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# A Family of Holomorphic and $m$-Fold Symmetric Bi-Univalent Functions Endowed with Coefficient Estimate Problems

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**Abstract:** This paper presents a new general subfamily $\mathcal{N}^{u,v}_{\Sigma_m}(\eta,\mu,\gamma,\ell)$ of the family $\Sigma_m$ that contains holomorphic normalized $m$-fold symmetric bi-univalent functions in the open unit disk $\mathbb{D}$ associated with the Ruscheweyh derivative operator. For functions belonging to the family introduced here, we find estimates of the Taylor–Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$, and the consequences of the results are discussed. The current findings both extend and enhance certain recent studies in this field, and in specific scenarios, they also establish several connections with known results.

**Keywords:** holomorphic functions; univalent functions; $m$-fold symmetric bi-univalent functions; bi-starlike functions; bi-convex functions; Ruscheweyh derivative operator

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## 1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be an open unit disk in the complex plane and $\mathcal{A}$ be a collection of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{D}), \tag{1}$$

which are holomorphic in $\mathbb{D}$ together with a normalization given by

$$f(0) = f'(0) - 1 = 0.$$

The Hadamard product $f(z) * l(z)$ of $f(z)$ and $l(z)$ is defined by

$$(f * l)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (l * f)(z) \quad (z \in \mathbb{D}),$$

where the function $l(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is also holomorphic in $\mathbb{D}$.

The Ruscheweyh derivative operator $\mathcal{R}^{\ell}:\mathcal{A}\to\mathcal{A}$ (see [1]) of $f\in\mathcal{A}$ is defined as

$$\mathcal{R}^{\ell}f(z)=\frac{z\left(z^{\ell-1}f(z)\right)^{(\ell)}}{\ell!}=\frac{z}{(1-z)^{\ell+1}}*f(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\ell+n)}{\Gamma(n)\Gamma(\ell+1)}a_nz^n,$$
$$(\ell\in\mathbb{N}_0=\{0,1,2,\dots\}=\mathbb{N}\cup\{0\},z\in\mathbb{D}).$$

Denote the sub-collection of $\mathcal{A}$ by $\mathcal{S}$, consisting of univalent functions in $\mathbb{D}$, and consider the sub-collection $\mathcal{P}$ of functions

$$p(z)=1+\sum_{n=1}^{\infty}p_nz^n\quad(z\in\mathbb{D}),\tag{2}$$

that are holomorphic in $\mathbb{D}$ and the real part, $\Re(p(z))$, is positive.

According to the Koebe 1/4 Theorem (see [2]), the image of $\mathbb{D}$ under any univalent function consists of a disk of radius 1/4. As a consequence, every function $f\in\mathcal{S}$ has an inverse $f^{-1}$ such that

$$f^{-1}(f(z))=z\quad(z\in\mathbb{D})$$

and

$$f\left(f^{-1}(w)\right)=w\quad\left(|w|<r_0(f),r_0(f)\geq\frac{1}{4}\right).$$

The inverse of the function $f(z)$ has a series expansion in some disk about the origin of the following form:

$$f^{-1}(w)=w+\sum_{n=2}^{\infty}b_nw^n\tag{3}$$

A univalent function $f(z)$ in the neighborhood of the origin and its inverse $f^{-1}(w)$ satisfy the following condition:

$$f\left(f^{-1}(w)\right)=w$$

or, equivalently,

$$w=f^{-1}(w)+\sum_{n=2}^{\infty}a_n\left[f^{-1}(w)\right]^n.\tag{4}$$

Using (1) and (3) in (4), we get

$$g(w)=f^{-1}(w)=w-a_2w^2+\left(2a_2^2-a_3\right)w^3-\left(5a_2^3-5a_2a_3+a_4\right)w^4+\cdots.\tag{5}$$

If a function $f\in\mathcal{A}$ and its inverse are both univalent on $\mathbb{D}$, then $f$ is called a bi-univalent function. Denote the family of all bi-univalent functions in $\mathbb{D}$ by $\Sigma$.

Lewin [3] conducted a study on the family $\Sigma$ of bi-univalent functions and discovered that $|a_2|<1.51$ for the functions belonging to the family $\Sigma$. Later, Brannan and Clunie [4] proposed the conjecture that $|a_2|\leq\sqrt{2}$. Subsequently, Netanyahu [5] demonstrated that $\max|a_2|=\frac{4}{3}$ for $f\in\Sigma$. To explore various fascinating examples of $f\in\Sigma$, refer to the seminal work on this area by Srivastava et al. [6], which has revitalized the study of $f\in\Sigma$ functions in recent years.

Srivastava et al. [6] showed that the family $\Sigma$ is nonempty by providing some explicit examples, including the following function:

$$\frac{1}{2}\log\left(\frac{1+z}{1-z}\right),\quad-\log(1-z)\quad\text{and}\quad\frac{z}{1-z}$$

whose inverses are

$$\frac{e^{2w}-1}{e^{2w}+1},\quad\frac{e^{w}-1}{e^{w}}\text{ and }\frac{w}{1+w},$$

respectively. It worth noting that the Koebe function is not a member of $\Sigma$. Hence, $\Sigma$ is a proper subfamily of $\mathcal{A}$. In fact, this pioneering work of Srivastava et al. [6] actually revived the study of analytic and biunivalent functions in recent years. It was followed by a remarkably huge flood of sequels on the subject.

Let $0 \leq \vartheta < 1$. Brannan and Taha [7] introduced specific subfamilies of $\Sigma$, analogous to the well-known subfamilies, starlike functions $\mathcal{S}^*(\vartheta)$, and convex functions $\mathcal{K}(\vartheta)$ of order $\theta$. A function $f \in \Sigma$ is in the family $\mathcal{S}^*_\Sigma(\vartheta)$ of bi-starlike functions of order $\vartheta$ if both $f$ and its inverse are starlike functions of order $\vartheta$, or is in the family $\mathcal{K}_\Sigma(\vartheta)$ of bi-convex functions of order $\vartheta$ if both $f$ and its inverse are convex functions of order $\vartheta$. Moreover, for $0 < \vartheta \leq 1$, the function $f \in \mathcal{A}$ is classified as a strongly bi-starlike function, $\mathcal{S}^*_\Sigma[\vartheta]$ (see [7,8]), if it satisfies:

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\vartheta\pi}{2} \quad \text{and} \quad \left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\vartheta\pi}{2},$$

where $g = f^{-1}$ is defined by (5).

Recently, studying the family $\Sigma$ and deriving non-sharp bounds on $|a_2|$ and $|a_3|$, where $a_2$ and $a_3$ are the initial Taylor–Maclaurin coefficients, have become an active area of research. In particular, the pioneering work by Srivastava et al. [6] has crucially advanced the study of certain subfamilies within $\Sigma$ and identified constraints on $|a_2|$ and $|a_3|$. A substantial number of subsequent works have been published in the literature, building upon the groundbreaking research by Srivastava et al. [6] and focusing on coefficient problems for different subfamilies of $\Sigma$ (see, for example, [9,10] and the above-cited works). However, the general coefficient estimate bounds on $|a_n|(n \in \{4,5,6,\ldots\})$ for functions $f$ in the family $\Sigma$ remain an unsolved problem.

For $f \in \mathcal{S}$, the function

$$h(z) = (f(z^m))^{\frac{1}{m}}, \quad (m \in \mathbb{N}, z \in \mathbb{D}) \tag{6}$$

is univalent and maps $\mathbb{D}$ into an $m$-fold symmetric region. A function $f \in \mathcal{A}$ is called $m$-fold symmetric (see [11]) if it is of the form:

$$f(z) = z + \sum_{n=1}^{\infty} a_{nm+1} z^{nm+1}, \quad (m \in \mathbb{N}, z \in \mathbb{D}). \tag{7}$$

The family of all $m$-fold symmetric functions is denoted by $\mathcal{A}_m$. For a function $f \in \mathcal{A}_m$ defined by (7), analogous to the Ruscheweyh derivative operator, the $m$-fold Ruscheweyh derivative $D^\ell : \mathcal{A}_m \to \mathcal{A}_m$ is defined as follows (see [12]):

$$D^\ell f(z) = z + \sum_{n=1}^{\infty} \frac{\Gamma(\ell+n+1)}{\Gamma(n+1)\Gamma(\ell+1)} a_{nm+1} z^{nm+1}, \quad (\ell \in \mathbb{N}_0, m \in \mathbb{N}, z \in \mathbb{D}).$$

Let $\delta_m$ denote the family of $m$-fold symmetric univalent functions in $\mathbb{D}$ normalized by (7). Then, the functions $f \in \mathcal{S}$ are one-fold symmetric. As stated by Koepf [11], the $m$-fold symmetric $p$ in $\mathcal{P}$ has the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_{nm} z^{nm}, \quad (m \in \mathbb{N}, z \in \mathbb{D}). \tag{8}$$

Recently, Srivastava et al. [13] defined the family of $m$-fold symmetric bi-univalent functions $\Sigma_m$ analogous to the family $\Sigma$, and the inverse of functions $f$ given by (7) is specified as follows:

$$g(w) = w - a_{m+1}w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right]w^{2m+1} - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right]w^{3m+1} + \cdots. \tag{9}$$

For $m = 1$, the function in (9) coincides with (5) of the family $\Sigma$. Some examples of $m$-fold symmetric bi-univalent functions are given below:

$$\left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)\right]^{\frac{1}{m}}, \quad [-\log(1-z^m)]^{\frac{1}{m}} \quad \text{and} \quad \left[\frac{z^m}{1-z^m}\right]^{\frac{1}{m}}$$

with inverses of

$$\left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{\frac{1}{m}}, \quad \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}} \text{ and } \left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}},$$

respectively.

Recent research has been dedicated to analyzing the functions in the family $\Sigma_m$ and obtaining non-sharp bounds on $|a_{m+1}|$ and $|a_{2m+1}|$, where $a_{m+1}$ and $a_{m+2}$ are the initial Taylor–Maclaurin coefficients. In reality, Srivastava et al. [13] have greatly advanced the research on many subfamilies of the family $\Sigma_m$ and obtained restrictions on $|a_{m+1}|$ and $|a_{2m+1}|$ in recent years. Later on, some scholars followed them (see, for example, [14,15] and the above-cited works).

Motivated by the aforementioned works, the primary goal of this study is to propose a formula to determine the coefficients of the functions for the family $\Sigma_m$ utilizing the residue of calculus. As an example, we construct estimates of the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions belonging to a generic subfamily $\mathcal{N}_{\Sigma_m}^{u,v}(\eta, \mu, \gamma, \ell)$ of $\Sigma_m$ in $\mathbb{D}$, and additional links to previously known results are made. Furthermore, by sufficiently specializing the parameters, some consequences of this family are demonstrated.

## 2. The Family $\mathcal{N}_{\Sigma_m}^{u,v}(\lambda, \mu, \gamma, \ell)$ and Its Associated Coefficient Estimates

In this section, the following general family $\mathcal{N}_{\Sigma_m}^{u,v}(\eta, \mu, \gamma, \ell)$ is introduced and investigated.

**Definition 1.** *A function $f \in \Sigma_m$ given by (7) belongs to the family*

$$\mathcal{N}_{\Sigma_m}^{u,v}(\eta, \mu, \gamma, \ell) \ (\eta \geq 1, \mu \geq 0, \gamma \geq 0, \ell \in \mathbb{N}_0, m \in \mathbb{N} \text{ and } u, v : \mathbb{D} \to \mathbb{C})$$

*if the following conditions are satisfied:*

$$\min\{\Re(u(z)), \Re(v(z))\} > 0 \quad \text{and} \quad u(0) = v(0) = 1, \tag{10}$$

$$(1-\eta)\left(\frac{D^\ell f(z)}{z}\right)^\mu + \eta\left(D^\ell f(z)\right)'\left(\frac{D^\ell f(z)}{z}\right)^{\mu-1} + \frac{\gamma(\mu+2\eta)}{1+2\eta}z\left(D^\ell f(z)\right)'' \in u(\mathbb{D}), \tag{11}$$

*and*

$$(1-\eta)\left(\frac{D^\ell g(w)}{w}\right)^\mu + \eta\left(D^\ell g(w)\right)'\left(\frac{D^\ell g(w)}{w}\right)^{\mu-1} + \frac{\gamma(\mu+2\eta)}{1+2\eta}w\left(D^\ell g(w)\right)'' \in v(\mathbb{D}), \tag{12}$$

*where $z, w \in \mathbb{D}$, $u$ and $v$, holomorphic in $\mathbb{D}$, are defined by the expansion (8), and the function $g = f^{-1}$ is defined by (9).*

Many choices of the functions $u$ and $v$ can be used to create attractive subfamilies of the functions that are holomorphic in the family $\mathcal{A}_m$.

**Example 1.** *If we let*

$$u(z) = v(z) = \left(\frac{1 - z^m}{1 + z^m}\right)^\alpha; \quad 0 < \alpha \leq 1,$$

*it can be seen that the functions $u(z)$ and $v(z)$ satisfy the conditions of Definition 1. Thus, if $f \in \mathcal{N}_{\Sigma_m}^{u,v}(\eta, \mu, \gamma, \ell) \equiv \mathcal{N}_{\Sigma_m}(\eta, \mu, \gamma, \ell; \alpha)$, then $f \in \Sigma_m$ and*

$$\left|\arg\left\{(1-\eta)\left(\frac{D^\ell f(z)}{z}\right)^\mu + \eta\left(D^\ell f(z)\right)'\left(\frac{D^\ell f(z)}{z}\right)^{\mu-1} + \frac{\gamma(\mu+2\eta)}{1+2\eta} z\left(D^\ell f(z)\right)''\right\}\right| < \frac{\alpha\pi}{2}$$

*and*

$$\left|\arg\left\{(1-\eta)\left(\frac{D^\ell g(w)}{w}\right)^\mu + \eta\left(D^\ell g(w)\right)'\left(\frac{D^\ell g(w)}{w}\right)^{\mu-1} + \frac{\gamma(\mu+2\eta)}{1+2\eta} w\left(D^\ell g(w)\right)''\right\}\right| < \frac{\alpha\pi}{2},$$

*where the function $g = f^{-1}$ is defined by (9).*

This means that

$$\mathcal{N}_{\Sigma_m}(\eta, \mu, \gamma, \ell; \alpha) \subset \mathcal{N}_{\Sigma_m}^{u,v}(\eta, \mu, \gamma, \ell)$$

and the family $\mathcal{N}_{\Sigma_m}^{u,v}(\eta, \mu, \gamma, \ell)$ is not empty.

**Example 2.** *If we set*

$$u(z) = v(z) = \frac{1 - (1-2\beta)z^m}{1 + z^m}; \quad 0 \leq \beta < 1,$$

*then the conditions of Definition 1 are satisfied for both functions $u(z)$ and $v(z)$. Thus, if $f \in \mathcal{N}_{\Sigma_m}^{u,v}(\eta, \mu, \gamma, \ell) \equiv \mathcal{N}_{\Sigma_m}(\eta, \mu, \gamma, \ell; \beta)$, then $f \in \Sigma_m$,*

$$\Re\left\{(1-\eta)\left(\frac{D^\ell f(z)}{z}\right)^\mu + \eta\left(D^\ell f(z)\right)'\left(\frac{D^\ell f(z)}{z}\right)^{\mu-1} + \frac{\gamma(\mu+2\eta)}{1+2\eta} z\left(D^\ell f(z)\right)''\right\} > \beta$$

*and*

$$\Re\left\{(1-\eta)\left(\frac{D^\ell g(w)}{w}\right)^\mu + \eta\left(D^\ell g(w)\right)'\left(\frac{D^\ell g(w)}{w}\right)^{\mu-1} + \frac{\gamma(\mu+2\eta)}{1+2\eta} w\left(D^\ell g(w)\right)''\right\} > \beta,$$

*where the function $g = f^{-1}$ is defined by (9).*

This means that

$$\mathcal{N}_{\Sigma_m}(\eta, \mu, \gamma, \ell; \beta) \subset \mathcal{N}_{\Sigma_m}^{u,v}(\eta, \mu, \gamma, \ell).$$

It can be seen that, for symmetric one-fold bi-univalent functions, by specializing $\eta, \mu, \gamma$ and $\ell$, we get several known subfamilies of $\Sigma$ recently investigated by various authors. Let us present some examples.

**Example 3.** *Let $m = 1$ and $\ell = 0$. Then, the family $\mathcal{N}_{\Sigma_m}^{u,v}(\eta, \mu, \gamma, \ell)$ reduces to the family $\mathcal{B}_{\Sigma}^{u,v}(\eta, \mu, \gamma)$ inspected by Bulut [16], which is defined by the requirement that $f \in \Sigma$,*

$$\min\{\Re(u(z)), \Re(v(z))\} > 0 \text{ and } u(0) = v(0) = 1,$$

$$(1-\eta)\left(\frac{f(z)}{z}\right)^{\mu}+\eta(f(z))'\left(\frac{f(z)}{z}\right)^{\mu-1}+\frac{\gamma(\mu+2\eta)}{1+2\eta}z(f(z))''\in u(\mathbb{D}),$$

*and*

$$(1-\eta)\left(\frac{g(w)}{w}\right)^{\mu}+\eta(g(w))'\left(\frac{g(w)}{w}\right)^{\mu-1}+\frac{\gamma(\mu+2\eta)}{1+2\eta}w(g(w))''\in v(\mathbb{D}),$$

*where $u,v:\mathbb{D}\to\mathbb{C}$, holomorphic in $\mathbb{D}$, are given by (2), and the function $g=f^{-1}$ is defined by (5).*

**Example 4.** *Let $m=1,\gamma=0$ and $\ell=0$. Then, the family $\mathcal{N}_{\Sigma_m}^{u,v}(\eta,\mu,\gamma,\ell)$ reduces to the family $\mathcal{N}_{\Sigma}^{u,v}(\eta,\mu)$ considered by Srivastava et al. [17], which is defined by the requirement that $f\in\Sigma$,*

$$\min\{\Re(u(z)),\Re(v(z))\}>0 \text{ and } u(0)=v(0)=1,$$

$$(1-\eta)\left(\frac{f(z)}{z}\right)^{\mu}+\eta(f(z))'\left(\frac{f(z)}{z}\right)^{\mu-1}\in u(\mathbb{D}),$$

*and*

$$(1-\eta)\left(\frac{g(w)}{w}\right)^{\mu}+\eta(g(w))'\left(\frac{g(w)}{w}\right)^{\mu-1}\in v(\mathbb{D}),$$

*where $u,v:\mathbb{D}\to\mathbb{C}$, holomorphic in $\mathbb{D}$, are given by (2), and the function $g=f^{-1}$ is defined by (5).*

**Example 5.** *Let $m=1,\mu=1,\gamma=0$ and $\ell=0$. Then, the family $\mathcal{N}_{\Sigma_m}^{u,v}(\eta,\mu,\gamma,\ell)$ reduces to the family $\mathcal{B}_{\Sigma}^{u,v}(\eta)$ studied by Xu et al. [18], which is defined by the requirement that $f\in\Sigma$,*

$$\min\{\Re(u(z)),\Re(v(z))\}>0 \text{ and } u(0)=v(0)=1,$$

$$(1-\eta)\frac{f(z)}{z}+\eta f'(z)\in u(\mathbb{D}),$$

*and*

$$(1-\eta)\frac{g(w)}{w}+\eta g'(w)\in v(\mathbb{D}),$$

*where $u,v:\mathbb{D}\to\mathbb{C}$, holomorphic in $\mathbb{D}$, are given by (2), and the function $g=f^{-1}$ is defined by (5).*

**Example 6.** *Let $m=1,\eta=1,\mu=0,\gamma=0$ and $\ell=0$. Then, the family $\mathcal{N}_{\Sigma_m}^{u,v}(\eta,\mu,\gamma,\ell)$ reduces to the family $\mathcal{B}_{\Sigma}^{u,v}$ considered by Bulut [19], which is defined by the requirement that $f\in\Sigma$,*

$$\min\{\Re(u(z)),\Re(v(z))\}>0 \text{ and } u(0)=v(0)=1,$$

$$\frac{zf'(z)}{f(z)}\in u(\mathbb{D}),$$

*and*

$$\frac{wg'(w)}{g(w)}\in v(\mathbb{D}),$$

*where $u,v:\mathbb{D}\to\mathbb{C}$, holomorphic in $\mathbb{D}$, are given by (2), and the function $g=f^{-1}$ is defined by (5).*

**Example 7.** *Let $m=1,\eta=1,\mu=1,\gamma=0$ and $\ell=0$. Then, the family $\mathcal{N}_{\Sigma_m}^{u,v}(\eta,\mu,\gamma,\ell)$ reduces to the family $\mathcal{H}_{\Sigma}^{u,v}$ studied by Xu et al. [20], which is defined by the requirement that $f\in\Sigma$,*

$$\min\{\Re(u(z)),\Re(v(z))\}>0 \text{ and } u(0)=v(0)=1,$$

$$f'(z)\in u(\mathbb{D}),$$

*and*

$$g'(w) \in v(\mathbb{D}),$$

*where $u, v : \mathbb{D} \to \mathbb{C}$, holomorphic in $\mathbb{D}$, are given by (2), and the function $g = f^{-1}$ is defined by (5).*

Now, we are able to express bounds for $|a_{m+1}|$ and $|a_{2m+1}|$ for the subfamily $\mathcal{N}^{u,v}_{\Sigma_m}(\eta, \mu, \gamma, \ell)$ of the family $\Sigma_m$.

**Theorem 1.** *Let $f \in \mathcal{N}^{u,v}_{\Sigma_m}(\eta, \mu, \gamma, \ell)$ be given by (7). Then,*

$$|a_{m+1}| \leq \min\left\{ \frac{\left|u^{(m)}(0)\right|}{m!(\ell+1)\varphi_1}, \sqrt{\frac{\left|u^{(2m)}(0)\right| + \left|v^{(2m)}(0)\right|}{m(2m-1)!(\ell+1)[2\varphi_2 + \varphi_3 + 2\varphi_4]}} \right\}, \tag{13}$$

*and*

$$\begin{aligned} |a_{2m+1}| \leq \min\Bigg\{ & \frac{(m+1)\left|u^{(m)}(0)\right|^2}{2[m!(\ell+1)\varphi_1]^2} + \frac{\left|u^{(2m)}(0)\right| + \left|v^{(2m)}(0)\right|}{(2m)!(\ell+1)(\ell+2)\varphi_5}, \\ & \frac{[\varphi_2 + \varphi_3 + 2\varphi_4]\left|u^{(2m)}(0)\right| + (\ell+1)|1-\mu|(\mu+2\eta m)\left|v^{(2m)}(0)\right|}{m(2m-1)!(\ell+1)(\ell+2)\varphi_5[2\varphi_2 + \varphi_3 + 2\varphi_4]} \Bigg\}, \end{aligned} \tag{14}$$

*where*

$$\varphi_1 := \mu + \eta m + m(m+1)\frac{\gamma(\mu+2\eta)}{1+2\eta}, \tag{15}$$
$$\varphi_2 := (\mu-1)(\ell+1)(\mu+2\eta m), \tag{16}$$
$$\varphi_3 := (m+1)(\ell+2)(\mu+2\eta m), \tag{17}$$
$$\varphi_4 := m(m+1)(2m+1)(\ell+2)\frac{\gamma(\mu+2\eta)}{1+2\eta}, \tag{18}$$

*and*

$$\varphi_5 := \mu + 2\eta m + 2m(2m+1)\frac{\gamma(\mu+2\eta)}{1+2\eta}. \tag{19}$$

**Proof.** It is implied by (10) and (11) that

$$(1-\eta)\left(\frac{D^\ell f(z)}{z}\right)^\mu + \eta\left(D^\ell f(z)\right)'\left(\frac{D^\ell f(z)}{z}\right)^{\mu-1} + \frac{\gamma(\mu+2\eta)}{1+2\eta} z\left(D^\ell f(z)\right)'' = u(z), \tag{20}$$

and

$$(1-\eta)\left(\frac{D^\ell g(w)}{w}\right)^\mu + \eta\left(D^\ell g(w)\right)'\left(\frac{D^\ell g(w)}{w}\right)^{\mu-1} + \frac{\gamma(\mu+2\eta)}{1+2\eta} w\left(D^\ell g(w)\right)'' = v(w), \tag{21}$$

where $u(z)$ and $v(w)$ satisfy the conditions of (10) and have the series representations

$$u(z) = 1 + u_m z^m + u_{2m} z^{2m} + u_{3m} z^{3m} + \cdots, \tag{22}$$

and

$$v(w) = 1 + v_m w^m + v_{2m} w^{2m} + v_{3m} w^{3m} + \cdots. \tag{23}$$

Substituting the expansions (22) and (23) into (20) and (21), respectively, yields

$$(\ell+1)\left[\mu+\eta m+m(m+1)\frac{\gamma(\mu+2\eta)}{1+2\eta}\right]a_{m+1}=u_m, \tag{24}$$

$$\begin{aligned}(\ell+1)(\ell+2)&\left[\frac{1}{2}(\mu+2\eta m)+m(2m+1)\frac{\gamma(\mu+2\eta)}{1+2\eta}\right]a_{2m+1}\\&+\frac{1}{2}(\ell+1)^2(\mu\quad-1)(\mu+2\eta m)a_{m+1}^2=u_{2m},\end{aligned} \tag{25}$$

$$-(\ell+1)\left[\mu+\eta m+m(m+1)\frac{\gamma(\mu+2\eta)}{1+2\eta}\right]a_{m+1}=v_m, \tag{26}$$

and

$$\begin{aligned}-(\ell+1)(\ell+2)&\left[\frac{1}{2}(\mu+2\eta m)+m(2m+1)\frac{\gamma(\mu+2\eta)}{1+2\eta}\right]a_{2m+1}\\&+(\ell+1)\left[\frac{1}{2}(\ell+1)(\mu-1)(\mu+2\eta m)+\frac{1}{2}(m+1)(\ell+2)(\mu+2\eta m)\right.\\&\left.+m(m+1)(2m+1)(\ell+2)\frac{\gamma(\mu+2\eta)}{1+2\eta}\right]a_{m+1}^2=v_{2m}.\end{aligned} \tag{27}$$

In light of (24) and (26), we conclude that

$$u_m=-v_m, \tag{28}$$

and

$$2(\ell+1)^2\varphi_1^2a_{m+1}^2=u_m^2+v_m^2 \tag{29}$$

where $\varphi_1$ is given by (15).

If the equalities (25) and (27) are added, we obtain the relation

$$(\ell+1)\left[\varphi_2+\frac{1}{2}\varphi_3+\varphi_4\right]a_{m+1}^2=u_{2m}+v_{2m} \tag{30}$$

where $\varphi_2$, $\varphi_3$ and $\varphi_4$ are given by (16), (17) and (18), respectively.

Therefore, from (29) and (30), we have

$$a_{m+1}^2=\frac{u_m^2+v_m^2}{2(\ell+1)^2\varphi_1^2}, \tag{31}$$

and

$$a_{m+1}^2=\frac{2(u_{2m}+v_{2m})}{(\ell+1)[2\varphi_2+\varphi_3+2\varphi_4]}, \tag{32}$$

respectively. Therefore, taking the absolute value of (31) and (32), and using (28), we deduce that

$$|a_{m+1}|^2\le\frac{\left|u^{(m)}(0)\right|^2}{[m!(\ell+1)\varphi_1]^2}, \tag{33}$$

and

$$|a_{m+1}|^2\le\frac{\left|u^{(2m)}(0)\right|+\left|v^{(2m)}(0)\right|}{m(2m-1)!(\ell+1)[2\varphi_2+\varphi_3+2\varphi_4]}, \tag{34}$$

respectively. Thus, we have the desired result as asserted in (13).

Then, to obtain $|a_{2m+1}|$, subtract (27) from (25),

$$(\ell+1)(\ell+2)\varphi_5 a_{2m+1} - (\ell+1)\left[\frac{1}{2}\varphi_3 + \varphi_4\right]a_{m+1}^2 = u_{2m} - v_{2m} \tag{35}$$

where $\varphi_5$ is given by (19).

Now, putting the value of $a_{m+1}^2$ from (29) into (35), it follows that

$$a_{2m+1} = \frac{(m+1)\left(u_m^2 + v_m^2\right)}{4(\ell+1)^2\varphi_1^2} + \frac{u_{2m} - v_{2m}}{(\ell+1)(\ell+2)\varphi_5}. \tag{36}$$

Therefore, taking the absolute value of (36) and using the relation given by (28), we deduce that

$$|a_{2m+1}| \leq \frac{(m+1)\left|u^{(m)}(0)\right|^2}{2[m!(\ell+1)\varphi_1]^2} + \frac{\left|u^{(2m)}(0)\right| + \left|v^{(2m)}(0)\right|}{(2m)!(\ell+1)(\ell+2)\varphi_5}. \tag{37}$$

By putting the value of $a_{m+1}^2$ from (30) into (35), we obtain

$$a_{2m+1} = \frac{[\varphi_2 + \varphi_3 + 2\varphi_4]u_{2m} + (\ell+1)(1-\mu)(\mu+2\eta m)v_{2m}}{(\ell+1)(\ell+2)\varphi_5\left[\varphi_2 + \frac{1}{2}\varphi_3 + \varphi_4\right]}. \tag{38}$$

Therefore, taking the absolute value of (38), we conclude the following bound

$$|a_{2m+1}| \leq \frac{[\varphi_2 + \varphi_3 + 2\varphi_4]\left|u^{(2m)}(0)\right| + (\ell+1)|1-\mu|(\mu+2\eta m)\left|v^{(2m)}(0)\right|}{m(2m-1)!(\ell+1)(\ell+2)\varphi_5[2\varphi_2 + \varphi_3 + 2\varphi_4]}. \tag{39}$$

Finally, from (37) and (39), we get the relevant estimate as asserted in (14). This completes the proof. □

## 3. Corollaries and Consequences

If we put

$$u(z) = v(z) = \left(\frac{1-z^m}{1+z^m}\right)^{\alpha}; \quad (0 < \alpha \leq 1),$$

in Theorem 1, then Corollary 1 can be obtained.

**Corollary 1.** *Let $f(z) \in \mathcal{N}_{\Sigma_m}(\eta, \mu, \gamma, \ell; \alpha)$ be of the form (7). Then,*

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{(\ell+1)\left[(\ell+1)(1-\alpha)\varphi_1^2 + \alpha\left[\varphi_2 + \frac{1}{2}\varphi_3 + \varphi_4\right]\right]}},$$

*and*

$$|a_{2m+1}| \leq \frac{4\alpha^2}{(\ell+1)^2\varphi_1^2} + \frac{4\alpha}{(\ell+1)(\ell+2)\varphi_5},$$

*where $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ and $\varphi_5$ are given by (15), (16), (17), (18) and (19), respectively.*

If we set

$$u(z) = v(z) = \frac{1-(1-2\beta)z^m}{1+z^m}; \quad (0 \leq \beta < 1),$$

in Theorem 1, then Corollary 2 can be obtained.

**Corollary 2.** *Let $f(z) \in \mathcal{N}_{\Sigma_m}(\eta, \mu, \gamma, \ell; \beta)$ be of the form (7). Then,*

$$|a_{m+1}| \leq \min\left\{ \frac{2(1-\beta)}{(\ell+1)\varphi_1}, \sqrt{\frac{8(1-\beta)}{(\ell+1)[2\varphi_2+\varphi_3+2\varphi_4]}} \right\},$$

*and*

$$|a_{2m+1}| \leq \min\left\{ \frac{4(1-\beta)^2}{(\ell+1)^2\varphi_1^2} + \frac{4(1-\beta)}{(\ell+1)(\ell+2)\varphi_5}, \frac{8(1-\beta)[\varphi_2+\varphi_3+2\varphi_4]}{(\ell+1)(\ell+2)\varphi_5[2\varphi_2+\varphi_3+2\varphi_4]} \right\},$$

*where $\varphi_1$, $\varphi_2$, $\varphi_3$, $\varphi_4$ and $\varphi_5$ are given by (15), (16), (17), (18) and (19), respectively.*

By letting

$$u(z) = v(z) = \left(\frac{1-z}{1+z}\right)^{\alpha}; \quad (0 < \alpha \leq 1),$$

in Theorem 1 for the subfamily $\mathcal{N}_{\Sigma}^{u,v}(\eta,\mu,\gamma,\ell)$ of the family $\Sigma := \Sigma_1$ that contains normalized holomorphic and bi-univalent functions, then Corollary 3 can be derived.

**Corollary 3.** *Let $f(z) \in \mathcal{N}_{\Sigma}(\eta,\mu,\gamma,\ell;\alpha)$ be of the form (1). Then,*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(1-\alpha)(\ell+1)[(\ell+1)[\mu+\eta+2(\mu+2\eta)\tau]^2+\alpha(\mu+2\eta)[1+\mu(\ell+1)+6(\ell+2)\tau]]}},$$

*and*

$$|a_3| \leq \frac{4\alpha^2}{(\ell+1)^2[\mu+\eta+2(\mu+2\eta)\tau]^2} + \frac{4\alpha}{(\ell+1)(\ell+2)(\mu+2\eta)(1+6\tau)}$$

*where*

$$\tau := \frac{\gamma}{1+2\eta}$$

By putting

$$u(z) = v(z) = \frac{1-(1-2\beta)z}{1+z}; \quad (0 \leq \beta < 1)$$

in Theorem 1 for the subfamily $\mathcal{N}_{\Sigma}^{u,v}(\eta,\mu,\gamma,\ell)$ of the family $\Sigma$ that contains normalized holomorphic and bi-univalent functions, then Corollary 4 can be derived.

**Corollary 4.** *Let $f(z) \in \mathcal{N}_{\Sigma}(\eta,\mu,\gamma,\ell;\beta)$ be of the form (1). Then,*

$$|a_2| \leq \min\left\{ \frac{2(1-\beta)}{(\ell+1)\epsilon_1}, \sqrt{\frac{4(1-\beta)}{(\ell+1)\epsilon_2}} \right\},$$

*and*

$$|a_3| \leq \min\left\{ \frac{4(1-\beta)^2}{(\ell+1)^2\epsilon_1^2} + \frac{4(1-\beta)}{(\ell+1)(\ell+2)\epsilon_3}, \frac{4(1-\beta)\epsilon_4}{(\ell+1)(\ell+2)\epsilon_2\epsilon_3} \right\},$$

*where*

$$\epsilon_1 := \mu + \eta + \frac{2\gamma(\mu + 2\eta)}{1 + 2\eta},$$
$$\epsilon_2 := 1 + (\ell + 1)\mu + \frac{6\gamma(\ell + 2)}{1 + 2\eta},$$
$$\epsilon_3 := \mu + 2\eta + \frac{6\gamma(\mu + 2\eta)}{1 + 2\eta},$$

*and*

$$\epsilon_4 := 3 + \mu + \ell(\mu + 1) + \frac{12\gamma(\ell + 2)}{1 + 2\eta}.$$

The following corollary follows from Theorem 1 for one-fold symmetric bi-univalent functions.

**Corollary 5.** *Let $f(z) \in \mathcal{N}_{\Sigma}^{u,v}(\eta, \mu, \gamma, \ell)$ be of the form (1). Then,*

$$|a_2| \leq \min\left\{ \frac{|u'(0)|}{\Omega_1}, \sqrt{\frac{|u''(0)| + |v''(0)|}{2\Omega_2}} \right\}$$

*and*

$$|a_3| \leq \min\left\{ \frac{|u'(0)|^2}{\Omega_1^2} + \frac{|u''(0)| + |v''(0)|}{2(\ell + 1)\Omega_3}, \frac{[2\Omega_3 + \Omega_4]|u''(0)| + (\ell + 1)|1 - \mu|(\mu + 2\eta)|v''(0)|}{2\Omega_2\Omega_3} \right\},$$

*where*

$$\Omega_1 := (\ell + 1)\left[\mu + \eta + \frac{2\gamma(\mu + 2\eta)}{1 + 2\eta}\right],$$
$$\Omega_2 := (\ell + 1)(\mu + 2\eta)\left[1 + \mu(\ell + 1) + \frac{6\gamma(\ell + 2)}{1 + 2\eta}\right],$$
$$\Omega_3 := (\ell + 2)(\mu + 2\eta)\left[1 + \frac{6\gamma}{1 + 2\eta}\right],$$

*and*

$$\Omega_4 := (\ell + 1)(\mu - 1)(\mu + 2\eta).$$

By specializing the parameters in Corollary 3, it can be seen that several estimate bounds for known subfamilies of $\Sigma$ can be attained as special cases.

**Example 8.** *Put $\ell = 0$ in Corollary 5. Then, the family $\mathcal{N}_{\Sigma}^{u,v}(\eta, \mu, \gamma, \ell)$ reduces to the family $\mathcal{B}_{\Sigma}^{u,v}(\eta, \mu, \gamma)$ studied by Bulut [16], and for a function $f \in \mathcal{B}_{\Sigma}^{u,v}(\eta, \mu, \gamma)$ of the form (1), we have*

$$|a_2| \leq \min\left\{ \frac{|u'(0)|}{\mu + \eta + \frac{2\gamma(\mu+2\eta)}{1+2\eta}}, \sqrt{\frac{|u''(0)| + |v''(0)|}{2(\mu + 2\eta)\left[1 + \mu + \frac{12\gamma}{1+2\eta}\right]}} \right\},$$

*and*

$$|a_3| \leq \min\left\{ \frac{|u'(0)|^2}{\left[\mu+\eta+\frac{2\gamma(\mu+2\eta)}{1+2\eta}\right]^2} + \frac{|u''(0)|+|v''(0)|}{4(\mu+2\eta)\left[1+\frac{6\gamma}{1+2\eta}\right]}, \frac{\left[3+\mu+\frac{24\gamma}{1+2\eta}\right]|u''(0)|+|1-\mu||v''(0)|}{4(\mu+2\eta)\left[1+\mu+\frac{12\gamma}{1+2\eta}\right]\left[1+\frac{6\gamma}{1+2\eta}\right]} \right\}.$$

**Example 9.** *Let $\ell = \gamma = 0$ in Corollary 5. Then, the family $\mathcal{N}_\Sigma^{u,v}(\eta,\mu,\gamma,\ell)$ reduces to the family $\mathcal{N}_\Sigma^{u,v}(\eta,\mu)$ considered by Srivastava et al. [17], and for a function of the form (1) in this family, we have*

$$|a_2| \leq \min\left\{ \frac{|u'(0)|}{\mu+\eta}, \sqrt{\frac{|u''(0)|+|v''(0)|}{2(\mu+1)(\mu+2\eta)}} \right\},$$

*and*

$$|a_3| \leq \min\left\{ \frac{|u'(0)|^2}{(\mu+\eta)^2} + \frac{|u''(0)|+|v''(0)|}{4(\mu+2\eta)}, \frac{(3+\mu)|u''(0)|+|1-\mu||v''(0)|}{4(\mu+1)(\mu+2\eta)} \right\}.$$

**Example 10.** *Set $\ell = \gamma = 0$ and $\mu = 1$ in Corollary 5. Then, the family $\mathcal{N}_\Sigma^{u,v}(\eta,\mu,\gamma,\ell)$ reduces to the family $\mathcal{N}_\Sigma^{u,v}(\eta)$ investigated by Xu et al. [18], and for $f \in \mathcal{B}_\Sigma^{u,v}(\eta)$ of the form (1), we have*

$$|a_2| \leq \min\left\{ \frac{|u'(0)|}{1+\eta}, \frac{1}{2}\sqrt{\frac{|u''(0)|+|v''(0)|}{(1+2\eta)}} \right\},$$

*and*

$$|a_3| \leq \min\left\{ \frac{|u'(0)|^2}{(1+\eta)^2} + \frac{|u''(0)|+|v''(0)|}{4(1+2\eta)}, \frac{|u''(0)|}{2(1+2\eta)} \right\}$$

**Example 11.** *Let $\ell = \gamma = \mu = 0$ and $\eta = 1$ in Corollary 5. Then, the family $\mathcal{N}_\Sigma^{u,v}(\eta,\mu,\gamma,\ell)$ reduces to the family $\mathcal{N}_\Sigma^{u,v}$ investigated by Bulut [19], and for a function of the form (1) in this family, we have*

$$|a_2| \leq \min\left\{ |u'(0)|, \frac{1}{2}\sqrt{|u''(0)|+|v''(0)|} \right\},$$

*and*

$$|a_3| \leq \min\left\{ |u'(0)|^2 + \frac{1}{8}[|u''(0)|+|v''(0)|], \frac{1}{8}[3|u''(0)|+|v''(0)|] \right\}.$$

**Example 12.** *Let $\ell = \gamma = 0$ and $\mu = \eta = 1$ in Corollary 5. Then, the family $\mathcal{N}_\Sigma^{u,v}(\eta,\mu,\gamma,\ell)$ reduces to the family $\mathcal{H}_\Sigma^{u,v}$ investigated by Xu et al. [20], and for a function $f \in \mathcal{H}_\Sigma^{u,v}$ of the form (1), we have*

$$|a_2| \leq \min\left\{ \frac{1}{2}|u'(0)|, \frac{1}{2}\sqrt{\frac{|u''(0)|+|v''(0)|}{3}} \right\},$$

*and*

$$|a_3| \leq \min\left\{ \frac{1}{4}\left[|u'(0)|^2 + \frac{1}{3}[|u''(0)|+|v''(0)|]\right], \frac{1}{6}|u''(0)| \right\}.$$

## 4. Conclusions

In this paper, a general family of holomorphic and $m$-fold symmetric bi-univalent functions was defined and studied. The coefficient bounds $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in this family were derived, showing how the results are generalized from some recent works. Furthermore, by sufficiently specializing the parameters, some consequences of this family were mentioned.

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