Article

# An Approach to Multidimensional Discrete Generating Series 

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#### Abstract

We extend existing functional relationships for the discrete generating series associated with a single-variable linear polynomial coefficient difference equation to the multivariable case.


Keywords: forward difference operator; difference equation; generating series; shift operator; characteristic polynomial; Cauchy problem

MSC: 05A15; 39A05; 39A06

## 1. Introduction

An approach to build the general theory of a discrete generating series of one variable and its connection with the linear difference equations was introduced in [1]. We extend those results to the multidimensional case. We define a discrete generating series for $f: \mathbb{Z}^{n} \rightarrow \mathbb{C}$ and derive functional relations for such series.

The general theory of linear recurrences with constant coefficients and the Stanley hierarchy $[2,3]$ of its generating functions (rational, algebraic, $D$-finite) depending on the initial data function was considered in [4]. Difference equations with polynomial coefficients is an effective means to study lattice paths with restriction [5,6]. Some properties of linear difference operators whose coefficients have the form of infinite two-sided sequences over a field of characteristic zero are considered in [7]. An effective method of obtaining explicit formulas for the coefficients of a generating function related to the Aztec diamond and a generating function related to the permutations with cycles was derived in $[8,9]$. Using the notion of amoeba [10] of the characteristic polynomial of a difference equation, a description for the solution space of a multidimensional difference equation with constant coefficients was obtained in [11]. A generalization to several variables of the classical Poincare theorem on the asymptotic behavior of solutions of a linear difference equation is presented in [12]. We can also note that the almost periodic and the almost automorphic solutions to the difference equations depending on several variables are not well explored in the existing literature [13].

Let $\mathbb{Z} \geqslant$ denote the non-negative integers, $\mathbb{Z}^{n}=\mathbb{Z} \times \cdots \times \mathbb{Z}$ be the $n$-dimensional integers, and $\mathbb{Z}_{\geqslant}^{n}=\mathbb{Z}_{\geqslant} \times \cdots \times \mathbb{Z}_{\geqslant}$for $n \in \mathbb{Z} \geqslant$ be its non-negative orthant. For any $z \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geqslant}$, we define the falling factorial $z^{\underline{n}}=z(z-1) \cdots(z-n+1)$ with $z^{0}=1$ and the Pochhammer symbol (or rising factorial) is defined by $(z)_{n}=z(z+1) \cdots(z+n-1)$ with $(z)_{0}=1$. Throughout, we will use the multidimensional notation for convience of expressions: $x=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{Z}_{\geqslant}^{n}, z=\left(z_{1}, \ldots, z_{n}\right) \in \mathbb{C}^{n}, \xi=\left(\xi_{1}, \ldots, \xi_{n}\right) \in \mathbb{C}^{n}$, $\xi^{x}=\xi_{1}^{x_{1}} \cdots \xi_{n}^{x_{n}}, z^{\underline{x}}=z_{1}^{\frac{x_{1}}{1}} \cdots z_{n}^{\frac{x_{n}}{n}}, \ell=\left(\ell_{1}, \ldots, \ell_{n}\right) \in \mathbb{Z}_{\geqslant}^{n}, x!=x_{1}!\ldots x_{n}!$. We also will use $x \leq y$ for $x, y \in \mathbb{Z}^{n}$ componentwise, i.e., that $x_{i} \leq y_{i}$ for all $i=1, \ldots, n$.

Given a function $f: \mathbb{Z}_{\gtrless}^{n} \rightarrow \mathbb{C}$, we define the associated multidimensional discrete generating series of $f$ as

$$
F(\xi ; \ell ; z)=\sum_{x \in \mathbb{Z}_{\underline{n}}^{n}} f(x) \xi^{x} z^{\underline{\ell x}}=\sum_{x_{1}=0}^{\infty} \cdots \sum_{x_{n}=0}^{\infty} f\left(x_{1}, \ldots, x_{n}\right) \xi_{1}^{x_{1}} \cdots \xi_{n}^{x_{n}} z_{1}^{\ell_{1} x_{1}} \cdots z \frac{\ell_{n} x_{n}}{n} .
$$

Let $p_{\alpha} \in \mathbb{C}[z]$ denote polynomials with complex coefficients. The difference equation under consideration in this work is

$$
\begin{equation*}
\sum_{\alpha \in A} p_{\alpha}(x) f(x-\alpha)=0, \tag{1}
\end{equation*}
$$

where set $A \subset \mathbb{Z}_{\geqslant}^{n}$ is finite and there is $m \in A$ such that for all $\alpha \in A$, the inequality $\alpha \leqslant m$, which means $\alpha_{j} \leqslant m_{j}, j=1, \ldots, n$, holds. Occasionally we will use an equivalent notation $0 \leqslant \alpha \leqslant m$, assuming that for some $\alpha$ coefficients, $p_{\alpha}(x)$ vanishes and only $p_{m}(x) \not \equiv 0$. In Section 2, we will particularly consider a homogeneous difference equation with constant coefficients.

The special case where each $p_{\alpha}=c_{\alpha}$ is a constant

$$
\begin{equation*}
\sum_{\alpha \in A} c_{\alpha} f(x-\alpha)=0 \tag{2}
\end{equation*}
$$

arises in a wide class of combinatorial analysis problems [3], for instance, in lattice path problems [4], the theory of digital recursive filters [14], and the wavelet theory [15]. The question about correctness and well-posedness of (2) was considered in [16-18].

We equip (1) with initial data on a set named $X_{m}$, which is used often enough. We introduce the notation $\mathbb{Z}_{\ngtr}$ as $X_{m}=\mathbb{Z}_{\geqslant}^{n} \backslash\left(m+\mathbb{Z}_{\geqslant}^{n}\right)=\left\{x \in \mathbb{Z}_{\geqslant}^{n}: x \ngtr m\right\}$ (see Figure 1) and we define the initial data function $\varphi: X_{m} \rightarrow \mathbb{C}$ so that

$$
\begin{equation*}
f(x)=\varphi(x), \quad x \in X_{m} . \tag{3}
\end{equation*}
$$


(a) $x \geqslant m$

(b) $x \leqslant m$

(c) $x \ngtr m$

Figure 1. Illustration of the sets $x \geqslant m, x \leqslant m$, and $x \ngtr m$.
For convenience, we extend $\varphi$ to the whole of $\mathbb{Z}^{n}$ by taking it to be identically zero outside of $X_{m}$. The Cauchy problem is to find a solution to difference Equation (1) that coincides with $\varphi$ on $X_{m}$, i.e., $f(x)=\varphi(x)$, for all $x \in X_{m}$.

In Section 2, functional equations for the discrete generating series are derived for the solution of the difference equations with constant coefficients. In Section 3, a case of difference equations with polynomial coefficients is considered. Section 4 contains two examples that illustrate our approach to discrete generating series.

## 2. Discrete Generating Series for Linear Difference Equations with Constant Coefficients

In this section, we consider a homogeneous difference equation with constant coefficients (2) and introduce the shift operator by

$$
\begin{equation*}
\mathcal{P}(\xi ; \ell ; z)=\sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \xi^{\alpha} z^{\ell \alpha} \rho^{\ell \alpha} . \tag{4}
\end{equation*}
$$

Also useful is its truncation for $\tau \in \mathbb{Z}^{n}$, defined by the formula

$$
\mathcal{P}_{\tau}(\xi ; \ell ; z)=\sum_{\substack{0 \leqslant \alpha \leqslant m \\ \alpha \neq \tau}} c_{\alpha} \xi^{\alpha} z^{\ell \alpha} \rho^{\ell \alpha},
$$

and the discrete generating series of the initial data for $\tau \in X_{m}$ by

$$
\begin{equation*}
\Phi_{\tau}(\xi ; \ell ; z)=\sum_{x \ngtr \tau} \varphi(x) \xi^{x} z^{\underline{\ell x}} . \tag{5}
\end{equation*}
$$

Let $\delta_{j}: x \rightarrow x+e^{j}$ be the forward shift operator for $j=1, \ldots, n$ with multidimensional notation $\delta^{\alpha}=\delta_{1}^{\alpha_{1}} \ldots \delta_{n}^{\alpha_{n}}$ and define the polynomial difference operator

$$
P(\delta)=\sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \delta^{\alpha} .
$$

With this notation, Equation (2) is represented compactly as

$$
P\left(\delta^{-1}\right) f(x)=0, \quad x \geqslant m .
$$

The case of generating series $\sum_{x} f(x) z^{x}$ and exponential generating series $\sum_{x} \frac{f(x)}{x!} z^{x}$ is well-studied for both one and several variables: one of the first convenient formulas to derive the generating series exploiting the characteristic polynomial and the initial data function was proven in [19]. We will prove analogues of these formulas for the discrete generating series $F(\xi ; \ell ; z)$.

Theorem 1. The discrete generating series $F(\xi ; \ell ; z)$ for the solution to the Cauchy problem for Equation (2) with initial data (3) satisfies the functional equations:

$$
\begin{align*}
\mathcal{P}(\xi ; \ell ; z) F(\xi ; \ell ; z) & =\sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \xi^{\alpha} z^{\underline{\ell \alpha}} \rho^{\ell \alpha} \Phi_{m-\alpha}(\xi ; \ell ; z)  \tag{6}\\
& =\sum_{x \ngtr m} P\left(\delta^{-1}\right) \varphi(x) \xi^{x} z^{\underline{\ell x}}  \tag{7}\\
& =\sum_{x \ngtr m} \mathcal{P}_{m-x}(\xi ; \ell ; z) \varphi(x) z^{\underline{\ell x}} . \tag{8}
\end{align*}
$$

Proof. By multiplying (2) by $\xi^{x} z^{\underline{\ell x}}$ and summing over $x \geqslant m$, we obtain

$$
\begin{aligned}
0 & =\sum_{x \geqslant m} \sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} f(x-\alpha) \xi^{x} z^{\underline{\ell x}} \\
& =\sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \sum_{x \geqslant m} f(x-\alpha) \xi^{x} z^{\underline{\ell x}} .
\end{aligned}
$$

Now, substituting $x$ with $x+\alpha$ yields

$$
\begin{aligned}
0 & =\sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \sum_{x \geqslant m-\alpha} f(x) \xi^{x+\alpha} z^{\underline{\ell(x+\alpha)}} \\
& =\sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \xi^{\alpha} z^{\underline{\ell \alpha}} \rho^{\ell \alpha} \sum_{x \geqslant m-\alpha} f(x) \xi^{x} z^{\underline{\ell x}} \\
& =\underbrace{\sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \xi^{\alpha} z^{\underline{\ell \alpha}} \rho^{\ell \alpha}\left(\sum_{x \geqslant 0} f(x) \xi^{x} z^{\underline{\ell x}}-\sum_{x \ngtr m-\alpha} \varphi(x) \xi^{x} z^{\underline{\ell x}}\right)}_{0 \leqslant \mathcal{P}(\xi ; \ell ; z)} \\
& =\underbrace{\sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \xi^{\alpha} z^{\ell \alpha} \rho^{\ell \alpha}}_{=F(\xi ; \ell ; z)} \underbrace{\sum_{x \geqslant 0} f(x) \xi^{x} z^{\underline{\ell x}}}_{x \geqslant 0}-\sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \xi^{\alpha} z^{\chi \alpha} \rho^{\ell \alpha} \underbrace{\sum_{x \neq m-\alpha} \varphi(x) \xi^{x} z^{\underline{\ell x}}}_{=\Phi_{m-\alpha}(\xi ; \ell ; z)} .
\end{aligned}
$$

Thus, by (5), we have established (6). Since

$$
\begin{aligned}
\sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \xi^{\alpha} z^{\ell \alpha} \rho^{\ell \alpha} \sum_{x \ngtr m-\alpha} \varphi(x) \xi^{x} z^{\underline{\ell x}} & =\sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \sum_{x \ngtr m-\alpha} \varphi(x) \xi^{x+\alpha} z \underline{\underline{\ell(x+\alpha)}} \\
& =\sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \sum_{x \ngtr m} \varphi(x-\alpha) \xi^{x} z^{\underline{\ell x}} \\
& =\sum_{x \ngtr m} \underbrace{\left[\sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \varphi(x-\alpha)\right]}_{=P\left(\delta^{-1}\right) \varphi(x)} \xi^{x} z^{\underline{\ell x}},
\end{aligned}
$$

which yields (7). Finally, collecting (6) by $\varphi(x)$ yields

$$
\sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \xi^{\alpha} z^{\underline{\ell \alpha}} \rho^{\ell \alpha} \sum_{x \ngtr m-\alpha} \varphi(x) \xi^{x} z^{\underline{\ell x}}=\sum_{x \ngtr m}^{\sum_{=\mathcal{P}_{m-x}(\xi ; \ell ; z)}^{\sum_{\substack{0 \leqslant \alpha \leqslant m \\ \alpha \neq m-x}} c_{\alpha} \xi^{\alpha} z^{\underline{\ell \alpha}} \rho^{\ell \alpha}} \varphi(x) z^{\underline{\ell x}}, ~}
$$

completing the proof of (8).
For $z=\left(z_{1}, \ldots, z_{n}\right)$, we denote the projection operator $\pi_{j} z=\left(z_{1}, \ldots, z_{j-1}, 0, z_{j+1}, \ldots, z_{n}\right)$ and we introduce the notation

$$
\pi_{j} F(\xi ; \ell ; z)=F\left(\xi ; \ell ; \pi_{j} z\right)=\sum_{\substack{x \geqslant 0 \\ x_{j}=0}} f(x) \xi^{x} z \underline{\ell x}
$$

and we define the combined projection $\Pi=\left(1-\pi_{1}\right) \circ \cdots \circ\left(1-\pi_{n}\right)$ as the composition of $1-\pi_{j}$ for all $j=1, \ldots, n$.

For the next result, we introduce the symbols $I=(1,1, \ldots, 1) \in \mathbb{Z}^{n}$ and the unit vectors $e_{j}=(0, \ldots, 0,1,0, \ldots, 0)$ for $j=1,2, \ldots, n$, which is nonzero only for the $j$ th component. In these two lemmas, we will prove some useful properties of the combined projection $\Pi$.

Lemma 1. The following formula holds:

$$
\Pi \sum_{x \geqslant 0} f(x) \xi^{x} z^{\underline{\ell x}}=\sum_{x \geqslant 1} f(x) \xi^{x} z^{\underline{\ell x}} .
$$

Proof. First, compute for any $j=1,2, \ldots, n$,

$$
\begin{aligned}
\left(1-\pi_{j}\right) \sum_{x \geqslant 0} f(x) \xi^{x} z^{\underline{\ell x}} & =\sum_{x \geqslant 0} f(x) \xi^{x} z^{\underline{\ell x}}-\pi_{j} \sum_{x \geqslant 0} f(x) \xi^{x} z^{\underline{\ell x}} \\
& =\sum_{x \geqslant e_{j}} f(x) \xi^{x} z^{\underline{\ell x}} .
\end{aligned}
$$

Thus, we see that applying $\Pi$ to $\sum_{x \geq 0} f(x) \xi^{x} z^{\underline{\underline{\ell x}}}$ yields the desired result.
We now obtain a similar result as Lemma 1 but for a shifted discrete generating series.
Lemma 2. The following formula holds:

$$
\Pi \xi_{j} z^{\frac{\ell_{j}}{j}} \rho_{j}^{\ell_{j}} F(\xi ; \ell ; z)=\sum_{x \geqslant I} f\left(x-e^{j}\right) \xi^{x} z^{\underline{\ell x}} .
$$

Proof. First, compute

$$
\begin{aligned}
\left(1-\pi_{j}\right) \xi_{j} z_{j}^{\ell_{j}} \rho^{\ell_{j}} F(\xi ; \ell ; z) & =\xi_{j} z_{j}^{\ell_{j}} \rho^{\ell_{j}} F(\xi ; \ell ; z)-\pi_{j} \xi_{j} z_{j}^{\ell_{j}} \rho^{\ell_{j}} F(\xi ; \ell ; z) \\
& =\xi_{j} z_{j}^{\ell_{j}} \rho^{\ell_{j}} \sum_{x \geqslant 0} f(x) \xi^{x} z^{\underline{\ell x}} \\
& =\sum_{x \geqslant 0} f(x) \xi^{x+e_{j}} z^{\ell\left(x+e^{j}\right)} \\
& =\sum_{x \geqslant e^{j}} f\left(x-e^{j}\right) \xi^{x} z^{\underline{\ell x}} .
\end{aligned}
$$

Thus, we see that applying $\Pi$ to $\xi_{j} z_{j}^{\ell_{j}} \rho^{\ell_{j}} F(\xi ; \ell ; z)$ completes the proof.
We introduce the inner product

$$
\left\langle c, \xi z^{\underline{\ell}} \rho^{\ell}\right\rangle=c_{1} \xi z z \frac{\ell_{1}}{1} \rho_{1}^{\ell_{1}}+\cdots+c_{n} \xi_{n} z_{n}^{\ell_{n}} \rho_{n}^{\ell_{n}}
$$

and

$$
\left\langle c, \delta^{-I}\right\rangle=c_{1} \delta_{1}^{-1}+\cdots+c_{n} \delta_{n}^{-1}
$$

We are now prepared to prove an analogue of [20], [Theorem 1.1].
Theorem 2. The following formula holds:

$$
\begin{equation*}
\Pi\left[\left(1-\left\langle c, \xi z^{\underline{\ell}} \rho^{\ell}\right\rangle\right) F(\xi ; \ell ; z)\right]=\sum_{x \geqslant I}\left(1-\left\langle c, \delta^{-I}\right\rangle\right) f(x) \xi^{x} z^{\underline{\ell}} . \tag{9}
\end{equation*}
$$

Proof. Applying $\Pi$ to $\left(1-\left\langle c, \xi z^{\underline{\ell}} \rho^{\ell}\right\rangle\right) F(\xi ; \ell ; z)$ yields

$$
\begin{aligned}
\Pi[(1- & \left.\left.\left\langle c, \xi z^{\underline{\ell}} \rho^{\ell}\right\rangle\right) F(\xi ; \ell ; z)\right]=\Pi F(\xi ; \ell ; z)-\left\langle c, \Pi \xi z^{\underline{\ell}} \rho^{\ell}\right\rangle F(\xi ; \ell ; z) \\
& =\Pi F(\xi ; \ell ; z)-c_{1} \Pi \xi \xi_{1} z^{\frac{\ell_{1}}{1}} \rho_{1}^{\ell_{1}} F(\xi ; \ell ; z)-\cdots-c_{n} \Pi \xi \xi_{n} z^{\frac{\ell_{n}}{n}} \rho_{n}^{\ell_{n}} F(\xi ; \ell ; z) \\
& =\sum_{x \geqslant I} f(x) \xi^{x} z^{\underline{\ell x}}-c_{1} \sum_{x \geqslant I} f\left(x-e^{1}\right) \xi^{x} z^{\underline{\ell x}}-\cdots-c_{n} \sum_{x \geqslant I} f\left(x-e^{n}\right) \xi^{x} z^{\underline{\ell x}} \\
\quad= & \sum_{x \geqslant I}\left(f(x)-c_{1} f\left(x-e^{1}\right)-\cdots-c_{n} f\left(x-e^{n}\right)\right) \xi^{x} z^{\underline{\ell}} \\
\quad= & \sum_{x \geqslant I}\left(1-\left\langle c, \delta^{-I}\right\rangle\right) f(x) \xi^{x} z^{\underline{\ell x}},
\end{aligned}
$$

thereby completing the proof.
The following corollary is straightforward.
Corollary 1. If $f$ solves $\left(1-\left\langle c, \delta^{-I}\right\rangle\right) f(x)=0$, then

$$
\begin{equation*}
\Pi[(1-\langle c, \zeta\rangle) F(\xi ; \ell ; z)]=0 . \tag{10}
\end{equation*}
$$

3. Discrete Generating Series for Linear Difference Equations with Polynomial Coefficients

We define the componentwise forward difference operators $\Delta_{j}$ by

$$
\Delta_{j} F(z)=F\left(z+e^{j}\right)-F(z), \quad j=1, \ldots, n
$$

If $z^{\underline{x}}=z_{1}^{\frac{x_{1}}{1}} \ldots z^{\frac{x_{n}}{n}}$, then $\Delta_{j} z^{\underline{x}}=x_{j} z^{\underline{x-e^{j}}}$. Thus, we can regard $\Delta_{j}$ as a discrete analogue of a partial derivative operator. Now, compute

$$
\begin{aligned}
\Delta_{j} F(\xi ; \ell ; z) & =\Delta_{j} \sum_{x \geqslant 0} f(x) \xi^{x} z^{\underline{\ell x}} \\
& =\sum_{x \geqslant 0} f(x) \xi^{x} \Delta_{j} z^{\underline{\ell x}} \\
& =\sum_{x \geqslant 0} \ell_{j} x_{j} f(x) \xi^{x} z^{\underline{\ell x-e^{j}}} .
\end{aligned}
$$

We denote the componentwise backward jump $\rho_{j}$ by

$$
\rho_{j} F(z)=F\left(z-e^{j}\right)
$$

and we define the componentwise operators $\theta_{j}=\ell_{j}^{-1} z_{j} \rho_{j} \Delta_{j}$, which generalizes the singlevariable one defined earlier in [21,22]. Now, we prove some useful properties of the operator $\theta^{k}:=\theta_{1}^{k} \ldots \theta_{n}^{k}$.

Lemma 3. If $k=\left(k_{1}, \ldots, k_{n}\right) \in \mathbb{Z}_{\geqslant}^{n}$, then the following formula holds:

$$
\begin{equation*}
\theta^{k} F(\xi ; \ell ; z)=\sum_{x \geqslant 0} x^{k} f(x) \xi^{x} z^{\underline{\ell x}} . \tag{11}
\end{equation*}
$$

Proof. We obtain:

$$
\begin{aligned}
\theta^{k} F(\xi ; \ell ; z) & =\theta_{1}^{k_{1}} \ldots \theta_{n}^{k_{n}} F(\xi ; \ell ; z) \\
& =\theta_{1}^{f k_{1}} \ldots \theta_{n-1}^{k_{n-1}}\left(\ell_{n}^{-1} z_{n} \rho_{n} \Delta_{n}\right)^{k_{n}-1} \ell_{n}^{-1} z_{n} \rho_{n} \Delta_{n} F(\xi ; \ell ; z) \\
& =\theta_{1}^{k_{1}} \ldots \theta_{n-1}^{k_{n-1}}\left(\ell_{n}^{-1} z_{n} \rho_{n} \Delta_{n}\right)^{k_{n}-1} \ell_{n}^{-1} z_{n} \rho_{n} \sum_{x \geqslant 0} \ell_{n} x_{n} f(x) \xi^{x} z^{\underline{\ell x-e^{n}}} \\
& =\theta_{1}^{k_{1}} \ldots \theta_{n-1}^{k_{n-1}}\left(\ell_{n}^{-1} z_{n} \rho_{n} \Delta_{n}\right)^{k_{n}-1} \sum_{x \geqslant 0} x_{n} f(x) \xi^{x} z^{\ell \underline{x}} .
\end{aligned}
$$

Continuing this process $k_{n}-1$ times for $\theta_{n}$ and in $k_{j}$ times in turn for the powers of $\theta_{j}$, $j=1, \ldots, n-1$ completes the proof.

The proof of the following lemma resembles the proof of Lemma 3 but for the operator $p(\theta)=\sum_{\alpha \in A \subset \mathbb{Z}_{\geqslant}^{n}} c_{\alpha} \theta^{\alpha}$, so we omit explicitly writing the proof.

Lemma 4. The following formula holds:

$$
\begin{equation*}
p(\theta) F(\xi ; \ell ; z)=\sum_{x \geqslant 0} p(x) f(x) \xi^{x} z^{\underline{\ell x}} \tag{12}
\end{equation*}
$$

We define an operator $\mathcal{P}_{A}$ by

$$
\mathcal{P}_{A}(\xi ; \ell ; z ; \theta ; \rho)=\sum_{\alpha \in A} p_{\alpha}(\theta+\alpha) \xi^{\alpha} z^{\ell \alpha} \rho^{\ell \alpha}
$$

Theorem 3. The discrete generating series $F(\xi ; \ell ; \cdot)$ of the Cauchy problem for Equation (1) with initial data (3) satisfies the functional equation

$$
\begin{equation*}
\mathcal{P}_{A}(\xi ; \ell ; z ; \theta ; \rho) F(\xi ; \ell ; z)=\sum_{\alpha \in A} \sum_{x \ngtr m-\alpha} p_{\alpha}(x-\alpha) \varphi(x) \xi^{x} z \underline{\ell(x+\alpha)} . \tag{13}
\end{equation*}
$$

Proof. Similar to the proof of Theorem 1, we multiply (1) by $\xi^{x} z^{\underline{\ell x}}$ and sum over $x \geqslant m$ to obtain

$$
0=\sum_{x \geqslant m} \sum_{\alpha \in A} p_{\alpha}(x) f(x-\alpha) \xi^{x} z^{\underline{\ell x}}=\sum_{\alpha \in A} \sum_{x \geqslant m-\alpha} p_{\alpha}(x-\alpha) f(x) \xi^{x+\alpha} z^{\underline{\ell(x+\alpha)}} .
$$

Replacing $x$ with $x+\alpha$ then leads to

$$
0=\sum_{\alpha \in A} \xi^{\alpha}\left(\sum_{x \geqslant 0} p_{\alpha}(x-\alpha) f(x) \xi^{x} z^{\underline{\ell(x+\alpha)}}-\sum_{x \ngtr m-\alpha} p_{\alpha}(x-\alpha) \varphi(x) \xi^{x} z^{\ell(x+\alpha)}\right)
$$

and routine algebraic manipulation completes the proof.

## 4. Examples

Example 1. We will derive the functional equation for the discrete generating series

$$
F\left(; ; z_{1}, z_{2}\right)=F\left(\xi_{1}, \xi_{2} ; \ell_{1}, \ell_{2} ; z_{1}, z_{2}\right)
$$

for the basic combinatorial recurrence

$$
\begin{equation*}
f\left(x_{1}, x_{2}\right)-f\left(x_{1}-1, x_{2}\right)-f\left(x_{1}, x_{2}-1\right)=0 . \tag{14}
\end{equation*}
$$



$$
\begin{aligned}
\sum_{\left(x_{1}, x_{2}\right) \geqslant(1,1)} f\left(x_{1}, x_{2}\right) \xi_{1}^{x_{1}} \xi_{2}^{x_{2}} z_{1}^{\ell_{1} x_{1}} z_{2}^{\ell_{2} x_{2}}- & \sum_{\left(x_{1}, x_{2}\right) \geqslant(1,1)} f\left(x_{1}-1, x_{2}\right) \xi_{1}^{x_{1}} \xi_{2}^{x_{2}} z_{1}^{\ell_{1} x_{1}} z_{2}^{\ell_{2} x_{2}} \\
& -\sum_{\left(x_{1}, x_{2}\right) \geqslant(1,1)} f\left(x_{1}, x_{2}-1\right) \xi_{1}^{x_{1}} \xi_{2}^{x_{2}} z_{1}^{\ell_{1} x_{1}} z_{2}^{\ell_{2} x_{2}}=0
\end{aligned}
$$

We consider each sum separately:

$$
\begin{aligned}
& \sum_{\left(x_{1}, x_{2}\right) \geqslant(1,1)} f\left(x_{1}, x_{2}\right) \xi_{1}^{x_{1}} \xi_{2}^{x_{2}} z_{1}^{\ell_{1} x_{1}} z \frac{\ell_{2} x_{2}}{2} \\
& =F\left(; ; z_{1}, z_{2}\right)-F\left(; ; 0, z_{2}\right)-F\left(; ; z_{1}, 0\right)+F(; ; 0,0) ; \\
& \sum_{\left(x_{1}, x_{2}\right) \geqslant(1,1)} f\left(x_{1}-1, x_{2}\right) \xi_{1}^{x_{1}} \xi_{2}^{x_{2}} z \frac{\ell_{1} x_{1}}{z_{2}} z_{2}^{\ell_{2}} \\
& =\sum_{\left(x_{1}, x_{2}\right) \geqslant(0,1)} f\left(x_{1}, x_{2}\right) \xi_{1}^{x_{1}+1} \xi_{2}^{x_{2}} z_{1}^{\frac{\ell_{1}\left(x_{1}+1\right)}{}} z_{2}^{\ell_{2} x_{2}}= \\
& =\xi_{1} z_{1}^{\frac{\ell_{1}}{1}} \rho_{1}^{\ell_{1}} \sum_{\left(x_{1}, x_{2}\right) \geqslant(0,1)} f\left(x_{1}, x_{2}\right) \xi_{1}^{x_{1}} \xi_{2}^{x_{2}} z_{1}^{\ell_{1} x_{1}} z_{2}^{\ell_{2} x_{2}} \\
& =\xi_{1} z_{1}^{\frac{\ell_{1}}{1}} \rho_{1}^{\ell_{1}}\left(F\left(; ; z_{1}, z_{2}\right)-F\left(; ; z_{1}, 0\right)\right) ; \\
& \sum_{\left(x_{1}, x_{2}\right) \geqslant(1,1)} f\left(x_{1}, x_{2}-1\right) \xi_{1}^{x_{1}} \xi_{2}^{x_{2}} z_{1}^{\ell_{1} x_{1}} z \frac{\ell_{2} x_{2}}{2} \\
& =\xi_{2} z_{2}^{\frac{\ell_{2}}{2}} \rho_{2}^{\ell_{2}}\left(F\left(; ; z_{1}, z_{2}\right)-F\left(; ; 0, z_{2}\right)\right) .
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
F\left(; ; z_{1}, z_{2}\right) & -F\left(; ; 0, z_{2}\right)-F\left(; ; z_{1}, 0\right)+F(; ; 0,0) \\
& -\xi_{1} z_{1}^{\ell_{1}} \rho_{1}^{\ell_{1}}\left(F\left(; ; z_{1}, z_{2}\right)-F\left(; ; z_{1}, 0\right)\right)-\xi_{2} z_{2}^{\frac{\ell_{2}}{2}} \rho_{2}^{\ell_{2}}\left(F\left(; ; z_{1}, z_{2}\right)-F\left(; ; 0, z_{2}\right)\right)=0,
\end{aligned}
$$

which yields the functional equation on $F\left(; ; z_{1}, z_{2}\right)$ :

$$
\begin{aligned}
\left(1-\xi_{1} z \frac{\ell_{1}}{1} \rho_{1}^{\ell_{1}}-\xi_{2} z \frac{\ell_{2}}{2} \rho_{2}^{\ell_{2}}\right) F(; & \left.; z_{1}, z_{2}\right) \\
& \quad-\left(1-\xi_{2} z \frac{\ell_{2}}{2} \rho_{2}^{\ell_{2}}\right) F\left(; ; 0, z_{2}\right)-\left(1-\xi_{1} z \frac{\ell_{1}}{1} \rho_{1}^{\ell_{1}}\right) F\left(; ; z_{1}, 0\right)+F(; ; 0,0)=0
\end{aligned}
$$

Example 2. We consider a difference equation with polynomial coefficients whose solution is a p-recursive series [23]:

$$
\begin{equation*}
f\left(x_{1}, x_{2}\right)-\left(1+x_{1} x_{2}\right) f\left(x_{1}-1, x_{2}\right)-x_{2}^{2} f\left(x_{1}, x_{2}-1\right)=0 . \tag{15}
\end{equation*}
$$

Multiplying both sides of (15) by $\xi_{1}^{x_{1}} \xi_{2}^{x_{2}} z_{1}^{\ell_{1} x_{1}} z_{2}^{\ell_{2} x_{2}}$ and summing over $\left(x_{1}, x_{2}\right) \geqslant(1,1)$ yields

$$
\begin{aligned}
& \sum_{\left(x_{1}, x_{2}\right) \geqslant(1,1)} f\left(x_{1}, x_{2}\right) \xi_{1}^{x_{1}} \xi_{2}^{x_{2}} z_{1}^{\ell_{1} x_{1}} z_{2}^{\ell_{2} x_{2}}- \\
&\left(\sum_{\left(x_{1}, x_{2}\right) \geqslant(1,1)}\left(1+x_{1} x_{2}\right) f\left(x_{1}-1, x_{2}\right) \xi_{1}^{x_{1}} \xi_{2}^{x_{2}} z_{1}^{\ell_{1} x_{1}} z_{2}^{\ell_{2} x_{2}}\right. \\
&-\sum_{\left(x_{1}, x_{2}\right) \geqslant(1,1)} x_{2}^{2} f\left(x_{1}, x_{2}-1\right) \xi_{1}^{x_{1}} \xi_{2}^{x_{2}} z_{1}^{\ell_{1} x_{1}} z_{2}^{\ell_{2} x_{2}}=0 .
\end{aligned}
$$

The first sum is the same as in the previous example. We consider the second and third sum separately:

$$
\begin{aligned}
& \sum_{\left(x_{1}, x_{2}\right) \geqslant(1,1)}\left(1+x_{1} x_{2}\right) f\left(x_{1}-1, x_{2}\right) \xi_{1}^{x_{1}} \xi_{2}^{x_{2}} z_{1}^{\ell_{1} x_{1}} z_{2}^{\ell_{2} x_{2}} \\
& =\sum_{\left(x_{1}, x_{2}\right) \geqslant(0,1)}\left(1+\left(x_{1}+1\right) x_{2}\right) f\left(x_{1}, x_{2}\right) \xi_{1}^{x_{1}+1} \xi_{2}^{x_{2}} z_{1}^{\ell_{1}\left(x_{1}+1\right)} z_{2}^{\ell_{2} x_{2}} \\
& =\left(1+\left(\theta_{1}+1\right) \theta_{2}\right) \xi z z \frac{\ell_{1}}{1} \rho_{1}^{\ell_{1}} \sum_{\left(x_{1}, x_{2}\right) \geqslant(0,1)} f\left(x_{1}, x_{2}\right) \xi_{1}^{x_{1}} \xi_{2}^{x_{2}} z_{1}^{\ell_{1} x_{1}} z_{2}^{\ell_{2} x_{2}} \\
& =\left(1+\left(\theta_{1}+1\right) \theta_{2}\right) \xi \xi_{1} z_{1}^{\ell_{1}} \rho_{1}^{\ell_{1}}\left(F\left(; ; z_{1}, z_{2}\right)-F\left(; ; z_{1}, 0\right)\right) ; \\
& \sum_{\left(x_{1}, x_{2}\right) \geqslant(1,1)} x_{2}^{2} f\left(x_{1}, x_{2}-1\right) \xi_{1}^{x_{1}} \xi_{2}^{x_{2}} z_{1}^{\ell_{1} x_{1}} z \frac{\ell_{2} x_{2}}{2} \\
& =\left(\theta_{2}+1\right)^{2} \xi_{2} z_{2}^{\ell_{2}} \rho_{2}^{\ell_{2}}\left(F\left(; ; z_{1}, z_{2}\right)-F\left(; ; 0, z_{2}\right)\right),
\end{aligned}
$$

which yields the functional equation

$$
\begin{aligned}
&\left(1-\left(1+\theta_{1} \theta_{2}+\theta_{2}\right) \xi_{1} z_{1}^{\ell_{1}} \rho_{1}^{\ell_{1}}-\left(\theta_{2}+1\right)^{2} \xi_{2} z_{2}^{\ell_{2}} \rho_{2}^{\ell_{2}}\right) F\left(; ; z_{1}, z_{2}\right) \\
&-\left(1-\left(1+\theta_{1} \theta_{2}+\theta_{2}\right) \xi_{1} z_{1}^{\frac{\ell_{1}}{1}} \rho_{1}^{\ell_{1}}\right) F\left(; ; 0, z_{2}\right) \\
& \quad-\left(1-\left(\theta_{2}+1\right)^{2} \xi_{2} z \frac{\ell_{2}}{2} \rho_{2}^{\ell_{2}}\right) F\left(; ; z_{1}, 0\right)+F(; ; 0,0)=0
\end{aligned}
$$

## 5. Conclusions

We have initiated the theory of discrete generating series for multidimensional polynomial coefficient difference equations. We introduced a multidimensional polynomial shift operator and established three functional equations that these new discrete generating series obey, revealing some of their structural properties. A strong direction for future research is to generalize to the time scales calculus [24]. The falling factorial functions here are called generalized $h_{k}$ polynomials in time scales. This suggests some directions for the time scales analogue of this research, which was arguably anticipated with the definition of a moment-generating series for distributions in [25]. One particularly interesting question is what the proper analogue of (1) is for an arbitrary time scale, and perhaps analysis from a generating series perspective would reveal new insights to this problem.

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