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A System of Coupled Impulsive Neutral Functional Differential Equations: New Existence Results Driven by Fractional Brownian Motion and the Wiener Process

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Abstract: Conditions for the existence and uniqueness of mild solutions for a system of semilinear impulsive differential equations with infinite fractional Brownian movements and the Wiener process are established. Our approach is based on a novel application of Burton and Kirk's fixed point theorem in extended Banach spaces. This paper aims to extend current results to a differential-inclusions scenario. The motivation of this paper for impulsive neutral differential equations is to investigate the existence of solutions for impulsive neutral differential equations with fractional Brownian motion and a Wiener process (topics that have not been considered and are the main focus of this paper).

Keywords: mathematical model; stochastic systems; wiener process; fractional derivatives; impulsive differential equations; matrix; generalized Banach space; iterative methods; differential equations

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1. Introduction

Stochastic differential equations describe many real and practically important problems in modern physics, biology, economics, cybernetics, etc. Impulse differential equations are a suitable mathematical model for financial processes. Among the various questions that arise when solving such problems, one of the most important is the question of the existence and stability of solutions to stochastic functional differential equations with impulse influences, see [1–4].

Stochastic time PDEs with variable deviating (delayed) arguments have long attracted the attention of researchers, with the first results dating back to the 18th century. To study the existence and uniqueness of mild solutions for such systems, a premise less strict than the Lipschitz condition in nonlinear terms was used [5,6].

Many social, biological, physical, and engineering problems can be modeled using random differential and integral equations [7–11]. For example, Tsokos and Padgett considered a stochastic distribution model of drugs in a biological problem in [7].

Neutral differential equations are prevalent in various areas of applied mathematics and are used to model numerous phenomena and evolutionary processes in the natural sciences, including population dynamics, chemical technology, and physics. These phenomena may experience brief perturbations or abrupt state changes, which can be conceptualized as impulses. It is also well known that numerous applications in communications, mechanics, electrical engineering, biology, medicine, and other fields involve impulsive elements. Following the initial consideration of differential equations with impulses by [12], there has been a period of vigorous research in this area. Due to this reason, these types of systems have received much attention in recent decades. The literature on ordinary neutral functional differential equations is abundant; we refer the reader to [13–16].

Studies on the existence of systems with impulsive functional differential equations have gained a broad scope after their emergence. Numerous works by mathematicians are dedicated to the study of these issues in impulse differential equations, employing various functional methods. However, for impulsive neutral differential equations, the exploration of solutions involving fractional Brownian motion and a Wiener process has not been previously considered; see [17–21]. Motivated by the previous works, in the present paper, we consider and analyze the impact of impulsive conditions, fractional Brownian motion (FBM), and the Wiener process on the existence of solutions to the system (1)–(3). In addition, to the best of our acknowledge, there are no results concerning coupled systems under impulsive conditions. Under suitable assumptions on the functions $(f_i)_{i\in\{1,2\}}$, $(g_i)_{i\in\{1,2\}}$, we prove the existence and uniqueness of solutions to the system (1)–(3). To this end, let $\mathcal{J}=[0,b]$, $\mathcal{J}_0=(-\infty,0]$, and

$$\mathcal{J}_\mu=\begin{cases}(t_{\mu-1},t_\mu], & \text{for } \mu=1,2,\dots,m\\ \mathcal{J}_0 & \text{if } \mu=0.\end{cases}$$

The existence of solutions for a set of the following kinds of systems (the stochastic impulsive differential equations) is the subject of this paper:

$$\begin{cases} d[x(t)-h^1(t,x_t,w_t)] & =[Ax(t)+f^1(t,x_t,w_t)]dt\\ & +\sum_{l=1}^{\infty}\varpi_l^1(t)dB_l^a(t)\\ & +g^1(t)dW(t),\ t\in\mathcal{J},\\ d[w(t)-h^2(t,x_t,w_t)] & =[Aw(t)+f^2(t,x_t,w_t)]dt\\ & +\sum_{l=1}^{\infty}\varpi_l^2(t)dB_l^a(t)\\ & +g^2(t)dW(t),\quad t\in\mathcal{J}_0,\end{cases} \tag{1}$$

$$\begin{cases}\Delta_x x|_{t=t_\mu} & =x(t_\mu^+)-x(t_\mu)=I_\mu(x(t_\mu),w(t_\mu)),\ \mu=1,\dots,m,\\ \Delta_x w|_{t=t_\mu} & =w(t_\mu^+)-w(t_\mu)=\bar{I}_\mu(x(t_\mu),w(t_\mu)),\ \mu=1,\dots,m,\end{cases} \tag{2}$$

$$\begin{cases} x(t)=\Phi(t),\\ w(t)=\bar{\Phi}(t).\end{cases}\ \text{for } a.e\ t\in\mathcal{J}_0, \tag{3}$$

Let $\mathcal{H}$ be the well-known real separable Hilbert space, and its inner product $\langle .,.\rangle$ is brought about by the norm $\|.\|$. Here, let $(S(t))_{t\in\mathbb{R}_+}$ be the linear operators in $\mathcal{H}$, which are bounded, where

(i) The operator A: $D(A)\subset\mathcal{H}\to\mathcal{H}$ is an infinitesimal generator of a strongly continuous semi-group of $S(t)$.

(ii) B_l^a is the infinite sequence of independent fractional Brownian motions, $l\in\mathbb{N}^*$, with the Hurst parameter, $\mathcal{H}$.

(iii) $I_\mu,\bar{I}_\mu\in\mathcal{C}(\mathcal{H},\mathcal{H}),\forall\mu=1,2,\dots,m$.

We denote by $L^0_{\chi_i}(\mathcal{H}',\mathcal{H})$ the space of all χ_i-Hilbert–Schmidt operators from $\mathcal{H}'$ into $\mathcal{H}$, and let $L^0(\mathcal{H}',\mathcal{H})=L_2(\chi^{1/2}\mathcal{H}',\mathcal{H})$ be a separable Hilbert space, defined with respect to the Hilbert–Schmidt norm $\|.\|_{L^0}$. Here, χ is a Wiener process on $(\Omega,\mathcal{F},\mathbb{P})$ with a linear bounded covariance operator χ, such that $\mathrm{tr}\chi<\infty$. Let

$$\{W(t) : t \in \mathbb{R}\},$$

be a standard cylindrical Wiener process valued in $\mathcal{H}'$ with a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ that is furnished with an increasing family of right-continuous ϖ-algebras $\{\mathcal{F}_t, t \in \mathcal{J}\}$, verifying $\mathcal{F}_t \subset \mathcal{F}$, where $\mathcal{D}_{\mathcal{F}_0}$ is a linear space of families of $\mathcal{F}_0$-measurable functions from $(-\infty, 0]$ into $\mathcal{H}$, which will be explained further in the following context. The fixed time t_μ satisfies $0 < t_1 < t_2 < \ldots < t_m < b$, where $x(t_\mu^+)$ denotes the right limits of $x(t)$ when $t = t_\mu$ and $x(t_\mu^-)$ denotes the left limits. Regarding x_t, it is understood as a segment solution, defined in a standard manner. Specifically, if $x(.,.) \in ((-\infty, b] \times \Omega, X)$, then $\forall 0 \leq t, x_t(.,.) : (-\infty, 0] \times \Omega \to X$ will be defined by

$$x_t(\varrho, \omega) = x(t + \varrho, \omega), \text{ for } \omega \in \Omega, -\infty < \varrho \leq 0,$$

$$\begin{cases} \varpi(t) = (\varpi_1(t), \varpi_2(t), \ldots), \\ \|\varpi(t)\|^2 = \sum_{l=1}^{\infty} \|\varpi_l(t)\|^2_{L^0_\chi} < \infty, \end{cases}$$

where $\varpi \in \ell^2$, with

$$\ell^2 = \left\{ \Phi = (\Phi_l)_{l \geq 1} : [0, T] \to L^0_\chi(\mathcal{H}', \mathcal{H}) : \|\Phi(t)\|^2 = \sum_{l=1}^{\infty} \|\Phi_l(t)\|^2_{L^0_\chi} < \infty \right\}.$$

System (1)–(3) can be viewed as a fixed point issue, as in [22]

$$\begin{cases} d(\theta(t) - & h(t, \theta_t)) = A_* \theta(t) + f(t, \theta_t) dt + g(t) dW(t) \\ & + \sum_{l=1}^{\infty} \varpi_l(t) dB^a_l(t), \ t \in \mathcal{J}, t \neq t_\mu, \\ \Delta_x \theta(t) & = I^*_\mu(\theta(t_\mu)), \quad t = t_\mu \quad \mu = 1, 2, \ldots, m \\ \theta(t) & = \theta_0, \quad t \in \mathcal{J}_0, \end{cases}$$

where

$$\theta_t = \begin{bmatrix} x_t \\ w_t \end{bmatrix}, A_* = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \ f(t, \theta_t) = \begin{bmatrix} f^1(t, x_t, w_t) \\ f^2(t, x_t, w_t) \end{bmatrix}, \ \varpi_l(t) = \begin{bmatrix} \varpi^1_l(t) \\ \varpi^2_l(t) \end{bmatrix}$$

and

$$\theta_0 = \begin{bmatrix} x_0 \\ w_0 \end{bmatrix}, \ h(t, \theta_t) = \begin{bmatrix} h^1(t, x_t, w_t) \\ h^2(t, x_t, w_t) \end{bmatrix}, \ g(t) = \begin{bmatrix} g^1(t) \\ g^2(t) \end{bmatrix}.$$

We will introduce some nomenclature and define certain spaces before discussing the conditions met by operators f^i, h^i, ϖ^i and $I_\mu, \bar{I}_\mu$.

In this study, we will use the Hale and Kato [23] axiomatic description of the phase space $\mathcal{D}_{\mathcal{F}_0}$.

Definition 1. *The following axioms must be met for $\mathcal{D}_{\mathcal{F}_0}$ to be a linear space of a family of $\mathcal{F}_0$-measurable functions from $(-\infty, 0] \to \mathcal{H}$ with the norm of $\|.\|_{\mathcal{D}_{\mathcal{F}_0}}$,*

(i) *If $x : (-\infty, b] \to X$, for $0 < b$ be so that $(\mathcal{H}_0, w_0) \in \mathcal{D}_{\mathcal{F}_0} \times \mathcal{D}_{\mathcal{F}_0}$, then for any $0 \leq t < b$, the following conditions hold:*

- (a) $x_t \in \mathcal{D}_{\mathcal{F}_0}$
- (b) $\|x(t)\| \leq L \|x_t\|_{\mathcal{D}_{\mathcal{F}_0}}$
- (c) $\|x_t\|_{\mathcal{D}} \leq \nu(t) \sup\{\|x(\tau)\| : 0 \leq s \leq t\} + N(t) \|x_0\|_{\mathcal{D}_{\mathcal{F}_0}}$,
  where $L > 0$ $\nu, N \in C([0, \infty), [0, \infty))$ are independent of $x(.)$, and N is locally bounded.

(ii) *The function $x(.)$ introduced in (i), x_t is a $\mathcal{D}_{\mathcal{F}_0}$-valued function on $[0, b)$.*

(iii) *The space $\mathcal{D}_{\mathcal{F}_0}$ is complete.*

We denote for $t \in \mathcal{J}$

$$\widehat{\nu} = \sup\{\nu(t)\},$$
$$\widehat{N} = \sup\{N(t)\}.$$

Then, we establish the value of $b > 0$ by

$$\mathcal{D}_{\mathcal{F}_b} = \Big\{x : (-\infty, b] \times \Omega \to X,\ x_\mu \in C(\mathcal{J}_\mu, X), \mu = 1, \dots m,\ x_0 \in \mathcal{D}_{\mathcal{F}_0},\ \text{there exist}$$
$$x(t_\mu^-) \text{ and } x(t_\mu^+) \text{ with } x(t_\mu) = x(t_\mu^-),\ \mu = 1, \dots, m, \text{ and } \sup_{t \in \mathcal{J}} E(|x(t)|^2) < \infty\Big\},$$

with the norm

$$\|x\|_{\mathcal{D}_{\mathcal{F}_b}} = \|\Phi\|_{\mathcal{D}_{\mathcal{F}_0}} + \sup_{0 \le \tau \le b} (\sqrt{\mathbb{E}\|x(\tau)\|^2}),$$

where the restriction of x to $\mathcal{J}_\mu$ is denoted by x_μ, with

$$\mathcal{J}_\mu = \begin{cases} (t_{\mu-1}, t_\mu], & \text{for } \mu = 1, 2, \dots, m \\ (-\infty, 0] & \text{if } \mu = 0. \end{cases}$$

Then, we shall think about our original data, $\Phi, \bar{\Phi} \in \mathcal{D}_{\mathcal{F}_0}$; we assume that $h^i : \mathcal{J} \times \mathcal{D}_{\mathcal{F}_0} \times \mathcal{D}_{\mathcal{F}_0} \to \mathcal{H}$, $f^i : \mathcal{J} \times \mathcal{D}_{\mathcal{F}_0} \times \mathcal{D}_{\mathcal{F}_0} \to X$ and $\varpi^i : \mathcal{J} \to L^0_{\chi_i}(\mathcal{H}', \mathcal{H})$ and $g^i : \mathcal{J} \to L^0(\mathcal{H}', \mathcal{H})$.

The structure of the essay is as follows: We review several crucial antecedents in Section 2. Based on the Burton and Kirk theorem, we demonstrate certain results related to the existence in Section 3. Finally, the main theorem is demonstrated with an example in Section 4.

2. Preliminaries and Tools

In this section, we introduce several notations, review definitions, and provide some background information that will be used throughout the study. Although we could refer to relevant references as needed, it is important to note that these notations and definitions are derived from multiple sources.

Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is an exhaustive probability space and that the filtering $(\mathcal{F} = \mathcal{F}_t)_{t \in \mathbb{R}_+}$ satisfies the standard requirements (i.e., right-continuous and having $\mathcal{F}_0$ contain all $\mathbb{P}$-null sets).

If there is no chance of confusion, we will consider $x(t)$ rather than $x(t, \omega)$ for the stochastic process

$$x(.,.) : [0, T] \times \Omega \to X.$$

Definition 2. *Given $a \in (0, 1)$, it is claimed that a continuous Gaussian process B^a is a mono-dimensional FBM with two sides and the Hurst parameter $\mathcal{H}$, if*

$$R_a(t, \tau) = E[B^a(t))B^a(\tau)], \tag{4}$$

meets the following condition:

$$R_a(t, \tau) = \frac{1}{2}(|t|^{2a} + |\tau|^{2a} - |t - \tau|^{2a}) \quad 0 \le t, \tau \le T.$$

where (4) is its covariance function.

The following Volterra representation is known to be admitted by $B^a(t)$ with $a > \frac{1}{2}$,

$$B^a(t) = \int_0^t \nu_a(t, \tau) dB(\tau) d\tau, \tag{5}$$

the usual Brownian motion represented by B is

$$B(t) = B^a((\nu_a^*)^{-1}\zeta_{[0,T]}).$$

Moreover, the well-known Volterra kernel $\nu_a(t,\tau)$ is provided as

$$\nu_a(t,\tau) = \begin{cases} c_a \tau^{1/2-a} \int_\tau^t (u-\tau)^{a-\frac{3}{2}} \left(\frac{u}{\tau}\right)^{a-\frac{1}{2}} du, & \text{if } t \geq \tau, \\ 0, & \text{if } t < \tau, \end{cases}$$

where $c_a = \sqrt{\frac{a(2a-1)}{\sigma(2a-2,a-\frac{1}{2})}}$. The following defines the kernel ν_a^*. The collection of step functions on $[0,T]$ is denoted by $\mathcal{E}$. Suppose that $\mathcal{H}$ represents the Hilbert space, which is the closure of the space $\mathcal{E}$ with regard to

$$\langle \zeta_{[0,T]}, \zeta_{[0,\tau]} \rangle_{\mathcal{H}} = R_a(t,\tau).$$

Likewise, take into account the linearity of the operator ν_a^* defined by $\mathcal{E}$ into $L^2([0,T])$, given as

$$(\nu_a^*\Phi)(t) = \int_\tau^T \Phi(s) \frac{\partial \nu_a}{\partial s}(s,\tau) ds.$$

We note that

$$(\nu_a^* \zeta_{[0,T]})(\tau) = \nu_a(t,\tau)\zeta_{[0,T]}(\tau).$$

When extended to a Hilbert space, $\mathcal{H}$, the operator, ν_a^*, is isometric between $\mathcal{E}$ and $L^2([0,T])$. For any $0 \leq s,t \leq T$, we actually have

$$\langle \nu_a^* \zeta_{[0,T]}, \nu_a^* \zeta_{[0,T]} \rangle_{L^2([0,T])} = \langle \zeta_{[0,T]}, \zeta_{[0,\tau]} \rangle_{\mathcal{H}} = R_a(t,\tau).$$

Additionally, $\forall \Phi \in \mathcal{H}$,

$$\int_0^T \Phi(s) dB^a(s) ds = \int_0^T (\nu_a^*\Phi)(s) dB(s) ds, \ \nu_a^*\Phi \in L^2([0,T]).$$

Given $(\mathcal{H}, \langle .,. \rangle_{\mathcal{H}})$ and $(\mathcal{H}', \langle .,. \rangle_{\mathcal{H}'})$ as separable Hilbert spaces, with $(e_n)_{n\in\mathbb{N}}$ being a complete orthonormal basis in $\mathcal{H}'$, let $\mathcal{L}(\mathcal{H}',\mathcal{H})$ denote the space of all linear bounded operators from $\mathcal{H}'$ into $\mathcal{H}$. Let $\chi \in \mathcal{L}(\mathcal{H}',\mathcal{H})$ be an operator given by

$$Qe_n = \alpha_n e_n, \text{ with } \alpha_n \geq 0, \forall n \in \mathbb{N}, \text{ and } \mathrm{tr}\chi = \sum_{n=1}^{\infty} \alpha_n < \infty.$$

Let $(\sigma_n^a)_{n\in\mathbb{N}}$ be a sequence of two-sided one-dimensional standard fractional Brownian motions, mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$. We define the infinite-dimensional fBm on $\mathcal{H}'$ with covariance χ, as follows:

$$B^a(t) = \sum_{n=1}^{\infty} \sqrt{\alpha_n} \sigma_n^a(t) e_n, \tag{6}$$

This can be accurately described as a $\mathcal{H}'$-valued χ-cylindrical FBM, in the style described in [8]; thus, we have

$$E\langle \sigma_l^a(t), x \rangle \langle \sigma_\mu^a(\tau), w \rangle = R_{a_{l\mu}}(t,\tau) \langle \chi(x), w \rangle, \forall x, w \in \mathcal{H}', \ 0 \leq s,t \leq T,$$

such that

$$R_{a_{l\mu}} = \frac{1}{2}\Big(|t|^{2a} + |\tau|^{2a} + |t-\tau|^{2a}\Big)\delta_{l\mu}, \quad 0 \leq t,\tau \leq T,$$

where

$$\delta_{li} = \begin{cases} 1 & i = l, \\ 0, & i \neq l. \end{cases}$$

We establish the space $L_\chi^0 := L_\chi^0(\mathcal{H}', \mathcal{H})$ of all χ-Hilbert–Schmidt operators

$$\varphi : \mathcal{H}' \to \mathcal{H},$$

to define the Wiener integrals with regard to the χ-fractional Brownian motion. We should not forget that $\varphi \in L(\mathcal{H}', \mathcal{H})$ is referred to as a χ-Hilbert–Schmidt operator, if

$$\|\varphi\|_{L_\chi^0}^2 = \|\varphi\chi^{1/2}\|_{aS}^2 = \mathrm{tr}(\varphi\chi\varphi^*) < \infty.$$

Definition 3. *A function having values in $L_\chi^0(\mathcal{H}', \mathcal{H})$) is defined as Φ. According to (6), the Wiener integral, with regard to fBm, is defined as*

$$\int_0^T \Phi(\tau) dB^a(\tau) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\alpha_n} \Phi(\tau) e_n d\sigma_n^a = \sum_{n=1}^{\infty} \int_0^T \sqrt{\alpha_n} \nu_a^*(\Phi e_n)(\tau) d\sigma_n.$$

If we have

$$\sum_{n=1}^{\infty} \|\Phi\chi^{1/2} e_n\|_{L^{1/a}([0,T];X)} < \infty, \tag{7}$$

the following outcome guarantees the series convergence in the preceding definition.

Lemma 1 ([24])**.** *If*

$$\Phi : [0, T] \to L_\chi^0(\mathcal{H}', \mathcal{H}),$$

so that (7) holds, and $\forall 0 \leq \kappa, \sigma \leq T$ with $\kappa > \sigma$, we have

$$\mathbb{E}\left|\int_\kappa^\sigma \Phi(\tau) dB^a(\tau)\right|_X^2 \leq c_a a(2a-1)(\kappa - \sigma)^{2a-1} \sum_{n=1}^{\infty} \int_\kappa^\sigma \left|\Phi(\tau)\chi^{1/2} e_n\right|_X^2 d\tau.$$

For $0 \leq t \leq T$, if

$$\sum_{n=1}^{\infty} |\Phi\chi^{1/2} e_n|_X,$$

is uniformly convergent. Then

$$\mathbb{E}\left|\int_\kappa^\sigma \Phi(\tau) dB^a(\tau)\right|_X^2 \leq c_a a(2a-1)(\kappa - \sigma)^{2a-1} \int_\kappa^\sigma \|\Phi(\tau)\|_{L_\chi^0}^2 d\tau.$$

Let us now state the well-known Lemma [8], which will be used in the key result proofs in the following section.

Lemma 2. *For each $i = 1, 2$, for every $r \geq 1$, and for any $g^i(.)$, which is an L_0^2-valued predictable process, the following holds:*

$$\mathbb{E} \sup_{0 \leq s \leq T} \left|\int_0^\tau g(u) dW(u)\right|_X^{2r} \leq (r(2r-1))^r \left|\int_0^t \left|g^i(\tau)\right|_{L_0^2}^{2r} d\tau\right|^r.$$

3. Fixed Point Results

As in [25–27], the classical Banach contraction principle was expanded for contracted maps on spaces endowed with vector-valued metric space. Now, let us review some definitions and outcomes that are helpful.

Definition 4. *If and only if a square matrix of $\mathbb{R}$ has a spectral radius $\rho(M)$ that is strictly less than 1, it is said to be convergent to zero. In other words, all M eigenvalues are contained within an open unit disc, with $det(M - \alpha I) = 0$ and I standing for the unit matrix of $\mathcal{M}_{n\times n}(\mathbb{R}), \forall\alpha \in \mathbb{C}$.*

The following fixed point theorem attributed to [28] serves as the foundation for our major conclusion.

Theorem 1. *Let X be a Banach space, and $P_1, P_2 : X^2 \to X^2$ denote two operators, satisfying the following:*

1. *P_1 is a contraction,*
2. *P_2 is completely continuous*

Then, either

$$(x, w) = P_1(x, w) + P_2(x, w),$$

possesses a solution, or the set

$$\mathcal{M} = \left\{ x, w \in \mathcal{H} : \alpha P_1(\frac{(x, w)}{\alpha}) + \alpha P_2(x, w) = (x, w), \text{ for } 0 < \alpha < 1 \right\},$$

is unbounded.

The semigroup $S(t)$ must be uniformly bounded to obtain the continuity and the boundness of the operator in order to have the equi-continuity. We assume that the semigroup $S(t)$ is uniformly bounded; that is,

$$\exists \bar{M}_1 \geq 1 : \|S(t)\| \leq \bar{M}_1, \forall t \in \mathbb{R}_+.$$

We also assume that $0 \in \rho(A)$ (the resolvent set of A) and the semigroup $S(t)$ are both continuous. The fractional power operator $(-a)^\kappa$ can be defined for $0 < \kappa \leq 1$ as a closed linear operator on its domain $\mathcal{D}((-a)^\kappa)$. The subspace $\mathcal{D}((-a)^\kappa)$ is additionally dense in X. We designate as X_κ the Banach space $\mathcal{D}((-a)^\kappa)$ endowed with the norm

$$\|x\|_\kappa = \|(-a)^\kappa x\|,$$

which defines a norm on $\mathcal{D}((-a)^\kappa)$ that is identical to the graph norm of $(-a)^\kappa$. In the follow-up, we use the norm.$\|.\|_\kappa$ to denote X_κ, which stands for the space $\mathcal{D}((-a)^\kappa)$. Then, we have the following well-known characteristics that are mentioned in [29].

Lemma 3. *We have*

(A) *If $0 < \sigma < \kappa \leq 1$, then $X_\kappa \subset X_\sigma$, and when the resolvent operator of A is compact, the embedding is also compact.*
(B) *For any $0 < \kappa \leq 1$, $C_\kappa > 0$ occurs in the following way:*

$$\|(-a)^\kappa S(t)\| \leq \frac{C_\kappa}{t^\kappa} \exp(-\alpha t), \quad 0 < t, \quad \alpha > 0.$$

Lemma 4 ([30]). *Let*

$$y, v : \mathcal{J} \to [0, \infty),$$

be two continuous functions. If $y(.)$ is non-decreasing and $\exists \varrho > 0$, $0 < \kappa < 1$, so that

$$v(t) \leq y(t) + \varrho \int_0^t \frac{v(\tau)}{(t-\tau)^{1-\kappa}} d\tau, \quad t \in \mathcal{J}.$$

Thus,

$$v(t) \leq \exp(\varrho^n \Gamma(\kappa)^n t^{n\kappa} / \Gamma(n\kappa)) \sum_{j=0}^{n-1} \left(\frac{\varrho b^\kappa}{\kappa} \right)^j y(t),$$

for any $t \in \mathcal{J}$ and all $n \in \mathbb{N}$, where $n\kappa > 1$, and $\Gamma(.)$ is the Gamma function.

We will now present the idea of a modest solution to our problem.

Definition 5. *Let $v = (x, w) \in \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b}$ be the $X-$valued stochastic process. We note that it is the solution on (1)–(3) in the probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$, if*

(1) *Function $v(t)$ is $\mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b}$-adapted $\forall t \in \mathcal{J}_\mu$, $\quad \mu = 1, 2, \ldots, m$;*
(2) *Function $v(t)$ is right-continuous and has a limit on the left, almost surely;*
(3) *Function $aS(t-\tau)h^i(\tau, v_\tau)$ is integrable;*
(4) *Function $v(t)$ satisfies the conditions $\forall t \in \mathcal{J}$ and almost surely, as expressed by the following equation:*

$$\begin{cases} x(t) &= \Phi(t) \in \mathcal{D}_{\mathcal{F}_b} \quad -\infty < t \leq 0 \\ x(t) &= S(t)(\Phi(0) + h^1(0, \Phi, \bar{\Phi})) + h^1(t, x_t, w_t) + \int_0^t aS(t-s)h^1(s, x_s, w_s)ds \\ &+ \int_0^t S(t-s) f^1(s, x(s), w(s))ds + \sum_{l=1}^{\infty} \int_0^t S(t-s)\varpi_l^1(t)dB_l^a(s) \\ &+ \sum_{0<t_\mu<t} S(t-t_\mu) I_\mu(x(t_\mu), w(t_\mu)) + \int_0^t S(t-s)g^1(s)dW(s) \\ w(t) &= \bar{\Phi}(t) \in \mathcal{D}_{\mathcal{F}_b} \quad t \in (-\infty, 0] \\ w(t) &= S(t)(\bar{\Phi}(0) + h^2(0, \Phi, \bar{\Phi})) + h^2(t, x_t, w_t) + \int_0^t aS(t-s)h^2(s, x_s, w_s)ds \\ &+ \int_0^t S(t-s) f^2(s, x(s), w(s))ds + \sum_{l=1}^{\infty} \int_0^t S(t-s)\varpi_l^2(t)dB_l^a(s) \\ &+ \sum_{0<t_\mu<t} S(t-t_\mu) \bar{I}_\mu(x(t_\mu), w(t_\mu)) + \int_0^t S(t-s)g^2(s)dW(s). \end{cases}$$

We will need to use the following hypotheses. In this section, we assume that $M > 0$ occurs in such a way that

$$\|S(t)\| \leq M, \quad \text{for any } t \in \mathcal{J}.$$

(H_1) There exists a constant M so that A can be an infinitesimal generator of the analytic semigroup of the linear, bounded operators $(S(t))_{0<t}$, such that

$$\|S(t)\|^2 \leq M \quad \text{and} \quad \|(-a)^{1-\sigma}S(t)\| \leq M_{1-\sigma}t^{\sigma-1}, \forall 0 < t.$$

(H_2) $\exists\, 0 < \sigma < 1$, $c_{h^i} \geq 0$ and continuous-bounded function

$$p_i : \mathcal{J} \to \mathbb{R}^+,$$

such that h^i is X_σ-valued, $(-a)^\sigma h^i$ is continuous and

$$\mathbb{E}|(-a)^\sigma h^i(t, x, w)|^2 \leq p_i(t)(\|x\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|y\|^2_{\mathcal{D}_{\mathcal{F}_0}}), \quad t \in \mathcal{J},\ x, w \in \mathcal{D}_{\mathcal{F}_0},$$

and

$$\mathbb{E}|(-a)^\sigma h^i(t, x, w) - (-a)^\sigma h^i(t, \bar{x}, \bar{w})|^2 \leq c_h^i \|x - \bar{x}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \bar{c}_h^i \|w - \bar{w}\|^2_{\mathcal{D}_{\mathcal{F}_0}}, \quad t \in \mathcal{J},$$

$\forall x, w, \overline{x}, \overline{w} \in \mathcal{D}_{\mathcal{F}_0}$, with

$$M_{b,c} = 3\begin{pmatrix} \sqrt{\alpha_1} & \sqrt{\alpha_2} \\ \sqrt{\bar{\alpha}_1} & \sqrt{\bar{\alpha}_2} \end{pmatrix},$$

where

$$\alpha_1 = 4c_h^1\left(\|(-a)^{-\sigma}\|^2 + \frac{(C_{1-\sigma}b^\sigma)^2}{2\sigma - 1}\right)\tilde{\nu}^2, \quad \alpha_2 = 4\bar{c}_h^1\left(\|(-a)^{-\sigma}\|^2 + \frac{(C_{1-\sigma}b^\sigma)^2}{2\sigma - 1}\right)\tilde{\nu}^2,$$

and

$$\bar{\alpha}_2 = 4\bar{c}_h^2\bigg(\|(-a)^{-\sigma}\|^2 + \frac{(C_{1-\sigma}b^{\sigma})^2}{2\sigma-1}\bigg)\tilde{\nu}^2, \quad \bar{\alpha}_2 = 4\bar{c}_h^2\bigg(\|(-a)^{-\sigma}\|^2 + \frac{(C_{1-\sigma}b^{\sigma})^2}{2\sigma-1}\bigg)\tilde{\nu}^2,$$

where $\alpha_i, \bar{\alpha}_i \geq 0$, for each $i = 1, 2$. If $M_{b,c}$ converges to zero

(H_3) There are constants $(d_\mu)_{\mu\in\mathbb{N}}, (\bar{d}_\mu)_{\mu\in\mathbb{N}} > 0$, such that

$$\sum_{\mu=0}^{\infty} d_\mu < \infty, \sum_{\mu=0}^{\infty} \bar{d}_\mu < \infty,$$

with

$$|I_\mu(x,w)| \leq d_\mu, \text{ and } |\bar{I}_\mu(x,w)| \leq \bar{d}_\mu, \forall \mu \in \mathbb{N}, x, w \in X.$$

(H_4) f^i is a L^2-Carathéodory map, and for any $t \in \mathcal{J}$ for each $i = 1,2$ the functions $t \mapsto f^i(t, x(t), w(t))$ and $t \mapsto f^i(t, x(t), w(t)), x(t), w(t)$ are measurable.

(H_5) The function $\varpi^i : \mathcal{J} \to L_{\chi^i}(Y, x)$ satisfies

$$\sum_{l=1}^{\infty} \int_0^T \|\varpi_l^i(\tau)\|_{L_{\chi^i}}^2 d\tau < \infty.$$

(H_6) There is a non-decreasing function $\Psi \in C([0,\infty),[0,\infty))$ and $m \in L^1(\mathcal{J}, \mathbb{R}_+)$, such that

$$E|f^i(t,x,w)|^2 \leq m(t)\Psi(\|x\|_{\mathcal{D}_{\mathcal{F}_0}}^2 + \|y\|_{\mathcal{D}_{\mathcal{F}_0}}^2), \forall t \in \mathcal{J}, x, w \in \mathcal{D}_{\mathcal{F}_0}.$$

(H_7) The function $g^i : \mathcal{J} \to L^0(Y, x)$ satisfies

$$\int_0^b \|g^i(\tau)\|_{L^0}^2 d\tau = C_1 < \infty, \quad t \in \mathcal{J}.$$

Theorem 2. *Assume that* (H_1)–(H_7) *are true. Thus, problems (1)–(3) possess a unique mild solution on* $(-\infty, b]$.

Proof. Models (1)–(3) can be transformed into the fixed-point system. Let us now consider the operator

$$N : \mathcal{D}_{\mathcal{F}_b}^2 \to \mathcal{D}_{\mathcal{F}_b}^2,$$

defined by

$$N_1(x,w)(t) = \begin{cases} \Phi(t), \text{ if } t \in (-\infty, \leq 0], \\ S(t)[\Phi(t=0) - h^1(t=0,\Phi,\bar{\Phi})] + h^1(t,x_t,w_t) + \int_0^t aS(t-s)h^1(s,x_s,w_s)ds, \\ + \int_0^t S(t-s)f^1(s,x_s,w_s)ds + \sum\limits_{l=1}^{\infty} \int_0^t S(t-s)\varpi_l^1(s)dB^a(s), \\ + \int_0^t S(t-s)g^1(s)dW(s) + \sum\limits_{0<t_\mu<t} S(t-t_\mu)I_\mu(x(t_\mu^-), w(t_\mu^-)), \end{cases}$$

and

$$N_2(x,w)(t) = \begin{cases} \bar{\Phi}(t), \text{ if } t \in (-\infty, \leq 0], \\ S(t)[\bar{\Phi}(t=0) - h^2(t=0,\Phi,\bar{\Phi})] + h^2(t,x_t,w_t) + \int_0^t aS(t-s)h^2(s,x_s,w_s)ds, \\ + \int_0^t S(t-s)f^2(s,x_s,w_s)ds + \sum\limits_{l=1}^{\infty} \int_0^t S(t-s)\varpi_l^2(s)dB^a(s), \\ + \int_0^t S(t-s)g^2(s)dW(s) + \sum\limits_{0<t_\mu<t} S(t-t_\mu)\bar{I}_\mu(x(t_\mu^-), w(t_\mu^-)). \end{cases}$$

Put $\forall(\Phi, \bar{\Phi}) \in \mathcal{D}_{\mathcal{F}_0} \times \mathcal{D}_{\mathcal{F}_0}$

$$\varrho(t) = \begin{cases} \Phi(t), & \text{if } -\infty < t \leq 0, \\ S(t)\Phi(0), & \text{if } t \in \mathcal{J}, \end{cases}$$

and

$$\bar{\varrho}(t) = \begin{cases} \bar{\Phi}(t), & \text{if } -\infty < t \leq 0, \\ S(t)\bar{\Phi}(0) & \text{if } t \in \mathcal{J}. \end{cases}$$

It is clear that $(\varrho, \bar{\varrho}) \in \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b}$.
Set $(x(t), w(t)) = (\theta(t) + \varrho(t), \bar{\theta}(t) + \bar{\varrho}(t))$ for each $t \in (-\infty, \leq b]$, where

$$\theta(t) = \begin{cases} \theta_0, \text{ if } -\infty < t \leq 0, \\ -S(t)h^1(0, \Phi, \bar{\Phi}) + h^1(t, \theta_t + \varrho_t, \bar{\theta}_\tau + \bar{\varrho}_\tau), \text{ if } t \in \mathcal{J}, \\ + \int_0^t aS(t-\tau)h^1(\tau, \theta_\tau + \varrho_\tau, \bar{\theta}_\tau + \bar{\varrho}_\tau)d\tau + \int_0^t S(t-\tau)f^1(\tau, \theta_\tau + \varrho_\tau, \bar{\theta}_\tau + \bar{\varrho}_\tau)d\tau, \\ + \sum_{l=1}^{\infty} \int_0^t S(t-\tau)\varpi_l^1(\tau)dB^a(\tau) + \int_0^t S(t-\tau)g^2(\tau)dW(\tau), \\ + \sum_{0<t_\mu<t} S(t-t_\mu)I_\mu(\theta(t_\mu^-) + \varrho(t_\mu^-), \bar{\theta}(t_\mu^-) + \bar{\varrho}(t_\mu^-)), \end{cases}$$

and

$$\bar{\theta}(t) = \begin{cases} \bar{\theta}_0, \text{ if } t \in (-\infty, \leq 0], \\ -S(t)h^2(0, \Phi, \bar{\Phi}) + h^2(t, \theta_t + \varrho_t, \bar{\theta}_t + \bar{\varrho}_t) + \text{ if } t \in \mathcal{J}. \\ \int_0^t aS(t-\tau)h^2(\tau, \theta_\tau + \varrho_\tau, \bar{\theta}_\tau + \bar{\varrho}_\tau)d\tau + \int_0^t S(t-\tau)f^2(\tau, \theta_\tau + \varrho_\tau, \bar{\theta}_\tau + \bar{\varrho}_\tau)d\tau \\ + \sum_{l=1}^{\infty} \int_0^t S(t-\tau)\varpi_l^2(\tau)dB^a(\tau) + \int_0^t S(t-\tau)g^2(\tau)dW(\tau) + \\ \sum_{0<t_\mu<t} S(t-t_\mu)\bar{I}_\mu(\theta(t_\mu^-) + \varrho(t_\mu^-), \bar{\theta}(t_\mu^-) + \bar{\varrho}(t_\mu^-)). \end{cases}$$

Set $\widehat{\mathcal{D}}_{\mathcal{F}_b} = \{\theta, \bar{\theta} \in \mathcal{D}_{\mathcal{F}_b} : (\theta_0, \bar{\theta}_0) = (0,0)\}$, we have

$$\|\theta\|_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} = \|\theta_0\|_{\mathcal{D}_{\mathcal{F}_0}} + \sup_{t \in \mathcal{J}} \sqrt{\mathbb{E}\|\theta(t)\|^2} = \sup_{t \in \mathcal{J}} \sqrt{\mathbb{E}\|\theta(t)\|^2}.$$

It is not hard to verify that $(\widehat{\mathcal{D}}_{\mathcal{F}_b}, \|.\|_{\widehat{\mathcal{D}}_{\mathcal{F}_b}})$ is a Banach space. The set

$$\mathcal{B}_q = \left\{ x, w \in \widehat{\mathcal{D}}_{\mathcal{F}_b}, \quad \|x\|^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} \leq q \quad \text{and} \quad \|y\|^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} \leq q, \ q \geq 0 \right\},$$

is a closed bounded convex in $\widehat{\mathcal{D}}_{\mathcal{F}_b}$ for $0 \leq q$ and $x \in \mathcal{B}_q$, we have

$$\begin{aligned} & \|\theta_t + \varrho_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{\theta}_t + \bar{\varrho}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\ \leq \ & 2(\|\theta_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\varrho_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}) + 2(\|\bar{\theta}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{\varrho}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}) \\ \leq \ & 4(\widetilde{N}^2(\|\Phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{\Phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}) + \tilde{\nu}^2(2q + M(E|\Phi(0)|^2 + E|\bar{\Phi}(0)|^2)) \\ = \ & C_{std}. \end{aligned}$$

We consider the operator

$$\widehat{N} : \widehat{\mathcal{D}}^2_{\mathcal{F}_b} \to \widehat{\mathcal{D}}^2_{\mathcal{F}_b},$$

defined by

$$\widehat{N}(\theta, \bar{\theta}) = (\widehat{N}_1(\theta, \bar{\theta}), \widehat{N}_2(\theta, \bar{\theta}), \ (\theta, \bar{\theta}) \in \widehat{\mathcal{D}}_{\mathcal{F}_b} \times \widehat{\mathcal{D}}_{\mathcal{F}_b},$$

here

$$\widehat{N}_1(\theta,\bar{\theta}) = \begin{cases} 0, & \text{if } t \in (-\infty, \leq 0], \\ -S(t)h^1(0,\Phi,\bar{\Phi}) + h^1(t,\theta_t+\varrho_t,\bar{\theta}_\tau+\bar{\varrho}_\tau) \\ +\int_0^t aS(t-\tau)h^1(\tau,\theta_\tau+\varrho_\tau,\bar{\theta}_\tau+\bar{\varrho}_\tau)d\tau \\ +\int_0^t S(t-\tau)f^1(\tau,\theta_\tau+\varrho_\tau,\bar{\theta}_\tau+\bar{\varrho}_\tau)d\tau+ & \text{if } t \in \mathcal{J}, \\ +\sum_{l=1}^{\infty}\int_0^t S(t-\tau)\varpi_l^1(\tau)dB^a(\tau) + \int_0^t S(t-\tau)g^1(\tau)dW(\tau) \\ +\sum_{0<t_\mu<t} S(t-t_\mu)I_\mu(\theta(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}(t_\mu^-)+\bar{\varrho}(t_\mu^-)), \end{cases}$$

and

$$\widehat{N}_2(\theta,\bar{\theta}) = \begin{cases} 0, & \text{if } t \in (-\infty, \leq 0], \\ -S(t)h^2(0,\Phi,\bar{\Phi}) + h^2(t,\theta_t+\varrho_t,\bar{\theta}_t+\bar{\varrho}_t) \\ +\int_0^t aS(t-\tau)h^2(\tau,\theta_\tau+\varrho_\tau,\bar{\theta}_\tau+\bar{\varrho}_\tau)d\tau \\ +\int_0^t S(t-\tau)f^2(\tau,\theta_\tau+\varrho_\tau,\bar{\theta}_\tau+\bar{\varrho}_\tau)d\tau+ & \text{if } t \in \mathcal{J}, \\ +\sum_{l=1}^{\infty}\int_0^t S(t-\tau)\varpi_l^2(\tau)dB^a(\tau) + \int_0^t S(t-\tau)g^2(\tau)dW(\tau)+ \\ +\sum_{0<t_\mu<t} S(t-t_\mu)\bar{I}_\mu(\theta(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}(t_\mu^-)+\bar{\varrho}(t_\mu^-)). \end{cases}$$

Now, consider the four operators, $\widehat{N}_{11}, \widehat{N}_{12}$, and $\widehat{N}_{21}, \widehat{N}_{22}$

$$\widehat{N}_{11}(\theta,\bar{\theta}) = \begin{cases} 0, & \text{if } t \in (-\infty, \leq 0], \\ -S(t)h^1(0,\Phi,\bar{\Phi}) + h^1(t,\theta_t+\varrho_t,\bar{\theta}_s+\bar{\varrho}_s) \\ +\int_0^t aS(t-s)h^1(s,\theta_s+\varrho_s,\bar{\theta}_s+\bar{\varrho}_s)ds & \text{if } t \in \mathcal{J}. \\ +\int_0^t S(t-s)g^1(s)dW(s) + \sum_{l=1}^{\infty}\int_0^t S(t-s)\varpi_l^1(s)dB^a(s), \end{cases}$$

and

$$\widehat{N}_{12}(\theta,\bar{\theta}) = \begin{cases} 0, & \text{if } -\infty < t \leq 0, \\ \int_0^t S(t-s)f^1(s,\theta_s+\varrho_s,\bar{\theta}_s+\bar{\varrho}_s)ds & \text{if } t \in \mathcal{J}, \\ +\sum_{0<t_\mu<t} S(t-t_\mu)I_\mu(\theta(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}(t_\mu^-)+\bar{\varrho}(t_\mu^-)). \end{cases}$$

Put

$$\widehat{N}_1 = \widehat{N}_{11} + \widehat{N}_{12},$$

with

$$\widehat{N}_{21}(\theta,\bar{\theta}) = \begin{cases} 0, & \text{if } t \in (-\infty, \leq 0], \\ -S(t)h^2(0,\Phi,\bar{\Phi}) + h^2(t,\theta_t+\varrho_t,\bar{\theta}_s+\bar{\varrho}_s) \\ +\int_0^t aS(t-s)h^2(s,\theta_s+\varrho_s,\bar{\theta}_s+\bar{\varrho}_s)ds & \text{if } t \in \mathcal{J}. \\ +\int_0^t S(t-s)g^2(s)dW(s) + \sum_{l=1}^{\infty}\int_0^t S(t-s)\varpi_l^2(s)dB^a(s), \end{cases}$$

and

$$\widehat{N}_{22}(\theta,\bar{\theta}) = \begin{cases} 0, & \text{if } -\infty < t \leq 0, \\ \int_0^t S(t-\tau)f^2(\tau,\theta_\tau+\varrho_\tau,\bar{\theta}_\tau+\bar{\varrho}_\tau)d\tau \\ +\sum_{0<t_\mu<t} S(t-t_\mu)\bar{I}_\mu(\theta(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}(t_\mu^-)+\bar{\varrho}(t_\mu^-)), & \text{if } t \in \mathcal{J}. \end{cases}$$

It is clear that

$$\widehat{N}_{21} + \widehat{N}_{22} = \widehat{N}_2.$$

Then, solving (1)–(3) is reduced to find the solution on

$$(\theta(t),\bar{\theta}(t)) = (\widehat{N}_1(\theta,\bar{\theta})(t),\widehat{N}_2(\theta,\bar{\theta})(t)), -\infty < t \leq b.$$

We will show that $\widehat{N}_{11}, \widehat{N}_{21}$ and $\widehat{N}_{12}, \widehat{N}_{22}$ satisfy all assumptions of Theorem 2. We will provide our proof in several steps.
Part 1 : $\widehat{N}_{11}, \widehat{N}_{21}$ are contractions.
For $(v,u), (\bar{v},\bar{u}) \in \widehat{\mathcal{D}}_{\mathcal{F}_b} \times \widehat{\mathcal{D}}_{\mathcal{F}_b}$ and $t \in \mathcal{J}$, we have

$$\begin{aligned}
& \mathbb{E}|\widehat{N}_{11}(u,v)(t) - \widehat{N}_{11}(\bar{u},\bar{v})(t)|^2 \\
\leq \; & 2E|h^1(t, u_t + \varrho_t, v_t + \bar{\varrho}_t) - h^1(t, \bar{u}_t + \varrho_t, \bar{v}_t + \bar{\varrho}_t)|^2 \\
& + 2E\left|\int_0^t aS(t-\tau)(h^1(\tau, u_\tau + \varrho_\tau, v_\tau + \bar{\varrho}_\tau) - h^1(\tau, \bar{u}_\tau + \varrho_\tau, \bar{v}_\tau + \bar{\varrho}_\tau))d\tau\right|^2 \\
\leq \; & 2\|(-a)^{-\sigma}\|^2\Big(c_h^1\|u-\bar{u}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \bar{c}_h^1\|v-\bar{v}\|^2_{\mathcal{D}_{\mathcal{F}_0}}\Big) \\
& + 2b\int_0^t \frac{C^2_{1-\sigma}}{(t-\tau)^{2(1-\sigma)}}\Big(c_h^1\|u(\tau)-\bar{u}(\tau)\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \bar{c}_h^1\|v(\tau)-\bar{v}(\tau)\|^2_{\mathcal{D}_{\mathcal{F}_0}}\Big)d\tau, \\
\leq \; & 2c_h^1\left(\|(-a)^{-\sigma}\|^2 + \frac{(C_{1-\sigma}b^\sigma)^2}{2\sigma-1}\right) \times \Bigg[2\widetilde{v}^2 \sup_{0\leq\tau\leq b} E|u(\tau)-\bar{u}(\tau)|^2 \\
& + 2\widetilde{N}\|u_0\|^2_{\mathcal{D}_{\mathcal{F}_0}} + 2\widetilde{N}\|\bar{u}_0\|^2_{\mathcal{D}_{\mathcal{F}_0}}\Bigg] + 2\bar{c}_h^1\left(\|(-a)^{-\sigma}\|^2 + \frac{(C_{1-\sigma}b^\sigma)^2}{2\sigma-1}\right) \\
& \times\left(2\widetilde{v}^2 \sup_{0\leq s\leq b} E|v(s)-\bar{v}(s)|^2 + 2\widetilde{N}\|v_0\|^2_{\mathcal{D}_{\mathcal{F}_0}} + 2\widetilde{N}\|\bar{v}_0\|^2_{\mathcal{D}_{\mathcal{F}_0}}\right) \\
\leq \; & \alpha_1 \sup_{0\leq s\leq b} E|u(s)-\bar{u}(s)|^2 + \alpha_2 \sup_{0\leq s\leq b} E|v(s)-\bar{v}(s)|^2.
\end{aligned}$$

Then,

$$\mathbb{E}|\widehat{N}_{11}(u,v)(t) - \widehat{N}_{11}(\bar{u},\bar{v})(t)|^2 \leq \alpha_1 \sup_{0\leq\tau\leq b} E|u(\tau)-\bar{u}(\tau)|^2 + \alpha_2 \sup_{0\leq\tau\leq b} E|v(\tau)-\bar{v}(\tau)|^2.$$

Since $\|u_0\|^2_{\mathcal{D}_{\mathcal{F}_0}} = \|\bar{u}_0\|^2_{\mathcal{D}_{\mathcal{F}_0}} = 0$, $\|v_0\|^2_{\mathcal{D}_{\mathcal{F}_0}} = \|\bar{v}_0\|^2_{\mathcal{D}_{\mathcal{F}_0}} = 0$. Then, we deduce

$$\|\widehat{N}_{11}(u,v)(t) - \widehat{N}_{11}(\bar{u},\bar{v})(t)\|^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} \leq \alpha_1\|u(\tau)-\bar{u}(\tau)\|^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} + \alpha_2\|v(\tau)-\bar{v}(\tau)\|^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}},$$

where

$$\alpha_1 = 4c_h^1\left(\|(-a)^{-\sigma}\|^2 + \frac{(C_{1-\sigma}b^\sigma)^2}{2\sigma-1}\right)\widetilde{v}^2, \quad \text{and} \quad \alpha_2 = 4\bar{c}_h^1\left(\|(-a)^{-\sigma}\|^2 + \frac{(C_{1-\sigma}b^\sigma)^2}{2\sigma-1}\right)\widetilde{v}^2.$$

Similarly,

$$\|\widehat{N}_{21}(u,v)(t) - \widehat{N}_{21}(\bar{u},\bar{v})(t)\|^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} \leq \bar{\alpha}_1\|u(\tau)-\bar{u}(\tau)\|^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} + \bar{\alpha}_2\|v(\tau)-\bar{v}(\tau)\|^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}},$$

where

$$\bar{\alpha}_1 = 4c_h^2\left(\|(-a)^{-\sigma}\|^2 + \frac{(C_{1-\sigma}b^\sigma)^2}{2\sigma-1}\right)\widetilde{v}^2, \quad \text{and} \quad \bar{\alpha}_2 = 4\bar{c}_h^2\left(\|(-a)^{-\sigma}\|^2 + \frac{(C_{1-\sigma}b^\sigma)^2}{2\sigma-1}\right)\widetilde{v}^2.$$

$\forall 0 \leq e, f$ we have

$$\sqrt{f+e} \leq \sqrt{f} + \sqrt{e},$$

and then,

$$\|\widehat{N}_{11}(u,v)(t) - \widehat{N}_{11}(\bar{u},\bar{v})(t)\|_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} \leq \sqrt{\alpha_1}\|u(\tau)-\bar{u}(\tau)\|_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} + \sqrt{\alpha_2}\|v(\tau)-\bar{v}(\tau)\|_{\widehat{\mathcal{D}}_{\mathcal{F}_b}}.$$

Similar computations for N_{21} yield:

$$\|\widehat{N}_{21}(u,v)(t) - \widehat{N}_{21}(\bar{u},\bar{v})(t)\|_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} \leq \sqrt{\bar{\alpha}_1}\|u(\tau)-\bar{u}(\tau)\|_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} + \sqrt{\bar{\alpha}_2}\|v(\tau)-\bar{v}(\tau)\|_{\widehat{\mathcal{D}}_{\mathcal{F}_b}}.$$

Thus,

$$\begin{pmatrix} \|\widehat{N}_{11}((u,v)-\widehat{N}_{11}(\bar{u},\bar{v})\|_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} \\ \|\widehat{N}_{21}(u,v)-\widehat{N}_{21}(\bar{u},\bar{v})\|_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} \end{pmatrix} \le \begin{pmatrix} \sqrt{\alpha_1} & \sqrt{\alpha_2} \\ \sqrt{\bar{\alpha}_1} & \sqrt{\bar{\alpha}_2} \end{pmatrix} \begin{pmatrix} \|u-\bar{u}\|_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} \\ \|v-\bar{v}\|_{\widehat{\mathcal{D}}_{\mathcal{F}_b}}. \end{pmatrix}$$

$M_{b,c}$ converges to zero.

Part 2. It remains to be proven that $\widehat{N}_{12}, \widehat{N}_{22}$ are completely continuous.

Step 1: $\widehat{N}_{12}, \widehat{N}_{22}$ is continuous.

Set $(\theta^n, \bar{\theta}^n)$ as a sequence, so that $(\theta^n, \bar{\theta}^n) \to (\theta, \bar{\theta})$ in $\widehat{\mathcal{D}}_{\mathcal{F}_b}$. Then, for $t \in \mathcal{J}$, by (H_1), (H_3) and (H_6), we have that $I_\mu, \bar{I}_\mu$, $\mu = 1, 2, \ldots, m$ are continuous; owing to the the dominated convergence theorem, we have

$$\begin{aligned}
&\mathbb{E}|\widehat{N}_{12}(\theta^n,\bar{\theta}^n)(t)-\widehat{N}_{12}(\theta,\bar{\theta})(t)|^2 \\
&\le 2\mathbb{E}\Big|\int_0^t S(t-\tau)(f^1(\tau,(\theta_\tau^n+\varrho_\tau)+(\bar{\theta}_\tau^n+\bar{\varrho}_\tau))-f^1(\tau,\theta_\tau+\varrho_\tau)+(\bar{\theta}_\tau+\bar{\varrho}_\tau))d\tau\Big| \\
&+2\mathbb{E}\Bigg|\sum_{0\le t_\mu\le t}|S(t-t_\mu)|^2|I_\mu(\theta^n(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}^n(t_\mu^-)+\bar{\varrho}(t_\mu^-)) \\
&-I_\mu(\theta(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}(t_\mu^-)+\bar{\varrho}(t_\mu^-))\Bigg|^2 \\
&\le 2Mb\int_0^t E\Big|(f^1(\tau,(\theta_\tau^n+\varrho_\tau),(\bar{\theta}_\tau^n+\bar{\varrho}_\tau))-f^1(\tau,\theta_\tau+\varrho_\tau),(\bar{\theta}_\tau+\bar{\varrho}_\tau))\Big|^2 d\tau \\
&+2mM\sum_{0<t_\mu<t}E\Big|I_\mu(\theta^n(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}^n(t_\mu^-),\bar{\varrho}(t_\mu^-)) \\
&-I_\mu(\theta(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}(t_\mu^-)+\bar{\varrho}(t_\mu^-))\Big|^2 \to 0 \text{ as } n\to+\infty,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}|\widehat{N}_{22}(\theta^n,\bar{\theta}^n)(t)-\widehat{N}_{22}(\theta,\bar{\theta})(t)|^2 \\
&\le 2\mathbb{E}\Big|\int_0^t S(t-\tau)(f^2(\tau,(\theta_\tau^n+\varrho_\tau)+(\bar{\theta}_\tau^n+\bar{\varrho}_\tau))-f^2(\tau,\theta_\tau+\varrho_\tau)+(\bar{\theta}_\tau+\bar{\varrho}_\tau))d\tau\Big|^2 \\
&+2E\Bigg|\sum_{0\le t_\mu\le t}|S(t-t_\mu)|^2|\bar{I}_\mu(\theta^n(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}^n(t_\mu^-)+\bar{\varrho}(t_\mu^-)) \\
&-\bar{I}_\mu(\theta(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}(t_\mu^-)+\bar{\varrho}(t_\mu^-))\Bigg|^2 \\
&\le 2Mb\int_0^t E\Big|(f^2(\tau,(\theta_\tau^n+\varrho_\tau)+(\bar{\theta}_\tau^n+\bar{\varrho}_\tau))-f^2(\tau,\theta_\tau+\varrho_\tau)+(\bar{\theta}_\tau+\bar{\varrho}_\tau))\Big|^2 d\tau \\
&+2mM\sum_{0<t_\mu<t}E\Big|\bar{I}_\mu(\theta^n(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}^n(t_\mu^-)+\bar{\varrho}(t_\mu^-)) \\
&-\bar{I}_\mu(\theta(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}(t_\mu^-)+\bar{\varrho}(t_\mu^-))\Big|^2 \to 0 \text{ as } n\to+\infty.
\end{aligned}$$

Thus, $\widehat{N}_{12}, \widehat{N}_{22}$ is continuous.

Step 2. $\widehat{N}_{12}, \widehat{N}_{22}$ maps bounded sets into bounded sets in $\widehat{\mathcal{D}}_{\mathcal{F}_b}$.

It suffices to prove that for all $q > 0$, there exists $l_1, l_2 > 0$ so that for any

$$\theta, \bar{\theta} \in \mathcal{B}_q = \{\Big\{\theta, \bar{\theta} \in \widehat{\mathcal{D}}_{\mathcal{F}_b} : \|\theta\|^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} \le q, \|\bar{\theta}\|^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} \le q\Big\},$$

we have

$$||\widehat{N}_{12}(\theta,\bar{\theta})||^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} \le l_1,\ ||\widehat{N}_{22}(\theta,\bar{\theta})||^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} \le l_2.$$

Let $\theta, \bar{\theta} \in \mathcal{B}_q$, then for any $t \in \mathcal{J}$, we have

$$
\begin{aligned}
|\widehat{N}_{12}(\theta,\bar{\theta})(t)|^2 &\leq \Big| \textstyle\int_0^t S(t-\tau) f^1(\tau,\theta_\tau+\varrho_\tau,\bar{\theta}_\tau+\bar{\varrho}_\tau)d\tau \\
&\quad + \sum_{0\leq t_\mu \leq t} S(t-t_\mu) I_\mu(\theta(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}(t_\mu^-)+\bar{\varrho}(t_\mu^-))) \Big|^2 \\
&\leq 2M \Big| \textstyle\int_0^t f^1(\tau,\theta_\tau+\varrho_\tau,\bar{\theta}_\tau+\bar{\varrho}_\tau)d\tau \Big|^2 \\
&\quad +2M \sum_{0\leq t_\mu \leq t} \Big| I_\mu(\theta(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}(t_\mu^-)+\bar{\varrho}(t_\mu^-))) \Big|^2. \\
&\leq 2bM \textstyle\int_0^t m(\tau)\Psi(\|\theta_\tau+\varrho_\tau\|^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} + \|\bar{\theta}_\tau+\bar{\varrho}_\tau\|^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}})d\tau \\
&\quad +2Mm \sum_{0\leq t_\mu \leq t} \Big| I_\mu(\theta(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}(t_\mu^-)+\bar{\varrho}(t_\mu^-))) \Big|^2 \\
&\leq 2Mb\Psi(2C_{\tau t}) \textstyle\int_0^b m(\tau)d\tau + 2Mm \left(\sum_{\mu=1}^m d_\mu \right)^2.
\end{aligned}
$$

Then, we have

$$
\mathbb{E}|\widehat{N}_{12}(\theta,\bar{\theta})(t)|^2 \leq 2Mb\Psi(2C_{\tau t})\|m\|_{L^1} + 2M\left(\sum_{\mu=1}^m d_\mu \right)^2 = l_1.
$$

Similarly, we have

$$
\begin{aligned}
&|\widehat{N}_{22}(\theta,\bar{\theta})(t)|^2 \\
&\leq \Big| \textstyle\int_0^t S(t-\tau) f^2(\tau,\theta_\tau+\varrho_\tau,\bar{\theta}_\tau+\bar{\varrho}_\tau)d\tau \\
&\quad + \sum_{0\leq t_\mu \leq t} S(t-t_\mu) \bar{I}_\mu(\theta(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}(t_\mu^-)+\bar{\varrho}(t_\mu^-)))^2 \\
&\leq 2M \Big| \textstyle\int_0^t f^2(\tau,\theta_\tau+\varrho_\tau,\bar{\theta}_\tau+\bar{\varrho}_\tau)d\tau \Big|^2 \\
&\quad +2Mm \sum_{0\leq t_\mu \leq t} \Big| \bar{I}_\mu(\theta(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}(t_\mu^-)+\bar{\varrho}(t_\mu^-))) \Big|^2. \\
&\leq 2bM \textstyle\int_0^t m(\tau)\Psi(\|\theta_\tau+\varrho_\tau\|^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} + \|\bar{\theta}_\tau+\bar{\varrho}_\tau\|^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}})d\tau \\
&\quad +2M \sum_{0\leq t_\mu \leq t} \Big| \bar{I}_\mu(\theta(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}(t_\mu^-)+\bar{\varrho}(t_\mu^-))) \Big|^2 \\
&\leq 2Mb\Psi(2C_{\tau t}) \textstyle\int_0^b m(\tau)d\tau + 2Mm \left(\sum_{\mu=1}^m \bar{d}_\mu \right)^2.
\end{aligned}
$$

Then, we have

$$
E|\widehat{N}_{22}(\theta,\bar{\theta})(t)|^2 \leq 2Mb\Psi(2C_{\tau t})\|m\|_{L^1} + 2M\left(\sum_{\mu=1}^m \bar{d}_\mu \right)^2 = l_2.
$$

Step 3: $\widehat{N}_{12}, \widehat{N}_{22}$ maps bounded sets into equicontinuous sets of $\widehat{\mathcal{D}}_{\mathcal{F}_b}$. Let us suppose that for $\tau_1, \tau_2 \neq t_j, j = 1, \ldots, m$, where $0 < \varepsilon \leq \tau_1 < \tau_2 \in \mathcal{J}$, the set $\mathcal{B}_q$ is a bounded set of $\mathcal{D}'_{\mathcal{F}_T}$. Let $\theta \in \mathcal{B}_q$, then we have the estimates

$$
\begin{aligned}
&E|(\widehat{N}_{12}(\theta,\bar{\theta})(\tau_2) - (\widehat{N}_{12}(\theta,\bar{\theta}))(\tau_1)|^2 \\
&\leq 5b \textstyle\int_0^{\tau_1-\varepsilon} |S(\tau_2-\tau)-S(\tau_1-\tau)|^2 E|f^1(\tau,\theta_\tau+\varrho_\tau,\bar{\theta}_\tau+\bar{\varrho}_\tau)|^2 d\tau \\
&\quad +5b \textstyle\int_{\tau_1-\varepsilon}^{\tau_1} |S(\tau_2-\tau)-S(\tau_1-\tau)|^2 E|f^1(\tau,\theta_\tau+\varrho_\tau,\bar{\theta}_\tau+\bar{\varrho}_\tau)|^2 d\tau \\
&+5b \textstyle\int_{\tau_1}^{\tau_2} |S(\tau_2-\tau)|^2 E|f^1(\tau,\theta_\tau+\varrho_\tau,\bar{\theta}_\tau+\bar{\varrho}_\tau)|^2 d\tau \\
&\quad +5 \sum_{0<t_\mu<\tau_1} |S(\tau_2-t_\mu)-S(\tau_1-t_\mu)|^2 E|I_\mu(\theta^n(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}(t_\mu^-)+\bar{\varrho}(t_\mu^-))|^2 \\
&+5 \sum_{\tau_1\leq t_\mu<\tau_2} |S(\tau_2-t_\mu)|^2 |I_\mu(\theta(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}(t_\mu^-)+\bar{\varrho}(t_\mu^-))|^2.
\end{aligned}
$$

This implies, for any $t \in \mathcal{J}$,

$$\begin{aligned}
&E|(\widehat{N}_{12}(\theta,\bar{\theta})(\tau_2) - (\widehat{N}_{12}(\theta,\bar{\theta}))(\tau_1)|^2 \\
&\leq 6b\int_0^{\tau_1-\varepsilon}|S(\tau_2-\tau) - S(\tau_1-\tau)|^2\sigma^1_{q'}(\tau)d\tau \\
&\quad +6b\int_{\tau_1-\varepsilon}^{\tau_1}|S(\tau_2-\tau) - S(\tau_1-\tau)|^2\sigma^1_{C_{\tau t}}(\tau)d\tau + 6bM\int_{\tau_1}^{\tau_2}\sigma^1_{C_{\tau t}}(\tau)d\tau \\
&+4\sum_{0<t_\mu<\tau_1}|S(\tau_2-t_\mu) - S(\tau_1-t_\mu)d_\mu|^2 + 4Mm\left(\sum_{\tau_1\leq t_\mu<\tau_2} d_\mu\right)^2.
\end{aligned}$$

Similar computations for $\widehat{N}_{22}$ yield

$$\begin{aligned}
&\mathbb{E}|(\widehat{N}_{22}(\theta,\bar{\theta})(\tau_2) - (\widehat{N}_{22}(\theta,\bar{\theta}))(\tau_1)|^2 \\
&\leq 6b\int_0^{\tau_1-\varepsilon}|S(\tau_2-\tau) - S(\tau_1-\tau)|^2\sigma^2_{q'}(\tau)d\tau \\
&\quad +6b\int_{\tau_1-\varepsilon}^{\tau_1}|S(\tau_2-\tau) - S(\tau_1-\tau)|^2\sigma^2_{C_{\tau t}}(\tau)d\tau + 6bM\int_{\tau_1}^{\tau_2}\sigma^2_{C_{\tau t}}(\tau)d\tau \\
&+4\sum_{0<t_\mu<\tau_1}|S(\tau_2-t_\mu) - S(\tau_1-t_\mu)\bar{d}_\mu|^2 + 4Mm\left(\sum_{\tau_1\leq t_\mu<\tau_2} \bar{d}_\mu\right)^2.
\end{aligned}$$

The RHS tends to zero as $\tau_2 - \tau_1 \to 0$, and ε is small enough; owing to the compactness of $S(t)$ for $0 < t$, we have continuity, as in [29], which implies the equi-continuity of the operator. It remains to consider the case when $b \geq \tau_2 > \tau_1 > 0$, since the cases $0 \geq \tau_2 > \tau_1$ or $T \geq \tau_2 \geq 0 \geq \tau_1$ are relatively straightforward to handle.
Step 4. $(\widehat{N}_{12}\mathcal{B}_q)(t), (\widehat{N}_{22}\mathcal{B}_q)(t)$ is pre-compact in the space X, which is a consequence of the first step to the third step; with the use of the Arzelá-Ascoli theorem, it suffices to prove that $\widehat{N}_{12}, \widehat{N}_{22}$ maps $\mathcal{B}_q$ into a pre-compact subset of X. Set $b > t > 0$ to be fixed and set $\varepsilon \in \mathbb{R}$, satisfying $0 < \varepsilon < t$. For $\theta, \bar{\theta} \in \mathcal{B}_q$, we give

$$\begin{aligned}
\widehat{N}_{12}(\theta,\bar{\theta})(t) &= \int_0^{t-\varepsilon} S(t-\tau)f^1(\tau,\theta_\tau+\varrho_\tau,\bar{\theta}_\tau+\bar{\varrho}_\tau)d\tau \\
&+ \sum_{0<t_\mu<t-\varepsilon} S(t-t_\mu)I_\mu(\theta(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}(t_\mu^-)+\bar{\varrho}(t_\mu^-)) \\
&= S(\varepsilon)\int_0^{t-\varepsilon} S(t-\tau-\varepsilon)f^1(\tau,\theta_\tau+\varrho_\tau,\bar{\theta}_\tau+\bar{\varrho}_\tau)d\tau \\
&+ S(\varepsilon)\sum_{0<t_\mu<t-\varepsilon} S(t-t_\mu-\varepsilon)I_\mu(\theta(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}(t_\mu^-)+\bar{\varrho}(t_\mu^-)),
\end{aligned}$$

and

$$\begin{aligned}
\widehat{N}_{22}(\theta,\bar{\theta})(t) &= S(\varepsilon)\int_0^{t-\varepsilon} S(t-\tau-\varepsilon)f^2(\tau,\theta_\tau+\varrho_\tau,\bar{\theta}_\tau+\bar{\varrho}_\tau)d\tau \\
&\quad +S(\varepsilon)\sum_{0<t_\mu<t-\varepsilon} S(t-t_\mu-\varepsilon)\bar{I}_\mu(\theta(t_\mu^-)+\varrho(t_\mu^-),\bar{\theta}(t_\mu^-)+\bar{\varrho}(t_\mu^-)).
\end{aligned}$$

As $S(t)$ is compact, we have

$$A_\varepsilon(t) = \{\widehat{N}^\varepsilon_{12}(\theta,\bar{\theta})(t), \widehat{N}^\varepsilon_{22}(\theta,\bar{\theta})(t) : \theta,\bar{\theta} \in \mathcal{B}_q\},$$

as pre-compact in the space X $\forall 0 < \varepsilon < t$. In addition, $\forall \theta, \bar{\theta} \in \mathcal{B}_q$ we have

$$\mathbb{E}|\widehat{N}_{12}(\theta,\bar{\theta})(t) - (\widehat{N}^\varepsilon_{12}(\theta,\bar{\theta})(t)|^2 \leq 4bM\int_{t-\varepsilon}^t \sigma^1_{C_{\tau t}}(\tau)d\tau + 4Mm\left(\sum_{0<t_\mu<t-\varepsilon} d_\mu\right)^2.$$

Similarly,

$$\mathbb{E}|\widehat{N}_{22}(\theta,\bar{\theta})(t)-(\widehat{N}^{\varepsilon}_{22}(\theta,\bar{\theta})(t)|^2 \leq 4bM\int_{t-\varepsilon}^{t}\sigma^2_{C_{\tau t}}(\tau)d\tau+4Mm\left(\sum_{0<t_\mu<t-\varepsilon}\bar{d}_\mu\right)^2.$$

Therefore, there are pre-compact sets arbitrarily close to the set

$$A_\varepsilon(t)=\{\widehat{N}^{\varepsilon}_{12}(\theta,\bar{\theta})(t),\widehat{N}^{\varepsilon}_{22}(\theta,\bar{\theta})(t):\theta,\bar{\theta}\in\mathcal{B}_q\}.$$

Hence, the set

$$A(t)=\{\widehat{N}_{12}(\theta,\bar{\theta})(t),\widehat{N}_{22}(\theta,\bar{\theta})(t):\theta,\bar{\theta}\in\mathcal{B}_q\},$$

is pre-compact in X, and then, the two operators $\widehat{N}_{12},\widehat{N}_{22}:\widehat{\mathcal{D}}_{\mathcal{F}_b}\times\widehat{\mathcal{D}}_{\mathcal{F}_b}\to\widehat{\mathcal{D}}_{\mathcal{F}_b}$ are completely continuous.
Step 5: The a priori bounds:

$$U=\{\theta,\bar{\theta}\in\widehat{\mathcal{D}}_{\mathcal{F}_b}:(\theta,\bar{\theta})=\alpha\widehat{N}_{11}(\theta,\bar{\theta})+\alpha\widehat{N}_{12}\left(\frac{\theta}{\alpha},\frac{\bar{\theta}}{\alpha}\right),\text{for some } 0<\alpha<1\}$$

is bounded.

$$\begin{aligned}
\theta(t) &= -S(t)h^1(0,\Phi,\bar{\Phi})+h^1(t,\theta_t+\varrho_t,\bar{\theta}_\tau+\bar{\varrho}_\tau)+\int_0^t aS(t-\tau)h^1(\tau,\theta_\tau+\varrho_\tau,\bar{\theta}_\tau+\bar{\varrho}_\tau)d\tau\\
&+ \int_0^t S(t-\tau)f^1(\tau,\theta_\tau+\varrho_\tau,\bar{\theta}_\tau+\bar{\varrho}_\tau)d\tau+\sum_{l=1}^{\infty}\int_0^t S(t-\tau)\varpi^1_l(\tau)dB^a(\tau)\\
&+ \int_0^t S(t-\tau)g^1(\tau)dW(\tau)+\sum_{0<t_\mu<t}S(t-t_\mu)I_\mu(\theta(t^-_\mu)+\varrho(t^-_\mu),\bar{\theta}(t^-_\mu)+\bar{\varrho}(t^-_\mu)),
\end{aligned}$$

and

$$\begin{aligned}
\bar{\theta}(t) &= -S(t)h^2(0,\Phi,\bar{\Phi})+h^2(t,\theta_t+\varrho_t,\bar{\theta}_\tau+\bar{\varrho}_\tau)+\int_0^t aS(t-\tau)h^2(\tau,\theta_\tau+\varrho_\tau,\bar{\theta}_\tau+\bar{\varrho}_\tau)d\tau\\
&+ \int_0^t S(t-\tau)f^2(\tau,\theta_\tau+\varrho_\tau,\bar{\theta}_\tau+\bar{\varrho}_\tau)d\tau+\sum_{l=1}^{\infty}\int_0^t S(t-\tau)\varpi^2_l(\tau)dB^a(\tau)\\
&+ \int_0^t S(t-\tau)g^2(\tau)dW(\tau)+\sum_{0<t_\mu<t}S(t-t_\mu)\bar{I}_\mu(\theta(t^-_\mu)+\varrho(t^-_\mu),\bar{\theta}(t^-_\mu)+\bar{\varrho}(t^-_\mu)).
\end{aligned}$$

Thus, for $t\in\mathcal{J}$, namely

$$\begin{aligned}
&E|\theta(t)|^2\\
\leq\ & 7E|S(t)h^1(0,\Phi,\bar{\Phi})|^2+7E|h^1(t,\theta_t+\varrho_t,\bar{\theta}_\tau+\bar{\varrho}_\tau)|^2\\
&+7E\left|\int_0^t aS(t-\tau)h^1(\tau,\theta_\tau+\varrho_\tau,\bar{\theta}_\tau+\bar{\varrho}_\tau)d\tau\right|^2\\
&+7E\left|\int_0^t S(t-\tau)f^1(\tau,\theta_\tau+\varrho_\tau,\bar{\theta}_\tau+\bar{\varrho}_\tau)d\tau\right|^2+7E\left|\sum_{l=1}^{\infty}\int_0^t S(t-\tau)\varpi^1_l(\tau)dB^a(\tau)\right|^2\\
&+7E\left|\int_0^t S(t-\tau)g^1(\tau)dW(\tau)\right|^2+7E\left|\sum_{0<t_\mu<t}S(t-t_\mu)I_\mu(\theta(t^-_\mu)+\varrho(t^-_\mu),\bar{\theta}(t^-_\mu)+\bar{\varrho}(t^-_\mu))\right|^2.
\end{aligned}$$

This, together with (H_1)-(H_7), Lemma 2, and Lemma 1, yields the following:

$$\begin{aligned}
&\mathbb{E}|\theta(t)|^2 \\
\leq\ & 6(-a)^{-\sigma}Mp_1(t)(\|\Phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|\bar{\Phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}})+6(-a)^{-\sigma}p_1(t)(\|\theta_t+\varrho_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|\bar{\theta}_t+\bar{\varrho}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}) \\
& +6b\int_0^t C^2_{1-\sigma}(t-\tau)^{2(1-\sigma)}p_1(\tau)(\|\theta_\tau+\varrho_\tau\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|\bar{\theta}_\tau+\bar{\varrho}_\tau\|^2_{\mathcal{D}_{\mathcal{F}_0}})d\tau \\
& +6Mb\int_0^t m(\tau)\Psi(\|\theta_\tau+\varrho_\tau\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|\bar{\theta}_\tau+\bar{\varrho}_\tau\|^2_{\mathcal{D}_{\mathcal{F}_0}})d\tau \\
+\ & 6Mc_a a(2a-1)t^{2a-1}\int_0^t \|\varpi^1(\tau)\|^2 d\tau \\
& +\int_0^t \|g^1(\tau)\|^2_{L^0}d\tau+6Mm\left(\sum_{\mu=1}^m d_\mu\right)^2,
\end{aligned}$$

which immediately yields

$$\begin{aligned}
\mathbb{E}|\theta(t)|^2 \leq\ & A_1+7(-a)^{-\sigma}p_1(t)(\|\theta_t+\varrho_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|\bar{\theta}_t+\bar{\varrho}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}) \\
& +7b\int_0^t C^2_{1-\sigma}(t-\tau)^{2(1-\sigma)}p_1(\tau)(\|\theta_\tau+\varrho_\tau\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|\bar{\theta}_\tau+\bar{\varrho}_\tau\|^2_{\mathcal{D}_{\mathcal{F}_0}})d\tau \\
& +7Mb\int_0^t m(\tau)\Psi(\|\theta_\tau+\varrho_\tau\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|\bar{\theta}_\tau+\bar{\varrho}_\tau\|^2_{\mathcal{D}_{\mathcal{F}_0}})d\tau.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{E}|\bar{\theta}(t)|^2 \leq\ & A_2+7(-a)^{-\sigma}p_2(t)(\|\theta_t+\varrho_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|\bar{\theta}_t+\bar{\varrho}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}) \\
+\ & 7b\int_0^t C^2_{1-\sigma}(t-\tau)^{2(1-\sigma)}p_2(\tau)(\|\theta_\tau+\varrho_\tau\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|\bar{\theta}_\tau+\bar{\varrho}_\tau\|^2_{\mathcal{D}_{\mathcal{F}_0}})d\tau \\
+\ & 7Mb\int_0^t m(\tau)\Psi(\|\theta_\tau+\varrho_\tau\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|\bar{\theta}_\tau+\bar{\varrho}_\tau\|^2_{\mathcal{D}_{\mathcal{F}_0}})d\tau,
\end{aligned}$$

where

$$\begin{aligned}
A_1 =\ & 7C_1+7(-a)^{-\sigma}Mp_{1*}(\|\Phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|\bar{\Phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}) \\
+\ & 6M\left(\sum_{\mu=1}^m d_\mu\right)^2+7Mc_a a(2a-1)t^{2a-1}\int_0^t \|\varpi(\tau)\|^2 d\tau.
\end{aligned}$$

and

$$\begin{aligned}
A_2 =\ & 7C_1+7(-a)^{-\sigma}Mp_{2*}(\|\Phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}+\|\bar{\Phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}) \\
+\ & 6M\left(\sum_{\mu=1}^m \bar{d}_\mu\right)^2+7Mc_a a(2a-1)t^{2a-1}\int_0^t \|\varpi(\tau)\|^2 d\tau. \qquad (8)
\end{aligned}$$

Put

$$p_{1*}=\sup_{t\in\mathcal{J}}|p_1(t)| \quad \text{and} \quad p_{2*}=\sup_{t\in\mathcal{J}}|p_2(t)|.$$

Adding to obtain

$$\begin{aligned}
& \mathbb{E}|\theta(t)|^2 + E|\bar{\theta}(t)|^2 \\
\leq\ & B_* + 7(-a)^{-\sigma} p_*(t)(\|\theta_t + \varrho_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{\theta}_t + \bar{\varrho}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}) \\
& +7b \int_0^t C^2_{1-\sigma}(t-\tau)^{2(1-\sigma)} p_*(\tau)(\|\theta_\tau + \varrho_\tau\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{\theta}_\tau + \bar{\varrho}_\tau\|^2_{\mathcal{D}_{\mathcal{F}_0}}) d\tau \\
& +7Mb \int_0^t m_*(\tau)\Psi(\|\theta_\tau + \varrho_\tau\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{\theta}_\tau + \bar{\varrho}_\tau\|^2_{\mathcal{D}_{\mathcal{F}_0}}) d\tau,
\end{aligned}$$

where

$$B_* = A_1 + A_2 \quad \text{and} \quad p_*(t) = \sup\{p_{1*}(t), p_{2*}(t)\}.$$

That is to say,

$$\begin{aligned}
& \|\theta_t + \varrho_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{\theta}_t + \bar{\varrho}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\
\leq\ & 4\tilde{v}^2 \sup_{s\in\mathcal{J}} \Big(E|\theta(\tau)|^2 + E|\bar{\theta}(\tau)|^2 \Big) + 4\tilde{v}^2 M \Big(E|\Phi(0)|^2 + E|\bar{\Phi}(0)|^2 \Big) \\
& +4\tilde{N}^2 \Big(\|\Phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{\Phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}} \Big).
\end{aligned}$$

If we set $v(t)$, the RHS of the above inequality, we have

$$\|\theta_t + \varrho_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{\theta}_t + \bar{\varrho}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} \leq v(t). \tag{9}$$

Consequently,

$$\begin{aligned}
& \mathbb{E}|\theta(t)|^2 + E|\bar{\theta}(t)|^2 \\
\leq\ & B_* + 7(-a)^{-\sigma} p_*(t)(\|\theta_t + \varrho_t\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{\theta}_t + \bar{\varrho}_t\|^2_{\mathcal{D}_{\mathcal{F}_0}}) \\
& +7b \int_0^t C^2_{1-\sigma}(t-\tau)^{2(1-\sigma)} p_*(\tau)(\|\theta_\tau + \varrho_\tau\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{\theta}_\tau + \bar{\varrho}_\tau\|^2_{\mathcal{D}_{\mathcal{F}_0}}) d\tau \\
& +7Mb \int_0^t m_*(\tau)\Psi(\|\theta_\tau + \varrho_\tau\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{\theta}_\tau + \bar{\varrho}_\tau\|^2_{\mathcal{D}_{\mathcal{F}_0}}) d\tau \\
\leq\ & B_* + 7(-a)^{-\sigma} p_*(t) v(t) + 7b \int_0^t C^2_{1-\sigma}(t-\tau)^{2(1-\sigma)} p_*(\tau) v(\tau) d\tau \\
& +7Mb \int_0^t m_*(\tau)\Psi(v(\tau)) d\tau.
\end{aligned}$$

Using (9) in the definition of v, we have

$$\begin{aligned}
v(t) \ \leq\ & 4\tilde{v}^2 \bigg(B_* + 6(-a)^{-\sigma} p_*(t) v(t) + b \int_0^t C^2_{1-\sigma}(t-\tau)^{2(1-\sigma)} p_*(\tau) v(\tau) d\tau \\
& +6Mb \int_0^t m_*(\tau)\Psi(v(\tau) d\tau \bigg) + 4\tilde{v}^2 M \Big(E|\Phi(0)|^2 + E|\bar{\Phi}(0)|^2 \Big) \\
+\ & 4\tilde{N}^2 \Big(\|\Phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{\Phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}} \Big) \\
\leq\ & B_1 + B_2 \int_0^t v(\tau)(t-\tau)^{2(1-\sigma)} d\tau + B_3 \int_0^t m_*(\tau)\Psi(v(\tau)) d\tau,
\end{aligned} \tag{10}$$

where

$$B_1 = \frac{4\tilde{v}^2 M(E|\Phi(0)|^2 + E|\bar{\Phi}(0)|^2) + 4\tilde{N}^2(\|\Phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\bar{\Phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}) + 4\tilde{v}^2 B_*}{1 - 24\tilde{v}^2(-a)^{-\sigma}\bar{p}_*},$$

$$B_2 = \frac{24b\tilde{v}^2 C^2_{1-\sigma}\bar{p}_*}{1 - 24\tilde{v}^2(-a)^{-\sigma}\bar{p}_*},$$

and

$$B_3 = \frac{24b\widetilde{\nu}^2 M}{1 - 24\widetilde{\nu}^2(-a)^{-\sigma}\bar{p}_*}.$$

Now, from Lemma 4, we have

$$v(t) \leq L_1 + L_2 \int_0^t m_*(\tau)\Psi(v(\tau))d\tau, \tag{11}$$

where

$$L_1 = B_1 \exp\Big(B_2(\Gamma(2\sigma-1))^n b^{n(2\sigma-1)}/\Gamma(n(2\sigma-1))\Big) \sum_{\mathcal{J}=0}^{n-1} \left(\frac{B_2 b^{2\sigma-1}}{2\sigma-1}\right),$$

and

$$L_2 = B_3 \exp\Big(B_2(\Gamma(2\sigma-1))^n b^{n(2\sigma-1)}/\Gamma(n(2\sigma-1))\Big) \sum_{j=0}^{n-1} \left(\frac{B_2 b^{2\sigma-1}}{2\sigma-1}\right).$$

We denote the RHS of (11) by v. Then,

$$y(0) = L_1, \quad v(t) \leq y(t), \qquad t \in \mathcal{J},$$

and

$$y'(t) = L_2 m_*(t)\Psi(v(t)), \qquad t \in \mathcal{J}.$$

Owing to the increasing property of Ψ, we have

$$y'(t) \leq L_2 m_*(t)\Psi(y(t)), \qquad \text{for } a.e.\ t \in \mathcal{J}.$$

Then, for each $t \in \mathcal{J}$, we have

$$\int_{y(0)}^{y(t)} \frac{d\tau}{\Psi(\tau)} \leq L_2 \int_0^b m_*(\tau)d\tau < \int_{L_1}^{\infty} \frac{d\tau}{\Psi(\tau)}.$$

As a result, there is a constant $C_{\tau t}$, such that

$$v(t) \leq y(t) \leq C_{\tau t}, \qquad t \in \mathcal{J},$$

where $C_{\tau t}$ depends only on b and on the functions $m_*(.)$ and $\Psi(.)$. Then

$$\begin{aligned} \mathbb{E}|\theta(t)|^2 + E|\bar{\theta}(t)|^2 \quad \leq \quad & B_* + 6(-a)^{-\sigma}\bar{p}_*(t)C_{\tau t} + 6b\int_0^t C_{1-\sigma}^2 (t-\tau)^{2(1-\sigma)}\bar{p}_*(\tau)C_{\tau t}d\tau \\ & + 6Mb\int_0^t m_*(\tau)\Psi(C_{\tau t})d\tau. \end{aligned}$$

Thus,

$$\|\theta\|^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} \leq R \qquad \text{and} \qquad \|\bar{\theta}\|^2_{\widehat{\mathcal{D}}_{\mathcal{F}_b}} \leq R.$$

As a consequence of Theorem 2, we deduce that N has a fixed point, since

$$(x(t), w(t)) = (\theta(t) + \varrho(t), \bar{\theta}(t) + \bar{\varrho}(t)) - \infty < t \leq b.$$

Then (x, w) is a fixed point of the operator $N = (N_1, N_2)$, which is a mild solution on the problems (1)–(3). □

4. Example

In this part, we offer examples to show the applicability of our results. We will look at the infinite fractional Brownian motion.

Example 1. *We introduce, for $0 \leq \Xi \leq \pi$, the following stochastic partial differential equation with impulsive effects:*

$$\begin{cases} du(t,\Xi) & + & A^1(t,u(t-h,\Xi))] = \frac{\partial^2}{\partial \Xi^2}u(t,\Xi) + F(t,u(t,\Xi),v(t,\Xi)) \\ & & +f^1(t)dW(t) + \varpi^1(t)\frac{dB_\chi^a}{dt}, \quad t \in \mathbb{R}_+, \quad t \neq t_\mu, \\ dv(t,\Xi) & + & A^2(t,u(t-h,\Xi))] = \frac{\partial^2}{\partial \Xi^2}v(t,\Xi) + G(t,u(t,\Xi),v(t,\Xi)) \\ & & +f^2(t)dW(t) + \varpi^2(t)\frac{dB_\chi^a}{dt}, \quad t \in \mathbb{R}_+, \quad t \neq t_\mu, \\ u(t_\mu^+,\Xi) & - & u(t_\mu^-,\Xi) = \kappa_\mu u(t_\mu^-,\Xi), \quad \mu = 1,\ldots,m, \\ v(t_\mu^+,\Xi) & - & v(t_\mu^-,\Xi) = \bar{\kappa}_\mu v(t_\mu^-,\Xi), \quad \mu = 1,\ldots,m, \\ u(t,\Xi=0) & = & u(t,\pi) = 0, t \in \mathbb{R}_+, \\ v(t,\Xi=0) & = & v(t,\pi) = 0, t \in \mathbb{R}_+, \\ u(t=0,\Xi) & = & u_0(\Xi), \\ v(t=0,\Xi) & = & v_0(\Xi), \\ u(t,\Xi) & = & u_0(\Xi),\ t \in \mathbb{R}_-, \\ v(t,\Xi) & = & u_0(\Xi),\ t \in \mathbb{R}_-, \end{cases} \tag{12}$$

here, $0 < \kappa_\mu > 0$, B_χ^a represents an FBM, where the functions $G, F \in C([0,\pi] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. For $t \in \mathcal{J}, \quad \Xi \in [0,\pi], \varrho \in (-\infty, 0]$, let

$$x(t)(\Xi) = u(t,\Xi), w(t)(\Xi) = v(t,\Xi),$$

$$I_\mu(x(t_\mu))(\Xi) = \kappa_\mu u(t_\mu^-,\Xi), \bar{I}_\mu(w(t_\mu))(\Xi) = \kappa_\mu v(t_\mu^-,\Xi), \quad \Xi \in [0,\pi], \quad \mu = 1,\ldots,m,$$

$$f(t,\Phi,\bar{\Phi})(\Xi) = F(t,\Phi(-h,\Xi),\bar{\Phi}(-h,\Xi)),$$

$$g(t,x(t),w(t))(\Xi) = G(t,\Phi(-h,\Xi),\bar{\Phi}(-h,\Xi)),$$

$$h^1(t,\Phi,\bar{\Phi})(\Xi) = A^1(t,\Phi(-h,\Xi),\bar{\Phi}(-h,\Xi)),$$

$$h^2(t,\Phi,\bar{\Phi})(\Xi) = A^2(t,\Phi(-h,\Xi),\bar{\Phi}(-h,\Xi)),$$

$$u_0(\Xi) = u(t=0,\Xi),\ v_0(\Xi) = v(t=0,\Xi),$$

$$\Phi(\varrho)(\Xi) = \Phi(\varrho,\Xi), \quad \varrho \in (-\infty,0],$$

$$\bar{\Phi}(\varrho)(\Xi) = \bar{\Phi}(\varrho,\Xi),$$

Let $X = Y = L^2([0,\pi])$. We introduce the operator A by $Au = u''$, by

$$\mathcal{D}(A) = \{u \in X : u'' \in X, u(0) = u(\pi) = 0\}.$$

Then,

$$A\theta = -\sum_{n=1}^{\infty} \exp(-n^2 t)\langle \theta, e_n \rangle e_n, \theta \in X,$$

and A is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t \in \mathbb{R}_+}$ on X, defined by

$$S(t)u = \sum_{n=1}^{\infty} \exp(-n^2 t)\langle u, e_n \rangle e_n, \forall u \in \mathcal{H}.$$

Let $(e_n)_{n \in \mathbb{N}}$ be given by

$$e_n(u) = (2/\pi)^{1/2} \sin(nu), \forall n \in \mathbb{N} \text{ and } u \in \mathcal{H}.$$

Then $(e_n)_{n \in \mathbb{N}}$ is the orthogonal set of eigenvectors of A. The analytic semigroup $\{S(t)\}_{0<t}$ is compact, and

$$\exists M > 0 : \|S(t)\|^2 \leq M.$$

To define the operator $\chi : Y \to Y$, we should choose a sequence $\{(\varpi_n)_{n\geq 1} \subset \mathbb{R}^+$, set $Qe_n = \varpi_n e_n$, and let

$$\mathrm{tr}(\chi) = \sum_{n=1}^{\infty} \sqrt{\varpi_n} < \infty.$$

The process $B_\chi^a(\tau)$ is defined by

$$B_\chi^a = \sum_{n=1}^{\infty} \sqrt{\varpi_n} \gamma_n^a(t) e_n,$$

where $a \in (1/2, 1)$, and $\{(\gamma_n^a)\}_{n\in\mathbb{N}}$ is a sequence of mono-dimensional FBMs with two sides that are mutually independent. We set

(i) $\exists (d_\mu)_{\mu\in\{1,\ldots,m\}}, (\bar{d}_\mu)_{\mu\in\{1,\ldots,m\}} > 0$, *such that*

$$|I_\mu(\Xi)| \leq d_\mu, \quad |\bar{I}_\mu(\Xi)| \leq \bar{d}_\mu, \ \forall \Xi \in \mathbb{R}.$$

(ii) *We define $g \in C([0,T] \times X, X)$ by $g(t,u)(.) = G(t,u(.))$. Imposing appropriate conditions on the functional G to satisfy (H_2).*

(iii) *Assume that there exists an integral function $\eta \in C^0([0T], \mathbb{R}^+)$, so that*

$$\mathbb{E}|F(t,x,w)|^2 \leq \eta(t)\Psi(\mathbb{E}(|x|^2 + |y|^2)), \quad \mathbb{E}|G(t,x,w)|^2 \leq \eta(t)\Psi(\mathbb{E}(|x|^2 + |y|^2)),$$

$\forall x, w \in \mathbb{R}, 0 \leq t \leq T$, *where $\Psi \in C([0,\infty),(0,\infty))$ is a non-decreasing function with*

$$\int_1^{\infty} \frac{d\tau}{\Psi(\tau)} = +\infty.$$

Example 2. *The function $(\varpi^i)_{i=1,2} : \mathcal{J} \to L_\chi^2(Y, x)$ is bounded, which means that there exists a positive constant $L > 0$, such that*

$$\int_0^b \|\varpi^i(\tau)\|_{L_\chi^2}^2 d\tau < L, \forall i = 1,2.$$

The function $(f_i)_{i=1,2} : \mathcal{J} \to L^0(Y, x)$ is bounded and there exists $C > 0$, such that

$$\int_0^b \|f^i(\tau)\|_{L^0}^2 d\tau < C, \forall i = 1,2.$$

As a result, the problem (12) can be expressed in abstract form

$$\begin{cases} d(x(t) & -h^1(t,x,w)) = [Ax(t) + f(t,x,w)]dt \\ & +f^1(t)dW(t) + \varpi(t)dB_\chi^a(t), \quad t \in \mathcal{J}, \\ d(w(t) & -h^2(t,x,w)) = [Aw(t) + g(t,x,w)]dt \\ & +f^2(t)dW(t) + \varpi(t)dB_\chi^a(t), \quad t \in \mathcal{J}, \\ x(t_\mu^+) & -x(t_\mu) = I_\mu(x(t_\mu)), \quad \mu = 1,\ldots,m, \\ w(t_\mu^+) & -w(t_\mu) = I_\mu(w(t_\mu)), \quad \mu = 1,\ldots,m, \\ x(0) & = x_0, \\ w(0) & = w_0. \end{cases} \tag{13}$$

We are now in a position to confirm that (H_1)-(H_7) hold, provided that the assumptions in Theorem 2 yield, and we may conclude that the system (12) has a mild solution on $(-\infty, b]$.

5. Conclusions and Discussion of the Results

Among the main results of this paper is the derivation of sufficient conditions for the existence of solutions for systems of impulsive neutral functional differential equations, employing Burton and Kirk's fixed point theorems. These theorems are particularly effective for studying such cases in extended Banach spaces. To enhance the reader's understanding, it would be beneficial to outline some real-world physical applications where our findings could prove useful. To our knowledge, there are no results in the literature that deal with the system of functional differential equations involving fractional Brownian motion with a Wiener process. Impulsive effects exist widely and have become an important area of investigation in recent years, motivated by their numerous applications. Practical applications of systems involving neutral and impulsive functional differential equations span fields such as medicine, economics, epidemics, biology, mechanics, electronics, population dynamics, and telecommunications. The novelty of Burton and Kirk's fixed point theorem lies in decomposing the main operator into two operators, making them suitable for our proposed model. This allows us to prove the first results that process the fixed point using the Banach theorem, and the second one using the Leray-–Schauder theorem.

Extending these results to consider the question of stability (qualitative studies) will make it possible to advance the study in this direction, which will be our next project; see [31–37].

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