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Abstract: We extend existing functional relationships for the discrete generating series associated with a single-variable linear polynomial coefficient difference equation to the multivariable case.

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1. Introduction

An approach to build the general theory of a discrete generating series of one variable and its connection with the linear difference equations was introduced in [1]. We extend those results to the multidimensional case. We define a discrete generating series for $f: \mathbb{Z}^n \to \mathbb{C}$ and derive functional relations for such series.

The general theory of linear recurrences with constant coefficients and the Stanley hierarchy [2,3] of its generating functions (rational, algebraic, *D*-finite) depending on the initial data function was considered in [4]. Difference equations with polynomial coefficients is an effective means to study lattice paths with restriction [5,6]. Some properties of linear difference operators whose coefficients have the form of infinite two-sided sequences over a field of characteristic zero are considered in [7]. An effective method of obtaining explicit formulas for the coefficients of a generating function related to the Aztec diamond and a generating function related to the permutations with cycles was derived in [8,9]. Using the notion of amoeba [10] of the characteristic polynomial of a difference equation, a description for the solution space of a multidimensional difference equation with constant coefficients was obtained in [11]. A generalization to several variables of the classical Poincaré theorem on the asymptotic behavior of solutions of a linear difference equation is presented in [12]. We can also note that the almost periodic and the almost automorphic solutions to the difference equations depending on several variables are not well explored in the existing literature [13].

Let \mathbb{Z}_{\geq} denote the non-negative integers, $\mathbb{Z}^n = \mathbb{Z} \times \cdots \times \mathbb{Z}$ be the *n*-dimensional integers, and $\mathbb{Z}_{\geq}^n = \mathbb{Z}_{\geq} \times \cdots \times \mathbb{Z}_{\geq}$ for $n \in \mathbb{Z}_{\geq}$ be its non-negative orthant. For any $z \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq}$, we define the falling factorial $z^{\underline{n}} = z(z-1)\cdots(z-n+1)$ with $z^{\underline{0}} = 1$ and the Pochhammer symbol (or rising factorial) is defined by $(z)_n = z(z+1)\cdots(z+n-1)$ with $(z)_0 = 1$. Throughout, we will use the multidimensional notation for convience of expressions: $x = (x_1, \ldots, x_n) \in \mathbb{Z}_{\geq}^n$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, $\xi^x = \xi_1^{x_1} \cdots \xi_n^{x_n}$, $z^{\underline{x}} = z_1^{\underline{x}_1} \cdots z_n^{\underline{x}_n}$, $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{Z}_{\geq}^n$, $x! = x_1! \ldots x_n!$. We also will use $x \leq y$ for $x, y \in \mathbb{Z}^n$ componentwise, i.e., that $x_i \leq y_i$ for all $i = 1, \ldots, n$.



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Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). Given a function $f: \mathbb{Z}_{\geq}^n \to \mathbb{C}$, we define the associated multidimensional discrete generating series of *f* as

$$F(\xi;\ell;z) = \sum_{x \in \mathbb{Z}_{\geq}^{n}} f(x)\xi^{x} z^{\ell x} = \sum_{x_{1}=0}^{\infty} \dots \sum_{x_{n}=0}^{\infty} f(x_{1},\dots,x_{n})\xi_{1}^{x_{1}} \dots \xi_{n}^{x_{n}} z^{\ell_{1}x_{1}}_{1} \dots z^{\ell_{n}x_{n}}_{n}$$

Let $p_{\alpha} \in \mathbb{C}[z]$ denote polynomials with complex coefficients. The difference equation under consideration in this work is

$$\sum_{\alpha \in A} p_{\alpha}(x) f(x - \alpha) = 0,$$
(1)

where set $A \subset \mathbb{Z}_{\geq}^{n}$ is finite and there is $m \in A$ such that for all $\alpha \in A$, the inequality $\alpha \leq m$, which means $\alpha_{j} \leq m_{j}, j = 1, ..., n$, holds. Occasionally we will use an equivalent notation $0 \leq \alpha \leq m$, assuming that for some α coefficients, $p_{\alpha}(x)$ vanishes and only $p_{m}(x) \neq 0$. In Section 2, we will particularly consider a homogeneous difference equation with constant coefficients.

The special case where each $p_{\alpha} = c_{\alpha}$ is a constant

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$$\sum_{\alpha \in A} c_{\alpha} f(x - \alpha) = 0$$
⁽²⁾

arises in a wide class of combinatorial analysis problems [3], for instance, in lattice path problems [4], the theory of digital recursive filters [14], and the wavelet theory [15]. The question about correctness and well-posedness of (2) was considered in [16–18].

We equip (1) with initial data on a set named X_m , which is used often enough. We introduce the notation \mathbb{Z}_{\geq} as $X_m = \mathbb{Z}_{\geq}^n \setminus (m + \mathbb{Z}_{\geq}^n) = \{x \in \mathbb{Z}_{\geq}^n : x \geq m\}$ (see Figure 1) and we define the initial data function $\varphi : X_m \to \mathbb{C}$ so that

$$f(x) = \varphi(x), \quad x \in X_m. \tag{3}$$



Figure 1. Illustration of the sets $x \ge m$, $x \le m$, and $x \ge m$.

For convenience, we extend φ to the whole of \mathbb{Z}^n by taking it to be identically zero outside of X_m . The Cauchy problem is to find a solution to difference Equation (1) that coincides with φ on X_m , i.e., $f(x) = \varphi(x)$, for all $x \in X_m$.

In Section 2, functional equations for the discrete generating series are derived for the solution of the difference equations with constant coefficients. In Section 3, a case of difference equations with polynomial coefficients is considered. Section 4 contains two examples that illustrate our approach to discrete generating series.

2. Discrete Generating Series for Linear Difference Equations with Constant Coefficients

In this section, we consider a homogeneous difference equation with constant coefficients (2) and introduce the shift operator by

$$\mathcal{P}(\xi;\ell;z) = \sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \xi^{\alpha} z^{\underline{\ell\alpha}} \rho^{\ell\alpha}.$$
(4)

Also useful is its truncation for $\tau \in \mathbb{Z}^n$, defined by the formula

$$\mathcal{P}_{\tau}(\xi;\ell;z) = \sum_{\substack{0 \leqslant \alpha \leqslant m \\ \alpha \not\geqslant \tau}} c_{\alpha} \xi^{\alpha} z^{\underline{\ell} \alpha} \rho^{\ell \alpha},$$

and the discrete generating series of the initial data for $\tau \in X_m$ by

$$\Phi_{\tau}(\xi;\ell;z) = \sum_{x \not\ge \tau} \varphi(x)\xi^{x} z^{\underline{\ell} x}.$$
(5)

Let $\delta_j : x \to x + e^j$ be the forward shift operator for j = 1, ..., n with multidimensional notation $\delta^{\alpha} = \delta_1^{\alpha_1} \dots \delta_n^{\alpha_n}$ and define the polynomial difference operator

$$P(\delta) = \sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \delta^{\alpha}.$$

With this notation, Equation (2) is represented compactly as

$$P(\delta^{-1})f(x) = 0, \quad x \ge m.$$

The case of generating series $\sum_{x} f(x)z^{x}$ and exponential generating series $\sum_{x} \frac{f(x)}{x!}z^{x}$ is well-studied for both one and several variables: one of the first convenient formulas to derive the generating series exploiting the characteristic polynomial and the initial data function was proven in [19]. We will prove analogues of these formulas for the discrete generating series $F(\xi; \ell; z)$.

Theorem 1. The discrete generating series $F(\xi; \ell; z)$ for the solution to the Cauchy problem for Equation (2) with initial data (3) satisfies the functional equations:

$$\mathcal{P}(\xi;\ell;z)F(\xi;\ell;z) = \sum_{0 \leqslant \alpha \leqslant m} c_{\alpha}\xi^{\alpha} z^{\underline{\ell\alpha}} \rho^{\ell\alpha} \Phi_{m-\alpha}(\xi;\ell;z)$$
(6)

$$=\sum_{\substack{x \not\ge m}} P(\delta^{-1})\varphi(x)\xi^{x} z^{\underline{\ell} \underline{x}}$$
(7)

$$=\sum_{x \neq m} \mathcal{P}_{m-x}(\xi;\ell;z)\varphi(x)z^{\ell x}.$$
(8)

Proof. By multiplying (2) by $\xi^x z^{\underline{\ell}x}$ and summing over $x \ge m$, we obtain

$$0 = \sum_{x \ge m} \sum_{0 \le \alpha \le m} c_{\alpha} f(x - \alpha) \xi^{x} z^{\underline{\ell} \underline{x}}$$
$$= \sum_{0 \le \alpha \le m} c_{\alpha} \sum_{x \ge m} f(x - \alpha) \xi^{x} z^{\underline{\ell} \underline{x}}.$$

Now, substituting *x* with $x + \alpha$ yields

$$0 = \sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \sum_{x \geqslant m-\alpha} f(x)\xi^{x+\alpha} z^{\underline{\ell}(x+\alpha)}$$

=
$$\sum_{0 \leqslant \alpha \leqslant m} c_{\alpha}\xi^{\alpha} z^{\underline{\ell}\alpha} \rho^{\ell\alpha} \sum_{x \geqslant m-\alpha} f(x)\xi^{x} z^{\underline{\ell}x}$$

=
$$\sum_{0 \leqslant \alpha \leqslant m} c_{\alpha}\xi^{\alpha} z^{\underline{\ell}\alpha} \rho^{\ell\alpha} \left(\sum_{x \geqslant 0} f(x)\xi^{x} z^{\underline{\ell}x} - \sum_{x \not\geqslant m-\alpha} \varphi(x)\xi^{x} z^{\underline{\ell}x} \right)$$

=
$$\underbrace{\sum_{0 \leqslant \alpha \leqslant m} c_{\alpha}\xi^{\alpha} z^{\underline{\ell}\alpha} \rho^{\ell\alpha}}_{=\mathcal{P}(\xi;\ell;z)} \underbrace{\sum_{x \geqslant 0} f(x)\xi^{x} z^{\underline{\ell}x}}_{=F(\xi;\ell;z)} - \underbrace{\sum_{0 \leqslant \alpha \leqslant m} c_{\alpha}\xi^{\alpha} z^{\underline{\ell}\alpha} \rho^{\ell\alpha}}_{=\Phi_{m-\alpha}(\xi;\ell;z)} \underbrace{\sum_{x \geqslant m-\alpha} \varphi(x)\xi^{x} z^{\underline{\ell}x}}_{=\Phi_{m-\alpha}(\xi;\ell;z)}$$

Thus, by (5), we have established (6). Since

$$\sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \xi^{\alpha} z^{\underline{\ell} \alpha} \rho^{\ell \alpha} \sum_{x \not\geqslant m-\alpha} \varphi(x) \xi^{x} z^{\underline{\ell} x} = \sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \sum_{x \not\geqslant m-\alpha} \varphi(x) \xi^{x+\alpha} z^{\underline{\ell}(x+\alpha)}$$
$$= \sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \sum_{x \not\geqslant m} \varphi(x-\alpha) \xi^{x} z^{\underline{\ell} x}$$
$$= \sum_{x \not\geqslant m} \underbrace{\left[\sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \varphi(x-\alpha) \right]}_{=P(\delta^{-1})\varphi(x)} \xi^{x} z^{\underline{\ell} x},$$

which yields (7). Finally, collecting (6) by $\varphi(x)$ yields

$$\sum_{0 \leqslant \alpha \leqslant m} c_{\alpha} \xi^{\alpha} z^{\underline{\ell}\alpha} \rho^{\ell\alpha} \sum_{x \not\geqslant m-\alpha} \varphi(x) \xi^{x} z^{\underline{\ell}x} = \sum_{x \not\geqslant m} \sum_{\substack{0 \leqslant \alpha \leqslant m \\ \alpha \not\geqslant m-x \\ = \mathcal{P}_{m-x}(\xi;\ell;z)}} c_{\alpha} \xi^{\alpha} z^{\underline{\ell}\alpha} \rho^{\ell\alpha} \varphi(x) z^{\underline{\ell}x}$$

completing the proof of (8). \Box

For $z = (z_1, ..., z_n)$, we denote the projection operator $\pi_j z = (z_1, ..., z_{j-1}, 0, z_{j+1}, ..., z_n)$ and we introduce the notation

$$\pi_j F(\xi;\ell;z) = F(\xi;\ell;\pi_j z) = \sum_{\substack{x \ge 0\\ x_j = 0}} f(x)\xi^x z^{\underline{\ell}x},$$

and we define the combined projection $\Pi = (1 - \pi_1) \circ \cdots \circ (1 - \pi_n)$ as the composition of $1 - \pi_i$ for all j = 1, ..., n.

For the next result, we introduce the symbols $I = (1, 1, ..., 1) \in \mathbb{Z}^n$ and the unit vectors $e_j = (0, ..., 0, 1, 0, ..., 0)$ for j = 1, 2, ..., n, which is nonzero only for the *j*th component. In these two lemmas, we will prove some useful properties of the combined projection Π .

Lemma 1. The following formula holds:

$$\prod \sum_{x \ge 0} f(x)\xi^{x} z^{\underline{\ell x}} = \sum_{x \ge I} f(x)\xi^{x} z^{\underline{\ell x}}.$$

Proof. First, compute for any j = 1, 2, ..., n,

$$(1 - \pi_j) \sum_{x \ge 0} f(x)\xi^x z^{\underline{\ell x}} = \sum_{x \ge 0} f(x)\xi^x z^{\underline{\ell x}} - \pi_j \sum_{x \ge 0} f(x)\xi^x z^{\underline{\ell x}}$$
$$= \sum_{x \ge e_j} f(x)\xi^x z^{\underline{\ell x}}.$$

Thus, we see that applying Π to $\sum_{x \ge 0} f(x)\xi^x z^{\underline{\ell}x}$ yields the desired result. \Box

We now obtain a similar result as Lemma 1 but for a shifted discrete generating series.

Lemma 2. The following formula holds:

$$\Pi \xi_j z_j^{\ell_j} \rho^{\ell_j} F(\xi;\ell;z) = \sum_{x \ge I} f(x-e^j) \xi^x z^{\ell x}.$$

Proof. First, compute

$$(1 - \pi_j)\xi_j z_j^{\ell_j} \rho^{\ell_j} F(\xi; \ell; z) = \xi_j z_j^{\ell_j} \rho^{\ell_j} F(\xi; \ell; z) - \pi_j \xi_j z_j^{\ell_j} \rho^{\ell_j} F(\xi; \ell; z)$$
$$= \xi_j z_j^{\ell_j} \rho^{\ell_j} \sum_{x \ge 0} f(x) \xi^x z^{\ell x}$$
$$= \sum_{x \ge 0} f(x) \xi^{x+e_j} z^{\ell(x+e^j)}$$
$$= \sum_{x \ge e^j} f(x-e^j) \xi^x z^{\ell x}.$$

Thus, we see that applying Π to $\xi_j z_j^{\ell_j} \rho^{\ell_j} F(\xi; \ell; z)$ completes the proof. \Box We introduce the inner product

$$\langle c, \xi z^{\underline{\ell}} \rho^{\ell} \rangle = c_1 \xi_1 z_1^{\underline{\ell}_1} \rho_1^{\ell_1} + \dots + c_n \xi_n z_n^{\underline{\ell}_n} \rho_n^{\ell_n}$$

and

$$\langle c, \delta^{-I} \rangle = c_1 \delta_1^{-1} + \dots + c_n \delta_n^{-1}$$

We are now prepared to prove an analogue of [20], [Theorem 1.1].

Theorem 2. *The following formula holds:*

$$\Pi\Big[(1-\langle c,\xi z^{\underline{\ell}}\rho^{\ell}\rangle)F(\xi;\ell;z)\Big] = \sum_{x\geqslant I}(1-\langle c,\delta^{-I}\rangle)f(x)\xi^{x}z^{\underline{\ell}x}.$$
(9)

Proof. Applying Π to $(1 - \langle c, \xi z^{\underline{\ell}} \rho^{\ell} \rangle) F(\xi; \ell; z)$ yields

$$\begin{split} \Pi[(1-\langle c,\xi z^{\underline{\ell}}\rho^{\ell}\rangle)F(\xi;\ell;z)] &= \Pi F(\xi;\ell;z) - \langle c,\Pi\xi z^{\underline{\ell}}\rho^{\ell}\rangle F(\xi;\ell;z) \\ &= \Pi F(\xi;\ell;z) - c_{1}\Pi\xi_{1}z_{1}^{\underline{\ell_{1}}}\rho_{1}^{\ell_{1}}F(\xi;\ell;z) - \cdots - c_{n}\Pi\xi_{n}z_{n}^{\underline{\ell_{n}}}\rho_{n}^{\ell_{n}}F(\xi;\ell;z) \\ &= \sum_{x\geqslant I}f(x)\xi^{x}z^{\underline{\ell_{x}}} - c_{1}\sum_{x\geqslant I}f(x-e^{1})\xi^{x}z^{\underline{\ell_{x}}} - \cdots - c_{n}\sum_{x\geqslant I}f(x-e^{n})\xi^{x}z^{\underline{\ell_{x}}} \\ &= \sum_{x\geqslant I}\left(f(x) - c_{1}f(x-e^{1}) - \cdots - c_{n}f(x-e^{n})\right)\xi^{x}z^{\underline{\ell_{x}}} \\ &= \sum_{x\geqslant I}(1-\langle c,\delta^{-I}\rangle)f(x)\xi^{x}z^{\underline{\ell_{x}}}, \end{split}$$

thereby completing the proof. \Box

The following corollary is straightforward.

Corollary 1. If f solves $(1 - \langle c, \delta^{-I} \rangle)f(x) = 0$, then

$$\Pi[(1 - \langle c, \zeta \rangle)F(\xi; \ell; z)] = 0.$$
⁽¹⁰⁾

3. Discrete Generating Series for Linear Difference Equations with Polynomial Coefficients We define the componentwise forward difference operators Δ_i by

$$\Delta_j F(z) = F(z + e^j) - F(z), \qquad j = 1, \dots, n.$$

If $z^{\underline{x}} = z_1^{\underline{x}_1} \dots z_n^{\underline{x}_n}$, then $\Delta_j z^{\underline{x}} = x_j z^{\underline{x}-e^j}$. Thus, we can regard Δ_j as a discrete analogue of a partial derivative operator. Now, compute

$$\begin{split} \Delta_j F(\xi;\ell;z) &= \Delta_j \sum_{x \ge 0} f(x) \xi^x z^{\underline{\ell} \underline{x}} \\ &= \sum_{x \ge 0} f(x) \xi^x \Delta_j z^{\underline{\ell} \underline{x}} \\ &= \sum_{x \ge 0} \ell_j x_j f(x) \xi^x z^{\underline{\ell} \underline{x} - e^j}. \end{split}$$

We denote the componentwise backward jump ρ_i by

$$\rho_j F(z) = F(z - e^j)$$

and we define the componentwise operators $\theta_j = \ell_j^{-1} z_j \rho_j \Delta_j$, which generalizes the singlevariable one defined earlier in [21,22]. Now, we prove some useful properties of the operator $\theta^k := \theta_1^k \dots \theta_n^k$.

Lemma 3. If $k = (k_1, ..., k_n) \in \mathbb{Z}_{\geq}^n$, then the following formula holds:

$$\theta^k F(\xi;\ell;z) = \sum_{x \ge 0} x^k f(x) \xi^x z^{\underline{\ell} x}.$$
(11)

Proof. We obtain:

$$\begin{aligned} \theta^{k} F(\xi;\ell;z) &= \theta_{1}^{k_{1}} \dots \theta_{n}^{k_{n}} F(\xi;\ell;z) \\ &= \theta_{1}^{fk_{1}} \dots \theta_{n-1}^{k_{n-1}} (\ell_{n}^{-1} z_{n} \rho_{n} \Delta_{n})^{k_{n}-1} \ell_{n}^{-1} z_{n} \rho_{n} \Delta_{n} F(\xi;\ell;z) \\ &= \theta_{1}^{k_{1}} \dots \theta_{n-1}^{k_{n-1}} (\ell_{n}^{-1} z_{n} \rho_{n} \Delta_{n})^{k_{n}-1} \ell_{n}^{-1} z_{n} \rho_{n} \sum_{x \ge 0} \ell_{n} x_{n} f(x) \xi^{x} z^{\ell x - e^{n}} \\ &= \theta_{1}^{k_{1}} \dots \theta_{n-1}^{k_{n-1}} (\ell_{n}^{-1} z_{n} \rho_{n} \Delta_{n})^{k_{n}-1} \sum_{x \ge 0} x_{n} f(x) \xi^{x} z^{\ell x}. \end{aligned}$$

Continuing this process $k_n - 1$ times for θ_n and in k_j times in turn for the powers of θ_j , j = 1, ..., n - 1 completes the proof. \Box

The proof of the following lemma resembles the proof of Lemma 3 but for the operator $p(\theta) = \sum_{\alpha \in A \subset \mathbb{Z}^n_{\geq}} c_{\alpha} \theta^{\alpha}$, so we omit explicitly writing the proof.

Lemma 4. The following formula holds:

$$p(\theta)F(\xi;\ell;z) = \sum_{x \ge 0} p(x)f(x)\xi^{x}z^{\underline{\ell}\underline{x}}.$$
(12)

We define an operator \mathcal{P}_A by

$$\mathcal{P}_A(\xi;\ell;z;\theta;
ho) = \sum_{\alpha\in A} p_\alpha(\theta+\alpha)\xi^\alpha z^{\underline{\ell\alpha}}\rho^{\ell\alpha}.$$

Theorem 3. The discrete generating series $F(\xi; \ell; \cdot)$ of the Cauchy problem for Equation (1) with *initial data* (3) *satisfies the functional equation*

$$\mathcal{P}_{A}(\xi;\ell;z;\theta;\rho)F(\xi;\ell;z) = \sum_{\alpha \in A} \sum_{x \not\ge m-\alpha} p_{\alpha}(x-\alpha)\varphi(x)\xi^{x} z^{\underline{\ell}(x+\alpha)}.$$
(13)

Proof. Similar to the proof of Theorem 1, we multiply (1) by $\xi^x z^{\underline{\ell}x}$ and sum over $x \ge m$ to obtain

$$0 = \sum_{x \ge m} \sum_{\alpha \in A} p_{\alpha}(x) f(x-\alpha) \xi^{x} z^{\underline{\ell x}} = \sum_{\alpha \in A} \sum_{x \ge m-\alpha} p_{\alpha}(x-\alpha) f(x) \xi^{x+\alpha} z^{\underline{\ell}(x+\alpha)}.$$

Replacing *x* with $x + \alpha$ then leads to

$$0 = \sum_{\alpha \in A} \xi^{\alpha} \left(\sum_{x \ge 0} p_{\alpha}(x - \alpha) f(x) \xi^{x} z^{\underline{\ell}(x + \alpha)} - \sum_{x \ge m - \alpha} p_{\alpha}(x - \alpha) \varphi(x) \xi^{x} z^{\underline{\ell}(x + \alpha)} \right)$$

and routine algebraic manipulation completes the proof. \Box

4. Examples

Example 1. We will derive the functional equation for the discrete generating series

$$F(;;z_1,z_2) = F(\xi_1,\xi_2;\ell_1,\ell_2;z_1,z_2)$$

for the basic combinatorial recurrence

$$f(x_1, x_2) - f(x_1 - 1, x_2) - f(x_1, x_2 - 1) = 0.$$
 (14)

Multiplying both sides of (14) by $\xi_1^{x_1}\xi_2^{x_2}z_1^{\ell_1x_1}z_2^{\ell_2x_2}$ and summing over $(x_1, x_2) \ge (1, 1)$ yields

$$\sum_{(x_1,x_2)\geqslant(1,1)} f(x_1,x_2)\xi_1^{x_1}\xi_2^{x_2}z_1^{\ell_1x_1}z_2^{\ell_2x_2} - \sum_{(x_1,x_2)\geqslant(1,1)} f(x_1-1,x_2)\xi_1^{x_1}\xi_2^{x_2}z_1^{\ell_1x_1}z_2^{\ell_2x_2} - \sum_{(x_1,x_2)\geqslant(1,1)} f(x_1,x_2-1)\xi_1^{x_1}\xi_2^{x_2}z_1^{\ell_1x_1}z_2^{\ell_2x_2} = 0.$$

We consider each sum separately:

$$\begin{split} \sum_{(x_1,x_2) \ge (1,1)} f(x_1,x_2) \xi_1^{x_1} \xi_2^{x_2} z_1^{\ell_1 x_1} z_2^{\ell_2 x_2} \\ &= F(;;z_1,z_2) - F(;;0,z_2) - F(;;z_1,0) + F(;;0,0); \\ \sum_{(x_1,x_2) \ge (1,1)} f(x_1 - 1,x_2) \xi_1^{x_1} \xi_2^{x_2} z_1^{\ell_1 x_1} z_2^{\ell_2 x_2} \\ &= \sum_{(x_1,x_2) \ge (0,1)} f(x_1,x_2) \xi_1^{x_1 + 1} \xi_2^{x_2} z_1^{\ell_1 (x_1 + 1)} z_2^{\ell_2 x_2} = \\ &= \xi_1 z_1^{\ell_1} \rho_1^{\ell_1} \sum_{(x_1,x_2) \ge (0,1)} f(x_1,x_2) \xi_1^{x_1} \xi_2^{x_2} z_1^{\ell_1 x_1} z_2^{\ell_2 x_2} \\ &= \xi_1 z_1^{\ell_1} \rho_1^{\ell_1} (F(;;z_1,z_2) - F(;;z_1,0)); \\ \sum_{(x_1,x_2) \ge (1,1)} f(x_1,x_2 - 1) \xi_1^{x_1} \xi_2^{x_2} z_1^{\ell_1 x_1} z_2^{\ell_2 x_2} \\ &= \xi_2 z_2^{\ell_2} \rho_2^{\ell_2} (F(;;z_1,z_2) - F(;;0,z_2)). \end{split}$$

Finally, we obtain

$$F(;;z_1,z_2) - F(;;0,z_2) - F(;;z_1,0) + F(;;0,0) - \xi_1 z_1^{\ell_1} \rho_1^{\ell_1} (F(;;z_1,z_2) - F(;;z_1,0)) - \xi_2 z_2^{\ell_2} \rho_2^{\ell_2} (F(;;z_1,z_2) - F(;;0,z_2)) = 0,$$

which yields the functional equation on $F(;;z_1,z_2)$:

$$(1 - \xi_1 z_1^{\ell_1} \rho_1^{\ell_1} - \xi_2 z_2^{\ell_2} \rho_2^{\ell_2}) F(;;z_1, z_2) - (1 - \xi_2 z_2^{\ell_2} \rho_2^{\ell_2}) F(;;0, z_2) - (1 - \xi_1 z_1^{\ell_1} \rho_1^{\ell_1}) F(;;z_1, 0) + F(;;0, 0) = 0.$$

Example 2. We consider a difference equation with polynomial coefficients whose solution is a *p*-recursive series [23]:

$$f(x_1, x_2) - (1 + x_1 x_2) f(x_1 - 1, x_2) - x_2^2 f(x_1, x_2 - 1) = 0.$$
(15)

Multiplying both sides of (15) by $\xi_1^{x_1}\xi_2^{x_2}z_1^{\ell_1x_1}z_2^{\ell_2x_2}$ and summing over $(x_1, x_2) \ge (1, 1)$ yields

$$\sum_{(x_1,x_2)\geqslant(1,1)} f(x_1,x_2)\xi_1^{x_1}\xi_2^{x_2}z_1^{\ell_1x_1}z_2^{\ell_2x_2} - \sum_{(x_1,x_2)\geqslant(1,1)} (1+x_1x_2)f(x_1-1,x_2)\xi_1^{x_1}\xi_2^{x_2}z_1^{\ell_1x_1}z_2^{\ell_2x_2} - \sum_{(x_1,x_2)\geqslant(1,1)} x_2^2f(x_1,x_2-1)\xi_1^{x_1}\xi_2^{x_2}z_1^{\ell_1x_1}z_2^{\ell_2x_2} = 0.$$

The first sum is the same as in the previous example. We consider the second and third sum separately:

$$\begin{split} \sum_{(x_1,x_2)\geqslant(1,1)} (1+x_1x_2)f(x_1-1,x_2)\xi_1^{x_1}\xi_2^{x_2}z_1^{\ell_1x_1}z_2^{\ell_2x_2} \\ &= \sum_{(x_1,x_2)\geqslant(0,1)} (1+(x_1+1)x_2)f(x_1,x_2)\xi_1^{x_1+1}\xi_2^{x_2}z_1^{\ell_1(x_1+1)}z_2^{\ell_2x_2} \\ &= (1+(\theta_1+1)\theta_2)\xi_1z_1^{\ell_1}\rho_1^{\ell_1}\sum_{(x_1,x_2)\geqslant(0,1)} f(x_1,x_2)\xi_1^{x_1}\xi_2^{x_2}z_1^{\ell_1x_1}z_2^{\ell_2x_2} \\ &= (1+(\theta_1+1)\theta_2)\xi_1z_1^{\ell_1}\rho_1^{\ell_1}\left(F(;;z_1,z_2)-F(;;z_1,0)\right); \\ \sum_{(x_1,x_2)\geqslant(1,1)} x_2^2f(x_1,x_2-1)\xi_1^{x_1}\xi_2^{x_2}z_1^{\ell_1x_1}z_2^{\ell_2x_2} \\ &= (\theta_2+1)^2\xi_2z_2^{\ell_2}\rho_2^{\ell_2}\left(F(;;z_1,z_2)-F(;;0,z_2)\right), \end{split}$$

which yields the functional equation

$$\begin{split} \big(1 - (1 + \theta_1 \theta_2 + \theta_2) \xi_1 z_1^{\ell_1} \rho_1^{\ell_1} - (\theta_2 + 1)^2 \xi_2 z_2^{\ell_2} \rho_2^{\ell_2} \big) F(;;z_1, z_2) \\ &- \big(1 - (1 + \theta_1 \theta_2 + \theta_2) \xi_1 z_1^{\ell_1} \rho_1^{\ell_1} \big) F(;;0, z_2) \\ &- \big(1 - (\theta_2 + 1)^2 \xi_2 z_2^{\ell_2} \rho_2^{\ell_2} \big) F(;;z_1, 0) + F(;;0, 0) = 0. \end{split}$$

5. Conclusions

We have initiated the theory of discrete generating series for multidimensional polynomial coefficient difference equations. We introduced a multidimensional polynomial shift operator and established three functional equations that these new discrete generating series obey, revealing some of their structural properties. A strong direction for future research is to generalize to the time scales calculus [24]. The falling factorial functions here are called generalized h_k polynomials in time scales. This suggests some directions for the time scales analogue of this research, which was arguably anticipated with the definition of a moment-generating series for distributions in [25]. One particularly interesting question is what the proper analogue of (1) is for an arbitrary time scale, and perhaps analysis from a generating series perspective would reveal new insights to this problem.

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