the calculation process of $P_{n+1|n+1}$ is as follows:

$$\begin{aligned}P_{n+1|n+1} =&\mathbb{E}[(e_{n+1|n} - K_{n+1}z_{n+1} + K_{n+1}z_{n+1})(e_{n+1|n} - K_{n+1}z_{n+1} + K_{n+1}z_{n+1})'|\hat{\mathcal{I}}_{n+1}]\\ =&\mathbb{E}[(e_{n+1|n} - K_{n+1}z_{n+1})(e_{n+1|n} - K_{n+1}z_{n+1})'|\hat{\mathcal{I}}_{n+1}] + K_{n+1}\mathbb{E}[z_{n+1}z'_{n+1}|\hat{\mathcal{I}}_{n+1}]\\ &\cdot K'_{n+1} + K_{n+1}\mathbb{E}[z_{n+1}(e_{n+1|n} - K_{n+1}z_{n+1})'|\hat{\mathcal{I}}_{n+1}] + \mathbb{E}[(e_{n+1|n} - K_{n+1}z_{n+1})\\ &\cdot z'_{n+1}K'_{n+1}|\hat{\mathcal{I}}_{n+1}].\end{aligned}$$

Make some substitutions based on (13)–(15) and (19), $P_{n+1|n+1}$ is written in the form as:

$$P_{n+1|n+1} = P_{n+1|n} - \beta(\delta)K_{n+1}C_{n+1}P_{n+1|n}. \tag{20}$$

Finally, combine (18) and (20) , one obtains:

$$P_{n+1|n+1} = P_{n+1|n} - [(1-\gamma_{n+1})\beta(\delta) + \gamma_{n+1}]K_{n+1}C_{n+1}P_{n+1|n}. \tag{21}$$

This completes the proof. □

**Remark 1.** *Exactly as [25], $P_{n+1|n+1}$ is about the function of $\gamma_{n+1}$ and $\beta(\delta)$, and both are affected by the value of $\delta$. It can be adjusted appropriately to achieve an ideal balance between the communication rate and the estimated performance. The complete algorithm for event triggering is given in Algorithm 1.*

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**Algorithm 1** Event-triggered EKF scheduler.

1. Prior estimate and error covariance matrix:
   $\hat{x}_{0|0} = \hat{x}_0$, $P_{0|0} = P_0$.
2. Time update:
   given
   $\hat{x}_{n|n}$, $P_{n|n}$,
   do
   $\hat{x}_{n+1|n} = f(\hat{x}_{n|n})$, $P_{n+1|n} = A_kP_{n|n}A_k^T + Q_k$.
   Sensor scheduling: Let the scheduling variable be given by:
   $$\gamma_n = \begin{cases}0, & \|\epsilon_k\|_\infty \le \delta,\\ 1, & \text{others}.\end{cases}$$
   Data transmission: If $\gamma_n = 1$, send $y_n$ to the estimator.
3. Measurement update:
   let
   $K_{n+1} = P_{n+1|n}C_{n+1}^T(C_{n+1}P_{n+1|n}C_{n+1}^T + R_{n+1})^{-1}$,
   do
   $\hat{x}_{n+1|n+1} = \hat{x}_{n+1|n} + \gamma_{n+1}K_{n+1}(y_{n+1} - \hat{y}_{n+1|n})$,
   $P_{n+1|n+1} = P_{n+1|n} - [(1-\gamma_{n+1})\beta(\delta) + \gamma_{n+1}]K_{n+1}C_{n+1}P_{n+1|n}$,
   where $\beta(\delta) = \frac{2}{\sqrt{2\pi}}\delta e^{-\frac{\delta^2}{2}}[1 - 2Q(\delta)]^{-1}$, $Q(\delta) \triangleq \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\mathrm{d}x$.

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## 4. Estimation Error Analysis

### *4.1. Boundedness of the Estimation Error*

According to (3) and (4), we derive the real estimation errors as:

$$\begin{aligned}
e_{n|n} &= (I_n - \gamma_n K_n C_n)e_{n|n-1} - \gamma_n K_n \psi(x_n, \hat{x}_{n|n-1}) - \gamma_n K_n v_n, \\
e_{n+1|n} &= A_n(x_n - \hat{x}_{n|n}) + \phi(x_n, \hat{x}_{n|n}) + \omega_n \\
&= A_n(I_l - \gamma_n K_n C_n)e_{n|n-1} + m_n + s_n, \qquad (22)
\end{aligned}$$

where

$$m_n = \phi(x_n, \hat{x}_{n|n}) - \gamma_n A_n K_n \psi(x_n, \hat{x}_{n|n-1}), \qquad (23)$$

$$s_n = \omega_n - \gamma_n A_n K_n v_n. \qquad (24)$$

Next, we will prove that under the event-triggered nonlinear filter, the generated estimation error is bounded if the following hypothesis holds:

**Assumption 1.** *Considering the event-triggered EKF for nonlinear systems, assume that the linearization of the nonlinear systems (1) and (2) satisfies the uniform observability condition. There are positive real constants $\bar{a}$, $\bar{c}$, $\bar{p}$, $\underline{p}$, $\bar{q}$, $\underline{q}$, $\bar{r}$, $\underline{r}$ such that the various matrices have the following boundaries*

$$\begin{aligned}
&\|A_n\| \le \bar{a}, \\
&\|C_n\| \le \bar{c}, \\
&\underline{q} I_l \le Q_n \le \bar{q} I_l, \\
&\underline{r} I_m \le R_n \le \bar{r} I_m, \\
&\underline{p} I_l \le P_{n+1|n+1} \le P_{n+1|n} \le \bar{p} I_l. \qquad (25)
\end{aligned}$$

For all $\varepsilon_\phi, \varepsilon_\psi > 0$ there are $\eta_\phi, \eta_\psi > 0$, such that for all $\|x_n - \hat{x}_{n|n}\| \le \eta_\phi$ and $\|x_n - \hat{x}_{n|n-1}\| \le \eta_\psi$, there hold

$$\|\phi(x_n, \hat{x}_{n|n})\| \le \varepsilon_\phi \|x_n - \hat{x}_{n|n}\|^2, \qquad (26)$$

$$\|\psi(x_n, \hat{x}_{n|n-1})\| \le \varepsilon_\psi \|x_n - \hat{x}_{n|n}\|^2. \qquad (27)$$

**Theorem 2.** *Under Assumption 1, there is a constant $\kappa > 0$, the random noise covariance matrix has a bound $\varepsilon > 0$, i.e., $\mathbb{E}[\omega_n \omega_n'] \le \varepsilon^2 I_l$, $\mathbb{E}[v_n v_n'] \le \varepsilon^2 I_m$, and a bound $\eta > 0$ for the initial estimation error, i.e., $\mathbb{E}[\|e_{1|0}\|] \le \eta$, so that the estimation error $e_{n+1|n}$ is bounded and satisfies the following inequality*

$$\mathbb{E}[\|e_{n+1|n}\|] \le \kappa. \qquad (28)$$

**Proof of Theorem 2.** Define $V_n(e_{n|n-1}) = e'_{n|n-1} P^{-1}_{n|n-1} e_{n|n-1}$, one obtains

$$\begin{aligned}
V_n(e_{n+1|n}) &= e'_{n+1|n} P^{-1}_{n+1|n} e_{n+1|n} \\
&= e'_{n|n-1}(I_l - \gamma_n K_n C_n)' A'_n P^{-1}_{n+1|n} A_n (I_l - \gamma_n K_n C_n) e_{n|n-1} + m'_n P^{-1}_{n+1|n}[2A_n(I_l \\
&\quad - \gamma_n K_n C_n)e_{n|n-1} + m_n] + 2s'_n P^{-1}_{n+1|n}[A_n(I_l - \gamma_n K_n C_n)e_{n|n-1} + m_n] \\
&\quad + s'_n P^{-1}_{n+1|n} s_n. \qquad (29)
\end{aligned}$$

Next, we will prove that there exist constants $d$, $\kappa_1$, $\kappa_2$ such that the following inequality is satisfied

$$\mathbb{E}[V_{n+1}(e_{n+1|n})] \le (1-d)\mathbb{E}[V_n(e_{n|n-1})] + \kappa_1 \|e_{n|n-1}\|^3 + \kappa_2 \varepsilon. \qquad (30)$$

First of all, let us prove:

$$(I_l - \gamma_n K_n C_n)' A_n' P_{n|n-1}^{-1} A_n (I_l - \gamma_n K_n C_n) \leq (1-d) P_{n|n-1}^{-1}. \tag{31}$$

Under the Assumption 1, one obtains:

$$P_{n+1|n} = A_n P_{n|n} A_n' + Q_n \geq (1 + \frac{\underline{q}}{\bar{a}^2 \bar{p}}) A_n P_{n|n} A_n. \tag{32}$$

Then, by using (12) and (17), we have:
(1) When $\gamma_n = 1$, it can be noted that $P_{n|n-1} C_n' = K_n (C_n P_{n|n-1} C_n' + R_n)$, one has

$$\begin{aligned} P_{n|n} &= P_{n|n-1} - K_n C_n P_{n|n-1} + P_{n|n-1} C_n' K_n' - P_{n|n-1} C_n' K_n' \\ &= P_{n|n-1} - K_n C_n P_{n|n-1} + K_n (C_n P_{n|n-1} C_n' + R_n) K_n' - P_{n|n-1} C_n' K_n' \\ &= (I_l - K_n C_n) P_{n|n-1} (I_l - K_n C_n)' + K_n R_n K_n'. \end{aligned}$$

(2) When $\gamma_n = 0$, $P_{n|n} = P_{n|n-1} - \beta(\delta) K_n C_n P_{n|n-1}$. Notice that $\beta(\delta) \in (0,1)$ leads to $\beta(\delta)^2 < \beta(\delta)$, then it follows:

$$\begin{aligned} P_{n|n} &= P_{n|n-1} - \beta(\delta) K_n C_n P_{n|n-1} + \beta(\delta) P_{n|n-1} C_n' K_n' - \beta(\delta) P_{n|n-1} C_n' K_n' \\ &= P_{n|n-1} - \beta(\delta) K_n C_n P_{n|n-1} + \beta(\delta) K_n C_n P_{n|n-1} (K_n C_n)' - \beta(\delta) P_{n|n-1} C_n' K_n' \\ &\quad + \beta(\delta) K_n R_n K_n' \\ &> P_{n|n-1} - \beta(\delta) K_n C_n P_{n|n-1} + \beta(\delta)^2 K_n C_n P_{n|n-1} (K_n C_n)' - \beta(\delta) P_{n|n-1} C_n' K_n' \\ &\quad + \beta(\delta) K_n R_n K_n' \\ &= (I_l - \beta(\delta) K_n C_n) P_{n|n-1} (I_l - \beta(\delta) K_n C_n)' + \beta(\delta) K_n R_n K_n' \\ &> (I_l - K_n C_n) P_{n|n-1} (I_l - K_n C_n)'. \end{aligned}$$

According to the results of the above two cases and the validity of (32), the desired inequality relation is obtained as:

$$P_{n+1|n} > (1 + \frac{\underline{q}}{\bar{a}^2 \bar{p}}) A_n (I_l - K_n C_n) P_{n|n-1} (I_l - K_n C_n)' A_n'.$$

Setting $A_n(I_l - K_n C_n) = A$, $(1 + \frac{\underline{q}}{\bar{a}^2 \bar{p}}) P_{n|n-1} = B$, $P_{n+1|n} = C$. From Lemma 6.1 in [15], suppose that $C - ABA' > 0$, then $B^{-1} - A' C^{-1} A > 0$. We have $(1 + \frac{\underline{q}}{\bar{a}^2 \bar{p}})^{-1} = 1 - \frac{\underline{q}}{\bar{a}^2 \bar{p} + \underline{q}} < 1$, one can let $d = \frac{\underline{q}}{\bar{a}^2 \bar{p} + \underline{q}}$, further we can obtain (31).

In the next part, let us show that

$$m_n' P_{n+1|n}^{-1} [2 A_n (I_l - \gamma_n K_n C_n) e_{n|n-1} + m_n] \leq \kappa_1 \| e_{n|n-1} \|^3. \tag{33}$$

We know that (17) holds, we derive $\|K_n\| \leq \frac{\bar{p}\bar{c}}{\underline{r}}$ under hypothetical conditions. Setting $\frac{\|x_n - \hat{x}_{n|n}\|}{\|x_n - \hat{x}_{n|n-1}\|} = \sigma$, according to (26) and (27), notice that (23) leads to

$$\begin{aligned} \|m_n\| &\leq \|\phi(x_n, \hat{x}_{n|n})\| + \gamma_n \frac{\bar{a}\bar{p}\bar{c}}{\underline{r}} \|\psi(x_n, \hat{x}_{n|n-1})\| \\ &\leq (\sigma^2 \varepsilon_\phi + \gamma_n \frac{\bar{a}\bar{p}\bar{c}}{\underline{r}} \varepsilon_\psi) \|e_{n|n-1}\|^2. \end{aligned}$$

Let $\kappa' = \sigma^2 \varepsilon_\phi + \gamma_n \frac{\bar{a}\bar{p}\bar{c}}{\underline{r}} \varepsilon_\psi$, thus $\|m_n\| \leq \kappa' \|e_{n|n-1}\|^2$. Since $\|e_{n|n-1}\| \leq \eta_\psi$, further calculation as follows:

$$m_n' P_{n+1|n}[2A_n(I_l - \gamma_n K_n C_n)e_{n|n-1} + m_n] \leq \kappa' \|e_{n|n-1}\|^2 \frac{1}{\underline{p}}[2\bar{a}(1 + \bar{p}\bar{c}\frac{1}{\underline{r}}\bar{c})\|e_{n|n-1}\| + \kappa' \eta_\psi \|e_{n|n-1}\|],$$

take $\kappa_1 = \kappa' \frac{1}{\underline{p}}[2\bar{a}(1 + \bar{p}\bar{c}\frac{1}{\underline{r}}\bar{c}) + \kappa' \eta_\psi]$ one can obtain (33).

The last part we have to prove

$$\mathbb{E}[s_n' P_{n+1|n}^{-1} s_n] \leq \kappa_2 \varepsilon. \tag{34}$$

Since $\omega_n$ and $v_n$ are uncorrelated, the expectation value of the crossterms containing both will vanish. Using (24), we have:

$$\begin{aligned}
\mathbb{E}[s_n' P_{n+1|n}^{-1} s_n] =& \mathbb{E}[\omega_n' P_{n+1|n}^{-1} \omega_n - \gamma_n \omega_n' P_{n+1|n}^{-1} A_n K_n v_n - \gamma_n v_n' K_n' A_n' P_{n+1|n}^{-1} \omega_n \\
&+ \gamma_n^2 v_n' A_n' K_n' P_{n+1|n}^{-1} K_n A_n v_n] \\
=& \mathbb{E}[\omega_n' P_{n+1|n}^{-1} \omega_n + \gamma_n v_n' A_n' K_n^v P_{n+1|n}^{-1} K_n A_n v_n] \\
\leq& \frac{1}{\underline{p}} \mathbb{E}[\omega_n' \omega_n] + \frac{1}{\underline{p}} (\frac{\bar{a}\bar{p}\bar{c}}{\underline{r}})^2 \mathbb{E}[v_n' v_n].
\end{aligned}$$

Let $\kappa_2 = \frac{\bar{q}}{\underline{p}} + \frac{\bar{r}}{\underline{p}} (\frac{\bar{a}\bar{p}\bar{c}}{\underline{r}})^2$, one can obtain (34). Finally, it follows with (31), (33) and (34) that (30) holds, which further implies that (28) is satisfied. To sum up, complete the proof. □

**Remark 2.** *By Theorem 2, the initial error and noise terms are bounded, then the estimation error remains bounded. The error bounds are quantified in the proof. State estimation is reliable if the numerically calculated values satisfy the required boundaries.*

### *4.2. Boundedness of the Error Covariance Matrices*

We present the following theorem to discuss the bounds of the error covariance matrices:

**Theorem 3.** *Assume that $m = n$, for all $n$, $C_n$ is invertible and its inverse satisfies $\|C_n^{-1}\| \leq \underline{c}^{-1}$, the arrival probability satisfies $\theta > 1 - \bar{a}^{-2}$, there are constants $\bar{p}, \underline{p} > 0$, such that*

$$\underline{p} I_l \leq P_{n+1|n+1} \leq P_{n+1|n}, \tag{35}$$

$$\mathbb{E}[P_{n+1|n+1}] \leq \mathbb{E}[P_{n+1|n}] \leq \bar{p} I_l. \tag{36}$$

**Proof of Theorem 3.** From (11)–(13), let $(1 - \gamma_n)\beta(\delta) + \gamma_n \triangleq \alpha_n(\gamma_n, \delta)$, then

$$\begin{aligned}
P_{n+1|n} &= A_n P_{n|n-1} A_n' - \alpha_n(\gamma_n, \delta) A_n K_n C_n P_{n|n-1} A_n' + Q_n \\
&= A_n P_{n|n-1} A_n' - \alpha_n(\gamma_n, \delta) A_n P_{n|n-1} C_n' (C_n P_{n|n-1} C_n' + R_n)^{-1} C_n P_{n|n-1} A_n' + Q_n \\
&= A_n [P_{n|n-1} - \alpha_n(\gamma_n, \delta) P_{n|n-1} C_n' (C_n P_{n|n-1} C_n' + R_n)^{-1} C_n P_{n|n-1}] A_n' + Q_n.
\end{aligned}$$

Let $C_n P_{n|n-1} = M$ and $R_n = N$, on the basis of Lemma 6.3 in [15], $(M+N)^{-1} \geq M^{-1} - M^{-1}NM^{-1}$, we obtain:

$$\begin{aligned} P_{n+1|n} \leq\ & A_n\{P_{n|n-1} - \alpha_n(\gamma_n,\delta)P_{n|n-1}C_n'[(C_nP_{n|n-1}C_n')^{-1} \\ & - (C_nP_{n|n-1}C_n')^{-1}R_n(C_nP_{n|n-1}C_n')^{-1}]C_nP_{n|n-1}\}A_n' + Q_n \\ \leq\ & [1-\alpha_n(\gamma_n,\delta)]A_nP_{n|n-1}A_n' + \alpha_n(\gamma_n,\delta)A_n(C_nR_nC_n')^{-1}A_n' + Q_n. \end{aligned}$$

Observing that $C_n^{-1}R_nC_n^{-T} \leq \bar{r}C_n^{-1}C_n^{-T} \leq \bar{r}\underline{c}^{-2}I_l$, $Q_n \leq \bar{q}I_l$, then:

$$P_{n+1|n} \leq [1-\alpha_n(\gamma_n,\delta)]A_nP_{n|n-1}A_n' + \alpha_n(\gamma_n,\delta)\bar{r}\underline{c}^{-2}A_nA_n' + \bar{q}I_n.$$

We now show that for all $n$, one has $\mathbb{E}[P_{n+1|n}] \leq p\sum_{j=0}^{n}[(1-\theta)\bar{a}^2]^jI_l$. We prove it by mathematical induction. First, notice that $(1-\theta)\bar{a}^2 < 1$ leads to $\theta > 1-\bar{a}^{-2}$, setting $p = \max\{\|P_{1|0}\|, \frac{\theta\bar{r}\bar{a}^2}{\underline{c}^2} + \bar{q}\}$, it follows:

$$\begin{aligned} \mathbb{E}[P_{2|1}] &\leq \mathbb{E}\{[1-\alpha_0(\gamma_0,\delta)]A_0P_{1|0}A_0' + \alpha_0(\gamma_0,\delta)\bar{r}\underline{c}^{-2}A_0A_0' + \bar{q}I_l\} \\ &\leq (1-\theta)\bar{a}^2P_{1|0} + \theta\bar{r}\underline{c}^{-2}\bar{a}^2I_l + \bar{q}I_l, \\ &\leq (1-\theta)\bar{a}^2pI_l + pI_l. \end{aligned}$$

Assume that $\mathbb{E}[P_{n|n-1}]$ satisfies the inequality $\mathbb{E}[P_{n|n-1}] \leq p\sum_{j=0}^{n}[(1-\theta)\bar{a}^2]^jI_l$, next we calculate $\mathbb{E}[P_{n+1|n}]$,

$$\begin{aligned} \mathbb{E}[P_{n+1|n}] &\leq \{[1-\alpha_n(\gamma_n,\delta)]A_nP_{n|n-1}A_n' + \alpha_n(\gamma_n,\delta)\bar{r}\underline{c}^{-2}A_nA_n' + \bar{q}I_l\} \\ &\leq (1-\theta)\mathbb{E}[\bar{a}^2P_{n|n-1}] + \theta\bar{r}\underline{c}^{-2}\bar{a}^2I_l + \bar{q}I_l \\ &\leq (1-\theta)\bar{a}^2p\sum_{j=0}^{n}[(1-\theta)\bar{a}^2]^jI_l + pI_l \\ &= p\sum_{j=0}^{n}[(1-\theta)\bar{a}^2]^jI_l, \end{aligned}$$

which completes the proof. □

## 5. Numerical Simulation

A nonlinear system is shown below:

$$\begin{aligned} x_{1,n} &= x_{1,n-1} + tx_{2,n-1} + \omega_{1,n-1}, \\ x_{2,n} &= -10t\sin x_{1,n-1} + (1-t)x_{2,n-1} + \omega_{2,n-1}, \\ y_{1,n} &= 2\sin\frac{x_{1,n}}{2} + v_{1,n}, \\ y_{2,n} &= \frac{x_{1,n}}{2} + v_{2,n}, \end{aligned}$$

where $x_n$ is the state and $y_n$ is the observation , $\omega_n$ and $v_n$ are noises. Covariance matrices $Q = \text{diag}(0.01, 0.0001)$ and $R = \text{diag}(0.1, 0.1)$. Let $P_{0|0} = \text{diag}(1,1)$ and the initial state be $[1,0]'$, filtering cycle $t = 0.05s$, and take $\delta = 1.70$. In Figure 2, we can see the results of the estimation by the Theorem 1 and the event trigger time, and the estimation value is closer to the real value than the measurement value. Compared with the measurement value, the estimation value result is more accurate. We trigger 43 times out of 200 moments, saving the communication energy. In the upper part of Figure 3, we use the Monte Carlo method to obtain the mean of the results of 10,000 runs. Traces of $P_{n|n-1}$ and $P_{n|n}$ are shown in the middle part of Figure 3. It shows the boundedness of the estimation error covariance and real estimation error, which verifies the effectiveness of Theorem 1 in this paper. It can be seen from the above simulation results that the designed nonlinear filter has a better estimation effect on nonlinear systems.

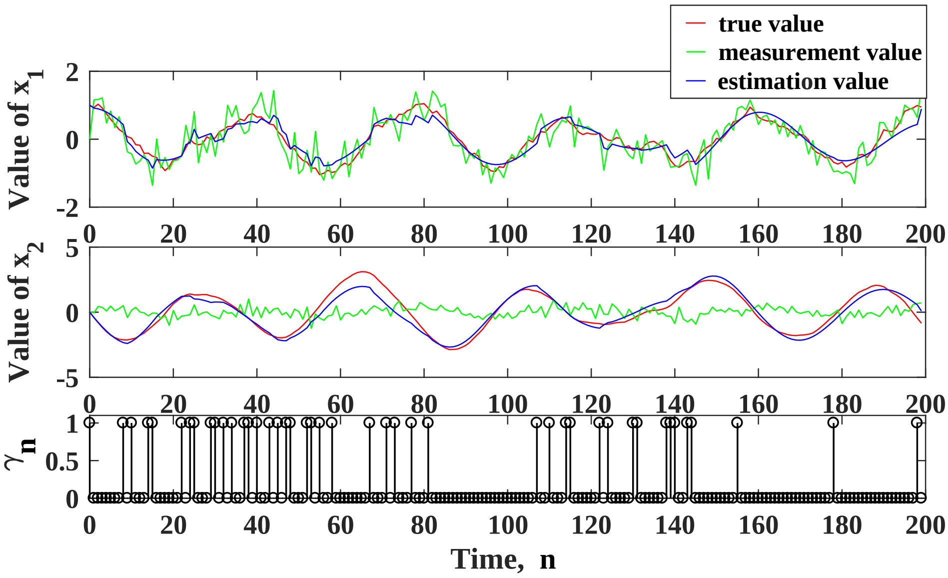


**Figure 2.** The actual state, estimation and event trigger time.

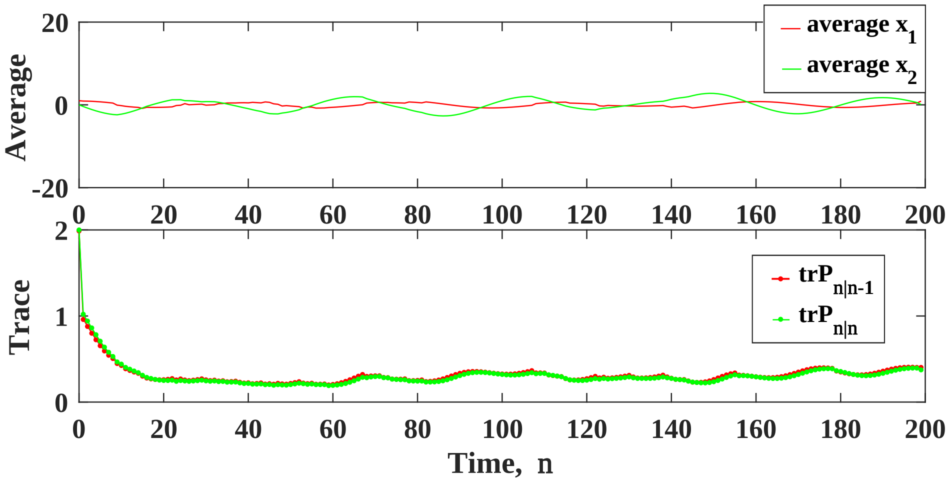


**Figure 3.** The boundedness of the estimation error and covariance matrix.

## 6. Conclusions

Aiming at the event-triggered mechanism in a nonlinear networked system, a nonlinear filtering algorithm based on EKF is proposed. The event-triggered data transmission scheme reduces the quantity of measurement transmissions on the bandwidth-limited network. The convergence of the designed filter is also analyzed, and sufficient conditions are established to ensure the convergence of the nonlinear filter. Finally, an example is given to illustrate the feasibility of the method.

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## Abbreviations

The following abbreviations are used in this manuscript:

| | |
|---|---|
| WSN | wireless sensor network |
| KF | Kalman filter |
| MMSE | minimum mean square error |
| EKF | extended Kalman filter |
| UKF | untraced Kalman filter |
| ETC | event-triggered control |

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