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Evolution Equations with Liouville Derivative on $\mathbb{R}$ without Initial Conditions

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Abstract: New classes of evolution differential equations with the Liouville derivative in Banach spaces are studied. Equations are considered on the whole real line and are not endowed by the initial conditions. Using the methods of the Fourier transform theory, we prove the unique solvability in the sense of classical solutions for the equation solved with respect to the Liouville fractional derivative with a bounded operator at the unknown function. This allows us to obtain the analogous result for the equation with a linear degenerate operator at the fractional derivative and with a spectrally bounded pair of operators. Abstract results are applied to obtain new results on the unique solvability of systems of ordinary differential equations, boundary problems to partial differential equations, and systems of equations.

Keywords: Liouville derivative; differential equation without initial conditions; Fourier transform

MSC: 34G10; 35R11; 34A08

1. Introduction

Equations with fractional derivatives have been under the close attention of researchers in recent decades. This has happened by virtue of their increasing importance in mathematical modeling problems [1–4] and the presence of many unexplored problems of fractional integro-differential calculus [5–7].

It has long been noticed that there are phenomena (for mathematicians, solutions of equations) that do not depend on the initial data of the process [8,9]. This is possible after a sufficiently long time has elapsed after its start. The corresponding solutions have a well-defined physical meaning and are called intermediate asymptotics [8,9]. In [10,11], a class of equations in Banach spaces with several Marchaud fractional derivatives and their intermediate asymptotics were studied. The well-posedness of the equation without initial data was proved in a suitable weighted function space.

As is well known, one of the effective approaches to the study of the existence and uniqueness of a solution and qualitative properties of solutions to differential equations is the investigation of differential equations in Banach spaces [12–15]. It allows the obtained general result to be used when considering various initial boundary value problems for equations and systems of equations having a common structure and being reduced to the corresponding abstract equation in a Banach space. In this work, we consider the differential equation with the Liouville fractional derivative on $\mathbb{R}$:

$$D^m \int_{-\infty}^{t} \frac{(t-s)^{m-\alpha-1}}{\Gamma(m-\alpha)} Lx(s)ds = Mx(t) + g(t) \tag{1}$$

without initial conditions, where $m-1<\alpha\leq m$, $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, $L:\mathcal{X}\to\mathcal{Y}$ is a linear bounded operator, and $M: D_M \to \mathcal{Y}$ is a linear closed operator with a dense in $\mathcal{X}$

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domain D_M. Note that in the vast majority of works devoted to the study of equations with fractional derivatives, the initial value problems for these equations are considered on a segment or on a half-axis. In contrast, in this paper, we study new classes of equations with fractional derivatives on the real line without initial conditions, which, as already noted, have well-defined physical interpretations. We use properties of the Liouville differential operator, which are discussed in [6,7], and apply the Fourier transform instead of the Laplace transform, which is usually used for problems with initial conditions.

First, under the suppositions that $\mathcal{X} = \mathcal{Y}$, $L = I$, and M is a bounded operator, the unique solvability of Equation (1) is proved in the sense of classical solutions. Such result for the integer order equation on the real line is obtained in the first section, and that for the fractional order equation is obtained in the second one. Then, in the third section, this result is applied for the study of the uniqueness and existence of a classical solution for Equation (1) with $\ker L \neq \{0\}$ under the condition of (L, p)-boundedness [14] of the operator M. We use the methods of the Fourier transform and the existence of pairs of invariant subspaces for operators L, M in the case when an operator M is (L, p)-bounded [14]. Abstract results are applied to the investigation of systems of ordinary differential equations (in the fourth section), some classes of boundary problems for partial differential equations (in the fifth section), and the Scott–Blair system of equations with the Liouville time fractional derivative (in the sixth section).

2. Integer Order Equation on $\mathbb{R}$

Let $\mathcal{Z}$ be a Banach space, where $z \in L_1(\mathbb{R}; \mathcal{Z})$. The Fourier transform [6,7] of z is

$$\mathcal{F}z(\omega) = \int_{\mathbb{R}} z(t)e^{i\omega t}dt.$$

The inverse Fourier transform [6,7] of $h \in L_1(\mathbb{R}; \mathcal{Z})$ has the form

$$\mathcal{F}^{-1}h(t) = \frac{1}{2\pi}\int_{\mathbb{R}} h(\omega)e^{-i\omega t}d\omega.$$

Firstly, we study the integer order linear equations on the real line without initial conditions. The Banach space of all linear bounded operators on $\mathcal{Z}$ are denoted by $\mathcal{L}(\mathcal{Z})$. For $A \in \mathcal{L}(\mathcal{Z})$, $m \in \mathbb{N}$, consider the linear homogeneous equation

$$D^m z(t) = Az(t), \quad t \in \mathbb{R}, \tag{2}$$

where D^m is the integer order differentiation operator. A solution of Equation (2) is a function $z \in C^m(\mathbb{R}; \mathcal{Z})$, such that equality (2) is valid for all $t \in \mathbb{R}$.

Denote by $\rho(A) := \{\mu \in \mathbb{C} : (\mu I - A)^{-1} \in \mathcal{L}(\mathcal{Z})\}$ the resolvent set of an operator A and by $\sigma(A) := \mathbb{C} \setminus \rho(A)$ the spectrum of the operator; $\mathbb{R}_- := \{a \in \mathbb{R} : a < 0\}$, $\mathbb{R}_+ := \{a \in \mathbb{R} : a > 0\}$.

Theorem 1. *Let* $m \in \mathbb{N}$, $A \in \mathcal{L}(\mathcal{Z})$, $\mathrm{mes}(\{(-i\omega)^m : \omega \in \mathbb{R}\} \cap \sigma(A)) = 0$. *Then, function* $z \equiv 0$ *is a unique solution of Equation (2).*

Proof. Take a solution z of Equation (2), and define for some $T \in \mathbb{R}$ $z_T(t) = z(t)$ for $|t| < T$, $z_T(t) = 0$ for $|t| \geq T$. Then, $z_T \in L_1(\mathbb{R}; \mathcal{Z})$, where z_T satisfies Equation (2) for $t \in \mathbb{R}$ excluding, possibly, the points $t = T$ and $t = -T$. Apply the Fourier transform on the both sides of the equality $D^m z_T(t) = Az_T(t)$, and obtain the equality $((-i\omega)^m - A)\mathcal{F}z_T(\omega) = 0$. Since $\{(-i\omega)^m : \omega \in \mathbb{R}\} \subset \rho(A)$ for almost all $\omega \in \mathbb{R}$, for these $\omega \in \mathbb{R}$, there exists a bounded operator $((-i\omega)^m - A)^{-1} \in \mathcal{L}(\mathcal{Z}))$. Apply this operator, and obtain $\mathcal{F}z_T(\omega) = 0$ a. e. on $\mathbb{R}$; hence, $\mathcal{F}z_T(\omega) \equiv 0$ because of continuity of $\mathcal{F}z_T$. By the inverse Fourier transform, we obtain $z_T \equiv 0$. Because of the arbitrariness of $T > 0$, we have $z \equiv 0$. □

Lemma 1. *Let $A \in \mathcal{L}(\mathcal{Z})$, $m \in \mathbb{N}$, $\{(-i\omega)^m : \omega \in \mathbb{R}\} \cap \sigma(A) = \emptyset$, for $t \in \mathbb{R}$*

$$Z_m(t) := \frac{1}{2\pi} \int_{\mathbb{R}} ((-i\omega)^m - A)^{-1} e^{-i\omega t} d\omega.$$

Then, $Z_m \in C^{m-1}(\mathbb{R}; \mathcal{L}(\mathcal{Z})) \cap C^\infty(\mathbb{R}_- \cup \mathbb{R}_+; \mathcal{L}(\mathcal{Z}))$, $D^k Z_m \in L_1(\mathbb{R}; \mathcal{L}(\mathcal{Z}))$ for $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Proof. Note that for $|\omega|^m > 2\|A\|_{\mathcal{L}(\mathcal{Z})}$,

$$\|((-i\omega)^m - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} = |\omega|^{-m} \|(1 - (-i\omega)^{-m} A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leq$$

$$\leq |\omega|^{-m} \sum_{k=0}^{\infty} \frac{\|A\|^k_{\mathcal{L}(\mathcal{Z})}}{|\omega|^{mk}} \leq 2|\omega|^{-m}. \tag{3}$$

For $R \geq R_0 > \|A\|_{\mathcal{L}(\mathcal{Z})}$, take the arcs $C^1_R := \{Re^{i\varphi} : \varphi \in (-\pi, 0)\}$ and $C^2_R := \{Re^{i\varphi} : \varphi \in (0, \pi)\}$ and the contours $\Gamma^1_R := [-R, R] \cup C^{1,-}_R$ and $\Gamma^2_R := [-R, R] \cup C^2_R$, where the minus in the upper index of C^1_R means the negative orientation of the arc. Note that for $t > 0$, $k = 0, 1, \ldots, m-1$,

$$\int_{-R}^{R} (-i\omega)^k ((-i\omega)^m - A)^{-1} e^{-i\omega t} d\omega =$$

$$= \int_{\Gamma^1_R} (-i\omega)^k ((-i\omega)^m - A)^{-1} e^{-i\omega t} d\omega + \int_{C^1_R} (-i\omega)^k ((-i\omega)^m - A)^{-1} e^{-i\omega t} d\omega.$$

As $R \to \infty$, we have

$$\int_{\mathbb{R}} (-i\omega)^k ((-i\omega)^m - A)^{-1} e^{-i\omega t} d\omega = \int_{\Gamma^1_{R_0}} (-i\omega)^k ((-i\omega)^m - A)^{-1} e^{-i\omega t} d\omega,$$

owing to the analyticity of $(-i\omega)^k ((-i\omega)^m - A)^{-1} e^{-i\omega t}$ outside of the disk $\{|\omega| \leq \|A\|_{\mathcal{L}(\mathcal{Z})}\}$ and by the equality

$$\lim_{R \to \infty} \int_{C^1_R} (-i\omega)^k ((-i\omega)^m - A)^{-1} e^{-i\omega t} d\omega = 0,$$

which is valid per the Jordan Lemma since

$$\lim_{|\omega| \to \infty} \|(-i\omega)^k ((-i\omega)^m - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} = 0.$$

Analogously, we can prove that for $t < 0$,

$$\int_{\mathbb{R}} (-i\omega)^k ((-i\omega)^m - A)^{-1} e^{-i\omega t} d\omega = \int_{\Gamma^2_{R_0}} (-i\omega)^k ((-i\omega)^m - A)^{-1} e^{-i\omega t} d\omega.$$

The families of operators

$$X^1_m(t) = \frac{1}{2\pi} \int_{\Gamma^1_{R_0}} ((-i\omega)^m - A)^{-1} e^{-i\omega t} d\omega, \quad t \in \mathbb{C},$$

$$X_m^2(t) = \frac{1}{2\pi} \int\limits_{\Gamma_{R_0}^2} ((-i\omega)^m - A)^{-1} e^{-i\omega t} d\omega, \quad t \in \mathbb{C},$$

are analytic in $\mathbb{C}$. Thus, for $k \in \mathbb{N}$,

$$D^k Z_m(t) = D^k X_m^1(t) = \frac{1}{2\pi} \int\limits_{\Gamma_{R_0}^1} (-i\omega)^k ((-i\omega)^m - A)^{-1} e^{-i\omega t} d\omega, \quad t > 0,$$

$$D^k Z_m(t) = D^k X_m^2(t) = \frac{1}{2\pi} \int\limits_{\Gamma_{R_0}^2} (-i\omega)^k ((-i\omega)^m - A)^{-1} e^{-i\omega t} d\omega, \quad t < 0,$$

We have $D^k Z_m(0) = D^k X_m^1(0) = D^k X_m^2(0)$, $k = 0, 1, \ldots, m-1$. Indeed, since, for $k = 0, 1, \ldots, m-1$, the infinity is the removable singular point of $(-i\omega)^k ((-i\omega)^m - A)^{-1}$ due to (3), we have

$$0 = \int\limits_{C_{R_0}^1 \cup C_{R_0}^2} (-i\omega)^k ((-i\omega)^m - A)^{-1} d\omega =$$

$$= \int\limits_{\Gamma_{R_0}^2} (-i\omega)^k ((-i\omega)^m - A)^{-1} d\omega - \int\limits_{\Gamma_{R_0}^1} (-i\omega)^k ((-i\omega)^m - A)^{-1} d\omega = D^k X_m^2(0) - D^k X_m^1(0).$$

Thus, $Z_m \in C^{m-1}(\mathbb{R}; \mathcal{L}(\mathcal{Z})) \cap C^\infty(\mathbb{R}_- \cup \mathbb{R}_+; \mathcal{L}(\mathcal{Z}))$.

In addition, the spectrum $\sigma(A)$ is compact, and $\{(-i\omega)^m : \omega \in \mathbb{R}\} \cap \sigma(A) = \emptyset$. Therefore, there is a small enough $\varepsilon > 0$ such that

$$D^k Z_m(t) = D^k X_m^1(t) = \frac{1}{2\pi} \int\limits_{\Gamma_{R_0}^1 - i\varepsilon} (-i\omega)^k ((-i\omega)^m - A)^{-1} e^{-i\omega t} d\omega, \quad t > 0,$$

$$D^k Z_m(t) = D^k X_m^2(t) = \frac{1}{2\pi} \int\limits_{\Gamma_{R_0}^2 + i\varepsilon} (-i\omega)^k ((-i\omega)^m - A)^{-1} e^{-i\omega t} d\omega, \quad t < 0,$$

where $\Gamma_{R_0}^k \pm i\varepsilon = \{\mu \pm i\varepsilon : \mu \in \Gamma_{R_0}^k\}$, $k = 1, 2$. Hence, there exists $C_k > 0$ such that

$$\|D^k Z_m(t)\|_{\mathcal{L}(\mathcal{Z})} \le C_k e^{-\varepsilon |t|}, \quad t \in \mathbb{R} \setminus \{0\}, \quad k \in \mathbb{N}. \tag{4}$$

Therefore, $D^k Z_m \in L_1(\mathbb{R}; \mathcal{L}(\mathcal{Z}))$. $\square$

Now, consider the inhomogeneous equation

$$D^m z(t) = Az(t) + f(t), \quad t \in \mathbb{R}, \tag{5}$$

with some $f \in C(\mathbb{R}; \mathcal{Z})$.

Theorem 2. *Let $m \in \mathbb{N}$, $A \in \mathcal{L}(\mathcal{Z})$, $\{(-i\omega)^m : \omega \in \mathbb{R}\} \cap \sigma(A) = \emptyset$, $f \in C(\mathbb{R}; \mathcal{Z}) \cap L_2(\mathbb{R}; \mathcal{Z})$. Then, function*

$$z_f(t) = \int\limits_{\mathbb{R}} Z_m(t-s) f(s) ds$$

is a unique solution of Equation (5); moreover, $z_f \in C^\infty(\mathbb{R}; \mathcal{Z})$.

Proof. Since $f \in L_2(\mathbb{R}; \mathcal{Z})$ and $D^k Z_m \in L_1(\mathbb{R}; \mathcal{L}(\mathcal{Z}))$ for $k = 0, 1, \ldots, m$ per Lemma 1, the convolutions $D^k z_f = D^k Z_m * f$, $k = 0, 1, \ldots, m$, are defined and belong to $L_2(\mathbb{R}; \mathcal{Z})$,

$$\mathcal{F}z_f(\omega) = \mathcal{F}Z_m(\omega)\mathcal{F}f(\omega) = ((-i\omega)^m - A)^{-1}\mathcal{F}f(\omega) \in L_2(\mathbb{R}; \mathcal{Z}),$$

since operator function $((-i\omega)^m - A)^{-1}$ is bounded on $\mathbb{R}$. Hence, $((-i\omega)^m - A)\mathcal{F}z_f(\omega) = \mathcal{F}(D^m z_f - Az_f)(\omega) = \mathcal{F}f(\omega) \in L_2(\mathbb{R}; \mathcal{Z})$. By the inverse Fourier transform, we obtain (5).

Note that for every $k \in \mathbb{N}_0$, there exists some $\xi \in [0, 1]$ such that, per (4),

$$\|D^k z_f(t+\delta) - D^k z_f(t)\|_{\mathcal{Z}} \leq \int\limits_{\mathbb{R}} \|D^k Z_m(t+\delta-s) - D^k Z_m(t-s)\|_{\mathcal{L}(\mathcal{Z})} \|f(s)\|_{\mathcal{Z}} ds \leq$$

$$\leq |\delta| \int\limits_{\mathbb{R}\setminus\{t\}} \|D^{k+1} Z_m(t+\xi-s)\|_{\mathcal{L}(\mathcal{Z})} \|f(s)\|_{\mathcal{Z}} ds \leq$$

$$\leq |\delta| C_{k+1} \left(\int\limits_{\mathbb{R}} e^{-2\varepsilon|t+\xi-s|} ds \right)^{1/2} \|f\|_{L_2(\mathbb{R};\mathcal{Z})} \to 0, \quad \delta \to 0.$$

Therefore, $z_f \in C^\infty(\mathbb{R}; \mathcal{Z})$.

From Theorem 1, the uniqueness of a solution follows. □

3. Fractional Order Equation on $\mathbb{R}$

For $z \in L_1(\mathbb{R}; \mathcal{Z})$, denote the Liouville fractional integral of the order $\beta \in (0, 1)$ by

$$J^\beta z(t) := \frac{1}{\Gamma(\beta)} \int\limits_{-\infty}^{t} (t-s)^{\beta-1} z(s) ds, \quad t \in \mathbb{R},$$

$J^0 z(t) := z(t)$. Let $m - 1 < \alpha \leq m \in \mathbb{N}$ and D^α be the Liouville fractional derivative, i.e., $D^\alpha z(t) := D^m J^{m-\alpha} z(t)$ [6,7]. For $\beta \in (-1, 0]$, $D^\beta z(t) := J^{-\beta} z(t)$.

The Fourier transforms of the Liouville fractional integral and the Liouville fractional derivative have the forms ([7], p. 90) $\mathcal{F}J^\beta h(\omega) = (-i\omega)^{-\beta}\mathcal{F}h(\omega)$, $\mathcal{F}D^\alpha h(\omega) = (-i\omega)^\alpha \mathcal{F}h(\omega)$, $\beta \in (-1, 0]$, $\alpha > 0$. Hereafter, by the fractional power of a complex number, we mean the value of the principal branch of the power function.

Let $A \in \mathcal{L}(\mathcal{Z})$. Consider the linear homogeneous equation

$$D^\alpha z(t) = Az(t), \quad t \in \mathbb{R}. \tag{6}$$

A solution of Equation (6) is a function $z \in C(\mathbb{R}; \mathcal{Z})$ such that $J^{m-\alpha} z \in C^m(\mathbb{R}; \mathcal{Z})$ and for all $t \in \mathbb{R}$ equality (6) is valid.

Theorem 3. *Let $A \in \mathcal{L}(\mathcal{Z})$, $\alpha > 0$, $\mathrm{mes}(\{(-i\omega)^\alpha : \omega \in \mathbb{R}\} \cap \sigma(A)) = 0$. Then, function $z \equiv 0$ is a unique solution of Equation (6).*

Proof. Take a solution z of Equation (5), and define for some $T \in \mathbb{R}$ $z_T(t) = z(t)$ for $|t| < T$, $z_T(t) = 0$ for $|t| \geq T$, then $z_T \in L_1(\mathbb{R}; \mathcal{Z})$. Note that z_T satisfies Equation (6) excluding the points $t = T$ and $t = -T$, possibly. By the Fourier transform for $D^\alpha z_T(t) - Az_T(t)$, we obtain the equality $(-i\omega)^\alpha \mathcal{F}z_T(\omega) - A\mathcal{F}z_T(\omega) \equiv 0$. We have $\{(-i\omega)^\alpha : \omega \in \mathbb{R}\} \subset \rho(A)$ for almost all $\omega \in \mathbb{R}$; therefore, $\mathcal{F}z_T(\omega) = 0$ a. e. on $\mathbb{R}$. The continuity of $\mathcal{F}z_T$ implies that $\mathcal{F}z_T(\omega) \equiv 0$. Using the inverse Fourier transform, we obtain $z_T \equiv 0$. It remains to note that $T > 0$ is arbitrary; hence, $z \equiv 0$. □

Lemma 2. *Let $A \in \mathcal{L}(\mathcal{Z})$, $\alpha > 0$, $\{(-i\omega)^\alpha : \omega \in \mathbb{R}\} \cap \sigma(A) = \emptyset$; for $t \in \mathbb{R}$,*

$$Z_\alpha(t) := \frac{1}{2\pi}\int_{\mathbb{R}} ((-i\omega)^\alpha - A)^{-1} e^{-i\omega t} d\omega.$$

Then, $Z_\alpha \in C^{m-1}(\mathbb{R}; \mathcal{L}(\mathcal{Z})) \cap C^\infty(\mathbb{R}_- \cup \mathbb{R}_+; \mathcal{L}(\mathcal{Z}))$, $D^k Z_\alpha \in L_1(\mathbb{R}; \mathcal{L}(\mathcal{Z}))$ for $k \in \mathbb{N}$,

$$J^{m-\alpha} Z_\alpha(t) = \int_{\mathbb{R}} (-i\omega)^{\alpha-m}((-i\omega)^\alpha - A)^{-1} e^{-i\omega t} d\omega,$$

$J^{m-\alpha} Z_\alpha \in C^{m-1}(\mathbb{R}; \mathcal{L}(\mathcal{Z})) \cap C^\infty(\mathbb{R}_- \cup \mathbb{R}_+; \mathcal{L}(\mathcal{Z}))$, $D^k J^{m-\alpha} Z_\alpha \in L_1(\mathbb{R}; \mathcal{L}(\mathcal{Z}))$ for $k \in \mathbb{N}$.

Proof. Per (3), we can obtain for $|\omega|^\alpha > 2\|A\|_{\mathcal{L}(\mathcal{Z})}$ the inequality

$$\|((-i\omega)^\alpha - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \le 2|\omega|^{-\alpha}. \tag{7}$$

Since $0 \in \rho(A)$ and $\rho(A)$ is an open set, there exists a small enough ρ_0 such that there exists $((-i\omega)^\alpha - A)^{-1} \in \mathcal{L}(\mathcal{Z})$ for all ω with $|\omega| \le \rho_0$.

Take, as in the proof of Lemma 1, the arcs $C_R^1 := \{Re^{i\varphi} : \varphi \in (-\pi, 0)\}$, $C_R^2 := \{Re^{i\varphi} : \varphi \in (0, \pi)\}$, and in the complex plane cut along the negative axis (we denote it by $\mathbb{C}^*$), we choose the contours $\Gamma_{R,\rho}^1 := \{re^{-i\pi} : r \in [\rho, R]\}^- \cup C_\rho^1 \cup [\rho, R] \cup C_R^{1,-}$ and $\Gamma_{R,\rho}^2 := \{re^{i\pi} : r \in [\rho, R]\}^- \cup C_\rho^{2,-} \cup [\rho, R] \cup C_R^2$ for $0 < \rho \le \rho_0 < \|A\|_{\mathcal{L}(\mathcal{Z})} < R_0 \le R$, where the minus in the upper index means the negative orientation of a curve. As in the proof of Lemma 1, it is easy to prove for $t > 0$, $k = 0, 1, \ldots, m-1$ the equalitites

$$\int_{-R}^{-\rho} (-i\omega)^k((-i\omega)^\alpha - A)^{-1} e^{-i\omega t} d\omega + \int_{\rho}^{R} (-i\omega)^k((-i\omega)^\alpha - A)^{-1} e^{-i\omega t} d\omega =$$

$$= \int_{\Gamma_{R,\rho}^1} (-i\omega)^k((-i\omega)^\alpha - A)^{-1} e^{-i\omega t} d\omega - \int_{C_\rho^1} (-i\omega)^k((-i\omega)^\alpha - A)^{-1} e^{-i\omega t} d\omega +$$

$$+ \int_{C_R^1} (-i\omega)^k((-i\omega)^\alpha - A)^{-1} e^{-i\omega t} d\omega.$$

Pass to the limit as $R \to \infty$, $\rho \to 0$ and obtain

$$\int_{\mathbb{R}} (-i\omega)^k((-i\omega)^\alpha - A)^{-1} e^{-i\omega t} d\omega = \int_{\Gamma_{R_0,\rho_0}^1} (-i\omega)^k((-i\omega)^\alpha - A)^{-1} e^{-i\omega t} d\omega, \quad t > 0$$

since

$$\left\| \int_{C_\rho^1} (-i\omega)^k((-i\omega)^\alpha - A)^{-1} e^{-i\omega t} d\omega \right\|_{\mathcal{L}(\mathcal{Z})} \le C\rho^{k+1} \to 0$$

as $\rho \to 0$. Analogously, we obtain that

$$\int_{\mathbb{R}} (-i\omega)^k((-i\omega)^\alpha - A)^{-1} e^{-i\omega t} d\omega = \int_{\Gamma_{R_0,\rho_0}^2} (-i\omega)^k((-i\omega)^\alpha - A)^{-1} e^{-i\omega t} d\omega, \quad t < 0.$$

The families of operators

$$X_\alpha^1(t) = \frac{1}{2\pi} \int_{\Gamma_{R_0,\rho_0}^1} ((-i\omega)^\alpha - A)^{-1} e^{-i\omega t} d\omega, \quad t \in \mathbb{C},$$

$$X_\alpha^2(t) = \frac{1}{2\pi} \int\limits_{\Gamma^2_{R_0,\rho_0}} ((-i\omega)^\alpha - A)^{-1} e^{-i\omega t} d\omega, \quad t \in \mathbb{C},$$

are analytic in $\mathbb{C}$. For $k \in \mathbb{N}$,

$$D^k Z_\alpha(t) = D^k X_\alpha^1(t) = \frac{1}{2\pi} \int\limits_{\Gamma^1_{R_0,\rho_0}} (-i\omega)^k ((-i\omega)^\alpha - A)^{-1} e^{-i\omega t} d\omega, \quad t > 0,$$

$$D^k Z_\alpha(t) = D^k X_\alpha^2(t) = \frac{1}{2\pi} \int\limits_{\Gamma^2_{R_0,\rho_0}} (-i\omega)^k ((-i\omega)^\alpha - A)^{-1} e^{-i\omega t} d\omega, \quad t < 0.$$

For $k = 0, 1, \ldots, m-1$, the infinity is the removable singular point of the functions $(-i\omega)^k((-i\omega)^\alpha - A)^{-1}$ in $\mathbb{C}^*$ per (7); hence, for the contour $\Gamma = C^1_{R_0} \cup C^2_{R_0} \cup \{re^{i\pi} : r \in [0, R]\}^- \cup C^{2,-}_\rho \cup C^{1,-}_\rho \cup \{re^{-i\pi} : r \in [0, R)\} \subset \mathbb{C}^*$, we have

$$0 = \int\limits_\Gamma (-i\omega)^k ((-i\omega)^\alpha - A)^{-1} d\omega = \int\limits_{\Gamma^2_{R_0,\rho_0}} (-i\omega)^k ((-i\omega)^\alpha - A)^{-1} d\omega -$$

$$- \int\limits_{\Gamma^1_{R_0,\rho_0}} (-i\omega)^k ((-i\omega)^\alpha - A)^{-1} d\omega = D^k X_\alpha^2(0) - D^k X_\alpha^1(0).$$

Therefore, $D^k Z_\alpha(0) = D^k X_\alpha^1(0) = D^k X_\alpha^2(0)$, $Z_\alpha \in C^{m-1}(\mathbb{R}; \mathcal{L}(\mathcal{Z})) \cap C^\infty(\mathbb{R}_- \cup \mathbb{R}_+; \mathcal{L}(\mathcal{Z}))$.

Since $\sigma(A)$ is compact and $\{(-i\omega)^\alpha : \omega \in \mathbb{R}\} \cap \sigma(A) = \emptyset$, we can choose a small enough $\varepsilon > 0$ such that

$$D^k Z_\alpha(t) = D^k X_\alpha^1(t) = \frac{1}{2\pi} \int\limits_{\Gamma^1_{R_0,\rho_0} - i\varepsilon} (-i\omega)^k ((-i\omega)^\alpha - A)^{-1} e^{-i\omega t} d\omega, \quad t > 0,$$

$$D^k Z_\alpha(t) = D^k X_\alpha^2(t) = \frac{1}{2\pi} \int\limits_{\Gamma^2_{R_0,\rho_0} + i\varepsilon} (-i\omega)^k ((-i\omega)^\alpha - A)^{-1} e^{-i\omega t} d\omega, \quad t < 0,$$

where $\Gamma^k_{R_0,\rho_0} \pm i\varepsilon = \{\mu \pm i\varepsilon : \mu \in \Gamma^k_{R_0,\rho_0}\}$, $k = 1, 2$. Consequently, for every $k \in \mathbb{N}$, there is $C_k > 0$ such that $\|D^k Z_\alpha(t)\|_{\mathcal{L}(\mathcal{Z})} \leq C_k e^{-\varepsilon|t|}$ for all $t \in \mathbb{R} \setminus \{0\}$. Therefore, $D^k Z_\alpha \in L_1(\mathbb{R}; \mathcal{L}(\mathcal{Z}))$, $k \in \mathbb{N}$.

Using the formula of the Laplace transformation for the power function, we have for $\omega \neq 0$

$$\int\limits_0^\infty \frac{y^{m-\alpha-1}}{\Gamma(m-\alpha)} e^{i\omega y} dy = (-i\omega)^{\alpha-m}.$$

Therefore,

$$J^{m-\alpha} Z_\alpha(t) = \int\limits_{-\infty}^t \frac{(t-s)^{m-\alpha-1}}{\Gamma(m-\alpha)} Z_\alpha(s) dt = \int\limits_{\mathbb{R}} ((-i\omega)^\alpha - A)^{-1} e^{-i\omega t} \int\limits_0^\infty \frac{y^{m-\alpha-1}}{\Gamma(m-\alpha)} e^{i\omega y} dy d\omega =$$

$$= \int\limits_{\mathbb{R}} (-i\omega)^{\alpha-m} ((-i\omega)^\alpha - A)^{-1} e^{-i\omega t} d\omega.$$

As for Z_α, it can be shown that $J^{m-\alpha}Z_\alpha \in C^{m-1}(\mathbb{R};\mathcal{L}(\mathcal{Z})) \cap C^\infty(\mathbb{R}_- \cup \mathbb{R}_+;\mathcal{L}(\mathcal{Z}))$; moreover, $D^kJ^{m-\alpha}Z_\alpha \in L_1(\mathbb{R};\mathcal{L}(\mathcal{Z}))$ for $k \in \mathbb{N}$,

$$\|D^kJ^{m-\alpha}Z_\alpha(t)\|_{\mathcal{L}(\mathcal{Z})} \le C_k e^{-\varepsilon|t|}, \quad t \in \mathbb{R}\setminus\{0\}, \quad k \in \mathbb{N}. \tag{8}$$

□

Consider the inhomogeneous equation

$$D^\alpha z(t) = Az(t) + f(t), \quad t \in \mathbb{R}, \tag{9}$$

with some function $f \in C(\mathbb{R};\mathcal{Z})$.

Theorem 4. *Let $A \in \mathcal{L}(\mathcal{Z})$, $\alpha > 0$, $\{(-i\omega)^\alpha : \omega \in \mathbb{R}\} \cap \sigma(A) = \emptyset$, $f \in C(\mathbb{R};\mathcal{Z}) \cap L_2(\mathbb{R};\mathcal{Z})$. Then, function*

$$z_f(t) = \int_{\mathbb{R}} Z_\alpha(t-s)f(s)ds \tag{10}$$

is a unique solution of Equation (9); moreover, $z_f, J^{m-\alpha}z_f \in C^\infty(\mathbb{R};\mathcal{Z})$.

Proof. We have $f \in L_2(\mathbb{R};\mathcal{Z})$, $Z_\alpha, D^kJ^{m-\alpha}Z_\alpha \in L_1(\mathbb{R};\mathcal{L}(\mathcal{Z}))$ for $k = 0,1,\dots,m$ by Lemma 2; hence, the convolutions $z_f = Z_\alpha * f$, $D^kJ^{m-\alpha}z_f = D^kJ^{m-\alpha}Z_\alpha * f$, $k = 0,1,\dots,m$, belong to $L_2(\mathbb{R};\mathcal{Z})$,

$$\mathcal{F}z_f(\omega) = \mathcal{F}Z_\alpha(\omega)\mathcal{F}f(\omega) = ((-i\omega)^\alpha - A)^{-1}\mathcal{F}f(\omega) \in L_2(\mathbb{R};\mathcal{Z}).$$

Therefore, $((-i\omega)^\alpha - A)\mathcal{F}z_f(\omega) = \mathcal{F}(D^\alpha z_f - Az_f)(\omega) = \mathcal{F}f(\omega) \in L_2(\mathbb{R};\mathcal{Z})$. Consequently, by the inverse Fourier transform, we obtain equality (9).

As in the proof of Theorem 2, we can show that $z_f \in C^\infty(\mathbb{R};\mathcal{Z})$. Analogously for $k \in \mathbb{N}_0$ and some $\xi \in [0,1]$ due to (8), we have

$$\|D^kJ^{m-\alpha}z_f(t+\delta) - D^kJ^{m-\alpha}z_f(t)\|_{\mathcal{Z}} \le$$

$$\le \int_{\mathbb{R}} \|D^kJ^{m-\alpha}Z_\alpha(t+\delta-s) - D^kJ^{m-\alpha}Z_\alpha(t-s)\|_{\mathcal{L}(\mathcal{Z})}\|f(s)\|_{\mathcal{Z}}ds \le$$

$$\le |\delta| \int_{\mathbb{R}} \|D^{k+1}J^{m-\alpha}Z_\alpha(t+\xi-s)\|_{\mathcal{L}(\mathcal{Z})}\|f(s)\|_{\mathcal{Z}}ds \le$$

$$\le |\delta|C_{k+1}\left(\int_{\mathbb{R}} e^{-2\varepsilon|t+\xi-s|}ds\right)^{1/2}\|f\|_{L_2(\mathbb{R};\mathcal{Z})} \to 0, \quad \delta \to 0.$$

Hence, $J^{m-\alpha}z_f \in C^\infty(\mathbb{R};\mathcal{Z})$.

Theorem 3 implies the uniqueness of a solution. □

4. Degenerate Evolution Equation of Fractional Order

Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces; denote by $\mathcal{L}(\mathcal{X};\mathcal{Y})$ the Banach space of all linear continuous operators from $\mathcal{X}$ into $\mathcal{Y}$ and by $\mathcal{C}l(\mathcal{X};\mathcal{Y})$ the set of all linear closed operators with dense domains in $\mathcal{X}$ acting into $\mathcal{Y}$. For $L \in \mathcal{L}(\mathcal{X};\mathcal{Y})$, $M \in \mathcal{C}l(\mathcal{X};\mathcal{Y})$, we also use the denotation $\rho^L(M)$ for the set of all $\mu \in \mathbb{C}$, such that the mapping $\mu L - M : D_M \to \mathcal{Y}$ is injective and $(\mu L - M)^{-1} \in \mathcal{L}(\mathcal{Y};\mathcal{X})$.

Let $m - 1 < \alpha \le m \in \mathbb{N}$. For $g \in C(\mathbb{R};\mathcal{Y})$, consider the linear equation

$$D^\alpha Lx(t) = Mx(t) + g(t), \quad t \in \mathbb{R}. \tag{11}$$

It is supposed that $\ker L \neq \{0\}$, therefore, Equation (11) is called a degenerate evolution equation.

A function $x : \mathbb{R} \to D_M$ is called a solution of Equation (11) if $Mx \in C(\mathbb{R};\mathcal{Y})$, $J^{m-\alpha}Lx \in C^m(\mathbb{R};\mathcal{Y})$, and for all $t \in \mathbb{R}$, equality (11) holds.

If there exists $a > 0$ such that $\{\lambda \in C : |\lambda| > a\} \subset \rho^L(M)$, then an operator M is called (L,σ)-bounded. In this case, there exist projectors

$$P := \frac{1}{2\pi i}\int_\gamma R^L_\mu(M)d\mu \in \mathcal{L}(\mathcal{X}), \quad Q := \frac{1}{2\pi i}\int_\gamma L^L_\mu(M)d\mu \in \mathcal{L}(\mathcal{Y}), \tag{12}$$

where $\gamma := \{\lambda \in \mathbb{C} : |\lambda| = r > a\}$. Hence, $\mathcal{X} = \mathcal{X}^0 \oplus \mathcal{X}^1$, $\mathcal{Y} = \mathcal{Y}^0 \oplus \mathcal{Y}^1$, where $\mathcal{X}^0 := \ker P$, $\mathcal{X}^1 := \mathrm{im} P$, $\mathcal{Y}^0 := \ker Q$, $\mathcal{Y}^1 := \mathrm{im} Q$. Denote $L_k := L|_{\mathcal{X}^k}$, $M_k := M|_{D_{M_k}}$, $D_{M_k} := D_M \cap \mathcal{X}^k$, $k = 0,1$.

Theorem 5 ([14], p. 90)**.** *Let an operator M be (L,σ)-bounded. Then, the following statements are satisfied:*

(i) $L_k \in \mathcal{L}(\mathcal{X}^k;\mathcal{Y}^k)$, $M_k \in \mathcal{Cl}(\mathcal{X}^k;\mathcal{Y}^k)$, $k = 0,1$;

(ii) *There exist the inverse operators* $L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1;\mathcal{X}^1)$, $M_0^{-1} \in \mathcal{L}(\mathcal{Y}^0;\mathcal{X}^0)$.

An (L,σ)-bounded operator M is called (L,p)-bounded for some $p \in \mathbb{N}_0$ if $H := M_0^{-1}L_0$ is nilpotent operator of the power p. Then, the operator $J = L_0M_0^{-1}$ is nilpotent of the power p also. Indeed, $J^{p+1} = M_0H^{p+1}M_0^{-1} = 0$.

Theorem 6. *Let $\alpha > 0$, $p \in \mathbb{N}_0$, an operator M be (L,p)-bounded,* $\mathrm{mes}(\{(-i\omega)^\alpha : \omega \in \mathbb{R}\} \cap \sigma^L(M)) = 0$*. Then, function $x \equiv 0$ is a unique solution of Equation (11).*

Proof. For a solution x of Equation (11) with $g \equiv 0$, take for some $T > 0$ $x_T(t) = x(t)$ for $|t| < T$, $x_T(t) = 0$ for $|t| \geq T$; hence, $x_T, Mx_T \in L_1(\mathbb{R};\mathcal{Z})$. The function x_T satisfies Equation (11) with $g \equiv 0$ excluding the points $t = T$ and $t = -T$, possibly. By the Fourier transform, we obtain the equality $(-i\omega)^\alpha L\mathcal{F}x_T(\omega) - M\mathcal{F}x_T(\omega) \equiv 0$. Since $\{(-i\omega)^\alpha : \omega \in \mathbb{R}\} \subset \rho^L(M)$ for almost all $\omega \in \mathbb{R}$, we have $\mathcal{F}x_T(\omega) = 0$ a. e. on $\mathbb{R}$; hence, $\mathcal{F}z_T(\omega) \equiv 0$. Thus, $x_T \equiv 0$ and $x \equiv 0$ since $T > 0$ is arbitrary. $\square$

Denote for $t \in \mathbb{R}$

$$X_\alpha(t) := \frac{1}{2\pi}\int_{\mathbb{R}}((-i\omega)^\alpha - L_1^{-1}M_1)^{-1}e^{-i\omega t}d\omega.$$

Theorem 7. *Let $\alpha > 0$, $p \in \mathbb{N}_0$, an operator M be (L,p)-bounded, $\{(-i\omega)^\alpha : \omega \in \mathbb{R}\} \cap \sigma^L(M) = \emptyset$, $g \in C(\mathbb{R};\mathcal{Y})$, $Qg \in L_2(\mathbb{R};\mathcal{Y})$, $(D^\alpha J)^k(I-Q)g \in C(\mathbb{R};\mathcal{X})$, $k = 0,1,\dots,p$. Then, function*

$$\int_{\mathbb{R}} X_\alpha(t-s)L_1^{-1}Qg(s)ds - \sum_{k=0}^{p}(D^\alpha H)^k M_0^{-1}(I-Q)g(t) \tag{13}$$

is a unique solution of Equation (11).

Proof. Set $x^0(t) := (I-P)x(t)$, $x^1(t) := Px(t)$. By Theorem 5 under the conditions of the present theorem, Equation (11) can be reduced to the system of equations

$$D^\alpha Hx^0(t) = x^0(t) + M_0^{-1}(I-Q)g(t), \quad t \in \mathbb{R}, \tag{14}$$

$$D^\alpha x^1(t) = L_1^{-1}M_1x^1(t) + L_1^{-1}Qg(t), \quad t \in \mathbb{R}, \tag{15}$$

with nilpotent operator $H \in \mathcal{L}(\mathcal{X}^0)$ and the operator $S \in \mathcal{L}(\mathcal{X}^1)$. Indeed, after applying the operator $M_0^{-1}(I-Q)$ to (11), we obtain Equation (14), and after applying $L_1^{-1}Q$ to Equation (11), we have (15). Here, we aslo use the equalities $LP = QL$, $MPv = QMv$ for $v \in D_M$, which follows immediately from the construction of the projectors P and Q [14].

Since

$$(\mu L - M)^{-1} = (\mu L_1 - M_1)^{-1}Q + (\mu L_0 - M_0)^{-1}(I-Q) =$$
$$= (\mu I - L_1^{-1}M_1)^{-1}L_1^{-1}Q + (\mu H - I)^{-1}M_0^{-1}(I-Q) =$$
$$= (\mu I - L_1^{-1}M_1)^{-1}L_1^{-1}Q - \sum_{k=0}^{p} \mu^k H^k M_0^{-1}(I-Q),$$

we have $\sigma^L(M) = \sigma(L_1^{-1}M_1)$. Therefore, by Theorem 4, there exists a unique solution of Equation (15).

Act on the right-hand side of (14) by $D^\alpha H$, and obtain

$$(D^\alpha H)^2 x^0(t) = D^\alpha H x^0(t) + D^\alpha H M_0^{-1}(I-Q)g(t) =$$
$$= x^0(t) + M_0^{-1}(I-Q)g(t) + D^\alpha H M_0^{-1}(I-Q)g(t).$$

Acting in a similar way, in the p-th step, we obtain

$$0 = (D^\alpha)^{p+1}H^{p+1}x^0(t) = (D^\alpha H)^{p+1}x^0(t) = x^0(t) + \sum_{k=0}^{p}(D^\alpha H)^k M_0^{-1}(I-Q)g(t).$$

Consequently,

$$x^0(t) = -\sum_{k=0}^{p}(D^\alpha H)^k M_0^{-1}(I-Q)g(t) = -M_0^{-1}\sum_{k=0}^{p}(D^\alpha J)^k(I-Q)g(t),$$

$$Mx^0(t) = -\sum_{k=0}^{p}(D^\alpha J)^k(I-Q)g(t), \quad D^\alpha L x^0(t) = -\sum_{k=1}^{p}(D^\alpha J)^k(I-Q)g(t)$$

since J is nilpotent of the power p. Thus, (13) is a solution of (11). □

5. Systems of Ordinary Differential Equations

Consider the system of ordinary differential equations

$$D^\alpha z(t) = Az(t) + f(t), \tag{16}$$

where $z(t) = (z_1(t), z_2(t), \ldots, z_n(t))^T$, $f(t) = (f_1(t), f_2(t), \ldots, f_n(t))^T$, A is an $(n \times n)$-matrix, which has no eigenvalues of the form $(-i\omega)^\alpha$ with $\omega \in \mathbb{R}$. We take $\mathcal{Z} = \mathbb{R}^n$, and by Theorem 4 in the case $f \in C(\mathbb{R};\mathbb{R}^n) \cap L_2(\mathbb{R};\mathbb{R}^n)$, we obtain the unique solution of system of Equations (16) on $\mathbb{R}$; it has the form (10).

For the system of ordinary differential equations

$$D^\alpha Lx(t) = Mx(t) + g(t), \tag{17}$$

with $(n \times n)$-matrices L and M, we take $\mathcal{X} = \mathcal{Y} = \mathbb{R}^n$. If $\det(\mu L - M) \not\equiv 0$ in $\mathbb{C}$, then the polynomial $\mu L - M$ has n roots in $\mathbb{C}$, taking into account their multiplicities. Then, operator M is (L,p)-bounded with some $p \in \{0, 1, \ldots, n-1\}$ (see [14]). Then, system (17) has a unique solution on $\mathbb{R}$ under the appropriate conditions on g per Theorem 7. It has the form (13).

6. Some Boundary Problems for Partial Differential Equations

Let $P_\varrho(\lambda) = \sum\limits_{j=0}^{\varrho} c_j\lambda^j$, $Q_\varrho(\lambda) = \sum\limits_{j=0}^{\varrho} d_j\lambda^j$, $c_j, d_j \in \mathbb{C}$, $j = 0, 1, \dots, \varrho \in \mathbb{N}_0$, $c_\varrho \neq 0$, $\Omega \subset \mathbb{R}^d$ be a bounded domain with a smooth boundary $\partial\Omega$,

$$(\Lambda u)(\xi) := \sum_{|q|\le 2r} a_q(\xi)\frac{\partial^{|q|}u(\xi)}{\partial\xi_1^{q_1}\partial\xi_2^{q_2}\dots\partial\xi_d^{q_d}}, \quad a_q \in C^\infty(\overline{\Omega}),$$

$$(B_l u)(\xi) := \sum_{|q|\le r_l} b_{lq}(\xi)\frac{\partial^{|q|}u(\xi)}{\partial\xi_1^{q_1}\partial\xi_2^{q_2}\dots\partial\xi_d^{q_d}}, \quad b_{lq} \in C^\infty(\partial\Omega),\ l = 1, 2, \dots, r,$$

where $q = (q_1, q_2, \dots, q_d) \in \mathbb{N}_0^d$, $|q| = q_1 + q_2 + \dots + q_d$ and operators pencil $\Lambda, B_1, \dots, B_r$ is regularly elliptic [16]. Let the operator $\Lambda_1 \in \mathcal{C}l(L_2(\Omega))$ with the domain $D_{\Lambda_1} = H^{2r}_{\{B_l\}}(\Omega) := \{v \in H^{2r}(\Omega) : B_l v(\xi) = 0,\ l = 1, 2, \dots, r,\ \xi \in \partial\Omega\}$ act as $\Lambda_1 u := \Lambda u$. Suppose that Λ_1 is a self-adjoint operator, then the spectrum $\sigma(\Lambda_1)$ of the operator Λ_1 is real and discrete and has finite multiplicities [16]. Let $\sigma(\Lambda_1)$ be bounded from the right and not contain zero, where $\{\varphi_k : k \in \mathbb{N}\}$ is an orthonormal in the $L_2(\Omega)$ system of operator Λ_1 eigenfunctions, numbered in the order of non-increase of the corresponding eigenvalues $\{\lambda_k : k \in \mathbb{N}\}$, taking into account their multiplicities.

Consider the boundary problem

$$B_l\Lambda^k u(\xi, t) = 0,\ k = 0, 1, \dots, \varrho - 1,\ l = 1, 2, \dots, r,\ (\xi, t) \in \partial\Omega \times \mathbb{R}, \tag{18}$$

$$D_t^\alpha P_\varrho(\Lambda)u(\xi, t) = Q_\varrho(\Lambda)u(\xi, t) + h(\xi, t),\ \ (\xi, t) \in \Omega \times \mathbb{R}, \tag{19}$$

where $\alpha > 0$, D_t^α is the Liouville fractional derivative with respect to the variable t; the function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is given.

Take spaces and operators

$$\mathcal{X} = \{v \in H^{2r\varrho}(\Omega) : B_l\Lambda^k v(s) = 0,\ k = 0, 1, \dots, \varrho - 1,\ l = 1, 2, \dots, r,\ \xi \in \partial\Omega\},$$

$$\mathcal{Y} = L_2(\Omega), \quad L = P_\varrho(\Lambda) \in \mathcal{L}(\mathcal{X}; \mathcal{Y}), \quad M = Q_\varrho(\Lambda) \in \mathcal{L}(\mathcal{X}; \mathcal{Y}).$$

Let $P_\varrho(\lambda_k) \neq 0$ for all $k \in \mathbb{N}$; then, there exists the inverse operator $L^{-1} \in \mathcal{L}(\mathcal{Y}; \mathcal{X})$, and problem (18), (19) may be reduced to Equation (9), where $\mathcal{Z} = \mathcal{X}$, $A = L^{-1}M \in \mathcal{L}(\mathcal{Z})$ has the spectrum $\sigma(L^{-1}M) = \{Q_\varrho(\lambda_k)/P_\varrho(\lambda_k) : k \in \mathbb{N}\}$, $f(t) = L^{-1}h(\cdot, t)$. By Theorem 4, we obtain the following unique solvability theorem.

Theorem 8. *Let $\alpha > 0$, $P_\varrho(\lambda_k) \neq 0$ for all $k \in \mathbb{N}$, $\{(-i\omega)^\alpha : \omega \in \mathbb{R}\} \cap \{Q_\varrho(\lambda_k)/P_\varrho(\lambda_k) : k \in \mathbb{N}\} = \varnothing$, $h \in C(\mathbb{R}; L_2(\Omega)) \cap L_2(\mathbb{R}; L_2(\Omega))$. Then, there exists a unique solution of problem* (18), (19). *It has the form*

$$u(\xi, t) = \frac{1}{2\pi}\int\limits_{\mathbb{R}}\sum_{k=1}^{\infty}\int\limits_{\mathbb{R}}\frac{e^{-i\omega(t-s)}d\omega}{(-i\omega)^\alpha P_\varrho(\lambda_k) - Q_\varrho(\lambda_k)}\langle h(\cdot, s), \varphi_k\rangle_{L_2(\Omega)}\varphi_k(\xi)ds.$$

Proof. Since $L^{-1} \in \mathcal{L}(\mathcal{Y}; \mathcal{X})$ and $h \in C(\mathbb{R}; L_2(\Omega)) \cap L_2(\mathbb{R}; L_2(\Omega))$, we have $L^{-1}h \in C(\mathbb{R}; \mathcal{X}) \cap L_2(\mathbb{R}; \mathcal{X})$. We also take into account here that

$$((-i\omega)^\alpha - L^{-1}M)^{-1}L^{-1}h(s) = \sum_{k=1}^{\infty}\frac{1}{(-i\omega)^\alpha - \frac{Q_\varrho(\lambda_k)}{P_\varrho(\lambda_k)}}\frac{\langle h(\cdot, s), \varphi_k\rangle_{L_2(\Omega)}}{P_\varrho(\lambda_k)}\varphi_k(\cdot).$$

□

For example, take $\varrho = 2$, $P_2(\lambda) = \lambda^2$, $Q_2(\lambda) = a_0 + a_1\lambda$, $d = 1$, $\Omega = (0,\pi)$, $r = 1$, $\Lambda u = \frac{\partial^2 u}{\partial \xi^2}$, $B_1 = I$. In this case, $\lambda_k = -k^2$, $\varphi_k(\xi) = \sin k\xi$, $k \in \mathbb{N}$, and problem (18), (19) have the form

$$D_t^\alpha \frac{\partial^4 u}{\partial \xi^4}(\xi,t) = a_0 u(\xi,t) + a_1 \frac{\partial^2 u}{\partial \xi^2}(\xi,t) + h(\xi,t), \quad (\xi,t) \in (0,\pi) \times \mathbb{R},$$

$$u(0,t) = u(\pi,t) = \frac{\partial^2 u}{\partial \xi^2}(0,t) = \frac{\partial^2 u}{\partial \xi^2}(\pi,t) = 0, \quad t \in \mathbb{R}.$$

Let $P_\varrho(\lambda_k) = 0$ for some $k \in \mathbb{N}$. If polynomials P_ϱ and Q_ϱ have no common roots among $\{\lambda_k\}$, then the operator M is $(L,0)$-bounded (see [17]), and projectors (12) are

$$P = \sum_{P_\varrho(\lambda_k) \neq 0} \langle \cdot, \varphi_k \rangle \varphi_k, \quad Q = \sum_{P_\varrho(\lambda_k) \neq 0} \langle \cdot, \varphi_k \rangle \varphi_k,$$

where $\langle \cdot, \varphi_k \rangle$ is the inner product in $L_2(\Omega)$. Then, problem (18), (19) may be presented as Equation (11) with the chosen above spaces $\mathcal{X}, \mathcal{Y}$ and operators L, M. If $\{(-i\omega)^\alpha : \omega \in \mathbb{R}\} \cap \{Q_\varrho(\lambda_k)/P_\varrho(\lambda_k) : P_\varrho(\lambda_k) \neq 0\} = \varnothing$ and the function $g = L^{-1}h$ satisfies the condition of Theorem 7, then by this theorem, there exists a unique solution of problem (18), (19).

Theorem 9. *Let $\alpha > 0$, $P_\varrho(\lambda_k) = 0$ for some $k \in \mathbb{N}$; polynomials P_ϱ and Q_ϱ have no common roots among $\{\lambda_k\}$, $\{(-i\omega)^\alpha : \omega \in \mathbb{R}\} \cap \{Q_\varrho(\lambda_k)/P_\varrho(\lambda_k) : P_\varrho(\lambda_k) \neq 0\} = \varnothing$, $h \in C(\mathbb{R}; L_2(\Omega)) \cap L_2(\mathbb{R}; L_2(\Omega))$. Then, there exists a unique solution of problem* (18), (19)*. It has the form*

$$u(\xi,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \sum_{P_\varrho(\lambda_k) \neq 0} \int_{\mathbb{R}} \frac{e^{-i\omega(t-s)} d\omega}{(-i\omega)^\alpha P_\varrho(\lambda_k) - Q_\varrho(\lambda_k)} \langle h(\cdot,s), \varphi_k \rangle_{L_2(\Omega)} \varphi_k(\xi) ds -$$

$$- \sum_{P_\varrho(\lambda_k) = 0} \frac{\langle h(\cdot,t), \varphi_k \rangle_{L_2(\Omega)}}{Q_\varrho(\lambda_k)} \varphi_k(\xi).$$

Proof. Here, $\sigma^L(M) = \{Q_\varrho(\lambda_k)/P_\varrho(\lambda_k) : P_\varrho(\lambda_k) \neq 0\}$ (see [17]),

$$((-i\omega)^\alpha - L_1^{-1}M_1)L_1^{-1}h(s) = \sum_{P_\varrho(\lambda_k) \neq 0} \frac{1}{(-i\omega)^\alpha - \frac{Q_\varrho(\lambda_k)}{P_\varrho(\lambda_k)}} \frac{\langle h(\cdot,s), \varphi_k \rangle_{L_2(\Omega)}}{P_\varrho(\lambda_k)} \varphi_k(\cdot).$$

By Theorem 7, we obtain the required assertion. □

Let $\varrho = 2$, $P_2(\lambda) \equiv \lambda(\lambda + 9)$, $Q_2(\lambda) = 1 + \lambda$, $d = 1$, $\Omega = (0,\pi)$, $r = 1$, $\Lambda u = \frac{\partial^2 u}{\partial \xi^2}$, $B_1 = I$. Then, degenerate problem (18), (19) have the form

$$D_t^\alpha \left(\frac{\partial^4 u}{\partial \xi^4} + 9 \frac{\partial^2 u}{\partial \xi^2} \right)(\xi,t) = \left(u + \frac{\partial^2 u}{\partial \xi^2} \right)(\xi,t), \quad (\xi,t) \in (0,\pi) \times \mathbb{R},$$

$$u(0,t) = u(\pi,t) = \frac{\partial^2 u}{\partial \xi^2}(0,t) = \frac{\partial^2 u}{\partial \xi^2}(\pi,t) = 0, \quad t \in \mathbb{R}.$$

Here, $P_2(0) = P_2(-9) = 0$, $0 \notin \sigma(\Lambda_1) = \{-k^2 : k \in \mathbb{N}\}$, $-9 = -3^2 \in \sigma(\Lambda_1)$; hence, $\mathcal{X}^0 = \mathcal{Y}^0 = \operatorname{span}\{\sin 3s\}$, $\mathcal{X}^1$ and $\mathcal{Y}^1$ are closures of $\operatorname{span}\{\sin ks : k \in \mathbb{N} \setminus \{3\}\}$ in the spaces $H^4(0,\pi)$ and $L_2(0,\pi)$ respectively.

7. Linearized Scott–Blair System

Consider a boundary problem for a linearized Scott–Blair system [18] of time-fractional order

$$v(\xi,t) = 0, \quad (\xi,t) \in \partial\Omega \times \mathbb{R}, \tag{20}$$

$$D_t^\alpha(1-\chi\Delta)v(\xi,t) = -(\tilde{v}\cdot\nabla)v(\xi,t) - (v\cdot\nabla)\tilde{v}(\xi,t) - r(\xi,t) = 0, \quad (\xi,t)\in\Omega\times\mathbb{R}, \quad (21)$$

$$\nabla\cdot v(\xi,t) = 0, \quad (\xi,t)\in\Omega\times\mathbb{R}, \quad (22)$$

with the Liouville fractional derivative D_t^α in t. Here, $\Omega\subset\mathbb{R}^d$ is a bounded domain with a smooth boundary $\partial\Omega$, and $\chi\in\mathbb{R}$, $\tilde{v}=(\tilde{v}_1,\tilde{v}_2,\dots,\tilde{v}_d)$ is a given vector function. Vector functions of the fluid velocity $v=(v_1,v_2,\dots,v_d)$ and its pressure gradient $r=(r_1,r_2,\dots,r_d)=\nabla p$ are unknown.

Denote $\mathbb{L}_2 := (L_2(\Omega))^d$, $\mathbb{H}^1 := (W_2^1(\Omega))^d$, $\mathbb{H}^2 := (W_2^2(\Omega))^d$. The closure of the subspace $\mathcal{L} := \{v\in(C_0^\infty(\Omega))^d : \nabla\cdot v = 0\}$ in the norm of $\mathbb{L}_2$ is denoted by $\mathbb{H}_\sigma$, and that in the norm of the space $\mathbb{H}^1$ is denotes by $\mathbb{H}_\sigma^1$.

We have the representation $\mathbb{L}_2 = \mathbb{H}_\sigma\oplus\mathbb{H}_\pi$, where $\mathbb{H}_\pi$ is the orthogonal complement $\mathbb{H}_\sigma$. Denote by $\Pi:\mathbb{L}_2\to\mathbb{H}_\pi$ the corresponding orthoprojector, $\Sigma = I-\Pi:\mathbb{L}_2\to\mathbb{H}_\sigma$, $\mathbb{H}_\sigma^2 = \mathbb{H}_\sigma^1\cap\mathbb{H}^2$.

Let the operator $A:=\Sigma\Delta$ have the domain $\mathbb{H}_\sigma^2$ in the space $\mathbb{H}_\sigma$ (see [19]). For $\tilde{v}\in\mathbb{H}^1$, define by the formula $Dw := -(\tilde{v}\cdot\nabla)w - (w\cdot\nabla)\tilde{v}$ an operator $D\in\mathcal{L}(\mathbb{H}_\sigma^2;\mathbb{L}_2)$. Let

$$\mathcal{X} = \mathbb{H}_\sigma^2\times\mathbb{H}_\pi, \quad \mathcal{Y} = \mathbb{L}_2 = \mathbb{H}_\sigma\times\mathbb{H}_\pi, \quad (23)$$

$$L = \begin{pmatrix} I-\chi A & \mathbb{O} \\ -\chi\Pi\Delta & \mathbb{O}\end{pmatrix}\in\mathcal{L}(\mathcal{X};\mathcal{Y}), \quad M = \begin{pmatrix}\Sigma D & \mathbb{O} \\ \Pi D & -I\end{pmatrix}\in\mathcal{L}(\mathcal{X};\mathcal{Y}). \quad (24)$$

We take into account that $r(\cdot,t) = \nabla p(\cdot,t)\in\mathbb{H}_\pi = \{\nabla\varphi : \varphi\in W_2^1(\Omega)\}$ for $t>0$. By the inclusion $v(\cdot,t)\in\mathbb{H}_\sigma^2$ for $t>0$, we take into account Equation (22) and boundary condition (20).

Lemma 3 ([20])**.** *Let $\chi\neq 0$, $\chi^{-1}\notin\sigma(A)$, the spaces $\mathcal{X}$, $\mathcal{Y}$ and the operators L, M be defined by Formulas (23) and (24), respectively. Then, the operator M is $(L,0)$-bounded, and projectors have the form*

$$P = \begin{pmatrix} I & \mathbb{O} \\ \chi\Pi\Delta(I-\chi A)^{-1}\Sigma D + \Pi D & \mathbb{O}\end{pmatrix}, \quad Q = \begin{pmatrix} I & \mathbb{O} \\ -\chi\Pi\Delta(I-\chi A)^{-1} & \mathbb{O}\end{pmatrix}.$$

The form of P and Q implies that

$$\mathcal{X}^0 = \{0\}\times\mathbb{H}_\pi, \quad \mathcal{X}^1 = \{(w_1,w_2)\in\mathbb{H}_\sigma^2\times\mathbb{H}_\pi : w_2 = (\chi\Pi\Delta(I-\chi A)^{-1}\Sigma D + \Pi D)w_1\},$$

$$\mathcal{Y}^0 = \{0\}\times\mathbb{H}_\pi, \quad \mathcal{Y}^1 = \{(w_1,w_2)\in\mathbb{H}_\sigma\times\mathbb{H}_\pi : w_2 = -\chi\Pi\Delta(I-\chi A)^{-1}w_1\}.$$

Therefore, the subspaces $\mathcal{X}^1$ and $\mathcal{Y}^1$ are isomorphic to $\mathbb{H}_\sigma^2$ and $\mathbb{H}_\sigma$, respectively.

Theorem 10. *Let $\alpha>0$, $\chi\neq 0$, $\chi^{-1}\notin\sigma(A)$, $\{(-i\omega)^\alpha : \omega\in\mathbb{R}\}\cap\sigma(\Sigma D(I-\chi A)^{-1}) = \varnothing$. Then, there exists a unique solution of problem (20)–(22).*

Proof. Consider problem (20)–(22) as Equation (11) with spaces $\mathcal{X}$, $\mathcal{Y}$ defined by (23) and operators L, M of form (24). In [20], it is shown also that $\sigma^L(M) = \sigma(\Sigma D(I-\chi A)^{-1})$. Thus, by Theorem 7, we obtain this assertion. □

8. Conclusions

The unique solvability of fractional differential equations in Banach spaces on $\mathbb{R}$ without initial conditions is studied in this work. We consider them to be solved with respect to the Liouville fractional derivative equations with a bounded operator at the unknown function and equations with a degenerate linear operator at this derivative and with a spectrally bounded pair of operators. General results are used for the investigation of concrete differential equations and systems of equations, ordinary and with partial derivatives.

The boundedness of the operator A (and spectral boundedness of the operator pair (L,M)) is an essential limitation in the considered problems. But, the results obtained will give us ideas on how to proceed in the study of more general classes of equations of this form with unbounded linear operators.

A separate topic of research is the development of methods for the numerical solution of equations on a real line without initial conditions. The results obtained in this work allow us to reduce the problem of a numerical solution search to the construction of numerical approximations for functions (10) and (13).

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