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Analysis on Controllability Results for Impulsive Neutral Hilfer Fractional Differential Equations with Nonlocal Conditions

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Abstract: In this paper, we investigate the controllability of the system with non-local conditions. The existence of a mild solution is established. We obtain the results by using resolvent operators functions, the Hausdorff measure of non-compactness, and Monch's fixed point theorem. We also present an example, in order to elucidate one of the results discussed.

Keywords: boundary condition; fractional calculus; impulsive condition; integro-differential system; controllability; fixed point theorem

MSC: 93B05; 47H10; 26A33; 93C27; 47H08; 34K40l

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1. Introduction

Fractional calculus primarily involves the description of fractional-order derivatives and integral operators [1]. It has grown in relevance in recent decades due to its vast range of applications in several scientific disciplines. There are research papers and books [2–20] on the topic of fractional differential equations, which are used to represent complicated physical and biological processes such as anomalous diffusion, signal processing, wave propagation, visco-elasticity behavior, power-laws, and automatic remote control systems. The evolution of a physical system in time is described by an initial and boundary value problem. In many cases, it is better to have more information on the conditions. Moreover, the non-local condition which is a generalization of the classical condition was motivated by physical problems. The pioneering work on non-local conditions is due to Byszewski [21]. Existence results for differential equations with non-local conditions were investigated by many authors [21–23]. Thereafter, by means of the non-compactness measure method Gu and Trujilo [24] defined the study of initial value problems with non-local conditions. The concept of controllability is critical in the study and application of control theory. Many authors [3,6,10,25,26] have investigated controllability with an impulsive condition. Hilfer [27] introduced another fractional derivative which includes the R-L derivative and Caputo fractional derivative. The Hilfer fractional operator is indeed intriguing and important, both in terms of its definition and its associated properties. Subsequently, many authors [28–30] studied the Hilfer neutral fractional differential equations. Recently, Subashini [31,32] obtained mild solutions for Hilfer integro-differential equations of fractional order by means of Monch's fixed point technique and measure of non-compactness [24,25]. The existing results for impulsive neutral Hilfer fractional differential equations with non-local conditions have the following form:

$$D^{\zeta,\eta}_{0^+}[v(\mathfrak{t}) - \mathfrak{F}_1(\mathfrak{t}, v(\mathfrak{t})] = \mathbb{A}v(\mathfrak{t}) + \mathfrak{F}_2\left(\mathfrak{t}, v_{\mathfrak{t}}, \int_0^{\mathfrak{t}} h(\mathfrak{t}, s, v_s)ds\right) + \mathbb{B}w(\mathfrak{t}),\ \mathfrak{t} \in \mathcal{I} = (0,p], \mathfrak{t} \neq \mathfrak{t}_k, \tag{1}$$

$$\Delta v|_{\mathfrak{t}=\mathfrak{t}_k} = I_k(v(\mathfrak{t}_k^-)), \quad k = 1,2,\ldots,m, \tag{2}$$

$$I^{(1-\zeta)(1-\eta)}_{0^+} v(0) = \phi(0) + \alpha(v_{\mathfrak{t}1}, v_{\mathfrak{t}2}, v_{\mathfrak{t}3}, \cdots, v_{\mathfrak{t}m})(0) \in \mathcal{P}_{\mathfrak{g}}, \mathfrak{t} \in (-\infty, 0] \tag{3}$$

where $D^{\zeta,\eta}_{0^+}$ denotes the Hilfer differential equation of order ζ and type η. Moreover, $0 \leq \zeta \leq 1; 0 \leq \eta \leq 1$ and $(v, \|\cdot\|)$ is a Banach space, $\mathbb{A}$ denotes the infinitesimal generator of strongly continuous functions of bounded linear operators $\{T(\mathfrak{t})\}_{\mathfrak{t}\geq 0}$ on $\mathfrak{X}$. A suitable function $\mathfrak{F}_2 : \mathcal{I} \times \mathcal{P}_h \times \mathfrak{X} \to \mathfrak{X}$ is connected with the Phase space $u_\theta(\mathfrak{t})$ with the mapping $u_{\mathfrak{t}} : (-\infty, 0] \to \mathcal{P}_{\mathfrak{g}}, u_{\mathfrak{t}}(d) = u(\mathfrak{t} + d), d \leq 0$. Now $\Delta = \{(\zeta, s) : 0 \leq s \leq \zeta \leq \mathfrak{t}\}$. For the purpose of brevity, we make use of $\mathfrak{g}v(\mathfrak{t}) = \int_0^\zeta h(\mathfrak{t}, s, v_s)ds$. Here, $w(\cdot)$ is provided in $L^2(\mathcal{I}, \mathfrak{X})$, a Banach space of admissible control functions. $0 < \mathfrak{t}_1 < \mathfrak{t}_2 < \mathfrak{t}_3 < \cdots < \mathfrak{t}_m \leq p$, $\alpha : \mathcal{P}^k_{\mathfrak{g}} \to \mathcal{P}_{\mathfrak{g}}$ is continuous functions.

The article is organized as follows: Section 2 introduces a few key notions and definitions related to our research that will be used throughout the discussion of this article. Section 3 discusses the controllability results with non-local conditions of the impulsive neutral Hilfer Fractional Differential Equations. Finally, Section 4 provides an example to illustrate the theory.

2. Preliminaries

Now we recall some definitions, concepts, and lemmas chosen to achieve the desired outcomes. Let $PC(\mathcal{I}, \mathfrak{X})$ be the Banach space of all continuous function spaces from $\mathcal{I} \to \mathfrak{X}$ where $\mathcal{I} = [0,p]$ and $\mathcal{I}' = [0,p]$ with $p > 0$. Now we define $C_{1-\zeta+\eta\zeta-\eta\vartheta}(\mathcal{I}, \mathfrak{X}) = \{v : \mathfrak{t}^{1-\zeta+\eta\zeta-\eta\vartheta} z(\mathfrak{t} \in PC(\mathcal{I}, \mathfrak{X}))\}.(\mathfrak{X}, \|\cdot\|)$ is a Banach space with $Z = \{v \in \mathbb{C} : \lim_{\mathfrak{t}\to 0} \mathfrak{t}^{1-\zeta+\eta\zeta-\eta\vartheta} v(\mathfrak{t})$ exists and finite$\}$. Let $v(\mathfrak{t}) = \mathfrak{t}^{1-\zeta+\eta\zeta-\eta\vartheta} x(\mathfrak{t}), \mathfrak{t} \in (0,p]$ then, $v \in Z iff x \in \mathbb{C} and \|v\|_Z = \|x\|$. Let us define $\mathfrak{F}_2 : \mathcal{I} \times \mathcal{P}_{\mathfrak{g}} \to \mathfrak{X}$ with $\|\mathfrak{F}_2\|_{L^\mu(\mathfrak{I},\mathbb{R}^+)}$.

We will now discuss some significant fractional calculus results (see Hilfer [27]).

Definition 1. *Let $\mathfrak{F} : [p, +\infty) \to \mathbb{R}$ and the integral*

$$I^\zeta_{p^+}\mathfrak{F}(\mathfrak{t}) = \frac{1}{\Gamma(\zeta)} \int_p^{\mathfrak{t}} \mathfrak{F}(\varrho)(\mathfrak{t} - \varrho)^{\zeta-1} d\varrho, \quad \mathfrak{t} > p, \zeta > 0$$

be called the left-sided R-L fractional integral of order ζ having lower limit p of a continuous function, where $\Gamma(\cdot)$ denotes the gamma function and provided that the right-hand side exists.

Definition 2. *Let $\mathfrak{F} : [p, +\infty) \to \mathbb{R}$ and the integral*

$$^{(R-L)}D^\zeta_{p^+}\mathfrak{F}(\mathfrak{t}) = \frac{1}{\Gamma(n-\zeta)} \left(\frac{d}{d\mathfrak{t}}\right)^n \int_p^{\mathfrak{t}} \frac{\mathfrak{F}(\varrho)}{(\mathfrak{t} - \varrho)^{\zeta+1-m}} d\varrho, \quad \mathfrak{t} > p, n-1 < \zeta < n.$$

be called the left-sided (R-L) FD of order $\eta \in [k-1, k)$, where $k \in \mathbb{R}$.

Definition 3. *Let $\mathfrak{F} : [p, +\infty) \to \mathbb{R}$ and the integral*

$$D^{\zeta,\eta}_{p^+}\mathfrak{F}(\mathfrak{t}) = \left(I^{\eta(1-\zeta)}_{p^+} D\left(I^{(1-\zeta)(1-\eta)}_{p^+}\mathfrak{F}\right)\right)(\mathfrak{t})$$

be called the left-sided Hilfer fractional derivative of order $0 \leq \zeta \leq 1$ and $0 < \eta < 1$ function of $\mathfrak{F}(t)$.

Definition 4. *Let $\mathfrak{F} : [p, +\infty) \to \mathbb{R}$ and the integral*

$$^{C}D^{\zeta}_{p^+}\mathfrak{F}(\mathfrak{t}) = \frac{1}{\Gamma(n-\zeta)}\int_p^{\mathfrak{t}} \frac{\mathfrak{F}^n(\mathfrak{t})}{(\mathfrak{t}-\varrho)^{\zeta+1-n}}d\mathfrak{t} = I^{n-\zeta}_{p^+}\mathfrak{F}^n(\mathfrak{t}), \quad \mathfrak{t} > p, n-1 < \zeta < n$$

be called the left-sided Caputo fractional derivative type of order $\eta \in (k-1, k)$, where $k \in \mathbb{R}$.

Remark 1. *(i) The Hilfer fractional derivative coincides with the standard (R-L) fractional derivative, if $\eta = 0, 0 < \zeta < 1$ and $p = 0$, then*

$$D^{\zeta,0}_{0^+}\mathfrak{F}(\mathfrak{t}) = \frac{d}{d\mathfrak{t}}I^{1-\zeta}_{0^+}\mathfrak{F}(\mathfrak{t}) =^{(R-L)} D^{\zeta}_{0^+}\mathfrak{F}(\mathfrak{t});$$

(ii) The Hilfer fractional derivative coincides with the standard Caputo derivative, if $\zeta = 1$, $0 < \eta < 1$ and $p = 0$, then

$$D^{\zeta,1}_{0^+}\mathfrak{F}(\mathfrak{t}) = I^{1-\zeta}_{0^+}\frac{d}{d\mathfrak{t}}\mathfrak{F}(\mathfrak{t}) =^{C} D^{\eta}_{0^+}\mathfrak{F}(\mathfrak{t}).$$

Let us characterize abstract phase space $\mathcal{P}_h$ and verify [23] for more details. Consider $g : (-\infty, 0] \to (0, +\infty)$ is continuous along $j = \int_{-\infty}^{0} h(\mathfrak{t})d\mathfrak{t} < +\infty$. For each $k > 0$,

$$\mathcal{P} = \{\lambda : [-i, 0] \to \mathfrak{X} \text{ such that } \lambda(\mathfrak{t}) \text{ is bounded and measurable}\},$$

along

$$\|\lambda\|_{[-i,0]} = \sup_{\mu\in[-i,0]} \|\psi(\delta)\|$$

for all $\lambda \in \mathcal{P}$.

Now, we define

$$\mathcal{P}_{\mathfrak{g}} = \left\{\lambda : (-\infty, 0] \to \mathfrak{X} \text{ such that for any } i > 0, \lambda|_{[-i,0]} \in \mathcal{P} \text{ and } \int_{-\infty}^{0} \mathfrak{g}(\mu)\|\lambda\|_{[\mu,0]}d\mu < +\infty\right\},$$

provided that $\mathcal{P}_{\mathfrak{g}}$ is endowed along

$$\|\psi\|_{\mathcal{P}_{\mathfrak{g}}} = \int_{-\infty}^{0} h(\delta)\|\psi\|_{[\delta,0]}d\delta$$

for all $\psi \in \mathcal{P}_{\mathfrak{g}}$; therefore, $(\mathcal{P}_{\mathfrak{g}}.\|\cdot\|)$ is a Banach space.

Now, we discuss

$$\mathcal{P}_{\mathfrak{g}}{}' = \{v : (-\infty, p) \to \mathfrak{X} \text{ such that } v|_{\mathcal{I}} \in C(\mathfrak{X}, v_0 = \psi \in \mathcal{P}_{\mathfrak{g}}\},$$

where v_k is limitation of v to $\mathfrak{X} = (\lambda_k, \lambda_{k+1}]$ for $k = 0, 1, \ldots, n$.

Set $\|\cdot\|_p$ be a semi-norm in $\mathcal{P}_{\mathfrak{g}}{}'$ defined by

$$\|v\|_p = \|\phi\|_{\mathcal{P}_{\mathfrak{g}}} + \sup\|v(\chi)\| : \chi \in [0, p]\}, v \in \mathcal{P}'_{\mathfrak{g}}.$$

Lemma 1. *Assuming $v \in \mathcal{P}_{\mathfrak{g}}{}'$, then for $\lambda \in \mathfrak{I}, v \in \mathcal{P}_{\mathfrak{g}}$. Moreover,*

$$j|v(\lambda)| \leq \|v_\lambda\|_{\mathcal{P}_{\mathfrak{g}}} \leq \|\phi\|_{\mathcal{P}_{\mathfrak{g}}} + j\sup_{\delta\in[0,\lambda]}\|u(r)\|,$$

where

$$j = \int_{-\infty}^{0} h(\lambda)d\lambda < +\infty.$$

Definition 5. *Assume $0 < \vartheta < 1$, $0 < \psi < \frac{\pi}{2}$, Let $\Theta_{\psi}^{-\vartheta}$ be the family of closed linear operators, $\mathbb{A} : D(\mathbb{A}) \subset \mathfrak{X} \to \mathfrak{X}$ such that the sectors $S_\psi = \theta \in C\ 0$ with $\|arg\theta\| \leq \psi$ and* *(i)* $\sigma(\mathbb{A}) \subseteq S_\psi$

(ii) *there exists* $\mathbb{N}_\lambda$ *as a constant,*

$$\|(\theta I - \mathbb{A})^{-1}\| \leq \mathbb{N}_\lambda |\mathfrak{t}|^{-\vartheta}, \text{for every } \psi < \lambda < \pi$$

then $\mathbb{A} \in \Theta_\psi^{-\vartheta}$ *called an infinitesimal generator of strongly continuous functions of bounded linear operators on* $\mathfrak{X}$.

Lemma 2. *Let* $0 < \vartheta < 1, 0 < \psi < \frac{\pi}{2}, \mathbb{A} \in \Theta_\psi^{-\vartheta}(\mathfrak{X})$. *Then*

(1) $T(s_1 + s_2) = T(s_1) + T(s_2))$, *for any* $s_1, s_2 \in S^0_{\frac{\pi}{2}-\varrho}$;

(2) *there exists* Λ_0 *is the constant such that* $\|T(\mathfrak{t}\|_C \leq \Lambda_0 \mathfrak{t}^{\vartheta-1}$, *for any* $\mathfrak{t} > 0$;

(3) *The range* $\mathbb{R}(T(\mathfrak{t}))$ *of* $T(\mathfrak{t})$, $z \in S^0_{\frac{\pi}{2}-\varrho}$ *is belong* $D(\mathbb{A}^\infty)$. *Especially,* $\mathbb{R}(T(\mathfrak{t})) \subset D(\mathbb{A}^\theta)$ *for all* $\theta \in C$ *with* $Re(\theta) > 0$,

$$\mathbb{A}^\theta T(\mathfrak{t})x = \frac{1}{2\pi i}\int_{\Gamma_\gamma} \mathfrak{t}^\theta e^{-\mathfrak{t}z} R(\mathfrak{t}; A) x dz, forall x \in \mathfrak{X},$$

and hence there exists a constant $\mu' = \Lambda'(\alpha, \theta) > 0$ *and satisfy* $\|\mathbb{A}^\theta T(\mathfrak{t})\|_{B(\mathfrak{X})} \leq \Lambda' \mathfrak{t}^{-\alpha - Re(\theta) - 1}$, *for all* $\mathfrak{t} > 0$;

(4) *If* $\theta > 1 - \vartheta$, *then* $D(\mathbb{A}^\theta) \subset \Sigma_T = \{x \in \mathfrak{X} : \lim_{\mathfrak{t}\to 0+} T(\mathfrak{t})x = x\}$;

(5) $\mathbb{R}(\zeta', \mathbb{A}) = \int_0^\infty e^{-\zeta' \mathfrak{t}} T(\mathfrak{t}) d\mathfrak{t}$, *for all* $\zeta' \in C$ *with* $Re(\zeta') > 0$.

we define the two operators $\{\mathbb{R}_\zeta(\mathfrak{t})\}_{\mathfrak{t}\in S_{\frac{\pi}{2}-\psi}}, \{\mathbb{S}_\zeta(\mathfrak{t})\}_{\mathfrak{t}\in S_{\frac{\pi}{2}-\psi}}$ *as follows*

$$\mathbb{R}_\zeta(\mathfrak{t}) = \int_0^\infty \mathbb{W}_\zeta(\theta) T(\mathfrak{t}^\zeta \theta) d\theta,$$

$$\mathbb{S}_\zeta(\mathfrak{t}) = \int_0^\infty \zeta\theta \mathbb{W}_\zeta(\theta) T(\mathfrak{t}^\zeta \theta) d\theta.$$

Let the Wright-type function

$$\mathbb{W}_\zeta(\alpha) = \Sigma_{n\in N} \frac{(-\alpha)^{n-1}}{\Gamma(1-\zeta n)(n-1)!}, \alpha \in \mathbb{C}. \tag{4}$$

The following are the properties of Wright-type functions

(a)

$$\mathbb{W}_\zeta(\theta) \leq 0, \mathfrak{t} > 0;$$

(b)

$$\int_0^\infty \theta^\iota \mathbb{W}_\zeta(\theta) d\theta = \frac{\Gamma(1+\iota)}{\Gamma(1+\zeta\iota)};$$

(c)

$$\int_0^\infty \frac{\zeta}{\theta^{(\zeta+1)}} e^{-p\theta} \mathbb{W}_\zeta(\frac{1}{\theta^\zeta}) d\theta = e^{-p^\zeta}.$$

Lemma 3. *The integral equation is equivalent to the (1)–(3)*

$$\begin{aligned} v(\mathfrak{t}) = {} & \frac{\phi(0) - \mathfrak{F}_1(0, \phi(0))}{\Gamma\eta(1-\zeta)} \mathfrak{t}^{(1-\zeta)(\eta-1)+\zeta} + \alpha(v_{\mathfrak{t}1}, v_{\mathfrak{t}2}, v_{\mathfrak{t}3}, \cdots, v_{\mathfrak{t}m})(0) \\ & + \frac{1}{\Gamma(\zeta)}\int_0^\mathfrak{t} (\mathfrak{t}-\varrho)^{\zeta-1}[\mathbb{A}\mathfrak{F}_1(\mathfrak{t}, v_\mathfrak{t}) \\ & + \mathbb{A}v_\varrho + \mathfrak{F}_2\left(\varrho, v_\varrho, \int_0^\mathfrak{t} h(\mathfrak{t}, s, v_s) ds\right) + \mathbb{B}w(\varrho)] d\varrho. \end{aligned}$$

$$+\sum_{0<\mathfrak{t}_i<\mathfrak{t}} S_{\zeta,\eta}(\mathfrak{t}-\mathfrak{t}_i)I_i(v(\mathfrak{t}_i^-)).$$

Definition 6.

$$\begin{aligned} v(\mathfrak{t}) = & R_{\zeta,\eta}(\mathfrak{t})[\phi(0) - \mathfrak{F}_1(0,\phi(0)) + \alpha(v_{\mathfrak{t}1}, v_{\mathfrak{t}2}, v_{\mathfrak{t}3}, \cdots, v_{\mathfrak{t}m})(0)] + \mathfrak{F}_1(\mathfrak{t}, v_\mathfrak{t}) \\ & + \int_0^t \mathbb{A}Q_\zeta(\mathfrak{t}-\varrho)\mathfrak{F}_1(\mathfrak{t}, v_\mathfrak{t})d\varrho \\ & + \int_0^\mathfrak{t} Q_\zeta(\mathfrak{t}-\varrho)\mathfrak{F}_2\left(\varrho, v_\varrho, \int_0^\varrho h(\varrho, s, v_s)ds\right)d\varrho \\ & + \int_0^\mathfrak{t} Q_\zeta(\mathfrak{t}-\varrho)\mathbb{B}w(\varrho)d\varrho, \mathfrak{t} \in \mathfrak{I}, \\ & + \sum_{0<\mathfrak{t}_i<\mathfrak{t}} S_{\zeta,\eta}(\mathfrak{t}-\mathfrak{t}_i)I_i(v(\mathfrak{t}_i^-)). \end{aligned} \tag{5}$$

The mild solutions of the Equations (1)–(3), is a function of $v(\mathfrak{t}) \in \mathbb{C}(\mathbb{I}', \mathfrak{X})$, that satisfies where $R_{\zeta,\eta}(\mathfrak{t}) = I_0^{\eta(1-\zeta)}Q_\zeta(\mathfrak{t}), Q_\zeta(\mathfrak{t}) = \mathfrak{t}^{\zeta-1}S_\eta(\mathfrak{t})$, i.e.,

$$\begin{aligned} v(\mathfrak{t}) = & R_{\zeta,\eta}(\mathfrak{t})[\phi(0) + \alpha(v_{\mathfrak{t}_1}, v_{\mathfrak{t}_2}, v_{\mathfrak{t}_3}, \cdots, v_{\mathfrak{t}_m}) - \mathfrak{F}_1(0,\phi(0))] + \mathfrak{F}_1(\mathfrak{t}, v_\mathfrak{t}) \\ & + \int_0^\mathfrak{t} (\mathfrak{t}-\varrho)^{\zeta-1}\mathbb{A}S_\zeta(\mathfrak{t}-\varrho)\mathfrak{F}_1(\mathfrak{t}, v_\mathfrak{t})d\varrho \\ & + \int_0^\mathfrak{t} (\mathfrak{t}-\varrho)^{\zeta-1}S_\zeta(\mathfrak{t}-\varrho)\mathfrak{F}_2\left(\varrho, v_\varrho, \int_0^\varrho h(\varrho, s, v_s)ds\right)d\varrho \\ & + \int_0^\mathfrak{t} (\mathfrak{t}-\varrho)^{\zeta-1}S_\zeta(\mathfrak{t}-\varrho)\mathbb{B}w(\varrho)d\varrho \\ & + \sum_{0<\mathfrak{t}_i<\mathfrak{t}} S_{\zeta,\eta}(\mathfrak{t}-\mathfrak{t}_i)I_i(v(\mathfrak{t}_i^-)). \end{aligned} \tag{6}$$

Lemma 4. *Let* $\{T(\mathfrak{t})\}_{\mathfrak{t}\geq 0}$ *is equi-continuous, then* $\{S_\zeta(\mathfrak{t})\}_\mathfrak{t} > 0$, $\{Q_\zeta(\mathfrak{t})\}_\mathfrak{t} > 0$ *and* $\{R_\zeta(\mathfrak{t})\}_\mathfrak{t} > 0$ *are the strongly continuous that is, for any* $x \in \mathfrak{X}$ *and* $\mathfrak{t}_2 > \mathfrak{t}_1 > 0$,

$$\|S_\zeta(\mathfrak{t}_2)v - S_\zeta(\mathfrak{t}_1)v\| \to 0, \|Q_\zeta(\mathfrak{t}_2)v - Q_\zeta(\mathfrak{t}_1)v\| \to 0, \|R_\zeta(\mathfrak{t}_2)v - R_\zeta(\mathfrak{t}_1)v\| \to 0, as\mathfrak{t}_2 \to \mathfrak{t}_1.$$

Lemma 5. *For any fixed* $\mathfrak{t} > 0, S_\zeta(\mathfrak{t}), Q_\zeta(\mathfrak{t})$ *and* $R_{\zeta,\eta}(\mathfrak{t})$ *are linear operators, and for any* $x \in \mathfrak{X}$,

$$\|S_\zeta(\mathfrak{t})v\| \leq L'\mathfrak{t}^{-\zeta+\zeta\vartheta}\|v\|, \|Q_\zeta(\mathfrak{t})v\| \leq L'\mathfrak{t}^{-1+\zeta\vartheta}\|v\|, \|R_\zeta(\mathfrak{t})v\| \leq L'\prime\mathfrak{t}^{-1+\epsilon-\zeta v+\zeta\vartheta}\|v\|,$$

where

$$M' = \Lambda_0\frac{\Gamma(\vartheta)}{\Gamma(\zeta\vartheta}), M'' = \Lambda_0\frac{\Gamma(\vartheta)}{\Gamma(\epsilon(1-\zeta)+\zeta\vartheta)}.$$

Lemma 6. *Let (1) and (2) is said to be controllable in* $\mathcal{I}$ *for every continuous initial value function* $\phi \in \mathcal{P}_\mathfrak{g}$,$v_1 \in \mathfrak{X}$, *there exists* $w \in L^2(\mathcal{I}, V)$ *and the mild solution* $v(\mathfrak{t})$ *satisfied with* $v(p) = v_1$.

Definition 7. *Suppose* E^+ *is the positive cone of a Banach space* $(E, \leq)$. *Let* Φ *be the function defined on the set of all bounded subsets of the Banach space* $\mathfrak{X}$ *with values in* E^+ *is known as a measure of non-compactness on* $\mathfrak{X}$ *iff* $\Phi(\bar{conv}(\Omega)) = \Phi(\Omega)$ *for every bounded subset* $\Omega \subset \mathfrak{X}$, *where* $conv(\Omega)$ *denoted the closed hull of* Ω.

We now present the basic result on measures of non-compactness.

Definition 8. *Let* P *be the bounded set in a Banach space* $\mathfrak{X}$, *the Hausdorff measures of non-compactness* μ *is defined as*

$$\gamma(P) = \inf\{r > 0 : \textit{P can be covered by a finite number of balls with radii } \theta\}. \tag{7}$$

Lemma 7. *Suppose $\mathfrak{X}$ is a Banach space and $P_1, P_2 \subseteq \mathfrak{X}$ are bounded. Then, the properties satisfy*

- *(i) P_1 is precompact iff $\mu(P_1) = 0$;*
- *(ii) $\gamma(P_1) = \gamma(\overline{P_1}) = \gamma(conv(P_1))$, where $conv(P_1)$ and $\overline{P_1}$ are denote the convex hull and closure of P_1, respectively;*
- *(iii) If $P_1 \subseteq P_2$ then $\gamma(P_1) \leq \gamma(P_2)$;*
- *(iv) $\gamma(P_1 + P_2) \leq \gamma(P_1) + \gamma(P_2)$, such that $P_1 + P_2 = \{= b_1 + b_2 : b_1 \in P_1, b_2 \in P_2\}$;*
- *(v) $\gamma(P_1 + P_2) \leq \max\{\gamma(P_1), \gamma(P_2)\}$;*
- *(vi) $\gamma(\mu P_1) = |\mu|\gamma(P_1 \forall \mu \in \mathbb{R}$, when $\mathfrak{X}$ be a Banach space;*
- *(vii) If the operator $\Phi : D(\phi) \subseteq \mathfrak{X} \to \mathfrak{X}_1$ is Lipschitz continuous, Λ_1 be the constant then we know $\tau(\Phi(P_1)) \leq \gamma(P_1) \forall$ bounded subset $P_1 \subset D(\Phi)$, where τ represent the Hausdorff measure of non-compactness in the Banach space $\mathfrak{X}_1$.*

Theorem 1. *If $\{v_n\}_{n=1}^{\infty}$ is a set of Bochner integrable functions from $\mathcal{I}$ to $\mathfrak{X}$ with the estimate property, $\|v_n(\mathfrak{t})\| \leq \gamma_1(\mathfrak{t})$ for almost all $\mathfrak{t} \in \mathcal{I}$ and every $n \geq 1$, where $\gamma_1 \in L^1(\mathbb{I}, \mathbb{R})$, then the function $\varpi(\mathfrak{t}) = \gamma(\{v_n((t) : n \geq 1\})$ be in $L^1(\mathcal{I}, \mathbb{R})$ and satisfies*

$$\gamma\left(\left\{\int_0^{\mathfrak{t}} v_n(\varrho)d\varrho : n \geq 1\right\}\right) \leq 2\int_0^{\mathfrak{t}} \varpi(\varrho)d\varrho.$$

Lemma 8. *If $P \subset C([a,b], \mathfrak{X})$ is bounded and equi-continuous, then $\gamma(P(\mathfrak{t}))$ is continuous for $a \leq \mathfrak{t} \leq b$ and*

$$\gamma(P) = \sup\{\mu(P(\mathfrak{t})), a \leq \mathfrak{t} \leq b\}, where P(\mathfrak{t}) = \{x(\mathfrak{t} : z \in P\} \subseteq \mathfrak{X}.$$

Lemma 9. *Let P be a closed convex subset of a Banach space $\mathfrak{X}$ and $0 \in P$. Assume that $\mathfrak{F}_2 : P \to \mathfrak{X}$ continuous map which satisfies Mönch's condition, i.e., if $P_1 \subset P$ is countable and $P_1 \subset conv(0 \cup \mathfrak{F}_1(P_1)) \Rightarrow \bar{P}_1$ is compact. Then, $\mathfrak{F}_2$ has a fixed point in P.*

3. Controllability Results

We require the succeeding hypothesis

Hypothesis 0 ($\mathbf{H_0}$). *Let $\mathbb{A}$ be the infinitesimal generator of strongly continuous functions of bounded linear operators of an analytic semigroup $\{T(\mathfrak{t}, \mathfrak{t} > 0)\}$ in $\mathfrak{X}$ such that $\|T(\mathfrak{t})\| \leq Q_1$ where $Q_1 \leq 0$ be the constant.*

Hypothesis 1 ($\mathbf{H_1}$). *The function $\mathfrak{F}_1 : \mathcal{I} \times \mathcal{P}_{\mathfrak{g}} \to \mathfrak{X}$ satisfies:*

(i) *Catheodary condition:$\mathfrak{F}_2(\cdot, s, u)$ is strongly measurable $\forall (s,u)\mathcal{P}_{\mathfrak{g}} \times \mathfrak{X}$ and$\mathfrak{F}_2(\mathfrak{t}, \cdot, \cdot)$ is continuous for a.e $\mathfrak{t} \in \mathcal{I}$, $\mathfrak{F}_2(\mathfrak{t}, \cdot, \cdot, v) : [0, p] \to \mathfrak{X}$ is strongly measurable.*

(ii) *$\exists$ a constants $0 < \zeta_1 < \zeta$ and $\theta_1 \in L^{\frac{1}{\eta_1}}(\mathcal{I}, \mathbb{R}^+)$ and non-decreasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|\mathfrak{F}_2(\mathfrak{t}, s, v)\| \leq \theta_1(\mathfrak{t})\psi(\|s\|_{P_{\mathfrak{g}}} + \|v\|), v \in \mathfrak{X}, \mathfrak{t} \in I$, where θ_1 satisfies $\liminf_{n\to\infty \frac{\theta_1(n)}{n}} = 0$.*

(iii) *$\exists$ a constant $0 < \zeta_2 < \zeta$ and $\theta_2 \in L^{\frac{1}{\eta_2}}(\mathcal{I}, \mathbb{R}^+)$ such that, for any bounded subset $D_1 \subset \mathfrak{X}$, $P_1 \subset P_{\mathfrak{g}}$,*

$$\gamma(\mathfrak{F}_2(\mathfrak{t}, P_1, D_1)) \leq \theta_2(\mathfrak{t}) \sup_{-\infty < \tau \leq 0} [\gamma(P_1(\rho)) + \gamma(D_1)].$$

(iv) *Let $I_i : F \mapsto F$ are continuous functions and there exists a constant $N > 0$ such that for all $\mathfrak{t} \in \mathfrak{X}$, we have $\|I_i(v_1) - I_i(v_2)\| \leq N\|v_1 - v_2\|$ for a.e $\mathfrak{t} \in \mathcal{I}$.*

Hypothesis 2 ($\mathbf{H_2}$). *The function $h : \mathcal{I} \times \mathcal{I} \times \mathcal{P}_{\mathfrak{g}} \to \mathfrak{X}$ satisfies the following:*

(i) *$h(\cdot, s, v)$ is measurable for all $h(s, v) \in \mathcal{P}_{\mathfrak{g}} \times \mathfrak{X}$, $\mathfrak{g}(\mathfrak{t}, \cdot, \cdot)$ is continuous for a.e $\mathfrak{t} \in \mathcal{I}$.*

(ii) *$\exists$ a constant $H_0 > 0$ such that $\|h(\mathfrak{t}, s, u)\| \leq H_0(1 + \|v\|_{P_w})$ for every $\mathfrak{t} \in \mathcal{I}$, $u \in \mathfrak{X}, s \in \mathcal{P}_{\mathfrak{g}}$.*

(iii) *There exists $\theta_3 \in L^1(\mathcal{I}, \mathbb{R}^+)$ such that, for any $D_2 \subset \mathfrak{X}$*

$$\gamma(h(\mathfrak{t}, s, D_2)) \leq \theta_3(\mathfrak{t}, s)\left[\sup_{-\infty<\rho\leq 0} \gamma(D_2(\tau))\right].$$

Hypothesis 3 ($\mathbf{H_3}$).

(i) *For any $\mathfrak{t} \in \mathcal{I}$, multivalued map $\mathfrak{F}_1 : \mathcal{I} \times \mathfrak{X} \to \mathfrak{X}$ is a continuous function and there exists $\beta \in (0,1)$ such that $\mathfrak{F}_1 \in D(\mathbb{A}^\beta)$ and for all $v \in \mathfrak{X}, \mathfrak{t} \in \mathcal{I}, \mathbb{A}^\beta \mathfrak{F}(\mathfrak{t}, \cdot)$ satisfies the following:*

$$\|\mathbb{A}^\beta \mathfrak{F}(\mathfrak{t}, v(\mathfrak{t}))\| \leq N_{\mathfrak{F}_1}\left(1 + \mathfrak{t}^{1-\epsilon+\zeta+\zeta\epsilon-\zeta\vartheta}\|v(\mathfrak{t}\|\right), (\mathfrak{t}, v) \in I \times \mathfrak{X}.$$

(ii) *$\mathfrak{F}_1$ is completely continuous and for any bounded set $D \subset C$ the set $\{\mathfrak{t} \to \mathfrak{F}_1(\mathfrak{t}, v_\mathfrak{t}), v \in D\}$ is equi-continuous in $\mathfrak{X}$.*

Hypothesis 4 ($\mathbf{H_4}$).

(i) *The linear operator $B : L^2(\mathcal{I}, V) \to L'(\mathcal{I}, \mathfrak{X})$ is bounded,$W : L^2(\mathcal{I}, V) \to \mathfrak{X}$ defined by $Wy = \int_0^p (p-\rho)^{\zeta-1} S_\zeta(p-\rho)\mathbb{B}y(\rho)d\rho$ has an inverse operator W^{-1} which take the values in $L^2(\mathcal{I}, V)/kerW$ and there exists two positive values Q_2 and Q_3 such that $\|\mathbb{B}\|_{L_b(U,\mathfrak{X})} \leq Q_2$, $\|W^{-1}\|_{L_p(\mathfrak{X},V)/kerW} \leq Q_3$.*

(ii) *$\exists$ a constants $\zeta_0 \in (0, \zeta)$ and $Q_W \in L^{\frac{1}{\zeta_0}}(\mathcal{I}, \mathbb{R}^+)$ such that, $\forall$ bounded set $Q \subset \mathfrak{X}, \gamma((W^{-1}Q))(\mathfrak{t})) \leq Q_W(\mathfrak{t})\gamma(Q)$.*

Hypothesis 5 ($\mathbf{H_5}$). *$\alpha : \mathcal{P}_\mathfrak{g}{}^i \to \mathcal{P}$ is continuous, there exists $\mathbb{L}_i(\alpha) > 0$,*
$\|\alpha(z_1, z_2, z_3 \cdots z_n) - \alpha(y_1, y_2, y_3 \cdots y_n)\| \leq \sum_{i=1}^k \mathbb{L}_n(\alpha)\|z_n - y_n\|_{P_\mathfrak{g}}$,
for every $z_n, y_n \in \mathcal{P}_\mathfrak{g}$
Take $\Delta_w = \sup\{\|\alpha(z_1, z_2, z_3 \cdots z_n)\| : z_n \in P_w\}$.

Let us define

$$Q_{\zeta_i} = [(\tfrac{1-\zeta_i}{\zeta\vartheta-1})p^{(\frac{\zeta\vartheta-1}{1-\zeta_i})}], i = 1,2$$
$$Q_4 = Q_{\zeta_1\|\theta_2\|_{L^{\frac{1}{\zeta_1}}}(\mathcal{I},\mathbb{R}^+)},$$
$$Q_5 = Q^2_{\zeta_1\|\theta_2\|_{L^{\frac{1}{\zeta_1}}}(\mathcal{I},\mathbb{R}^+)},$$
$$\|A^{-\beta}\| \leq N_0, and \|A^{1-\beta}\| \leq N_1.$$

Theorem 2. *Suppose H_1–H_5 holds, then the impulsive neutral Hilfer fractional differential Equations (1) and (2) has a solution on $[0, p]$ provided $\phi(0) \in D(\mathbb{A}^\theta)$ with $\hat{Q} = p^{1-\epsilon+\zeta\epsilon-\zeta\vartheta}(Q_4(1+\theta_3^*) + L'^2 Q_2 Q_W Q_5) < 1$ and $\theta > 1 + \vartheta$.*

Proof. Assume the operator $\Phi : \mathcal{P}'_\mathfrak{g} \to \mathcal{P}'_\mathfrak{g}$, with $\mathfrak{t} \in \mathcal{I}$ defined as

$$\Phi(v(\mathfrak{t})) = \begin{cases} \Phi_1(\mathfrak{t}), (-\infty, 0], \\ \mathfrak{t}^{1-\eta+\zeta\eta-\zeta\vartheta}[\mathbb{R}_{\zeta,\eta}[\phi(0) - \mathfrak{F}_1(0, \phi(0) + \alpha(v_{\mathfrak{t}_1},\ v_{\mathfrak{t}2}, v_{\mathfrak{t}3}, \cdots, v_{\mathfrak{t}m})(0))] \\ +\mathfrak{F}_1(\mathfrak{t}, v_\mathfrak{t}) + \int_0^\mathfrak{t}(\mathfrak{t}-\varrho)^{\zeta-1} A\, \mathbb{S}_\zeta(\zeta-\varrho)\mathfrak{F}_1(\varrho, v_\varrho)d\varrho \\ + \int_0^\mathfrak{t}(\mathfrak{t}-\varrho)^{\zeta-1}\mathbb{S}_\zeta(\mathfrak{t}-\varrho)\mathfrak{F}_2(\varrho, v_\varrho, \int_0^\varrho g(\varrho, s, v_s)ds)d\varrho + \int_0^\mathfrak{t}(\mathfrak{t}-\varrho)^{\zeta-1}\mathbb{S}_\zeta(\zeta-\varrho)\mathbb{B}w(\varrho)d\varrho] \\ + \sum_{0<\mathfrak{t}_i<\mathfrak{t}} S_{\zeta,\eta}(\mathfrak{t}-\mathfrak{t}_i)I_i(u(\mathfrak{t}_i^-)). \end{cases}$$

For $\Phi_1 \in \mathcal{P}_\mathfrak{g}$, we define $\widehat{\Phi}$ by

$$\widehat{\Phi}(\mathfrak{t}) = \begin{cases} \Phi_1(\mathfrak{t}), (-\infty, 0], \\ \mathbb{R}_{\zeta,\eta}\phi(0) + \alpha(v_{\mathfrak{t}1}, v_{\mathfrak{t}2}, v_{\mathfrak{t}3}, \cdots, v_{\mathfrak{t}m}), \mathfrak{t} \in \mathbb{I}, \end{cases}$$

then $\hat{\Phi} \in \mathcal{P}'_{\mathfrak{g}}$. Let $v(\mathfrak{t}) = x(\mathfrak{t}) + \hat{\Phi}_{\mathfrak{t}}$, $-\infty < \mathfrak{t} \leq p$, v satisfies from 6 iff x satisfies $x_0 = 0$ and

$$\begin{aligned}
x(\mathfrak{t}) = &-\mathbb{R}_{\zeta,\eta}\mathfrak{F}_1(0, \phi(0) + \alpha(v_{\mathfrak{t}1}, v_{\mathfrak{t}2}, v_{\mathfrak{t}3}, \cdots, v_{\mathfrak{t}m})(0)) + \mathfrak{F}_1(\mathfrak{t}, v_{\mathfrak{t}} + \hat{\Phi})d\varrho \\
&+ \int_0^{\mathfrak{t}} (\mathfrak{t}-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_\zeta(\zeta-\varrho)\mathfrak{F}_1(\varrho, x_\varrho + \hat{\Phi}_\varrho)d\varrho \\
&+ \int_0^{\mathfrak{t}} (\mathfrak{t}-\varrho)^{\zeta-1}\mathbb{S}_{\mathfrak{t}}(\mathfrak{t}-\varrho)\mathfrak{F}_2\left(\varrho, v_\varrho, \int_0^\varrho g(\varrho, s, v_s, \hat{\Phi}_s)ds\right)d\varrho \\
&+ \int_0^{\mathfrak{t}} (\mathfrak{t}-\varrho)^{\zeta-1}\mathbb{S}_{\mathfrak{t}}(\mathfrak{t}-\varrho)\mathbb{B}W^{-1}\big[v_p - \mathbb{R}_{\zeta,\eta}[\phi(0) + \alpha(v_{\mathfrak{t}_1}, v_{\mathfrak{t}_2}, v_{\mathfrak{t}_3}, \cdots, v_{\mathfrak{t}_m}) \\
&-\mathfrak{F}_1(0, \phi(0) + \alpha(v_{\mathfrak{t}_1}, v_{\mathfrak{t}_2}, v_{\mathfrak{t}_3}, \cdots, v_{\mathfrak{t}_m}))] - \mathfrak{F}_1(p, x_p + \hat{\Phi}_p) \\
&- \int_0^p (p-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_\zeta(p-\varrho)\mathfrak{F}_1(\varrho, x_\varrho + \hat{\Phi}_p) \\
&- \int_0^p (p-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_\zeta(p-\varrho)\mathfrak{F}_1(\varrho, x_\varrho + \hat{\Phi}_\varrho - \int_0^\varrho h(\varrho, s, x_\varrho + \hat{\Phi}_\varrho)ds)d\varrho\Big]d\varrho \\
&+ \sum_{0<\mathfrak{t}_i<\mathfrak{t}} S_{\zeta,\eta}(\mathfrak{t}-\mathfrak{t}_i)I_i(v(\mathfrak{t}_i^-))
\end{aligned}$$

where

$$\begin{aligned}
w(\mathfrak{t}) = &W^{-1}\big[v_p - \mathbb{R}_{\zeta,\eta}[\phi(0) - \mathfrak{F}_1(0, \phi(0) + \alpha(v_{\mathfrak{t}1}, v_{\mathfrak{t}2}, v_{\mathfrak{t}3}, \cdots, v_{\mathfrak{t}m}))] - \mathfrak{F}_1(p, x_p + \hat{\Phi}_\varrho) \\
&- \int_0^p (p-\varrho)^{\eta-1}\mathbb{A}\mathbb{S}_\zeta(b-\varrho)\mathfrak{F}_1(p, x_\varrho + \hat{\Phi}_\varrho)d\varrho \\
&- \int_0^p (p-\varrho)^{\eta-1}\mathbb{S}_\zeta(p-\varrho)\mathfrak{F}_2\big(\varrho, x_\varrho + \int_0^\varrho h(\varrho, s, x_\varrho + \hat{\Phi}_\varrho)ds\Big)d\varrho\Big]. \\
&+ \sum_{0<\mathfrak{t}_i<\mathfrak{t}} S_{\zeta,\eta}(\mathfrak{t}-\mathfrak{t}_i)I_i(v(\mathfrak{t}_i^-)).
\end{aligned}$$

Let $\mathcal{P}''_{\mathfrak{g}} = \{x \in \mathcal{P}'_{\mathfrak{g}} : x_0 \in \mathcal{P}_{\mathfrak{g}}\}$. For any $x \in \mathcal{P}'_{\mathfrak{g}}$,

$$\begin{aligned}
\|x\|_p &= \|x_0\|_{\mathcal{P}_{\mathfrak{g}}} + \sup\{x(\varrho) : 0 \leq \varrho \leq p\} \\
&= \sup\{\|x(\varrho)\| : 0 \leq \varrho \leq p\}.
\end{aligned}$$

Hence, $(\mathcal{P}''_{\mathfrak{g}}, \|\cdot\|_{\mathfrak{g}})$ is a Banach space. □

For $G > 0$, choose $\mathcal{P}_G = \{x \in \mathcal{P}''_{\mathfrak{g}} : \|x\|_p \leq G\}$, then $\mathcal{P}_G \subset \mathcal{P}''_{\mathfrak{g}}$ is uniformly bounded and for $x \in \mathcal{P}_G$ from Lemma 1,

$$\begin{aligned}
\|x_\varrho + \hat{\Phi}_\varrho\|_{\mathcal{P}_\varrho} &\leq \|x_{\mathfrak{t}}\|_{\mathcal{P}_G} + \|\hat{\Phi}_\varrho\|_{P_G} \\
&\leq l\Big(G + L''\mathfrak{t}^{-1+\zeta-\zeta\eta+\zeta\varrho}\Big) + \|\Phi_{1\mathcal{P}_g}\| \\
&= G'.
\end{aligned}$$

Let us introduce an operator $\Psi : \mathcal{P}_{\mathfrak{g}}'' \to \mathcal{P}_{\mathfrak{g}}''$, defined by

$$\Psi(x(\mathfrak{t})) = \begin{cases} 0, \mathfrak{t} \in (-\infty, 0], \\ -\mathfrak{F}_1(0, \phi(0) + \alpha(v_{\mathfrak{t}1}, v_{\mathfrak{t}2}, v_{\mathfrak{t}3}, \cdots, v_{\mathfrak{t}m})(0)) + \mathfrak{F}_1(\mathfrak{t}, x_{\mathfrak{t}} + \hat{\Phi}_\varrho) + \int_0^{\mathfrak{t}} (\mathfrak{t}-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_\zeta(\mathfrak{t}-\varrho)\mathfrak{F}_1(\varrho, v_\varrho)d\varrho \\ + \int_0^{\mathfrak{t}} (\mathfrak{t}-\varrho)^{\zeta-1}\mathbb{S}_\zeta(\mathfrak{t}-\varrho)\mathfrak{F}_2(\varrho, v_\varrho, \int_0^\varrho h(\varrho, s, v_s)ds)d\varrho + \int_0^{\mathfrak{t}} (\mathfrak{t}-\varrho)^{\zeta-1}\mathbb{S}_\zeta(\zeta-\varrho)\mathbb{B}w(\varrho)d\varrho \\ + \sum_{0<\mathfrak{t}_i<\mathfrak{t}} S_{\zeta,\eta}(\mathfrak{t}-\mathfrak{t}_i)I_i(v(\mathfrak{t}_i^-)). \mathfrak{t} \in \mathcal{I} \end{cases}$$

Next, to show that Ψ has a fixed point.

Step 1: we have to prove $\exists$ a positive value G such that $\Psi(\mathcal{P}_G) \subseteq \mathcal{P}_G$. Assume the statement is false, i.e., for each $G > 0$, there exists $x^G \in \mathcal{P}_G$, but $\Psi(x^G)$ not in $\mathcal{P}_G$,

$$\begin{aligned}
G &< \sup \mathfrak{t}^{1-\eta+\zeta\eta-\zeta\vartheta}\|\Psi(x^G(\mathfrak{t}))\| \\
&\leq p^{(1-\eta+\zeta\eta-\zeta\vartheta)}[\| - \mathbb{R}_{\zeta,\eta}\mathfrak{t}\mathfrak{F}_1(0,\phi(0)+\alpha(v_{\mathfrak{t}1},v_{\mathfrak{t}2},v_{\mathfrak{t}3},\cdots,v_{\mathfrak{t}m})(0))+\mathfrak{F}_1(\mathfrak{t},v_{\mathfrak{t}}^G+\hat{\Phi}_\varrho))d\varrho \\
&+\int_0^{\mathfrak{t}}(\mathfrak{t}-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_\zeta(\zeta-\varrho)\mathfrak{F}_1(\varrho,v_\varrho^G+\hat{\Phi}_\varrho)d\varrho \\
&+\int_0^{\mathfrak{t}}(\mathfrak{t}-\varrho)^{\zeta-1}\mathbb{S}_\zeta(\mathfrak{t}-\varrho)\mathfrak{F}_2\left(\varrho,v_\varrho,\int_0^s h(s,r,x_r^G+\hat{\Phi}_s)ds\right)d\varrho \\
&+\int_0^{\mathfrak{t}}(\mathfrak{t}-\varrho)^{\zeta-1}\mathbb{S}_\zeta(\zeta-\varrho)\mathbb{B}w^G(\varrho)\|d\varrho] \\
&+\sum_{0<\mathfrak{t}_i<\mathfrak{t}}S_{\zeta,\eta}(\mathfrak{t}-\mathfrak{t}_i)I_i(u(\mathfrak{t}_i^-)) \\
&\leq p^{(1-\eta+\zeta\eta-\zeta\vartheta)}[\| - \mathfrak{F}_1(0,\phi(0))+\alpha(v_{\mathfrak{t}1},v_{\mathfrak{t}2},v_{\mathfrak{t}3},\cdots,v_{\mathfrak{t}m})(0)\|+\|\mathfrak{F}_1(\mathfrak{t},v_{\mathfrak{t}}^G+\hat{\Phi}_\varrho)\| \\
&+\int_0^{\mathfrak{t}}(\mathfrak{t}-\varrho)^{\zeta-1}\|\mathbb{A}\mathbb{S}_\zeta(\zeta-\varrho)\mathfrak{F}_1(\varrho,v_\varrho^G+\hat{\Phi}_\varrho)\|d\varrho \\
&+\int_0^{\mathfrak{t}}(\mathfrak{t}-\varrho)^{\zeta-1}\|\mathbb{S}_\zeta(\mathfrak{t}-\varrho)\mathfrak{F}_2\left(\varrho,v_\varrho,\int_0^\varrho h(s,r,x_r^G+\hat{\Phi}_s)ds\right)\|d\varrho \\
&+\int_0^{\mathfrak{t}}(\mathfrak{t}-\varrho)^{\zeta-1}\|\mathbb{S}_\zeta(\zeta-\varrho)\|\|\mathbb{B}\|\|W^{-1}\|[v_p-\|S_{\zeta,\eta}(\mathfrak{t}[\phi(0) \\
&-\mathfrak{F}(0,\phi(0)+\alpha(v_{\mathfrak{t}1},v_{\mathfrak{t}2},v_{\mathfrak{t}3},\cdots,v_{\mathfrak{t}m})(0)]\| \\
&-\|\mathfrak{F}_1(p,x_p+\hat{\Phi}_p)-\int_0^p(p-\varrho)^{\zeta-1}\|\mathbb{A}S_\zeta(p-\varrho)\mathfrak{F}_1(p,x_\varrho+\hat{\Phi}_\varrho)\| \\
&-\int_0^p(p-\varrho)^{\zeta-1}\|\mathbb{S}_\zeta(\mathfrak{t}-\varrho)\mathfrak{F}_2\left(\varrho,v_\varrho,\int_0^\varrho h(\varrho,s,x_\varrho+\hat{\Phi}_\varrho)ds\right)\|d\varrho] \\
&+\sum_{0<\mathfrak{t}_i<\mathfrak{t}}\|I_i(v(\mathfrak{t}_i^-))\| \\
&+\leq p^{1-\eta+\zeta\eta-\zeta\vartheta}\big[(\mu_1+\mu_2+\mu_3)L'Q_2Q_3\,\frac{p^{\zeta\vartheta}}{\zeta\vartheta}\left(v_p-L''p^{-1+\eta-\zeta\eta+\zeta\vartheta}\ \phi(0)-(\mu_1+\mu_2+\mu_3)+(v(\mathfrak{t}_i^-))\right)\big]
\end{aligned}$$

where

$$\mu_1=[p^{-1+\eta-\zeta\eta+\zeta\vartheta}L''+(1+G')]N_0N_{\mathfrak{F}_1},$$

$$\mu_2=\frac{p^{\zeta\vartheta}}{\zeta\vartheta}L'N_1N_{\mathfrak{F}_1}(1+G'),$$

$$\mu_2=L'\psi(G'+H_0(1+G'))Q_{\zeta_1}\|\theta_1\|_{L^{\frac{1}{\zeta_1}}}.$$

The above inequality is divided by G and applying the limit as $G\to\infty$, we obtain $1\leq 0$, which is the contradiction. Therefore, $\Psi(P_G)\subseteq\mathcal{P}_G$.

Step 2: The operator Ψ is continuous on $\mathcal{P}_G$. For $\Psi:\mathcal{P}_G\to\mathcal{P}_G$, for any $x^k,x\in\mathcal{P}_G$, $k=0,1,2,...$ such that $\lim_{k\to\infty}x^k=x$, the we have $\lim_{k\to\infty}x^k\mathfrak{t}=x(\mathfrak{t})$ and $\lim_{k\to\infty}\mathfrak{t}^{1-\eta+\zeta\eta-\eta\vartheta}x^k(\mathfrak{t})=\mathfrak{t}^{1-\eta+\zeta\eta-\eta\vartheta}x(\mathfrak{t})$. From (H_2) and (H_3)

$$\mathfrak{F}\left(\mathfrak{t},v_{\mathfrak{t}}^k,\int_0^{\mathfrak{t}}h(\mathfrak{t},s,v_s^k)ds\right)=\mathfrak{F}\left(\mathfrak{t},x_{\mathfrak{t}}^k+\hat{\Phi}_{\mathfrak{t}},\int_0^{\mathfrak{t}}h(\mathfrak{t},s,x_s^k+\hat{\Phi}_{\mathfrak{t}})ds\right)\to\mathfrak{F}\left(\mathfrak{t},x_{\mathfrak{t}}^k+\hat{\Phi}_{\mathfrak{t}},\int_0^{\mathfrak{t}}h(\mathfrak{t},s,\hat{\Phi}_{\mathfrak{t}})ds\right)ask\to\infty.$$

Take

$$F_k(\varrho)=\mathfrak{F}_2\left(\varrho,x_\varrho^k+\hat{\Phi}_\varrho,\int_0^\varrho h(\mathfrak{t},s,x_s^k+\hat{\Phi}_\varrho)d\varrho\right)$$

$$F(\varrho)=\mathfrak{F}\left(\varrho,x_\varrho+\hat{\Phi}_\varrho,\int_0^\varrho h(\rho,s,x_s+\hat{\Phi}_\varrho)d\varrho\right).$$

Then, from Hypothesis 2 and 3 and Lebesgue's dominated convergence theorem, we can obtain

$$\int_0^{\mathfrak{t}} (\mathfrak{t}-\varrho)^{\zeta-1}\|F_k(\varrho)-F(\varrho)\|d\varrho \to 0 as k \to \infty, \mathfrak{t} \in \mathcal{I}. \tag{8}$$

Now, the Hypotheses (H_3),

$$\mathfrak{F}_1(\mathfrak{t}, v_{\mathfrak{t}}^k) = \mathfrak{F}_1(\mathfrak{t}, x_{\mathfrak{t}}^k + \hat{\Phi}_{\mathfrak{t}}) \to \mathfrak{F}_1(\mathfrak{t}, x_{\mathfrak{t}} + \hat{\Phi}_{\mathfrak{t}}) = \mathfrak{F}_1(\mathfrak{t}, v_{\mathfrak{t}})$$

so, we obtain

$$\int_0^{\mathfrak{t}} (\mathfrak{t}-\varrho)^{\zeta-1}\mathbb{A}S_\zeta(\mathfrak{t}-\varrho)\|\mathfrak{F}_1(\varrho, x_\varrho^k + \hat{\Phi}_\varrho - \mathfrak{F}_1(\varrho, x_\varrho + \hat{\Phi}_\varrho\|d\varrho ask \to \infty, \mathfrak{t} \in \mathcal{I}. \tag{9}$$

Next,

$$\begin{aligned}
w^k(\mathfrak{t}) = W^{-1}&[v_p - R_{\zeta,\eta}[\phi(0) - \mathfrak{F}_1(0, \phi(0) + \alpha(v_{\mathfrak{t}1}, v_{\mathfrak{t}2}, v_{\mathfrak{t}3}, \cdots, v_{\mathfrak{t}m})(0)] - \mathfrak{F}_1(p, x_p^k + \hat{\Phi}_p) \\
&- \int_0^p (p-\varrho)^{\zeta-1}\mathbb{A}S_\zeta(p-\varrho)\mathfrak{F}_1(\varrho, x_\varrho^k + \hat{\varrho}_\varrho)d\varrho \\
&- \int_0^p (p-\varrho)^{\varrho-1}S_\zeta(p-\varrho)\mathfrak{F}_2(\varrho, s, x_\varrho^k + \hat{\Phi}, \int_0^\varrho h(\varrho, s, x_\varrho^k + \hat{\Phi}_\varrho)ds)] \\
&+ \sum_{0<\mathfrak{t}_i<\mathfrak{t}} S_{\zeta,\eta}(\mathfrak{t}-\mathfrak{t}_i)I_i(v(\mathfrak{t}_i^-)),
\end{aligned}$$

$$\begin{aligned}
\|v^k(\mathfrak{t}) - v(\mathfrak{t})\| = W^{-1}&[\|\mathfrak{F}_1(p, x_p^k + \hat{\Phi}_p) - \mathfrak{F}_1(p, x_p + \hat{\Phi}_p)\| \\
&+ \int_0^p (p-\varrho)^{\zeta-1}\mathbb{A}S_\zeta(p-\varrho)\|\mathfrak{F}_1(\varrho, x_\varrho^k + \hat{\Phi}_\varrho) - \mathfrak{F}_1(\varrho, x_\varrho + \hat{\Phi}_\varrho)\|d\varrho \\
&+ \int_0^p (p-\varrho)^{\zeta-1}S_\zeta(p-\varrho)\|\mathfrak{F}_2(\varrho, x_\varrho^k + \hat{\Phi}_\varrho) - \mathfrak{F}_2(\varrho, x_\varrho + \hat{\Phi}_\varrho)\|d\varrho] \\
&+ \sum_{0<\mathfrak{t}_i<\mathfrak{t}} S_{\zeta,\eta}(\mathfrak{t}-\mathfrak{t}_i)I_i(v(\mathfrak{t}_i^-)).
\end{aligned} \tag{10}$$

From (8) and (9) the above term become converges to zero as $k \to \infty$. Now,

$$\begin{aligned}
\|\Phi x^k(\mathfrak{t}) - \Phi x(\mathfrak{t})\|_p \le \|\mathfrak{F}_1(\mathfrak{t}, x_{\mathfrak{t}}^k + \hat{\Phi}_{\mathfrak{t}}) - \mathfrak{F}_1(\mathfrak{t}, x_{\mathfrak{t}} + \hat{\Phi}_{\mathfrak{t}})\| + \int_0^{\mathfrak{t}} (\mathfrak{t}-\varrho)^{\zeta-1}S_\zeta(\mathfrak{t}-\varrho) \\
\times (\mathbb{A}\|\mathfrak{F}_1(\varrho, x_\varrho^k + \hat{\Phi}_\varrho) - \mathfrak{F}_1(\varrho, x_\varrho + \hat{\Phi}_\varrho)_{\mathfrak{t}}\|d\varrho + \|F_k(\varrho) - F(\varrho)\|d\varrho + \mathbb{B}\|w^k(\mathfrak{t}) - w(\mathfrak{t})\|)d\varrho.
\end{aligned}$$

Using (9) and (10), we obtain

$$\|\Psi x^k - \Psi x\|_p \to 0 \text{ as} k \to \infty.$$

Therefore, Ψ is continuous on $\mathcal{P}_G$.

Step 3: Now, we have to show Ψ is continuous. For $v \in \mathcal{P}_G$ and $0 \le \mathfrak{t}_1 \le \mathfrak{t}_2 \le p$, we have

$$\begin{aligned}
&\|\Phi x(\mathfrak{t}_2) - \Phi x(\mathfrak{t}_1)\| \\
= \;&\|\mathfrak{t}_2{}^{1-\eta+\zeta\eta-\zeta\vartheta}(-\mathbb{R}_{\zeta,\eta}(\mathfrak{t}_2)\mathfrak{F}_1(0, \phi(0)) + \alpha(v_{\mathfrak{t}1}, v_{\mathfrak{t}2}, v_{\mathfrak{t}3}, \cdots, v_{\mathfrak{t}m})(0) \\
&+ \mathfrak{F}_1(\mathfrak{t}_2, x_\varrho^G + \hat{\Phi}_\varrho) + \int_0^{\mathfrak{t}_2} (\mathfrak{t}_2-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(\mathfrak{t}_2-\varrho)\mathfrak{F}_1(\varrho, x_\varrho^b + \hat{\Psi}_\varrho)d\varrho \\
&+ \int_0^{\mathfrak{t}_2} (\mathfrak{t}_2-\varrho)^{\zeta-1}\mathbb{S}_k(\mathfrak{t}_2-\varrho)\mathfrak{F}_2(\varrho, x_\varrho^b + \hat{\Psi}_\varrho, \int_0^s h(s, r, x_r^G + \hat{\Phi}_s)ds)d\varrho \\
&+ \int_0^{\mathfrak{t}_2} (\mathfrak{t}_2-\varrho)^{\zeta-1}\mathbb{S}_k(\mathfrak{t}_2-\varrho)\mathbb{B}w^G(\varrho)\|d\varrho) + \sum_{0<\mathfrak{t}_i<\mathfrak{t}} S_{\zeta,\eta}(\mathfrak{t}-\mathfrak{t}_i)I_i(v(\mathfrak{t}_i^-))
\end{aligned}$$

$$
\begin{aligned}
&-\|\mathfrak{t}_1{}^{1-\eta+\zeta\eta-\zeta\vartheta}(-R_{\eta,\zeta}(\mathfrak{t}_1)\mathfrak{F}_1(0,\phi(0)+\alpha(v_{\mathfrak{t}1},v_{\mathfrak{t}2},v_{\mathfrak{t}3},\cdots,v_{\mathfrak{t}m})(0))+\mathfrak{F}_1(\mathfrak{t}_1,x_\varrho^p+\hat{\Phi}_\varrho)\\
&+\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varrho)^{\zeta-1}\mathbb{S}_k(\mathfrak{t}_1-\varrho)\mathfrak{F}_2(\varrho,x_\varrho^G+\hat{\Psi}_\varrho,\int_0^s h(s,r,x_r^b+\hat{\Phi}_s)ds)d\varrho\\
&+\int_0^{\mathfrak{t}_2}(\mathfrak{t}_2-\varrho)^{\zeta-1}\mathbb{S}_k(\mathfrak{t}_1-\varrho)Pw^G(\varrho)\|d\varrho\\
&-\sum_{0<\mathfrak{t}_i<\mathfrak{t}}S_{\zeta,\eta}(\mathfrak{t}-\mathfrak{t}_i)I_i(v(\mathfrak{t}_i^-))\\
&\leq\|\mathfrak{t}_2{}^{1-\eta+\zeta\eta-\zeta\vartheta}\mathbb{R}_{\zeta,\eta}(\mathfrak{t}_2)-\mathfrak{t}_1{}^{1-\eta+\zeta\eta-\zeta\vartheta}\mathbb{R}_{\zeta,\eta}(\mathfrak{t}_1)\|\mathfrak{F}_1(0,\phi(0))+\alpha(v_{\mathfrak{t}1},v_{\mathfrak{t}2},v_{\mathfrak{t}3},\cdots,v_{\mathfrak{t}m})(0))\\
&+\|\mathfrak{t}_2{}^{1-\eta+\zeta\eta-\zeta\vartheta}\mathfrak{F}_1(\mathfrak{t}_2,x_{\mathfrak{t}_2}+\hat{\Psi}_{\mathfrak{t}_2})-\mathfrak{t}_1{}^{1-\eta+\zeta\eta-\zeta\vartheta}\mathfrak{F}_1(\mathfrak{t}_2,x_{\mathfrak{t}_1}+\hat{\Psi}_{\mathfrak{t}_1})\|\\
&+\|\mathfrak{t}_2{}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{\mathfrak{t}_2}(\mathfrak{t}_1-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(\mathfrak{t}_1-\varrho)\mathfrak{F}_1(\varrho,x_\varrho^G+\hat{\Psi}_\varrho)d\varrho\\
&-\mathfrak{t}_1{}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{\mathfrak{t}_2}(\mathfrak{t}_1-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(\mathfrak{t}_1-\varrho)\mathfrak{F}_1(\varrho,x_\varrho^G+\hat{\Psi}_\varrho)d\varrho\|\\
&+\|\mathfrak{t}_2{}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{\mathfrak{t}_2}(\mathfrak{t}_1-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(\mathfrak{t}_1-\varrho)\mathfrak{F}_1(\varrho,x_\varrho^G+\hat{\Psi}_\varrho)d\varrho\\
&-\mathfrak{t}_1{}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(\mathfrak{t}_1-\varrho)\mathfrak{F}_1(\varrho,x_\varrho^G+\hat{\Psi}_\varrho)d\varrho\|\\
&+\|\mathfrak{t}_2{}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(\mathfrak{t}_2-\varrho)\mathfrak{F}_1(\mathfrak{t}_1,x_\varrho^G+\hat{\Psi}_\varrho)d\varrho\|\\
&+\|\mathfrak{t}_2{}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}(\mathfrak{t}_1-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(\mathfrak{t}_2-\varrho)\mathfrak{F}_1(\mathfrak{t}_2,x_\varrho+\hat{\Psi}_\varrho,\int_0^\varrho h(\varrho,s,x_\varrho+\hat{\Psi}_\varrho)ds)d\varrho\\
&-\mathfrak{t}_1{}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(\mathfrak{t}_2-\varrho)\mathfrak{F}_1(\mathfrak{t}_2,x_\varrho+\hat{\Psi}_\varrho,\int_0^\varrho h(\varrho,s,x_\varrho+\hat{\Psi}_\varrho)ds)d\varrho\|\\
&+\|\mathfrak{t}_1{}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(\mathfrak{t}_2-\varrho)\mathfrak{F}_1(\mathfrak{t}_2,x_\varrho+\hat{\Psi}_\varrho,\int_0^\varrho h(\varrho,s,x_\varrho+\hat{\Psi}_\varrho)ds)d\varrho\\
&-\mathfrak{t}_1{}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(\mathfrak{t}_2-\varrho)\mathfrak{F}_1(\mathfrak{t}_2,x_\varrho+\hat{\Psi}_\varrho,\int_0^\varrho h(\varrho,s,x_\varrho+\hat{\Psi}_\varrho)ds)d\varrho\|\\
&+\|\mathfrak{t}_2{}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}(\mathfrak{t}_2-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(\mathfrak{t}_2-\varrho)\mathfrak{F}_1(\mathfrak{t}_2,x_\varrho+\hat{\Psi}_\varrho,\int_0^\varrho h(\varrho,s,x_\varrho+\hat{\Psi}_\varrho)ds)d\varrho\|\\
&+\|\mathfrak{t}_2{}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{\mathfrak{t}_2}(\mathfrak{t}_2-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(\mathfrak{t}_2-\varrho)\mathbb{B}w(\varrho)d\varrho-\mathfrak{t}_1{}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(\mathfrak{t}_2-\varrho)\mathbb{B}w(\varrho)d\varrho\|\\
&+\|\mathfrak{t}_2{}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_2-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(\mathfrak{t}_1-\varrho)\mathbb{B}w(\varrho)d\varrho-\mathfrak{t}_1{}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(\mathfrak{t}_2-\varrho)\mathbb{B}w(\varrho)d\varrho\|\\
&+\|\mathfrak{t}_2{}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(\mathfrak{t}_1-\varrho)\mathbb{B}w(\varrho)\|+\|\mathfrak{t}_2^{1-\eta+\zeta\eta-\zeta\vartheta}(v(t)_2^- -v(t)_1^-))\|\\
&\leq\sum_{i=1}^{12}I_i.
\end{aligned}
$$

Now consider the following

$$I_1=\|\mathfrak{t}_2{}^{1-\eta+\zeta\eta-\zeta\vartheta}\mathbb{R}_{\zeta,\eta}(\mathfrak{t}_2)-\mathfrak{t}_1{}^{1-\eta+\zeta\eta-\zeta\vartheta}\mathbb{R}_{\zeta,\eta}(\mathfrak{t}_1)\|\mathfrak{F}_1(0,\phi(0)+\alpha(v_{\mathfrak{t}1},v_{\mathfrak{t}2},v_{\mathfrak{t}3},\cdots,v_{\mathfrak{t}m})(0),$$

from the strong continuity of $\mathbb{R}_{\zeta,\eta}(\mathfrak{t})$, $I_1\to 0$ as $\mathfrak{t}_2\to\mathfrak{t}_1$

$$I_2=\|\mathfrak{t}_2{}^{1-\eta+\zeta\eta-\zeta\vartheta}\mathfrak{F}_1(\mathfrak{t}_2,x_{\varrho 2}+\hat{\Psi}_{\varrho 2}-\mathfrak{t}_2{}^{1-\eta+\zeta\eta-\zeta\vartheta}\mathfrak{F}_1(\mathfrak{t}_1,x_{\varrho 1}+\hat{\Psi}_{\varrho 1})\|,$$

using the Hypotheses 3, I_2 becomes zero.

$$
\begin{aligned}
I_3=&\|\mathfrak{t}_2{}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(\mathfrak{t}_1-\varrho)\mathfrak{F}_1(\varrho,x_\varrho^G+\hat{\Psi}_\varrho)d\varrho-\mathfrak{t}_1{}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(\mathfrak{t}_1-\varrho)\\
\leq&\|\mathfrak{t}_2{}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varrho)^{\zeta-1}\mathbb{A}\mathfrak{F}_1(\varrho,x_\varrho^G+\hat{\Psi}_\varrho)(\mathbb{S}_k(\mathfrak{t}_2-\varrho)-\mathbb{S}_k(\mathfrak{t}_1-\varrho))d\varrho
\end{aligned}
$$

$$\leq p^{1-\eta+\zeta\eta-\zeta\vartheta}N_1N_{\mathfrak{F}_1}(1+G')\int_0^{t_1}(t_1-\varrho)^{\zeta-1}\|(\mathbb{S}_k(t_2-\varrho)-\mathbb{S}_k(t_1-\varrho))\|d\varrho.$$

$I_3\to 0$ as $t_2\to t_1$, since the strong continuity of $\mathbb{S}_k(t)$

$$\begin{aligned}I_4 &= \|{t_2}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{t_1}(t_1-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(t_1-\varrho)\mathfrak{F}_1(\varrho,x_\varrho^G+\hat{\Psi}_\varrho)d\varrho\\ &\quad-{t_1}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{t_1}(t_1-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(t_1-\varrho)\mathbb{S}_k(t_1-\varrho)\mathfrak{F}_1(\varrho,x_\varrho^G+\hat{\Psi}_\varrho)d\varrho\|\\ &\leq p^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{t_1}((t_1-\varrho)^{\zeta-1}-(t_1-\varrho)^{\zeta-1})\|\mathbb{A}(\mathbb{S}_k(t_2-\varrho)\mathfrak{F}_1(\varrho,x_\varrho^G+\hat{\Psi}_\varrho)\|d\varrho\end{aligned}$$

integrating and $t_2\to t_1$, then I_4 become zero.

$$\begin{aligned}I_5 &= \|{t_2}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_{t_1}^{t_2}(t_1-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(t_1-\varrho)\mathfrak{F}_1(\varrho,x_\varrho^G+\hat{\Psi}_\varrho)d\varrho\|\\ &\leq p^{1-\eta+\zeta\eta-\zeta\vartheta}N_1N_{\mathfrak{F}_1}L^1(1+G')\int_{t_1}^{t_2}((t_1-\varrho)^{\zeta\vartheta-1}d\varrho,\end{aligned}$$

integrating and $t_2\to t_1$, then I_5 become zero.

$$\begin{aligned}I_6 &= \|{t_2}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{t_1}(t_1-\varrho)^{\zeta-1}\mathbb{S}_k(t_2-\varrho)\mathfrak{F}_2(\varrho,x_\varrho+\hat{\Psi}_\varrho,\int_0^\varrho h(\varrho,s,x_\varrho+\hat{\Psi}_\varrho)ds)d\varrho\\ &\quad-{t_1}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{t_1}(t_1-\varrho)^{\zeta-1}\mathbb{S}_k(t_2-\varrho)\mathfrak{F}_2(\varrho,x_\varrho+\hat{\Psi}_\varrho,\int_0^\varrho h(\varrho,s,x_\varrho+\hat{\Psi}_\varrho)ds)d\varrho\|\\ &\leq \|(\int_0^{t_1}{t_2}^{1-\eta+\zeta\eta-\zeta\vartheta}(t_1-\varrho)^{\zeta-1}-{t_1}^{1-\eta+\zeta\eta-\zeta\vartheta}(t_1-\varrho)^{\zeta-1})(t_1-\varrho)\mathbb{S}_k(t_2-\varrho)\\ &\quad\times\mathfrak{F}_2(\varrho,x_\varrho+\hat{\Psi}_\varrho,\int_0^\varrho h(\varrho,s,x_\varrho+\hat{\Psi}_\varrho)ds)d\varrho\|\\ &\leq L'\|(\int_0^{t_1}{t_2}^{1-\eta+\zeta\eta+\zeta\vartheta}(t_1-\varrho)^{\zeta-1}-{t_1}^{1-\eta+\zeta\eta-\zeta\vartheta}(t_1-\varrho)^{\zeta-1})(t_2-\varrho)^{-\zeta-\zeta\vartheta}d\varrho\|\theta_1(p)\phi(B'+H_0(1+B')),\end{aligned}$$

implies $I_6\to 0$ as $t_2\to t_1$.

$$\begin{aligned}I_7 &= \|{t_1}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{t_1}(t_1-\varrho)^{\zeta-1}\mathbb{S}_k(t_2-\varrho)\mathfrak{F}_2(\varrho,x_\varrho+\hat{\Psi}_\varrho,\int_0^\varrho h(\varrho,s,x_\varrho+\hat{\Psi}_\varrho)ds)d\varrho\\ &\quad-{t_1}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{t_1}(t_1-\varrho)^{\zeta-1}\mathbb{S}_k(t_2-\varrho)\mathfrak{F}_2(\varrho,x_\varrho+\hat{\Psi}_\varrho,\int_0^\varrho h(\varrho,s,x_\varrho+\hat{\Psi}_\varrho)ds)d\varrho\|\\ &\leq \|{t_1}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{t_1}(t_1-\varrho)^{\zeta-1}[\mathbb{S}_k(t_2-\varrho)-\mathbb{S}_k(t_1-\varrho)]\mathfrak{F}_2(\varrho,x_\varrho+\hat{\Psi}_\varrho,\int_0^\varrho h(\varrho,s,x_\varrho+\hat{\Psi}_\varrho)ds)d\varrho\|\\ &\leq \|{t_1}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{t_1}(t_1-\varrho)^{\zeta-1}[\mathbb{S}_k(t_2-\varrho)-\mathbb{S}_k(t_1-\varrho)]\mathfrak{F}_2\|\theta_1(p)\phi(B'+H_0(1+B')).\end{aligned}$$

Since $\mathbb{S}_k(t)$ is uniformly continuous operator norm topology, we obtain $I_7\to 0$ as $t_2\to t_1$.

$$\begin{aligned}I_8 &= \|{t_1}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_{t_1}^{t_2}(t_2-\varrho)^{\zeta-1}\mathbb{S}_k(t_2-\varrho)\mathfrak{F}_2(\varrho,x_\varrho+\hat{\Psi}_\varrho,\int_0^\varrho h(\varrho,s,x_\varrho+\hat{\Psi}_\varrho)ds)d\varrho\\ &\leq L'\|{t_2}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_{t_1}^{t_2}(t_1-\varrho)^{\zeta\vartheta-1}d\varrho\|\theta_1(p)\phi(B'+H_0(1+B')),\end{aligned}$$

integrating and $t_2\to t_1$, then I_8 become zero.

$$\begin{aligned}I_9 &= \|{t_2}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{t_1}(t_1-\varrho)^{\zeta-1}\mathbb{S}_k(t_2-\varrho)\mathbb{B}w(\varrho)-{t_1}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{t_1}(t_1-\varrho)^{\zeta-1}\mathbb{A}\mathbb{S}_k(t_2-\varrho)\mathbb{B}w(\varrho)d\varrho\|\\ &\quad+\|({t_2}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{t_1}(t_1-\varrho)^{\zeta-1}-{t_1}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{t_1}(t_1-\varrho)^{\zeta-1})\mathbb{S}_k(t_2-\varrho)\mathbb{B}w(\varrho)d\varrho\|,\\ &\quad L'Q_2\|({t_2}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{t_1}(t_2-\varrho)^{\zeta-1}-{t_1}^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^{t_1}(t_1-\varrho)^{\zeta-1})(t_2-\varrho)^{-\zeta+\zeta\vartheta}w(\varrho)d\varrho\|,\end{aligned}$$

Implies $I_9 \to 0$,as $\mathfrak{t}_2 \to \mathfrak{t}_1$.

$$I_{10} = \|\mathfrak{t}_1{}^{1-\eta+\zeta\eta-\zeta\vartheta} \int_0^{\mathfrak{t}_1} (\mathfrak{t}_1-\varrho)^{\zeta-1}\mathbb{S}_k(\mathfrak{t}_2-\varrho)\mathbb{B}w(\varrho) - \mathfrak{t}_1{}^{1-\eta+\zeta\eta-\zeta\vartheta} \int_0^{\mathfrak{t}_1} (\mathfrak{t}_1-\varrho)^{\zeta-1}\mathbb{S}_k(\mathfrak{t}_2-\varrho)\mathbb{B}w(\varrho)d\varrho\|$$
$$+ \|(\mathfrak{t}_2{}^{1-\eta+\zeta\eta-\zeta\vartheta} \int_0^{\mathfrak{t}_1} (\mathfrak{t}_1-\varrho)^{\zeta-1}(\mathbb{S}_k(\mathfrak{t}_2-\varrho) - \mathbb{S}_k(\mathfrak{t}_2-\varrho))\mathbb{B}w(\varrho)d\varrho\|,$$

from the uniform continuity of $\mathbb{S}_k(\mathfrak{t})$, we obtain $I_{10} \to$ as $\mathfrak{t}_2 \to \mathfrak{t}_1$.

$$\begin{aligned} I_{11} &= \|\mathfrak{t}_2{}^{1-\eta+\zeta\eta-\zeta\vartheta} \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} (\mathfrak{t}_2-\varrho)^{\zeta-1}\mathbb{S}_k(\mathfrak{t}_2-\varrho)\mathbb{B}w(\varrho)d\varrho\| \\ &\le L'Q_2\|(\mathfrak{t}_2{}^{1-\eta+\zeta\eta-\zeta\vartheta} \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} (\mathfrak{t}_2-\varrho)^{\zeta\vartheta-1}w(\varrho)d\varrho\| \end{aligned}$$

integrating and applying limit $\Longrightarrow I_{11} = 0$. Therefore, Φ is equi-continuous on $\mathcal{I}$.

$$I_{12} = \|\mathfrak{t}_2^{1-\eta+\zeta\eta-\zeta\vartheta}(v(t)_2^- - v(t)_1^-))\|$$

using the Hypotheses H_5, $I_2 \Longrightarrow I_{12} = 0$.

Step 4: To show Mönch's condition. Suppose that $\mathcal{P}_0 \subset \mathcal{P}_G$ is a countable and $\mathcal{P}_0 \subset conv(0 \cup \Phi(\mathcal{P}_0))$. We prove $\gamma(\mathcal{P}_0) = 0$. For that, assume that $\mathcal{P}_0 = \{x^k + \hat{\Phi}\}_{k=1}\infty$. We need to prove that $\Phi(\mathcal{P}_0(\mathfrak{t}))$ is relatively compact in $\mathfrak{X}$ for each $\mathfrak{t} \in \mathcal{I}$.

$$\begin{aligned} \gamma(\Psi(\mathcal{P}_0)) &= \gamma(\Psi(x^k + \Psi_{k=1}^{\infty})) \\ &\le \gamma(\mathfrak{t}^{1-\eta+\zeta\eta-\zeta\vartheta}\{\int_0^{\mathfrak{t}} (\mathfrak{t}-\varrho)^{\zeta-1}\mathbb{S}_k(\mathfrak{t}-\varrho)\mathfrak{F}_2(\varrho, x_\varrho + \hat{\Psi}_\varrho, \int_0^\varrho h(\varrho, s, x_\varrho + \hat{\Psi}_\varrho)ds)d\varrho \\ &+ \int_0^{\mathfrak{t}} (\mathfrak{t}-\varrho)^{\zeta-1}\mathbb{S}_k(\mathfrak{t}-\varrho)\mathbb{B}w_{x^k}(\varrho)d\varrho\}_{k=1}^{\infty} \\ &\le \mathfrak{t}^{1-\eta+\zeta\eta-\zeta\vartheta}\gamma(\{\int_0^{\mathfrak{t}} (\mathfrak{t}-\varrho)^{\zeta-1}\mathbb{S}_k(\mathfrak{t}-\varrho)\mathfrak{F}_2(\varrho, x_\varrho + \hat{\Psi}_\varrho, \int_0^\varrho h(\varrho, s, x_\varrho + \hat{\Psi}_\varrho)ds)d\varrho \\ &+ \mathfrak{t}^{1-\eta+\zeta\eta-\zeta\vartheta}\gamma(\int_0^{\mathfrak{t}} (\mathfrak{t}-\varrho)^{\zeta-1}\mathbb{S}_k(\mathfrak{t}-\varrho)\mathbb{B}w_{x^k}(\varrho)d\varrho\}_{k=1}^{\infty}) \\ &= J_1 + J_2 \end{aligned}$$

where

$$\begin{aligned} J_1 &= \mathfrak{t}_1{}^{1-\eta+\zeta\eta-\zeta\vartheta}\gamma(\int_0^{\mathfrak{t}} (\mathfrak{t}-\varrho)^{\zeta-1}\mathbb{S}_k(\mathfrak{t}-\varrho)\mathfrak{F}_2(\varrho, x_\varrho + \hat{\Psi}_\varrho, \int_0^\varrho h(\varrho, s, \{x_\varrho + \hat{\Psi}_\varrho\}_{k=1}^{\infty})ds)d\varrho \\ &\le p^{1-\eta+\zeta\eta-\zeta\vartheta}\{\int_0^{\mathfrak{t}} (\mathfrak{t}-\varrho)^{\zeta-1}\mathbb{S}_k(\mathfrak{t}-\varrho)\mathfrak{F}_2(\varrho, x_\varrho + \hat{\Psi}_\varrho, \int_0^\varrho h(\varrho, s, \{x_\varrho + \hat{\Psi}_\varrho\}_{k=1}^{\infty})ds)d\varrho \\ &\le L'p^{1-\eta+\zeta\eta-\zeta\vartheta}\{\int_0^{\mathfrak{t}} (\mathfrak{t}-\varrho)^{\zeta\vartheta-1}\theta_2(\varrho)\gamma(P_0)[1+\theta_{\mathfrak{t}^*}]d\varrho \\ &\le L'p^{1-\eta+\zeta\eta-\zeta\vartheta}\gamma(P_0)[1+\theta_{\mathfrak{t}_1^*}]\{\int_0^{\mathfrak{t}} (\mathfrak{t}-\varrho)^{\frac{\zeta\vartheta-1}{\zeta}}d\varrho)^{\zeta_2-1}(\int_0^{\mathfrak{t}} \|\theta_2(\varrho)\|^{\zeta_2}d\varrho)^{\frac{1}{\zeta_2}} \\ &\le L'p^{1-\eta+\zeta\eta-\zeta\vartheta}Q_{\zeta_2}\|\theta_2\|_{L^{\frac{1}{\zeta_1}}}[1+\theta_3{}^*] \times \gamma(P_0) \end{aligned}$$

$$\begin{aligned} J_2 &= \mathfrak{t}^{1-\eta+\zeta\eta-\zeta\vartheta}\gamma\Big(\int_0^{\mathfrak{t}} (\mathfrak{t}-\varrho)^{\zeta-1}\mathbb{S}_k(\mathfrak{t}-\varrho)\mathbb{B}\{w_{x^k(\varrho)}\}_{k=1}^{\infty}\,d\varrho) \\ & L'^2Q_2p^{1-\eta+\zeta\eta-\zeta\vartheta}\int_0^p (p-\varrho)^{2\zeta\vartheta-2}[\gamma(W^{-1}\mathfrak{F}_2(\varrho, \{x_\varrho + \hat{\Psi}_\varrho\}_{k=1}^{\infty}, \int_0^\varrho h(\varrho, s, x_\varrho + \hat{\Psi}_{s_{k=1}}^{\infty}))d\varrho] \\ & L'^2Q_2p^{1-\eta+\zeta\eta-\zeta\vartheta}\{\int_0^p (\mathfrak{t}_1-\varrho)^{2\zeta\vartheta-2}\gamma(\mathfrak{F}_2(\varrho, \{x_\varrho + \hat{\Psi}_\varrho\}_{k=1}^{\infty}, \int_0^\varrho h(\varrho, s, x_\varrho + \hat{\Psi}_{s_{k=1}}^{\infty}))d\varrho \\ & L'^2Q_2Q_Wp^{1-\eta+\zeta\eta-\zeta\vartheta}\{\int_0^p (p-\varrho)^{\frac{\zeta\vartheta-1}{1-\zeta_2}}d\varrho)^{2-2\zeta_2}(\int_0^p \|\theta_2(\varrho)\|^{\zeta_2}d\varrho)^{\frac{1}{\zeta_2}} \times \gamma(P_0) \end{aligned}$$

$$\leq L' Q_2 Q_W p^{1-\eta+\zeta\eta-\zeta\vartheta} Q_{\zeta_2}^2 \|\theta_2\|_{L^{\frac{1}{\zeta_1}}}(\mathcal{I}, R^+) \times \gamma(P_0).$$

Now

$$\begin{aligned} J_1 + J_2 &\leq p^{1-\eta+\zeta\eta-\zeta\vartheta}(Q_{\zeta_2}^2 \|\theta_2\|_{L^{\frac{1}{\zeta_1}}}(\mathcal{I}, R^+)[1+\theta_3^*] + L'^2 Q_2 Q_W Q_{\zeta_2}^2 \|\theta_2\|_{L^{\frac{1}{\zeta_1}}}(x, R^+))\gamma(P_0) \\ &\leq p^{1-\eta+\zeta\eta-\zeta\vartheta}(Q_4(1+\theta_3^*) + L' Q_2 Q_W Q_5) \times \gamma(P_0). \end{aligned}$$

Therefore

$$\gamma(\Phi(\mathcal{P}_0)) \leq \hat{Q}\gamma(\mathcal{P}_0),$$

where $\hat{Q} = p^{1-\eta+\zeta\eta-\zeta\vartheta}(Q_4(1+\theta_3^*) + \leq L' Q_2 Q_W Q_5)$. Thus, from Mönch's condition, we obtain

$$\begin{aligned} \gamma(\mathcal{P}_0) \leq \gamma(Conv(0 \cup \Psi(\mathcal{P}_0))) = \gamma(\Phi(\mathcal{P}_0)) \leq \hat{Q}\gamma(\mathcal{P}_0) \\ \longrightarrow \gamma(\mathcal{P}_0) = 0. \end{aligned}$$

So, by Lemma 9, Ψ has a fixed point v in $\mathcal{P}_B$. Then $v = x + \hat{\Phi}$ is a mild solution of system (1)–(3) is controllable on $\mathfrak{X}$.

4. Example

Suppose the Hilfer fractional integro-differential system of the form,

$$\begin{aligned} D_{0+}^{\frac{2}{3},\eta}[v(\mathfrak{t},\mu)v(z,\mu)dz] = \frac{\partial^2}{\partial\mu^2}v(\mathfrak{t},\mu) + W\hat{\Phi}(\mathfrak{t},\mu) + \\ \Phi\left(\mathfrak{t}, \int_{-\infty}^{\mathfrak{t}} \Phi_1(\varrho - \mathfrak{t})v(\varrho,\mu)d\varrho, \int_0^{\mathfrak{t}}\int_{-\infty}^0 \Phi_2(\varrho,\mu,r-\varrho)v(r,\mu)d\mu d\varrho\right), \end{aligned} \tag{11}$$

$$\begin{aligned} \Delta v|_{\mathfrak{t}=\mathfrak{t}_k} = I_k(v(\mathfrak{t}_k^-)), \quad k = 1,2,\ldots,m, \\ I^{(1-\zeta)(1-\frac{2}{3})}[v(0,\mu)] = v_0(\mu), \mu \in [0,\pi], \\ v(\mathfrak{t},0) = v(\mathfrak{t},\pi) = 0, \mathfrak{t} \in \mathfrak{I}, \\ v(\mathfrak{t},\mu) = \phi(\mathfrak{t},\mu), 0 \leq \mu \leq \pi, \mathfrak{t} \in (-\infty,0] \end{aligned}$$

where $D_{0+}^{\frac{2}{3},\eta}$ denoted the Hilfer fractional derivative of order $\zeta = \frac{2}{3}$, type $\zeta \in [0,1]$ and $\Phi : \mathfrak{I} \times P_{\mathfrak{g}} \times \mathfrak{X} \to \mathfrak{X}$ is a continuous function. Moreover, ϕ is continuous and satisfies certain smoothness conditions, Φ_0, Φ_1 and Φ_2 are the appropriate functions. To change this system into an abstract structure, let $\mathfrak{X} = V = L^2[0,\pi]$ be endowed with the norm $\|\cdot\|_{L^2}$ and $\mathbb{A} : D(\mathbb{A}) \subset \mathfrak{X} \subset\!\!\to \mathfrak{X}$ is defined as $A\rho = \rho'$ with

$$D(\mathbb{A}) = \{\rho \in \mathfrak{X} : \rho, \frac{\partial}{\partial\rho} \text{ are absolutely continuous}, \frac{\partial^2}{\partial\rho^2} \in \mathfrak{X}, \rho(0) = \rho(\pi) = 0\}$$

and

$$A\rho = \sum_{i=1}^{\infty} i^2 \langle\rho,\rho_i\rangle\rho_i, \rho \in D(\mathbb{A})$$

where $\rho_k(y) = \sqrt{\frac{2}{\pi}}sin(i\rho), i \in \mathbb{N}$ is the orthogonal set of eigen vectors of $\mathbb{A}$.

We have $\mathbb{A}$ denotes the infinitesimal generator of strongly continuous functions of bounded linear operators $\{T(\mathfrak{t}), \mathfrak{t} \geq 0\}$ in $\mathfrak{X}$ and is given by $T(\mathfrak{t})P \leq \gamma(P)$, where γ denoted the Hausdorff measure of non-compactness and $Q_1 \geq 1$ is a constant, satisfy $\sup_{\mathfrak{t}\in\mathcal{I}} \|T(\mathfrak{t})\| \leq Q_1$, Furthermore, $\mathfrak{t} \to \rho(\mathfrak{t}^{\frac{2}{3}}\theta + \varrho)$ v is equi-continuous for $\mathfrak{t} \geq 0$ an $0 < \theta < \infty$. Define

$$v(\mathfrak{t}(\rho)) = v(\mathfrak{t},\rho)$$

$$\mathfrak{F}_2\left(\mathfrak{t}, v_{\mathfrak{t}}, \int_0^{\mathfrak{t}} h(\mathfrak{t}, s, u_{\mathfrak{t}})ds\right)(\rho) = \Phi\left(\mathfrak{t}, \int_{-\infty}^{\mathfrak{t}} \Phi_1(\varrho - \mathfrak{t})(\mathfrak{t}, \rho)d\varrho, \int_0^{\mathfrak{t}}\int_{-\infty}^{0} \Phi_2(\varrho, \rho, r-\varrho)v(r,\rho)d\rho d\varrho\right)$$

$$\mathfrak{F}_1(v, v_{\mathfrak{t}}) = \int_0^{\pi} \Phi_0(\mathfrak{t}, \rho)v(z, \rho)dz.$$

Let $\mathbb{B} : V \to \mathfrak{X}$

$$B(V(\rho)) = W_\gamma(\mathfrak{t}, \rho), 0 < \rho < \pi.$$

Given the entries $\mathbb{A}, \mathbb{B}, \mathfrak{F}_1$ and $\mathfrak{F}_2$, Equation (11) can be written as

$$D_0^{\zeta,\eta}[v(\mathfrak{t}) - \mathfrak{F}_1(\mathfrak{t}, v_{\mathfrak{t}})] = \mathbb{A}v(\mathfrak{F}_1) + \mathfrak{F}_2\left(\mathfrak{t}, v_{\mathfrak{t}}, \int_0^{\mathfrak{t}} h(\mathfrak{t}, s, u_{\mathfrak{t}})ds + \mathbb{B}w(\mathfrak{t});\ \zeta = \frac{2}{3} \in (0,1), \mathfrak{t} \in \mathcal{I}\right)$$

$$\Delta v|_{\mathfrak{t}=\mathfrak{t}_k} = I_k(v(\mathfrak{t}_k^-)), \quad k = 1, 2, \ldots, m,$$

$$I_{0^+}^{(1-\zeta)(1-\eta)} v_0 = \phi \in \mathcal{P}_{\mathfrak{g}}.$$

In order to validate the Theorem 2 assumptions, we additionally present some acceptable circumstances for the aforementioned functions, and we conclude that the impulsive neutral Hilfer fractional differential system (1)–(3) is controllable.

5. Conclusions

In this paper, we focused on the analysis of controllability for impulsive neutral Hilfer fractional differential equations with non-local conditions. Applying the findings and concepts from the infinitesimal generator of a strongly continuous function of bounded linear operators, fractional calculus, the measure of non-compactness, impulsive conditions, non-local conditions, and fixed point method, the main conclusion is established. Last but not least, we provided an example to illustrate the principle. Future research will concentrate on the many types of controllability of impulsive neutral Hilfer fractional differential systems with non-local conditions.

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