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Almost Automorphic Solutions to Nonlinear Difference Equations

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Abstract: In the present work, we concentrate on a certain class of nonlinear difference equations and propose sufficient conditions for the existence of their almost automorphic solutions. In our analysis, we invert an appropriate mapping and obtain the main existence outcomes by utilizing the contraction mapping principle. As the second objective of the manuscript, we reconsider one of the landmark results, namely the Bohr–Neugebauer theorem, in the qualitative theory of dynamical equations, and we investigate the relationship between the existence of almost automorphic solutions and the existence of solutions with a relatively compact range for the proposed difference equation type. Thus, we provide a discrete counterpart of the Bohr–Neugebauer theorem due to the almost automorphy notion under some technical conditions.

Keywords: discrete almost automorphic; discrete bi-almost automorphic; fixed point; contraction; Bohr–Neugebauer

MSC: 42A75; 43A60; 47D99

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1. Introduction

In the theory of dynamic equations, investigation of the existence and uniqueness of periodic solutions has become a very popular research topic for mathematicians, and there is a vast amount of literature on this research direction, which focuses on the real-life models constructed on continuous, discrete, or hybrid time domains with periodic structures. Indeed, the analysis of difference equations has taken as much of a prominent position as differential equations, and the studies based on periodicity for the solutions of differential equations have been carried on to discrete domains. Consequentially, the literature on differential and difference equations has grown simultaneously.

Conventional periodicity is a strong but relaxable condition for some classes of functions. The studies concentrating on the existence of conventionally periodic solutions of dynamic equations may not cover many mathematical models that involve not exactly periodic but nearly periodic arguments, roughly speaking. It is possible to see such real-life models in signal processing or in astrophysics (see [1–3]). As a relaxation of the conventional periodicity, the almost periodicity notion was first introduced by H. Bohr [4], and the theory of almost periodic functions has been developed by the contributions of several scientists including A.S. Besicovitch, S. Bochner, J. von Neumann, and W. Stepanoff who are very well-known in the mathematics community (see [5–8]). The first definition of an almost periodic function was introduced as a topological property; that is, a continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be almost periodic if the set

$$E(\varepsilon, f(t)) := \{\tau : |f(t+\tau) - f(t)| < \varepsilon \text{ for all } t \in \mathbb{R}\}$$

is relatively dense in $\mathbb{R}$ for all $\varepsilon > 0$. Subsequently, Bochner proposed a normality condition as an almost periodicity criterion, i.e., a continuous function $f(\cdot)$ is called almost periodic if for every real sequence $\{v'_n\}$ there exists a subsequence $\{v_n\}$ of $\{v'_n\}$ such that $\lim_{n\to\infty} f(t+v_n) = \bar{f}(t)$ uniformly for all t (see [6]). Afterwards, the theory of almost automorphic functions was introduced by S. Bochner ([9]) by relaxing the uniform convergence from the normality condition. That is, a continuous function $f : \mathbb{R} \to \mathbb{R}$ is called almost automorphic if for every real sequence $\{v'_n\}$ one can extract a subsequence $\{v_n\}$ of $\{v'_n\}$ such that $\lim_{n\to\infty} \lim_{m\to\infty} f(t+v_n-v_m) = f(t)$ for each $t \in \mathbb{R}$. Thus, the almost automorphy notion can be regarded as a weaker version of almost periodicity. It is obvious that the following relationship holds between the periodicity notions

$$\text{conventional periodicity} \Rightarrow \text{almost periodicity} \Rightarrow \text{almost automorphy},$$

while the inverse of the implication may not be correct. For example, the function

$$f(t) = \sin(2\pi t) + \sin\big(2\sqrt{2}\pi t\big),\ t \in \mathbb{R},$$

is almost periodic but not conventionally periodic, and

$$g(t) = \frac{2 + \exp(it) + \exp\left(i\sqrt{2}t\right)}{\left|2 + \exp(it) + \exp\left(i\sqrt{2}t\right)\right|},\ t \in \mathbb{R},$$

is an almost automorphic function that is not almost periodic (see [10,11]). In the recent past, the theories of almost periodic and almost automorphic functions have taken prominent attention from scholars, and the existence of almost periodic and almost automorphic solutions of dynamic equations has become a hot research topic on time domains with continuous, discrete, and hybrid structures. We refer readers to the monographs [10,12–15], papers [16–27], and references therein.

Analysis of the linkage between the existence of bounded and periodic solutions of dynamic equations has always been an interesting research topic in applied mathematics. Massera's theorem is the primary result for the qualitative theory of differential equations since it commentates on the boundedness and periodicity of the solutions (see [28]). Since then, various versions of Massera's theorem have been studied for linear and nonlinear dynamic equations over the last five decades. Undoubtedly, when the dynamic equation contains almost periodic or almost automorphic arguments, it becomes a grueling task to relate the existence of bounded and almost periodic (almost automorphic) solutions. In [29], Bohr and Neugebauer concentrated on the linear system

$$x'(t) = Ax(t) + f(t),$$

and showed that all bounded solutions of the almost periodic system of this form are almost periodic on $\mathbb{R}$. Actually, this crucial result can be regarded as an almost periodic analogue of the Massera's theorem. Moreover, it should be noted that when $A = A(t)$, and A is conventionally periodic, then it is possible to pursue a similar approach in the light of Floquet theory [30]. On the other hand, the nonautonomous linear system with almost periodic coefficients

$$x'(t) = A(t)x(t) + f(t), \quad t \in \mathbb{R},$$

is handled by Favard [31], and it is shown that the linear system has at least one almost periodic solution if it has a bounded solution under a separation assumption; that is, each bounded nontrivial solution of the system

$$x'(t) = B(t)x(t), \quad t \in \mathbb{R},$$

satisfies $\inf_{t\in\mathbb{R}}|x(t)| > 0$ where B is in the hull of A. This conception is known as Favard's theory in the existing literature. These milestone results have motivated researchers remarkably, and it is possible to find detailed literature providing Massera-, Bohr–Neugebauer-, and Favard-type theorems for various kind of dynamic equations based on conventional periodicity, almost periodicity, or almost automorphy notions. We refer to [21,32–40] as pioneering studies. However, we must point out that there is a poor research backlog on Massera- or Bohr–Neugebauer-type theorems on the almost automorphic solutions of difference equations unlike the enormous amount of literature on differential equations. Thus, one of the main objectives of this research is to make a new contribution to the qualitative theory of difference equations by filling the above-mentioned gap.

In this paper, we are inspired by the recent work [21] of A. Chávez, M. Pinto, and U. Zavaleta. We introduce a certain kind of nonlinear summation equation, namely a difference equation,

$$x(t+1) = a(t)x(t) + \sum_{j=-\infty}^{t-1} \Lambda_1(t,j,x(j)) + \sum_{j=t}^{\infty} \Lambda_2(t,j,x(j))$$

with discrete almost automorphic arguments. As the initial task of the study, we focus on the existence and uniqueness of discrete almost automorphic solutions of the nonlinear difference equation by employing fixed point theory. Then, we propose a Bohr–Neugebauer-type theorem that relates to the existence of bounded and discrete almost automorphic solutions. To the best of our knowledge, our study is the first of its kind since it introduces a discrete counterpart of the Bohr–Neugebauer theorem, which has not been considered so far, and consequently, it contributes to the ongoing theory of difference equations.

2. Background Material

In this section, we aim to give a precise review on discrete almost automorphic functions, and their basic characteristics. For the presentation of the preliminary content, we first assume that $\mathcal{X}$ stands for a real (or complex) Banach space endowed with the norm $\|\cdot\|_{\mathcal{X}}$.

Definition 1 (Discrete almost automorphy ([19])). *A function $f : \mathbb{Z} \to \mathcal{X}$ is said to be discrete almost automorphic if for every integer sequence $\{v'_n\}_{n\in\mathbb{Z}}$ there exists a subsequence $\{v_n\}_{n\in\mathbb{Z}}$ of $\{v'_n\}_{n\in\mathbb{Z}}$ such that*

$$\lim_{n\to\infty} f(t+v_n) =: \bar{f}(t) \tag{1}$$

is well defined for each $t \in \mathbb{Z}$, and

$$\lim_{n\to\infty} \bar{f}(t-v_n) = f(t) \tag{2}$$

for each $t \in \mathbb{Z}$.

As is underlined in [19] (Remark 2.2), if the convergence in Definition 1 is uniform, then the concept of discrete almost automorphy turns into a more specific notion, namely discrete almost periodicity. It is clear that every discrete almost periodic function is discrete almost automorphic; however, the inverse of the assertion may not be true. In the existing literature, it is possible to find some studies that propose examples of discrete almost automorphic functions that are not discrete almost periodic. For example, Bochner gave an example of a discrete almost automorphic function that is not discrete almost periodic

$$f(t) =: \operatorname{sgn}(\sin(2\pi t\Omega)),\ t \in \mathbb{Z},$$

for an irrational number Ω in his pioneering work [9] (see also [41]).

Definition 2 ([19]). *A function $g : \mathbb{Z} \times \mathcal{X} \to \mathcal{X}$ is said to be discrete almost automorphic in t for each $x \in \mathcal{X}$ if for every integer sequence $\{v'_n\}_{n\in\mathbb{Z}}$ there exists a subsequence $\{v_n\}_{n\in\mathbb{Z}}$ of $\{v'_n\}_{n\in\mathbb{Z}}$ such that*

$$\lim_{n\to\infty} g(t+v_n, x) =: \bar{g}(t,x)$$

is well defined for both $t \in \mathbb{Z}$ and $x \in \mathcal{X}$, and

$$\lim_{n\to\infty} \bar{g}(t - v_n, x) =: g(t, x)$$

for both $t \in \mathbb{Z}$ *and* $x \in \mathcal{X}$.

We refer to [19] (Theorems 2.4 and 2.9) (see also [14]) for a review of the well-known properties of discrete almost automorphic functions.

Next, we provide the notion of discrete bi-almost automorphy in the light of [21] (Definition 2.7) for multivariable functions.

Definition 3 (Discrete bi-almost automorphy)**.** *A function* $\Lambda : \mathbb{Z} \times \mathbb{Z} \times \mathcal{X} \to \mathcal{X}$ *is called discrete bi-almost automorphic in* $(t, s) \in \mathbb{Z} \times \mathbb{Z}$ *uniformly for* x *on bounded subsets of* $\mathcal{X}$ *if given any integer sequence* $\{v'_n\}_{n\in\mathbb{Z}}$ *and a bounded set* $B \subset \mathcal{X}$*; then, there exists a subsequence* $\{v_n\}_{n\in\mathbb{Z}}$ *of* $\{v'_n\}_{n\in\mathbb{Z}}$ *such that*

$$\lim_{n\to\infty} \Lambda(t + v_n, s + v_n, x) = \bar{\Lambda}(t, s, x)$$

is well defined for both $(t, s) \in \mathbb{Z} \times \mathbb{Z}$ *and* $x \in B$*, and*

$$\lim_{n\to\infty} \bar{\Lambda}(t - v_n, s - v_n, x) = \Lambda(t, s, x)$$

for both $(t, s) \in \mathbb{Z} \times \mathbb{Z}$ *and* $x \in B$.

Let $\mathcal{AA}(\mathbb{Z}, \mathcal{X})$ denote the set of all discrete almost automorphic functions defined on $\mathbb{Z}$. Then, $\mathcal{AA}(\mathbb{Z}, \mathcal{X})$ is a Banach space when it is endowed with the norm

$$\|f\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} := \sup_{t\in\mathbb{Z}} \|f(t)\|_{\mathcal{X}}. \tag{3}$$

The next result is crucial for the setup of the main outcomes.

Theorem 1 ([19])**.** *Let* $g : \mathbb{Z} \times \mathcal{X} \to \mathcal{X}$ *be discrete almost automorphic in* t*, for each* $x \in \mathcal{X}$*, and suppose that it satisfies the Lipschitz condition in* x *uniformly in* t*; that is,*

$$\|g(t, x) - g(t, y)\|_{\mathcal{X}} \le L\|x - y\|_{\mathcal{X}}, \; x, y \in \mathcal{X}.$$

Then, the function $g(t, \varphi(t))$ *is a discrete almost automorphic function whenever* $\varphi : \mathbb{Z} \to \mathcal{X}$ *is discrete almost automorphic.*

For more details about multidimensional almost automorphic sequences and their applications, we refer the reader to our recent research paper [24].

3. Setup and Main Results

Consider the following abstract nonlinear difference equation

$$x(t + 1) = a(t)x(t) + \sum_{j=-\infty}^{t-1} \Lambda_1(t, j, x(j)) + \sum_{j=t}^{\infty} \Lambda_2(t, j, x(j)), \tag{4}$$

where $a : \mathbb{Z} \to \mathbb{C}$, $a(t) \neq 0$ for all $t \in \mathbb{Z}$, and $\Lambda_{1,2} : \mathbb{Z} \times \mathbb{Z} \times \mathcal{X} \to \mathcal{X}$.

In the sequel, we give the following fundamental result, which is essential for the outcomes of the manuscript:

Lemma 1. *The function* $x(\cdot)$ *is a solution of* (4) *with the initial data* $x(t_0) = x_0$ *if and only if*

$$x(t) = x_0 \prod_{s=t_0}^{t-1} a(s) + \sum_{k=t_0}^{t-1} \left(\prod_{s=k+1}^{t-1} a(s) \right) \left(\sum_{j=-\infty}^{k} \Lambda_1(k, j, x(j)) + \sum_{j=k+1}^{\infty} \Lambda_2(k, j, x(j)) \right). \tag{5}$$

Proof. We multiply both sides of (4) with $\prod_{s=t_0}^{t-1} a^{-1}(s)$, and obtain

$$x(t+1)\prod_{s=t_0}^{t-1}a^{-1}(s)-a(t)x(t)\prod_{s=t_0}^{t-1}a^{-1}(s)=\prod_{s=t_0}^{t-1}a^{-1}(s)\left(\sum_{j=-\infty}^{t-1}\Lambda_1(t,j,x(j))+\sum_{j=t}^{\infty}\Lambda_2(t,j,x(j))\right).$$

By writing the above expression in the following form

$$\begin{aligned}&x(t+1)a(t)\prod_{s=t_0}^{t}a^{-1}(s)-a(t)x(t)\prod_{s=t_0}^{t-1}a^{-1}(s)\\&=\prod_{s=t_0}^{t-1}a^{-1}(s)\left(\sum_{j=-\infty}^{t-1}\Lambda_1(t,j,x(j))+\sum_{j=t}^{\infty}\Lambda_2(t,j,x(j))\right),\end{aligned}$$

we obtain

$$\Delta\left(x(t)\prod_{s=t_0}^{t-1}a^{-1}(s)\right)=\prod_{s=t_0}^{t-1}a^{-1}(s)\left(\sum_{j=-\infty}^{t-1}\Lambda_1(t,j,x(j))+\sum_{j=t}^{\infty}\Lambda_2(t,j,x(j))\right),$$

where Δ stands for the forward difference operator. Next, we take the summation from t_0 to $t-1$

$$\sum_{k=t_0}^{t-1}\Delta\left(x(k)\prod_{s=t_0}^{k-1}a^{-1}(s)\right)=\sum_{k=t_0}^{t-1}\left(\prod_{s=t_0}^{k}a^{-1}(s)\right)\left(\sum_{j=-\infty}^{k}\Lambda_1(k,j,x(j))+\sum_{j=k+1}^{\infty}\Lambda_2(k,j,x(j))\right).$$

This yields to

$$x(t)\prod_{s=t_0}^{t-1}a^{-1}(s)-x_0=\sum_{k=t_0}^{t-1}\left(\prod_{s=t_0}^{k}a^{-1}(s)\right)\left(\sum_{j=-\infty}^{k}\Lambda_1(k,j,x(j))+\sum_{j=k+1}^{\infty}\Lambda_2(k,j,x(j))\right),$$

and one may easily obtain (5). Since every step is reversible, the proof is complete. □

Henceforth, we assume that the following conditions are satisfied throughout the manuscript:

C1 The function $a(\cdot)$ is discrete almost automorphic.

C2 $\Lambda_{1,2}$ are discrete bi-almost automorphic in t and s, uniformly for x.

C3 For $u_{1,2}\in\mathcal{X}$, the Lipschitz inequalities

$$\|\Lambda_1(t,s,u_1)-\Lambda_1(t,s,u_2)\|_{\mathcal{X}}\le m_1(t,s)\|u_1-u_2\|_{\mathcal{X}}$$

and

$$\|\Lambda_2(t,s,u_1)-\Lambda_2(t,s,u_2)\|_{\mathcal{X}}\le m_2(t,s)\|u_1-u_2\|_{\mathcal{X}}$$

hold together with

$$\sup_{t\in\mathbb{Z}}\sum_{j=-\infty}^{t-1}m_1(t,j)=M_1<\infty,$$
$$\sup_{t\in\mathbb{Z}}\sum_{j=t}^{\infty}m_2(t,j)=M_2<\infty.$$

Subsequently, we introduce the mapping $H:\mathcal{X}\to\mathcal{X}$ given by

$$(Hx)(t):=x_0\prod_{s=t_0}^{t-1}a(s)+\sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}a(s)\right)(S_1(k,x(k))+S_2(k,x(k))),\tag{6}$$

where

$$S_1(k, x(k)) := \sum_{j=-\infty}^{k} \Lambda_1(k, j, x(j)), \tag{7}$$

and

$$S_2(k, x(k)) := \sum_{j=k+1}^{\infty} \Lambda_2(k, j, x(j)). \tag{8}$$

Lemma 2. *If $x \in \mathcal{AA}(\mathbb{Z}, \mathcal{X})$, then $S_1(\cdot, x(\cdot))$ and $S_2(\cdot, x(\cdot))$ are discrete almost automorphic.*

Proof. Suppose that $\xi,\ \varphi \in \mathcal{AA}(\mathbb{Z}, \mathcal{X})$. Then we have

$$\begin{aligned} \|S_1(k,\xi) - S_1(k,\varphi)\|_{\mathcal{X}} &= \left\| \sum_{j=-\infty}^{k} \Lambda_1(k, j, \xi(j)) - \sum_{j=-\infty}^{k} \Lambda_1(k, j, \varphi(j)) \right\|_{\mathcal{X}} \\ &\leq \sup_{k\in\mathbb{Z}} \sum_{j=-\infty}^{k} \|\Lambda_1(k, j, \xi(j)) - \Lambda_1(k, j, \varphi(j))\|_{\mathcal{X}} \\ &\leq \sup_{k\in\mathbb{Z}} \sum_{j=-\infty}^{k} m_1(k, j)\|\xi - \varphi\|_{\mathcal{X}} \\ &= M_1\|\xi - \varphi\|_{\mathcal{X}}. \end{aligned}$$

Similarly, we easily observe that

$$\|S_2(k,\xi) - S_2(k,\varphi)\|_{\mathcal{X}} \leq M_2\|\xi - \varphi\|_{\mathcal{X}}.$$

By Theorem 1, the proof of the assertion is complete. □

Lemma 3. *In addition to* ***C1*–*C3****, also assume that the condition*

C4 *For every integer sequence $\{v'_n\}_{n\in\mathbb{Z}}$ there exists a subsequence $\{v_n\}_{n\in\mathbb{Z}}$ of $\{v'_n\}_{n\in\mathbb{Z}}$ such that*

$$\lim_{n\to\infty} x(t_0 \pm v_n) = x(t_0) = x_0$$

holds. Then, H maps $\mathcal{AA}(\mathbb{Z}, \mathcal{X})$ into $\mathcal{AA}(\mathbb{Z}, \mathcal{X})$.

Proof. Suppose that $x \in \mathcal{AA}(\mathbb{Z}, \mathcal{X})$. By Lemma 2, the functions $S_1(t, x(t))$ and $S_2(t, x(t))$, which are defined in (7) and (8), are discrete almost automorphic functions in t for each x. That is, for every integer sequence $\{v'_n\}_{n\in\mathbb{Z}}$ there exists a subsequence $\{v_n\}_{n\in\mathbb{Z}}$ of $\{v'_n\}_{n\in\mathbb{Z}}$ such that

$$\begin{aligned} &\lim_{n\to\infty} S_1(t + v_n, x(t + v_n)) =: \overline{S_1}(t, \overline{x}(t)), \\ &\lim_{n\to\infty} \overline{S_1}(t - v_n, \overline{x}(t - v_n)) := S_1(t, x(t)) \end{aligned}$$

and

$$\begin{aligned} &\lim_{n\to\infty} S_2(t + v_n, x(t + v_n)) =: \overline{S_2}(t, \overline{x}(t)), \\ &\lim_{n\to\infty} \overline{S_2}(t - v_n, \overline{x}(t - v_n)) := S_2(t, x(t)) \end{aligned}$$

for each $t \in \mathbb{Z}$. Let us write

$$(Hx)(t+v_n) = x(t_0+v_n)\prod_{s=t_0+v_n}^{t+v_n-1} a(s) + \sum_{k=t_0+v_n}^{t+v_n-1}\left(\prod_{s=k+1}^{t+v_n-1} a(s)\right)(S_1(k,x(k)) + S_2(k,x(k)))$$

$$= x(t_0+v_n)\prod_{s=t_0}^{t-1} a(s+v_n)$$

$$+ \sum_{k=t_0}^{t-1}\left(\prod_{s=k+v_n+1}^{t+v_n-1} a(s)\right)(S_1(k+v_n, x(k+v_n)) + S_2(k+v_n, x(k+v_n)))$$

$$= x(t_0+v_n)\prod_{s=t_0}^{t-1} a(s+v_n)$$

$$+ \sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1} a(s+v_n)\right)(S_1(k+v_n, x(k+v_n)) + S_2(k+v_n, x(k+v_n))).$$

If we take the limit of $(Hx)(t+v_n)$ as $n \to \infty$ and utilize the Lebesgue convergence theorem, then we have

$$(\overline{Hx})(t) = x_0\prod_{s=t_0}^{t-1}\bar{a}(s) + \sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}\bar{a}(s)\right)\left(\overline{S_1}(k,\bar{x}(k)) + \overline{S_2}(k,\bar{x}(k))\right).$$

For the converse part, we can follow a similar procedure. Consider

$$(\overline{Hx})(t-v_n) = x(t_0-v_n)\prod_{s=t_0-v_n}^{t-v_n-1}\bar{a}(s) + \sum_{k=t_0-v_n}^{t-v_n-1}\left(\prod_{s=k+1}^{t-v_n-1}\bar{a}(s)\right)\left(\overline{S_1}(k,\bar{x}(k)) + \overline{S_2}(k,\bar{x}(k))\right)$$

$$= x(t_0-v_n)\prod_{s=t_0}^{t-1}\bar{a}(s-v_n)$$

$$+ \sum_{k=t_0}^{t-1}\left(\prod_{s=k-v_n+1}^{t-v_n-1}\bar{a}(s)\right)\left(\overline{S_1}(k-v_n,\bar{x}(k-v_n)) + \overline{S_2}(k-v_n,\bar{x}(k-v_n))\right),$$

which results in

$$(\overline{Hx})(t-v_n) = x(t_0-v_n)\prod_{s=t_0}^{t-1}\bar{a}(s-v_n)$$
$$+ \sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}\bar{a}(s-v_n)\right)\left(\overline{S_1}(k-v_n,\bar{x}(k-v_n)) + \overline{S_2}(k-v_n,\bar{x}(k-v_n))\right).$$

By taking the limit of $(\overline{Hx})(t-v_n)$ as $n \to \infty$, and using the Lebesgue convergence theorem, we obtain $\lim_{n\to\infty}(\overline{Hx})(t-v_n) = (Hx)(t)$. This completes the proof. □

Remark 1. *It should be highlighted that the condition* ***C4*** *is a compulsory technical condition for the construction of existence results. A similar condition can be found in the pioneering work of Bohner and Mesquita (see [20] (Theorem 3.10)). On the other hand, the main results of [21] do not require such an abstract condition since the authors concentrate on the solutions of integral equations rather than the solutions of integro-differential equations.*

3.1. Existence Results

Now, we are ready to present our first existence result.

Theorem 2. *Assume that* ***C1–C4*** *hold, and the condition*

C5
$$\sup_{t\in\mathbb{Z}}\sum_{k=t_0}^{t-1}\left\|\prod_{s=k+1}^{t-1}a(s)\right\|_{\mathcal{X}}(M_1+M_2)=\kappa<1$$

is satisfied. Then, the abstract difference Equation (4) has a unique discrete almost automorphic solution.

Proof. In addition to **C1–C4**, also suppose that **C5** holds. By taking Lemmas 2 and 3 into consideration, it remains to show that the mapping $H(\cdot)$ given in (6) is a contraction. Let $\xi,\ \varphi\in\mathcal{AA}(\mathbb{Z},\mathcal{X})$, then we have the following:

$$\begin{aligned}
&\|H\xi-H\varphi\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})}\\
&=\sup_{t\in\mathbb{Z}}\left\|\sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}a(s)\right)(S_1(k,\xi(k))-S_1(k,\varphi(k))+S_2(k,\xi(k))-S_2(k,\varphi(k)))\right\|_{\mathcal{X}}\\
&\le\sup_{t\in\mathbb{Z}}\sum_{k=t_0}^{t-1}\left\|\prod_{s=k+1}^{t-1}a(s)\right\|_{\mathcal{X}}(M_1+M_2)\|\xi-\varphi\|_{\mathcal{X}}\\
&\le\kappa\|\xi-\varphi\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})}.
\end{aligned}$$

This indicates that H is a contraction; by the Banach fixed point theorem, it has a unique fixed point. Thus, the nonlinear difference Equation (4) has a unique discrete almost automorphic solution. □

Theorem 3. *Assume that the conditions **C1–C5** hold. For a positive constant γ, we define the set*
$$W_\gamma=\left\{x\in\mathcal{AA}(\mathbb{Z},\mathcal{X}):\left\|x-x^0\right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})}\le\gamma\right\},\tag{9}$$

where
$$x^0(t)=\sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}a(s)\right)(S_1(k,0)+S_2(k,0)).\tag{10}$$

Let $\|x\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})}\le\gamma$ and

C6 $\left\|\prod_{s=t_0}^{t-1}a(s)\right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})}\le\psi$ *for all t.*

If
$$\|x_0\|_{\mathcal{X}}\psi+\kappa\gamma\le\gamma,\tag{11}$$
then the nonlinear difference Equation (4) has a unique discrete almost automorphic solution in W_γ.

Proof. Consider the operator H, which is defined in (6). In the proof of Theorem 2, it is already shown that H is a contraction when the condition **C5** holds. Thus, we have to prove that H maps W_γ into W_γ to conclude the proof. We suppose that $x\in W_\gamma$, and condition (11) holds. Then, we obtain

$$\begin{aligned}
&\left\|(Hx)(t)-x^0(t)\right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})}\\
&\le\|x_0\|_{\mathcal{X}}\left\|\prod_{s=t_0}^{t-1}a(s)\right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})}\\
&+\left\|\sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}a(s)\right)(S_1(k,x(k))-S_1(k,0)+S_2(k,x(k))-S_2(k,0))\right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})}
\end{aligned}$$

$$\leq \|x_0\|_{\mathcal{X}}\psi + \sup_{t\in\mathbb{Z}} \sum_{k=t_0}^{t-1} \left\| \prod_{s=k+1}^{t-1} a(s) \right\|_{\mathcal{X}} (M_1 + M_2)\|x\|_{\mathcal{X}}$$
$$\leq \|x_0\|_{\mathcal{X}}\psi + \kappa\gamma \leq \gamma.$$

Thus $H(W_\gamma) \subset W_\gamma$. This implies that H has a unique fixed point due to the contraction mapping principle, and consequentially, (4) has a unique almost automorphic solution in W_γ. □

Theorem 4. *Suppose that the conditions* ***C1–C6*** *hold, and x^0 is as in* (10)*. Consider the closed ball*

$$W_\phi = \left\{ x \in \mathcal{AA}(\mathbb{Z}, \mathcal{X}) : \left\| x - x^0 \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \leq \phi \right\}.$$

If

$$\|x_0\|_{\mathcal{X}}\psi + \kappa\phi + \left\| Hx^0 - x^0 \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \leq \phi, \tag{12}$$

then (4) *has a unique discrete almost automorphic solution in W_ϕ.*

Proof. Pick $x \in W_\phi$, and assume that (12) is satisfied. Then,

$$\begin{aligned}
&\left\| (Hx)(t) - x^0(t) \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \\
&\leq \left\| (Hx)(t) - \left(Hx^0\right)(t) \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} + \left\| \left(Hx^0\right)(t) - x^0(t) \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \\
&\leq \|x_0\|_{\mathcal{X}} \left\| \prod_{s=t_0}^{t-1} a(s) \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \\
&+ \left\| \sum_{k=t_0}^{t-1} \left(\prod_{s=k+1}^{t-1} a(s) \right) \left(S_1(k, x(k)) - S_1(k, x^0(k)) + S_2(k, x(k)) - S_2(k, x^0(k)) \right) \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \\
&+ \left\| \left(Hx^0\right)(t) - x^0(t) \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \\
&\leq \|x_0\|_{\mathcal{X}}\psi + \sup_{t\in\mathbb{Z}} \sum_{k=t_0}^{t-1} \left\| \prod_{s=k+1}^{t-1} a(s) \right\|_{\mathcal{X}} (M_1 + M_2) \left\| x - x^0 \right\|_{\mathcal{X}} + \left\| \left(Hx^0\right)(t) - x^0(t) \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})}.
\end{aligned}$$

This implies

$$\left\| (Hx)(t) - x^0(t) \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \leq \|x_0\|_{\mathcal{X}}\psi + \kappa\phi + \left\| Hx^0 - x^0 \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \leq \phi,$$

and consequentially, $H(W_\phi) \subset W_\phi$. Since the mapping H is a contraction, we deduce that (4) has a unique discrete almost automorphic solution in W_ϕ. □

Example 1. *Consider the nonlinear difference equation given by*

$$\begin{aligned}
x(t+1) &= \frac{1}{2}\operatorname{sgn}(\cos 2\pi t\Omega) x(t) \\
&+ \sum_{j=-\infty}^{t-1} \frac{1}{20} \left(\frac{1}{4} \left(\sin\left(\frac{\pi}{2} j\right) + \sin\left(\frac{\pi}{2} j\sqrt{2}\right) \right) \right)^{t-j} x(j) + \sum_{j=t}^{\infty} \frac{1}{20} \arctan\left(3^{t-j} x(j)\right),
\end{aligned} \tag{13}$$

where Ω is an irrational number, and $x(0) = x_0$. A comparison between (4) *and* (13) *results in*

$$a(t) = \frac{1}{2}\operatorname{sgn}(\cos 2\pi t\Omega),$$

$$\Lambda_1(t, s, x) = \frac{1}{20} \left(\frac{1}{4} \left(\sin\left(\frac{\pi}{2} s\right) + \sin\left(\frac{\pi}{2} s\sqrt{2}\right) \right) \right)^{t-s} x,$$

and

$$\Lambda_2(t,s,x) = \frac{1}{20}\arctan\left(3^{t-s}x\right).$$

The function $a(\cdot)$ is discrete almost automorphic for any irrational number Ω (see [41]). In addition to that, the function $f(t) = \sin\left(\frac{\pi}{2}t\right) + \sin\left(\frac{\pi}{2}t\sqrt{2}\right)$ is discrete almost periodic, and consequently, discrete almost automorphic. Thus, the function Λ_1 is discrete bi-almost automorphic. Despite the fact that the function Λ_2 does not contain any almost automorphic arguments, it can be considered as a discrete bi-almost automorphic function since it is a convolution term. Next, we analyze Λ_1 and Λ_2 in detail. We focus on

$$\|\Lambda_1(t,s,x_1) - \Lambda_1(t,s,x_2)\|_{\mathcal{X}} \leq \left|\frac{1}{20}\left(\frac{1}{4}\left(\sin\left(\frac{\pi}{2}s\right) + \sin\left(\frac{\pi}{2}s\sqrt{2}\right)\right)\right)^{t-s}\right| \|x_1 - x_2\|_{\mathcal{X}},$$

and set

$$m_1(t,s) = \left|\frac{1}{20}\left(\frac{1}{4}\left(\sin\left(\frac{\pi}{2}s\right) + \sin\left(\frac{\pi}{2}s\sqrt{2}\right)\right)\right)^{t-s}\right|.$$

Subsequently, we write

$$\sup_{t\in\mathbb{Z}} \sum_{j=-\infty}^{t-1} m_1(t,j) \leq \sup_{t\in\mathbb{Z}} \sum_{j=-\infty}^{t-1} \frac{1}{20}\left(\frac{1}{2}\right)^{t-j},$$

and obtain the constant $M_1 = \frac{1}{20}$. Similarly, we consider

$$\|\Lambda_2(t,s,x_1) - \Lambda_2(t,s,x_2)\|_{\mathcal{X}} \leq \frac{1}{20}3^{t-s}\|x_1 - x_2\|_{\mathcal{X}},$$

*and obtain $m_2(t,s) = \frac{1}{20}3^{t-s}$. Accordingly, we have the constant $M_2 = \frac{3}{40}$. Thus, the conditions **C1**–**C3** are satisfied. Furthermore, the condition **C5** holds since*

$$\sup_{t\in\mathbb{Z}} \sum_{k=0}^{t-1} \left\| \prod_{s=k+1}^{t-1} a(s) \right\|_{\mathcal{X}} (M_1 + M_2) = \sup_{t\in\mathbb{Z}} \sum_{k=0}^{t-1} \frac{1}{8} \left\| \prod_{s=k+1}^{t-1} \frac{1}{2}\operatorname{sgn}(\cos 2\pi s\Omega) \right\|_{\mathcal{X}} \leq \frac{1}{16}.$$

*Then, Theorem 2 implies that the nonlinear difference Equation (13) has a unique discrete almost automorphic solution whenever the technical condition **C4** holds.*

Furthermore, it is obvious that

$$\left\| \prod_{s=0}^{t-1} \frac{1}{2}\operatorname{sgn}(\cos 2\pi s\Omega) \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \leq 1.$$

If we concentrate on Theorem 3, then we obtain the existence of a unique discrete almost automorphic solution of (13) in the set

$$W_\gamma = \left\{ x \in \mathcal{AA}(\mathbb{Z},\mathcal{X}) : \left\| x - x^0 \right\|_{\mathcal{AA}(\mathbb{Z},\mathcal{X})} \leq \gamma \right\}$$

*for $\frac{16}{15}\|x_0\|_{\mathcal{X}} \leq \gamma$ by tacitly assuming that the condition **C4** holds.*

3.2. Bohr–Neugebauer Criterion

In this part of the manuscript, we focus on the connection between the existence of discrete almost automorphic solutions and bounded solutions of nonlinear difference equations with almost automorphic arguments. Since this result originated as the Bohr–Neugebauer theorem, the next result can be regarded as a discrete variant of the Bohr–Neugebauer theorem for a particular class of nonlinear difference equations.

Theorem 5. *Suppose that the conditions **C1–C5** are satisfied. Then, a bounded solution of a nonlinear abstract difference equation is discrete almost automorphic if and only if it has a relatively compact range.*

Proof. Suppose that $x(\cdot)$ is an almost automorphic solution of (4). This directly implies that its range $\mathcal{R}$ is relatively compact.

Assume that **C1–C5** hold, and $x(\cdot)$ is a bounded solution of (4) with a relatively compact range $\mathcal{R}$; that is, $\bar{\mathcal{R}}$ is compact. By **C1** and **C2**, for any arbitrary integer sequence $\{v_n''\}$, there exists a subsequence $\{v_n'\}$ of $\{v_n''\}$ such that the following limits hold:

$$\lim_{n\to\infty} a(t+v_n') = \bar{a}(t), \ \lim_{n\to\infty} \bar{a}(t-v_n') = a(t),$$

and

$$\lim_{n\to\infty} \Lambda_{1,2}\big(t+v_n', s+v_n', x\big) = \bar{\Lambda}_{1,2}(t,s,x), \ \lim_{n\to\infty} \bar{\Lambda}_{1,2}\big(t-v_n', s-v_n', x\big) = \Lambda_{1,2}(t,s,x).$$

Next, it is clear that $x(t+v_n')$ is a sequence in $\bar{\mathcal{R}}$, and by sequential compactness there exists a subsequence $\{v_n\}$ of $\{v_n'\}$ so that $x(t+v_n) \to \bar{x}(t)$ as $n \to \infty$. For the sequel, define

$$\varsigma(t) := x(t_0)\left(\prod_{s=t_0}^{t-1}\bar{a}(s)\right) + \sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}\bar{a}(s)\right)\big(\overline{S_1}(k,\overline{x}(k)) + \overline{S_2}(k,\overline{x}(k))\big), \tag{14}$$

where

$$\overline{S_1}(k,\overline{x}(k)) = \sum_{j=-\infty}^{k} \bar{\Lambda}_1(k,j,\bar{x}(j)),$$

and

$$\overline{S_2}(k,\overline{x}(k)) = \sum_{j=k+1}^{\infty} \bar{\Lambda}_2(k,j,\bar{x}(j)).$$

We have

$$\begin{aligned}
&\|x(t+v_n) - \varsigma(t)\|_{\mathcal{X}} \\
&= \Bigg\| x(t_0+v_n)\prod_{s=t_0+v_n}^{t+v_n-1} a(s) + \sum_{k=t_0+v_n}^{t+v_n-1}\left(\prod_{s=k+1}^{t+v_n-1} a(s)\right)(S_1(k,x(k)) + S_2(k,x(k))) \\
&-x(t_0)\prod_{s=t_0}^{t-1}\bar{a}(s) + \sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}\bar{a}(s)\right)\big(\overline{S_1}(k,\overline{x}(k)) + \overline{S_2}(k,\overline{x}(k))\big)\Bigg\|_{\mathcal{X}} \\
&\le \Bigg\| x(t_0+v_n)\prod_{s=t_0}^{t-1} a(s+v_n) - x(t_0)\prod_{s=t_0}^{t-1}\bar{a}(s)\Bigg\|_{\mathcal{X}} \\
&+ \Bigg\|\sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1} a(s+v_n)\right)(S_1(k+v_n, x(k+v_n)) + S_2(k+v_n, x(k+v_n))) \\
&-x(t_0)\left(\prod_{s=t_0}^{t-1}\bar{a}(s)\right) + \sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}\bar{a}(s)\right)\big(\overline{S_1}(k,\overline{x}(k)) + \overline{S_2}(k,\overline{x}(k))\big)\Bigg\|_{\mathcal{X}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\| x(t_0+v_n)\prod_{s=t_0}^{t-1} a(s+v_n) - x(t_0)\prod_{s=t_0}^{t-1}\bar{a}(s)\right\|_{\mathcal{X}} \\
&+ \left\| \sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1} a(s+v_n) - \prod_{s=t_0}^{t-1}\bar{a}(s)\right)(S_1(k+v_n,x(k+v_n)) + S_2(k+v_n,x(k+v_n)))\right\|_{\mathcal{X}} \\
&+ \left\| \sum_{k=t_0}^{t-1}\left(\prod_{s=t_0}^{t-1}\bar{a}(s)\right)(S_1(k+v_n,x(k+v_n)) + S_2(k+v_n,x(k+v_n)) \right.\\
&\left. -\overline{S_1}(k,\overline{x}(k)) - \overline{S_2}(k,\overline{x}(k)))\right\|_{\mathcal{X}} \\
&\leq \left\| x(t_0+v_n)\prod_{s=t_0}^{t-1} a(s+v_n) - x(t_0)\prod_{s=t_0}^{t-1}\bar{a}(s)\right\|_{\mathcal{X}} \\
&+ \sum_{k=t_0}^{t-1}\left\| \prod_{s=k+1}^{t-1} a(s+v_n) - \prod_{s=t_0}^{t-1}\bar{a}(s)\right\|_{\mathcal{X}} \|S_1(k+v_n,x(k+v_n)) + S_2(k+v_n,x(k+v_n))\|_{\mathcal{X}} \\
&+ \sum_{k=t_0}^{t-1}\left\| \prod_{s=t_0}^{t-1}\bar{a}(s)\right\| \left(\|S_1(k+v_n,x(k+v_n)) - \overline{S_1}(k,\overline{x}(k))\|_{\mathcal{X}} \right.\\
&\left. + \|S_2(k+v_n,x(k+v_n)) - \overline{S_2}(k,\overline{x}(k))\|_{\mathcal{X}}\right).
\end{aligned}
$$

In the light of Lebesgue convergence theorem, we obtain $\|x(t+v_n) - \varsigma(t)\|_{\mathcal{X}} \to 0$ as $n \to \infty$. Thus, $\bar{x}(t) = \varsigma(t)$, and $\bar{x}$ satisfies (14). Now, it remains to show that $\lim_{n\to\infty} \bar{x}(t-v_n) = x(t)$ for each $t \in \mathbb{Z}$. We focus on

$$
\begin{aligned}
&\|\bar{x}(t-v_n) - x(t)\|_{\mathcal{X}} \\
&\leq \left\| x(t_0-v_n)\prod_{s=t_0-v_n}^{t-v_n-1}\bar{a}(s) - x(t_0)\prod_{s=t_0}^{t-1} a(s)\right\|_{\mathcal{X}} \\
&+ \left\| \sum_{k=t_0-v_n}^{t-v_n-1}\left(\prod_{s=k+1}^{t-v_n-1}\bar{a}(s)\right)(\overline{S}_1(k,\overline{x}(k)) + \overline{S}_2(k,\overline{x}(k))) \right.\\
&\left. - \sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1} a(s)\right)(S_1(k,x(k)) + S_2(k,x(k)))\right\|_{\mathcal{X}} \\
&= \left\| x(t_0-v_n)\prod_{s=t_0-v_n}^{t-v_n-1}\bar{a}(s) - x(t_0)\prod_{s=t_0}^{t-1} a(s)\right\|_{\mathcal{X}} \\
&+ \left\| \sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}\bar{a}(s-v_n)\right)\left(\sum_{j=-\infty}^{k}\bar{\Lambda}_1(k-v_n,j-v_n,\bar{x}(j-v_n)) \right.\right.\\
&\left. + \sum_{j=k+1}^{\infty}\bar{\Lambda}_2(k-v_n,j-v_n,\bar{x}(j-v_n))\right) - \sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1} a(s)\right)\left(\sum_{j=-\infty}^{k}\Lambda_1(k,j,x(j)) \right.\\
&\left.\left. + \sum_{j=k+1}^{\infty}\Lambda_2(k,j,x(j))\right)\right\|_{\mathcal{X}} \\
&\leq \left\| x(t_0-v_n)\prod_{s=t_0}^{t-1}\bar{a}(s-v_n) - x(t_0)\prod_{s=t_0}^{t-1} a(s)\right\|_{\mathcal{X}}
\end{aligned}
$$

$$+\left\|\sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}\bar{a}(s-v_n)\right)\left(\sum_{j=-\infty}^{k}\bar{\Lambda}_1(k-v_n,j-v_n,\bar{x}(j-v_n))\right.\right.$$
$$\left.\left.+\sum_{j=k+1}^{\infty}\bar{\Lambda}_2(k-v_n,j-v_n,\bar{x}(j-v_n))-\sum_{j=-\infty}^{k}\Lambda_1(k,j,\bar{x}(j-v_n))-\sum_{j=k+1}^{\infty}\Lambda_2(k,j,\bar{x}(j-v_n))\right)\right\|_{\mathcal{X}}$$
$$+\left\|\sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}\bar{a}(s-v_n)-\prod_{s=k+1}^{t-1}a(s)\right)\left(\sum_{j=-\infty}^{k}\Lambda_1(k,j,\bar{x}(j-v_n))+\sum_{j=k+1}^{\infty}\Lambda_2(k,j,\bar{x}(j-v_n))\right)\right\|_{\mathcal{X}}$$
$$+\left\|\sum_{k=t_0}^{t-1}\left(\prod_{s=k+1}^{t-1}a(s)\right)\left(\sum_{j=-\infty}^{k}(\Lambda_1(k,j,\bar{x}(j-v_n))-\Lambda_1(k,j,x(j)))\right.\right.$$
$$\left.\left.+\sum_{j=k+1}^{\infty}(\Lambda_2(k,j,\bar{x}(j-v_n))-\Lambda_2(k,j,x(j)))\right)\right\|_{\mathcal{X}}$$
$$\leq\left\|x(t_0-v_n)\prod_{s=t_0}^{t-1}\bar{a}(s-v_n)-x(t_0)\prod_{s=t_0}^{t-1}a(s)\right\|_{\mathcal{X}} \tag{15}$$
$$+\sum_{k=t_0}^{t-1}\left\|\prod_{s=k+1}^{t-1}\bar{a}(s-v_n)\right\|_{\mathcal{X}}\left(\sum_{j=-\infty}^{k}\|\bar{\Lambda}_1(k-v_n,j-v_n,\bar{x}(j-v_n))-\Lambda_1(k,j,\bar{x}(j-v_n))\|_{\mathcal{X}}\right. \tag{16}$$
$$\left.+\sum_{j=k+1}^{\infty}\|\bar{\Lambda}_2(k-v_n,j-v_n,\bar{x}(j-v_n))-\Lambda_2(k,j,\bar{x}(j-v_n))\|_{\mathcal{X}}\right) \tag{17}$$
$$+\sum_{k=t_0}^{t-1}\left\|\prod_{s=k+1}^{t-1}\bar{a}(s-v_n)-\prod_{s=k+1}^{t-1}a(s)\right\|_{\mathcal{X}}\left\|\sum_{j=-\infty}^{k}\Lambda_1(k,j,\bar{x}(j-v_n))+\sum_{j=k+1}^{\infty}\Lambda_2(k,j,\bar{x}(j-v_n))\right\|_{\mathcal{X}} \tag{18}$$
$$+\sum_{k=t_0}^{t-1}\left\|\prod_{s=k+1}^{t-1}a(s)\right\|_{\mathcal{X}}\left(\sum_{j=-\infty}^{k}\|\Lambda_1(k,j,\bar{x}(j-v_n))-\Lambda_1(k,j,x(j))\|_{\mathcal{X}}\right. \tag{19}$$
$$\left.+\sum_{j=k+1}^{\infty}\|\Lambda_2(k,j,\bar{x}(j-v_n))-\Lambda_2(k,j,x(j))\|_{\mathcal{X}}\right). \tag{20}$$

The expressions in (15)–(18) converge to 0 as $n\to\infty$. On the other hand, from (19) and (20), we obtain

$$\sum_{k=t_0}^{t-1}\left\|\prod_{s=k+1}^{t-1}a(s)\right\|_{\mathcal{X}}$$
$$\times\left(\sum_{j=-\infty}^{k}\|\Lambda_1(k,j,\bar{x}(j-v_n))-\Lambda_1(k,j,x(j))\|_{\mathcal{X}}+\sum_{j=k+1}^{\infty}\|\Lambda_2(k,j,\bar{x}(j-v_n))-\Lambda_2(k,j,x(j))\|_{\mathcal{X}}\right)$$
$$\leq\sum_{k=t_0}^{t-1}\left\|\prod_{s=k+1}^{t-1}a(s)\right\|_{\mathcal{X}}\left(\sum_{j=-\infty}^{k}m_1(k,j)\|\bar{x}(j-v_n)-x(j)\|_{\mathcal{X}}+\sum_{j=k+1}^{\infty}m_2(k,j)\|\bar{x}(j-v_n)-x(j)\|_{\mathcal{X}}\right),$$

where we employed **C3**. Since x is bounded, $\|\bar{x}(j-v_n)-x(j)\|_{\mathcal{X}}$ forms a bounded sequence, and consequently, there exists a subsequence $\{v_p\}$ of $\{v_n\}$ so that

$$\|\bar{x}(t-v_p)-x(t)\|_{\mathcal{X}}\to\theta(t)$$

as $p\to\infty$. This implies the inequality

$$\theta(t)\leq\sum_{k=t_0}^{t-1}\left\|\prod_{s=k+1}^{t-1}a(s)\right\|_{\mathcal{X}}\left(\sum_{j=-\infty}^{k}m_1(k,j)\theta(j)+\sum_{j=k+1}^{\infty}m_2(k,j)\theta(j)\right),$$

and results in $\theta(t) = 0$ due to **C5**. Therefore, $x(\cdot)$ is a discrete almost automorphic solution of (4). The proof is complete. □

Remark 2. *As underlined in [21] (Remark 4.5), it is worth noting that relative compactness can be replaced with boundedness in the statement of Theorem 5 when $\mathcal{X}$ is finite dimensional.*

Remark 3. *As a direct consequence of Theorem 5, one may easily conclude that any solution of the nonlinear difference Equation (13) given in Example 1 with a relatively compact range is discrete almost automorphic.*

Example 2. *Let $\mathcal{X} = \mathbb{R}$ and Ω be an irrational number. Consider the following infinite delayed Volterra difference equation*

$$x(t+1) = \frac{1}{4}\sin\left(\frac{1}{2+\cos t+\cos(\Omega t)}\right) + \sum_{j=-\infty}^{t-1}\frac{1}{10}\left(\frac{1}{2}\cos\left(\frac{\pi j}{4}\right)\right)^{t-j}\frac{1}{1+x^2(j)}, \tag{21}$$

which is a particular form of (4). A direct comparison between (21) and (4) results in

$$a(t) = \frac{1}{4}\sin\left(\frac{1}{2+\cos t+\cos(\Omega t)}\right),$$

$$\Lambda_1(t,s,x) = \frac{1}{10}\left(\frac{1}{2}\cos\left(\frac{\pi j}{4}\right)\right)^{t-j}\frac{1}{1+x^2(j)},$$

and

$$\Lambda_2(t,s,x) = 0.$$

*Here we use the initial data $x(0) = x_0$ by assuming **C4** holds. Notice that a is discrete almost automorphic (see [42] (Remark 2.2) and [19] (Remark 2.2)) and Λ_1 is bi-periodic in t and s; therefore, it is discrete bi-almost automorphic in t and s, uniformly for x. Thus, the conditions **C1** and **C2** are satisfied. Moreover, we have*

$$\begin{aligned}|\Lambda_1(t,s,x)-\Lambda_1(t,s,y)| &\le \frac{1}{10}\left|\left(\frac{1}{2}\cos\left(\frac{\pi j}{4}\right)\right)^{t-j}\right|\left|\frac{1}{1+x^2}-\frac{1}{1+y^2}\right| \\ &\le \frac{1}{10}\left|\left(\frac{1}{2}\cos\left(\frac{\pi j}{4}\right)\right)^{t-j}\right||x-y|,\end{aligned}$$

where

$$m_1(t,s) = \frac{1}{10}\left|\left(\frac{1}{2}\cos\left(\frac{\pi j}{4}\right)\right)^{t-j}\right|.$$

*Consequentially, the condition **C3** is satisfied. In addition, one may easily verify that **C5** holds. Due to Lemma 1, the solution of (21) can be written as*

$$\begin{aligned}x(t) = {}& x_0\prod_{k=0}^{t-1}\left(\frac{1}{4}\sin\left(\frac{1}{2+\cos k+\cos(\Omega k)}\right)\right) \\ &+ \sum_{k=0}^{t-1}\left(\prod_{s=k+1}^{t-1}\left(\frac{1}{4}\sin\left(\frac{1}{2+\cos s+\cos(\Omega s)}\right)\right)\right)\sum_{j=-\infty}^{k}\frac{1}{10}\left(\frac{1}{2}\cos\left(\frac{\pi j}{4}\right)\right)^{k-j}\frac{1}{1+x^2(j)},\end{aligned}$$

and it is bounded. Then, Theorem 5 implies that it is discrete almost automorphic.

4. Conclusions

This study focuses on certain kinds of nonlinear difference equations, and provides an elaborative analysis on the existence of discrete almost automorphic solutions under sufficient conditions by fixed point theory. Utilization of the contraction mapping principle in the construction of the main results enables us to obtain the sufficient conditions regarding the existence and uniqueness of the solutions swiftly and elementarily. In addition to the main outcomes regarding the existence and uniqueness of almost automorphic solutions, the present work provides a discrete Bohr–Neugebauer-type theorem, and polishes the relationship between the existence of bounded and discrete almost automorphic solutions. To the best of our knowledge, our paper is the first to propose a Bohr–Neugebauer-type result for difference equations. Below, we exhibit our concluding remarks and future directions.

A. Difference equations are a very nice outlet to study some real-life models with a biological background. From our mathematical point of view, it may be an interesting task to relate almost periodic and almost automorphic functions with biological models in a similar direction as paper [43], which focuses on periodic functions. Moreover, the abstract outcomes of this study can be implemented on some particular models in real-life processes.

B. As is well-known, there are various classes of almost automorphic functions, such as asymptotically almost automorphic functions, Weyl almost automorphic functions, Besicovitch almost automorphic functions, and Stepanov almost automorphic functions. These specific notions can be adapted to the abstract difference equations that are considered in this work, and the outcomes of this manuscript can be reestablished.

C. The theory of time scales has become a hot topic in the last two decades since it avoids separate studies of differential and difference equations. In the existing literature, scholars have generalized already established theories on hybrid time domains. Motivated by this popularity, one may unify the main outcomes of this research on time scales that are translation invariant.

D. Quantum difference equations are an alternative to ordinary difference equations for discretization of differential equations. Indeed, q-difference equations provide better approximations for differential equations in particular cases. It is always an attracting task to focus on periodic and almost periodic structures on quantum domains since they are not translation invariant, i.e., quantum domains are not closed under addition. As a continuation of this study, it might be an interesting task to obtain a Bohr–Neugebauer-type theorem for q-difference equations by drawing on inspiration from the manuscripts [20,44].

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References

1. Avron, J.E.; Simon, B. Almost periodic Hill's equation and the rings of Saturn. *Phys. Rev. Lett.* **1981**, *46*, 1166–1168. [CrossRef]
2. Ohta, M.; Koizumi, T. Digital simulation of a white noise model formed of uniformly almost periodic functions. *Inf. Control* **1970**, *17*, 340–358. [CrossRef]
3. Ohta, M.; Hiromitsu, S. A trial of a new formation of the random noise model by use of arbitrary uniformly almost periodic functions. *Inf. Control* **1977**, *33*, 227–252. [CrossRef]
4. Bohr, H. Zur theorie der fastperiodischen funktionen I. *Acta Math.* **1925**, *45*, 29–127. [CrossRef]
5. Besicovitch, A.S. On generalized almost periodic functions. *Proc. Lond. Math. Soc.* **1926**, *s2-25*, 495–512. [CrossRef]
6. Bochner, S. Beitrage zur theorie der fastperiodischen funktionen. *Math. Ann.* **1926**, *96*, 119–147. [CrossRef]
7. Bochner, S.; von Neumann, J. Almost periodic function in a group II. *Trans. Am. Math. Soc.* **1935**, *37*, 21–50.
8. Stepanov, W. Über einige verallgemeinerungen der fastperiodischen funktionen. *Math. Ann.* **1925**, *45*, 473–498. [CrossRef]

9. Bochner, S. Continuous mappings of almost automorphic and almost periodic functions. *Proc. Nat. Acad. Sci. USA* **1964**, *52*, 907–910. [CrossRef]
10. Besicovitch, A.S. *Almost Periodic Functions*; Cambridge University Press: Cambridge, UK, 1954.
11. Veech, W.A. On a theorem of Bochner. *Ann. Math.* **1967**, *86*, 117–137. [CrossRef]
12. Bayliss, A. Almost Periodic Solutions to Difference Equations. Ph.D. Thesis, New York University, New York, NY, USA, 1975.
13. Diagana, T. *Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces*; Springer: Berlin/Heidelberg, Germany, 2013. [CrossRef]
14. N'Guerekata, G.M. *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*; Springer: Berlin/Heidelberg, Germany, 2001. [CrossRef]
15. Vesely, M. Constructions of Almost Periodic Sequences and Functions and Homogeneous Linear Difference and Differential Systems. Ph.D. Thesis, Masaryk University, Brno, Czech Republic, 2011.
16. Adıvar, M.; Koyuncuoglu, H.C. Almost automorphic solutions of discrete delayed neutral system. *J. Math. Anal. Appl.* **2016**, *435*, 532–550. [CrossRef]
17. Adıvar, M.; Koyuncuoglu, H.C.; Raffoul, Y.N. Almost automorphic solutions of delayed neutral dynamic systems on hybrid domains. *Appl. Anal. Discret. Math.* **2016**, *10*, 128–151. [CrossRef]
18. Koyuncuoglu, H.C.; Adıvar, M. Almost periodic solutions of Volterra difference systems. *Dem. Math.* **2017**, *50*, 320–329. [CrossRef]
19. Araya, D.; Castro, R.; Lizama, C. Almost automorphic solutions of difference equations. *Adv. Differ. Equ.* **2009**, *2009*, 1–69. [CrossRef]
20. Bohner, M.; Mesquita, J.G. Almost periodic functions in quantum calculus. *Electron. J. Differ. Equ.* **2018**, *2018*, 1–11.
21. Chávez, A.; Pinto, M.; Zavaleta, U. On almost automorphic type solutions of abstract integral equations, a Bohr-Neugebauer type property and some applications. *J. Math. Anal. Appl.* **2021**, *494*, 124395. . [CrossRef]
22. Castillo, S.; Pinto, M. Dichotomy and almost automorphic solution of difference system. *Electron. J. Qual. Theory Differ. Equ.* **2013**, *32*, 1–17. [CrossRef]
23. Diagana, T.; Elaydi, S.; Yakubu, A. Population models in almost periodic environments. *J. Differ. Equ. Appl.* **2007**, *13*, 239–260. [CrossRef]
24. Kostić, M.; Koyuncuoğlu, H.C. Multi-dimensional almost automorphic type sequences and applications. *Georgian Math. J.* **2024**, *in press*.
25. Lizama, C.; Mesquita, J.G. Almost automorphic solutions of dynamic equations on time scales. *J. Funct. Anal.* **2013**, *265*, 2267–2311. [CrossRef]
26. Lizama, C.; Mesquita, J.G. Almost automorphic solutions of non-autonomous difference equations. *J. Math. Anal. Appl.* **2013**, *407*, 339–349. [CrossRef]
27. Mishra, I.; Bahuguna, D.; Abbas, S. Existence of almost automorphic solutions of neutral functional differential equation. *Nonlinear Dyn. Syst. Theory* **2011**, *11*, 165–172.
28. Massera, J.L. The existence of periodic solutions of systems of differential equations. *Duke Math. J.* **1950**, *17*, 457–475. [CrossRef]
29. Bohr, H.; Neugebauer, O. Über lineare differentialgleichungen mit konstanten koeffizienten und fastperiodischer rechter seite. *Nachr. Ges. Wiss. Gött. Math.-Phys. Kl.* **1926**, *1926*, 8–22.
30. Floquet, G. Sur les équations différentielles linéaires à coefficients périodiques. *Ann. Sci. L'ÉCole Norm. SupÉRieure* **1883**, *12*, 47–88. [CrossRef]
31. Favard, J. Sur les équations différentielles linéaires à coefficients presque-périodiques. *Acta Math.* **1928**, *51*, 31–81. [CrossRef]
32. Benkhalti, R.; Es-sebbar, B.; Ezzinbi, K. On a Bohr-Neugebauer property for some almost automorphic abstract delay equations. *J. Integral Equ. Appl.* **2018**, *30*, 313–345. [CrossRef]
33. Dads, E.A.; Es-sebbar, B.; Lhachimi, L. On Massera and Bohr-Neugebauer type theorems for some almost automorphic differential equations. *J. Math. Anal. Appl.* **2023**, *518*. [CrossRef]
34. Drisi, N.; Es-sebbar, B. A Bohr-Neugebauer property for abstract almost periodic evolution equations in Banach spaces: Application to a size-structured population model. *J. Math. Anal. Appl.* **2017**, *456*, 412–428. [CrossRef]
35. Es-sebbar, B.; Ezzinbi, K.; N'Guérékata, G.M. Bohr-Neugebauer property for almost automorphic partial functional differential equations. *Appl. Anal.* **2019**, *98*, 381–407. [CrossRef]
36. Liu, J.; N'Guerekata, G.M.; Van Minh, N. A Massera type theorem for almost automorphic solutions of differential equations. *J. Math. Anal. Appl.* **2004**, *299*, 587–599. [CrossRef]
37. Liu, Q.; Van Minh, N.; N'Guerekata, G.M.; Yuan, R. Massera type theorems for abstract functional differential equations. *Funkcial. Ekvac.* **2008**, *51*, 329–350. [CrossRef]
38. Murakami, S.; Hino, Y.; Van Minh, N. Massera's theorem for almost periodic solutions of functional differential equations. *J. Math. Soc. Japan* **2004**, *56*, 247–268. [CrossRef]
39. Radova, L. Theorems of Bohr-Neugebauer-type for almost-periodic differential equations. *Math. Slovaca* **2004**, *54*, 191–207.
40. Van Minh, N.; Minh, H.B. A Massera-type criterion for almost periodic solutions of higher-order delay or advance abstract functional differential equations. *Abstr. Appl. Anal.* **2004**, *2004*, 881–896. [CrossRef]
41. Veech, W.A. Almost automorphic functions on groups. *Am. J. Math.* **1965**, *87*, 719–751. [CrossRef]

42. Ding, H.S.; Wan, S.M. Asymptotically almost automorphic solutions of differential equations with piecewise constant argument. *Open Math.* **2017**, *15*, 595–610. [CrossRef]
43. Cuchta, T.; Wintz, N. Periodic functions related to the Gompertz difference equation. *Math. Biosci. Eng.* **2022**, *19*, 8774–8785. [CrossRef]
44. Li, Y. Almost automorphic functions on the quantum time scale and applications. *Discrete Dyn. Nat. Soc.* **2017**, *2017*. [CrossRef]