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Asymptotic and Oscillatory Properties of Third-Order Differential Equations with Multiple Delays in the Noncanonical Case

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Abstract: This paper investigates the asymptotic and oscillatory properties of a distinctive class of third-order linear differential equations characterized by multiple delays in a noncanonical case. Employing the comparative method and the Riccati method, we introduce the novel and rigorous criteria to discern whether the solutions of the examined equation exhibit oscillatory behavior or tend toward zero. Our study contributes to the existing literature by presenting theories that extend and refine the understanding of these properties in the specified context. To validate our findings and demonstrate their applicability in a general setting, we offer two illustrative examples, affirming the robustness and validity of our proposed criteria.

Keywords: delay differential equations; asymptotic and oscillatory properties; third-order; noncanonical case

MSC: 34C10; 34K11

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1. Introduction

In this paper, our focus is on investigating the oscillatory characteristics exhibited by solutions to a linear third-order delay differential equation (DDE), given by the form

$$\left(r_2(s)\left(r_1(s)x'(s)\right)'\right)' + \sum_{i=1}^{n} q_i(s)x(\tau_i(s)) = 0,\ s \geq s_0, \tag{1}$$

where

Hypothesis 1. $r_1, r_2 \in C([s_0,\infty),\mathbb{R})$,

$$\int_{s_0}^{\infty} \frac{1}{r_1(\theta)}\mathrm{d}\theta < \infty \textit{ and } \int_{s_0}^{\infty} \frac{1}{r_2(\theta)}\mathrm{d}\theta < \infty; \tag{2}$$

Hypothesis 2. $q_i \in C([s_0,\infty), [0,\infty))$, $q_i(s) \geq 0$, $q_i(s)$ *does not vanish identically;*

Hypothesis 3. $\tau_i \in C^1([s_0,\infty),\mathbb{R})$, $\tau_i(s) \leq s$, *and* $\lim_{s\to\infty} \tau_i(s) = \infty$, $i = 1, 2, \ldots, n$.

We define the operators for the sake of clarity and brevity:

$$L_0 x = x,\ L_1 x = r_1 x',\ L_2 x = r_2 (r_1 x')',\ L_3 x = \left(r_2 (r_1 x')'\right)' \text{ on } [s_0, \infty).$$

A nontrivial function $x \in C^1([s_x, \infty), \mathbb{R})$, $s_x \geqslant s_0$, is said to be a solution of (1) which has the property $L_1 x, L_2 x \in C^1[s_x, \infty)$, and it satisfies (1) on $x \in [s_x, \infty)$. We consider only those solutions x of (1) which exist on some half-line $[s_x, \infty)$ and satisfy the condition

$$\sup\{|x(s)| : s \geqslant S\} > 0, \text{ for all } S \geq s_x.$$

A solution $x(s)$ of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory. The equation itself is termed oscillatory if all its solutions oscillate.

Differential equations form the backbone of mathematical modeling, offering a powerful framework to describe the behavior of various dynamic systems across diverse fields. These equations express relationships between a function and its derivatives, allowing for the exploration of how a system evolves over time. Their applications span physics, engineering, biology, economics, and more, making them an indispensable tool for understanding and predicting the behavior of complex phenomena, see [1–7].

In the realm of dynamic systems, delay differential equations of the third order introduce an additional layer of complexity by incorporating time delays into the modeling process. Unlike ordinary differential equations, these equations account for the influence of both current and past values of variables. The consideration of third-order delays enhances the ability to capture intricate temporal dependencies, providing a more accurate representation of systems exhibiting memory effects. The exploration of such equations is vital for unraveling the dynamics of real-world phenomena characterized by delayed responses, see [8–14].

Oscillatory theorems play a pivotal role in understanding the inherent vibrational patterns within dynamic systems. Investigating the oscillatory behavior of solutions to differential equations provides valuable insights into the stability and periodicity of the systems under consideration. Such theorems are essential in predicting and controlling oscillations, making them a cornerstone in the analysis of dynamic systems, see [15–20].

Although even-order delay differential equations have been more extensively investigated than their odd-order counterparts, the overall exploration of DDEs has experienced a notable surge in interest in recent years. For those interested, a wealth of literature exists, with significant contributions from researchers such as Baculikova et al. [21–23], Dzurina et al. [24,25], Chatzarakis et al. [26,27], Moaaz [28–32], Masood et al. [33,34], Alrashdi et al. [35], El-Gaber [36], and Hassan et al. [37,38]. Further details and additional references can be found in the works mentioned above, providing a robust foundation for delving into the expanding realm of DDE studies.

Hartman and Wintner [39], and Erbe [40] investigated a specific instance of (1), specifically, the third-order differential equation

$$x''' + q(s)x(\tau(s)) = 0.$$

Saker and Dzurina [41], Grace et al. [42], Baculíková and Džurina [22] explored the oscillatory behavior of

$$\left(r(s)\left(x''(s)\right)^{\alpha}\right)' + q(s)x^{\alpha}(\tau(s)) = 0$$

under the conditions

$$\int_{s_0}^{\infty} \frac{1}{r^{1/\alpha}(\theta)} \mathrm{d}\theta = \infty \text{ and } \int_{s_0}^{\infty} \frac{1}{r^{1/\alpha}(\theta)} \mathrm{d}\theta < \infty.$$

Jadlovska et al. [43] and Chatzarakis et al. [26] delved into a specific case of (1), a third-order delay differential equation

$$\left(r_2(s)\left(r_1(s)x'(s)\right)'\right)' + q(s)x(\tau(s)) = 0,$$

in the canonical scenario where

$$\int_{s_0}^{\infty} \frac{1}{r_1(\theta)}\mathrm{d}\theta = \infty \text{ and } \int_{s_0}^{\infty} \frac{1}{r_2(\theta)}\mathrm{d}\theta = \infty.$$

They discussed the criteria ensuring that all solutions oscillate or tend to zero. Subsequently, Masood et al. [44] extended this study to encompass the third-order quasilinear delay differential equation

$$\left(r_2(s)\left(\left(r_1(s)x'(s)\right)'\right)^{\alpha}\right)' + q(s)x^{\alpha}(\tau(s)) = 0,$$

in the canonical case

$$\int_{s_0}^{\infty} \frac{1}{r_1^{\alpha}(\theta)}\mathrm{d}\theta = \infty \text{ and } \int_{s_0}^{\infty} \frac{1}{r_2^{\alpha}(\theta)}\mathrm{d}\theta = \infty.$$

This paper explores the asymptotic and oscillatory characteristics of solutions to a delayed differential Equation (1). Employing both the comparison method and the Riccati method, we establish criteria that reveal whether the solutions to the examined equation exhibit oscillatory behavior or converge to zero. Our approach extends the investigation conducted in the literature [45], which specifically examined (1) under the case $i = 1$.

2. Main Results

In this paper, we assume that the functional inequalities discussed hold for sufficiently large values of s. To simplify the study without losing the generality, we focus only on the positive solutions of (1). Our analysis begins by examining the potential structure of non-oscillatory solutions.

For convenience, we define the following notations:

$$\pi_1(s) := \int_{s}^{\infty} \frac{1}{r_1(\theta)}\mathrm{d}\theta,\ \pi_2(s) := \int_{s}^{\infty} \frac{1}{r_2(\theta)}\mathrm{d}\theta,$$

$$\tau(s) = \min\{\tau_i(s),\ i = 1, 2, ..., n\},$$

$$\widetilde{\tau}(s) = \max\{\tau_i(s),\ i = 1, 2, ..., n\},$$

$$Q(s) = \sum_{i=1}^{n} q_i(s).$$

Definition 1 ([46])**.** *We say that (1) has property A if any solution x of (1) is either oscillatory or satisfies* $\lim_{s\to\infty} x(s) = 0$.

Lemma 1 ([45])**.** *Suppose that x is an eventually positive solution of (1). Then there exists* $s_1 \in [s_0, \infty)$ *such that the variable x satisfies one of the following cases:*

$$\begin{array}{lllll}
(\mathrm{C}_1): & x(s) > 0, & L_1x(s) < 0, & L_2x(s) < 0, & L_3x(s) < 0, \\
(\mathrm{C}_2): & x(s) > 0, & L_1x(s) < 0, & L_2x(s) > 0, & L_3x(s) < 0, \\
(\mathrm{C}_3): & x(s) > 0, & L_1x(s) > 0, & L_2x(s) > 0, & L_3x(s) < 0, \\
(\mathrm{C}_4): & x(s) > 0, & L_1x(s) > 0, & L_2x(s) < 0, & L_3x(s) < 0,
\end{array}$$

for $s \geq s_0$.

Lemma 2. *If $x' > 0$, then (1) implies*

$$L_3x(s) + Q(s)x(\tau(s)) \leq 0. \tag{3}$$

Proof. Since $x' > 0$, then x is increasing. From (1) we obtain

$$\begin{aligned} L_3x(s) &= \Big(r_2(s)\big(r_1(s)x'(s)\big)'\Big)' = -\sum_{i=1}^{n} q_i(s)x(\tau_i(s)) \\ &\leq -x(\tau(s))\sum_{i=1}^{n} q_i(s) = -Q(s)x(\tau(s)). \end{aligned}$$

□

Lemma 3. *If $x' < 0$, then (1) implies*

$$L_3x(s) + Q(s)x(\widetilde{\tau}(s)) \leq 0. \tag{4}$$

Proof. Since $x' < 0$, then x is decreasing. From (1) we have

$$\begin{aligned} L_3x(s) &= \Big(r_2(s)\big(r_1(s)x'(s)\big)'\Big)' = -\sum_{i=1}^{n} q_i(s)x(\tau_i(s)) \\ &\leq -x(\widetilde{\tau}(s))\sum_{i=1}^{n} q_i(s) = -Q(s)x(\widetilde{\tau}(s)). \end{aligned}$$

□

Theorem 1. *If*

$$\int_{s_0}^{\infty} \frac{1}{r_1(v)} \int_{s_0}^{v} \frac{1}{r_2(u)} \int_{s_0}^{u} Q(\theta)\mathrm{d}\theta\mathrm{d}u\mathrm{d}v = \infty, \tag{5}$$

then (1) possesses property A.

Proof. Firstly, it is crucial to emphasize that when both (H_1) and (5) are satisfied, then

$$\int_{s_0}^{\infty} \frac{1}{r_2(u)} \int_{s_0}^{u} Q(\theta)\mathrm{d}\theta\mathrm{d}u = \int_{s_0}^{\infty} Q(\theta)\mathrm{d}\theta = \infty. \tag{6}$$

Now, assume that for the sake of contradiction, that x is a nonoscillatory solution of (1) on $[s_0, \infty)$. Without loss of generality, we can choose $s_1 \geq s_0$ such that $x(s) > 0$ and $x(\tau_i(s)) > 0$ for $s \geq s_1$. According to Lemma 1 , there are four possible cases for $s \geq s_1$, and we will analyze each of these cases separately.

Suppose that (C_1) holds. In this scenario, due to $L_1x(s) < 0$, we observe that x is decreasing, that is, implying the existence of a finite constant $c \geq 0$ such that $\lim_{s\to\infty} x(s) = c$.

We claim that $c = 0$. Assuming the contrary, $c > 0$ would imply the existence of $s_2 \geq s_1$ such that $x(\tau_i(s)) \geq c$ for $s \geq s_2$, $i = 1, 2, ..., n$. Thus,

$$-L_3x(s) = \sum_{i=1}^{n} q_i(s)x(\tau_i(s)) \geq c\sum_{i=1}^{n} q_i(s) = cQ(s), \tag{7}$$

for $s \geq s_2$. Integrating (7) from s_2 to s, we find

$$\begin{aligned} -L_2x(s) &\geq -L_2x(s_2) + c\int_{s_2}^{s} Q(\theta)\mathrm{d}\theta \\ &\geq c\int_{s_2}^{s} Q(\theta)\mathrm{d}\theta. \end{aligned}$$

Hence,

$$-(L_1x)'(s) \geq \frac{c}{r_2(s)} \int_{s_2}^{s} Q(\theta)\mathrm{d}\theta. \tag{8}$$

Integrating (8) once more from s_2 to s, we obtain

$$\begin{aligned} -L_1x(s) &\geq -L_1x(s_2) + c\int_{s_2}^{s} \frac{1}{r_2(u)} \int_{s_2}^{u} Q(\theta)\mathrm{d}\theta\mathrm{d}u \\ &\geq c\int_{s_2}^{s} \frac{1}{r_2(u)} \int_{s_2}^{u} Q(\theta)\mathrm{d}\theta\mathrm{d}u. \end{aligned}$$

Hence,

$$-x'(s) \geq \frac{c}{r_1(s)} \int_{s_2}^{s} \frac{1}{r_2(u)} \int_{s_2}^{u} Q(\theta)\mathrm{d}\theta\mathrm{d}u \tag{9}$$

Integrating (9) from s_2 to s the final time and considering (5) into account, we obtain

$$x(s) \leq x(s_2) - c\int_{s_2}^{s} \frac{1}{r_1(v)} \int_{s_2}^{v} \frac{1}{r_2(u)} \int_{s_2}^{u} Q(\theta)\mathrm{d}\theta\mathrm{d}u\mathrm{d}v \to -\infty \text{ as } s \to \infty.$$

This contradicts the positivity of x. Therefore, we conclude that $\lim_{s\to\infty} x(s) = 0$.

Assume that (C_2) holds. We follow a similar procedure as in (C_1), to arrive at (7). Integrating (7) from s_2 to s, we observe that

$$L_2x(s) \leq L_2x(s_2) - \int_{s_2}^{s} Q(\theta)\mathrm{d}\theta \to -\infty \text{ as } s \to \infty, \tag{10}$$

where we utilized (6). This contradicts the positivity of $L_2x(s)$, and consequently, we conclude that $\lim_{s\to\infty} x(s) = 0$.

Assume that (C_3) holds. We define a function

$$w(s) := \frac{L_2x(s)}{x(\tau(s))},\ s \geq s_1.$$

Certainly, w is positive for $s \geq s_1$. According to (3), we find

$$\begin{aligned} w'(s) &= \frac{L_3x(s)}{x(\tau(s))} - \frac{L_2x(s)x'(\tau(s))\tau'(s)}{x^2(\tau(s))} \leq \frac{L_3x(s)}{x(\tau(s))} \\ &\leq \frac{-Q(s)x(\tau(s))}{x(\tau(s))} = -Q(s). \end{aligned}$$

Integrating the above inequality from s_1 to s and considering (7) into account, we have

$$w(s) \leq w(s_2) - \int_{s_1}^{s} Q(\theta)\mathrm{d}\theta \to -\infty \text{ as } s \to \infty,$$

which leads to a contradiction.

Assume that (C_4) holds. Since x is increasing, integration (3) from s_1 to s yields

$$\begin{aligned} -L_2x(s) &\geq -L_2x(s_1) + \int_{s_1}^{s} Q(\theta)x(\tau(\theta))\mathrm{d}\theta \\ &\geq x(\tau(s_1))\int_{s_1}^{s} Q(\theta)\mathrm{d}\theta. \end{aligned}$$

This leads to

$$-(L_1x)'(s) \geq \frac{k}{r_2(s)} \int_{s_1}^{s} Q(\theta)\mathrm{d}\theta,\ \text{where } k = x(\tau(s_1)). \tag{11}$$

Integrating (11) from s_1 to s and using (7), we have

$$L_1x(s) \leq L_1x(s_1) - k\int_{s_1}^{s} \frac{1}{r_2(u)} \int_{s_1}^{u} Q(\theta)\mathrm{d}\theta\mathrm{d}u \to -\infty \text{ as } s \to \infty, \tag{12}$$

This leads to a contradiction, completing the proof. □

Remark 1. *It is clear that any nonoscillatory solution mentioned in Theorem 1 satisfies either case* (C_1) *or* (C_2) *as stated in Lemma 1.*

In the subsequent result, we present more robust supplementary details regarding the monotonic behavior of solutions that adhere to (C_2).

Lemma 4. *Consider x satisfying* (C_2) *as described in Lemma 1 on the interval* $[s_1, \infty)$ *for some* $s_1 \geq s_0$. *Define the function*

$$\pi(s) := \int_{s}^{\infty} \frac{\pi_2(\theta)}{r_1(\theta)}\mathrm{d}\theta. \tag{13}$$

If the condition

$$\int_{s}^{\infty} Q(\theta)\pi(\widetilde{\tau}(\theta))\mathrm{d}\theta = \infty, \tag{14}$$

is satisfied, then there exists $s_2 \geq s_1$ *such that*

$$\frac{x(s)}{\pi(s)} \downarrow 0, \tag{15}$$

for $s \geq s_2$.

Proof. Assume that x satisfies (C_2) as stated in Lemma 1 on the interval $[s_1, \infty)$ for some $s_1 \geq s_0$. Firstly, we demonstrate that (11) implies

$$\lim_{s\to\infty} \frac{x(s)}{\pi(s)} = 0. \tag{16}$$

By applying L'Hôpital's rule, we obtain

$$\lim_{s\to\infty} \frac{x(s)}{\pi(s)} = \lim_{s\to\infty} \frac{-L_1x(s)}{\pi_2(s)} = \lim_{s\to\infty} L_2x(s).$$

As $L_2x(s)$ is decreasing, there is a finite constant $c_1 \geq 0$ such that $\lim_{s\to\infty} L_2x(s) = c$. We claim that $c > 0$. If not, then $L_2x(s) > c$ and consequently, $x(s) \geq c\pi(s)$ eventually, say for $s \geq s_2$ with $s_2 \in [s_1, \infty)$. Substituting this relation into (4) , we deduce that

$$-L_3x(s) \geq Q(s)x(\widetilde{\tau}(s)) \geq c_1Q(s)\pi(\widetilde{\tau}(s)).$$

Integrating the above inequality from s_2 to s, we obtain

$$L_2x(s) \leq L_2x(s_2) - c\int_{s_2}^{s} Q(\theta)\pi(\widetilde{\tau}(\theta))\mathrm{d}\theta \to -\infty \text{ as } s \to \infty.$$

This contradiction implies that (16) holds and consequently

$$\lim_{s\to\infty} x(s) = \lim_{s\to\infty} L_1x(s) = 0, \tag{17}$$

due to the decreasing nature of $\pi(s)$ and $\pi_2(s)$, respectively. Using the monotonicity of $L_2x(s)$ alongside (17), we derive

$$\begin{aligned} -L_1x(s) &= L_1x(\infty) - L_1x(s) \\ &= \int_s^\infty \frac{1}{r_2(\theta)} L_2x(\theta)\mathrm{d}\theta \\ &\le \pi_2(s)L_2x(s), \end{aligned}$$

which implies,

$$\left(\frac{L_1x(s)}{\pi_2(s)}\right)' = \frac{\pi_2(s)L_2x(s) + L_1x(s)}{r_2(s)\pi_2^2(s)} \ge 0.$$

Thus, $L_1x(s)/\pi_2(s)$ is increasing on $[s_3, \infty)$. Combining this information with (17) leads to

$$\begin{aligned} x(s) &= x(s) - x(\infty) \\ &= -\int_s^\infty \frac{\pi_2(\theta)}{r_1(\theta)} \frac{L_1x(\theta)}{\pi_2(\theta)}\mathrm{d}\theta \\ &\le -\frac{L_1x(s)}{\pi_2(s)}\pi(s). \end{aligned}$$

Therefore

$$\left(\frac{x(s)}{\pi(s)}\right)' = \frac{\pi(s)L_1x(s) + \pi_2(s)x(s)}{r_1(s)\pi^2(s)} \le 0,$$

and we conclude that $x(s)/\pi(s)$ is monotonically decreasing. This, along with (16), implies (15), completing the proof. □

Corollary 1. *Consider x satisfying (C_2) as described in Lemma 1 on the interval $[s_1, \infty)$ for some $s_1 \ge s_0$. Define the function $\pi(s)$ as given by (13). If (14) is satisfied, then there exists $s_2 \ge s_1$ such that*

$$x(s) \le c\pi(s),$$

for every constant $c > 0$ and $s \ge s_2$.

Theorem 2. *If*

$$\liminf_{s\to\infty} \int_{\widetilde{\tau}(s)}^s \frac{1}{r_1(v)} \int_{s_0}^v \frac{1}{r_2(u)} \int_{s_0}^u Q(\theta)\mathrm{d}\theta\mathrm{d}u\mathrm{d}v > \frac{1}{\mathrm{e}}, \tag{18}$$

and

$$\limsup_{s\to\infty} \int_{\widetilde{\tau}(s)}^s \frac{1}{r_1(v)} \int_v^s \frac{1}{r_2(u)} \int_u^s Q(\theta)\mathrm{d}\theta\mathrm{d}u\mathrm{d}v > 1, \tag{19}$$

then (1) is oscillatory.

Proof. Suppose for the sake of contradiction, that x is a nonoscillatory solution of (1) on $[s_0, \infty)$. Without the loss of generality, we can choose $s_1 \ge s_0$ such that $x(s) > 0$ and $x(\tau_i(s)) > 0$ for $s \ge s_1$. According to Lemma 1 , there are four possible cases for $s \ge s_1$, and and we will analyze each of these cases separately.

Assume that (C_1) holds. Integrating (4) from s_1 to s and using the fact that x is decreasing, we obtain

$$\begin{aligned} -L_2x(s) &\ge -L_2x(s_1) + \int_{s_1}^s Q(\theta)x(\widetilde{\tau}(\theta))\mathrm{d}\theta \\ &\ge x(\widetilde{\tau}(s))\int_{s_1}^s Q(\theta)\mathrm{d}\theta. \end{aligned} \tag{20}$$

This leads to

$$-(L_1x)'(s) \ge \frac{x(\widetilde{\tau}(s))}{r_2(s)} \int_{s_1}^s Q(\theta)\mathrm{d}\theta. \tag{21}$$

Integrating (21) from s_1 to s, we obtain

$$\begin{aligned} -L_1x(s) &\geq \int_{s_1}^{s} \frac{x(\widetilde{\tau}(u))}{r_2(u)} \int_{s_1}^{u} Q(\theta)\mathrm{d}\theta\mathrm{d}u \\ &\geq x(\widetilde{\tau}(s)) \int_{s_1}^{s} \frac{1}{r_2(u)} \int_{s_1}^{u} Q(\theta)\mathrm{d}\theta\mathrm{d}u. \end{aligned} \quad (22)$$

or

$$x'(s) + \left(\frac{1}{r_1(s)} \int_{s_1}^{s} \frac{1}{r_2(u)} \int_{s_1}^{u} Q(\theta)\mathrm{d}\theta\mathrm{d}u \right) x(\widetilde{\tau}(s)) \leq 0.$$

However, according to Theorem 2.1.1 in [15], condition (18) ensures that the above inequality does not have a positive solution, which contradicts our initial assumption.

Assume that (C_2) holds. Integrating (4) from u to s $(> u)$ and utilizing the monotonicity of x , we obtain

$$\begin{aligned} L_2x(u) &\geq L_2x(u) - L_2x(s) \\ &\geq \int_{u}^{s} Q(\theta)x(\widetilde{\tau}(\theta))\mathrm{d}\theta \\ &\geq x(\widetilde{\tau}(s)) \int_{u}^{s} Q(\theta)\mathrm{d}\theta. \end{aligned}$$

This leads to

$$(L_1x)'(u) \geq \frac{x(\widetilde{\tau}(s))}{r_2(s)} \int_{u}^{s} Q(\theta)\mathrm{d}\theta.$$

Iterating the integration process outlined above from u to s $(> u)$ twice, we derive

$$x(u) \geq x(\widetilde{\tau}(s)) \int_{u}^{s} \frac{1}{r_1(v)} \int_{v}^{s} \frac{1}{r_2(\varkappa)} \int_{\varkappa}^{s} Q(\theta)\mathrm{d}\theta\mathrm{d}\varkappa\mathrm{d}v. \quad (23)$$

Substituting $u = \widetilde{\tau}(s)$ in (23), we arrive at a contradiction with (19).

Lastly, by noting that (5) is necessary for the validity of (18), it follows immediately from Remark 1 that cases (C_3) and (C_4) are impossible. This concludes the proof. □

The next result is a straightforward consequence of Theorem 2 and Corollary 1. It is noteworthy that this result furnishes more robust information about solutions compared to property A.

Theorem 3. *If (14) and (18) are satisfied, then any positive solution of (1) satisfies (15) for every* $c > 0$ *when s is sufficiently large..*

In what follows, we present various results which can serve as alternatives to Theorem 2.

Theorem 4. *If*

$$\limsup_{s\to\infty} \pi_1(s) \int_{s_0}^{s} \frac{1}{r_2(u)} \int_{s_0}^{u} Q(\theta)\mathrm{d}\theta\mathrm{d}u > 1, \quad (24)$$

and (19) hold, then (1) is oscillatory.

Proof. Suppose for the sake of contradiction, that x is a nonoscillatory solution of (1) on $[s_0, \infty)$. Without loss of generality, we can choose $s_1 \geq s_0$ such that $x(s) > 0$ and $x(\tau_i(s)) > 0$ for $s \geq s_1$. According to Lemma 1, there are four possible cases for $s \geq s_1$.

Assume that (C_1) holds. Then

$$x(s) = x(\infty) - \int_{s}^{\infty} \frac{1}{r_1(\theta)} L_1x(\theta)\mathrm{d}\theta \geq -\pi_1(s)L_1x(s). \quad (25)$$

Employing the monotonicity of x and (25) in (22), we observe that

$$\begin{aligned} -L_1x(s) &\geq x(s)\int_{s_1}^{s}\frac{1}{r_2(u)}\int_{s_1}^{u}Q(\theta)\mathrm{d}\theta\mathrm{d}u \\ &\geq -\pi_1(s)L_1x(s)\int_{s_1}^{s}\frac{1}{r_2(u)}\int_{s_1}^{u}Q(\theta)\mathrm{d}\theta\mathrm{d}u. \end{aligned}$$

Taking lim sup on both sides of the above inequality, one obtains a contradiction with (24).

The proof of (C_2) follows a similar approach to that of Theorem 2. To establish the impossibility of (C_3) and (C_4), it is enough to note that (6) is necessary for the validity of (24). The rest of proof proceeds in the same manner as that of Theorem 1. The proof is complete. □

Theorem 5. *If (19) and*

$$\limsup_{s\to\infty}\int_{s_0}^{s}\left[\frac{\pi_1(\theta)}{r_2(\theta)}\int_{s_0}^{\theta}Q(u)\mathrm{d}u-\frac{1}{4r_1(\theta)\pi_1(\theta)}\right]\mathrm{d}\theta=\infty, \tag{26}$$

hold, then (1) is oscillatory.

Proof. Suppose for the sake of contradiction, that x is a nonoscillatory solution of (1) on $[s_0,\infty)$. Without loss of generality, we can choose $s_1 \geq s_0$ such that $x(s) > 0$ and $x(\tau_i(s)) > 0$ for $s \geq s_1$. According to Lemma 1, there are four possible cases for $s \geq s_1$.

Assume that (C_1) holds. Define the function

$$w(s)=\frac{L_1x(s)}{x(s)},\ s\geq s_1. \tag{27}$$

Clearly, $w < 0$ on $[s_0,\infty)$. Since $L_1x(s)$ is decreasing, we have

$$x(s)=x(l)-\int_{s}^{l}\frac{1}{r_1(\theta)}L_1x(\theta)\mathrm{d}\theta\geq -L_1x(s)\int_{s}^{l}\frac{1}{r_1(\theta)}\mathrm{d}\theta.$$

Putting $l\to\infty$ in the above inequality, we obtain

$$x(s)\geq -\pi_1(s)L_1x(s). \tag{28}$$

From this and the definition of w, it is easy to see that

$$-1\leq \pi_1(s)w(s)<0. \tag{29}$$

On the other hand, as in the proof of Theorem 2, we arrive at (20), which implies

$$\frac{L_2x(s)}{x(s)}\leq -\int_{s_1}^{s}Q(\theta)\mathrm{d}\theta, \tag{30}$$

Differentiating w and using (27) and (30), we have

$$\begin{aligned} w'(s) &= \frac{L_2x(s)}{r_2(s)x(s)}-\frac{x'(s)L_1x(s)}{x^2(s)} \\ &\leq \frac{-1}{r_2(s)}\int_{s_1}^{s}Q(\theta)\mathrm{d}\theta-\frac{1}{r_1(s)}w^2(s). \end{aligned} \tag{31}$$

Multiplying both sides of (31) by $\pi_1(s)$ and integrating the resulting inequality from s_1 to s, we have

$$\begin{aligned}
\pi_1(s)w(s) &\le \pi_1(s_1)w(s_1) - \int_{s_1}^{s} \frac{w(\theta)}{r_1(\theta)}\mathrm{d}\theta \\
&\quad - \int_{s_1}^{s} \frac{\pi_1(\theta)}{r_2(\theta)} \int_{s_1}^{\theta} Q(u)\mathrm{d}u\mathrm{d}\theta - \int_{s_1}^{s} \frac{\pi_1(\theta)}{r_1(\theta)} w^2(\theta)\mathrm{d}\theta \\
&= \pi_1(s_1)w(s_1) - \int_{s_1}^{s} \frac{\pi_1(\theta)}{r_2(\theta)} \int_{s_1}^{\theta} Q(u)\mathrm{d}u\mathrm{d}\theta \\
&\quad - \int_{s_1}^{s} \frac{\pi_1(\theta)}{r_1(\theta)} \left[\left(w(\theta) + \frac{1}{2\pi_1(\theta)} \right)^2 - \frac{1}{4\pi_1^2(\theta)} \right] \mathrm{d}\theta \\
&\le \pi_1(s_1)w(s_1) - \int_{s_1}^{s} \left[\frac{\pi_1(\theta)}{r_2(\theta)} \int_{s_1}^{\theta} Q(u)\mathrm{d}u - \frac{1}{4r_1(\theta)\pi_1(\theta)} \right] \mathrm{d}\theta.
\end{aligned}$$

However, in view of (26), this inequality contradicts (29).

Assume that (C_2) holds. As in the proof of Theorem 2, one arrives at contradiction with (19).

To show that (C_3) and (C_4) are impossible, it is sufficient to note that

$$\int_{s_0}^{s} \frac{\pi_1(\theta)}{r_2(\theta)} \int_{s_0}^{\theta} Q(u)\mathrm{d}u\mathrm{d}\theta = \infty \tag{32}$$

is necessary for the validity of (26). Furthermore, since $\pi_1(s)$ is decreasing due to (H_1), then (32) implies that the function

$$\int_{s_0}^{s} \frac{1}{r_2(\theta)} \int_{s_0}^{\theta} Q(u)\mathrm{d}u\mathrm{d}\theta,$$

is unbounded, and so (6) holds. The rest of proof proceeds in the same manner as that of Theorem 1. This completes the proof. □

Theorem 6. *Assume all conditions of Theorem 2 (Theorems 4 and 5) are met except for (19). If (14) and*

$$\limsup_{s\to\infty} \frac{1}{\pi(\tilde{\tau}(s))} \int_{\tilde{\tau}(s)}^{s} \frac{1}{r_1(v)} \int_{v}^{s} \frac{1}{r_2(u)} \int_{u}^{s} Q(\theta)\pi(\tilde{\tau}(\theta))\mathrm{d}\theta\mathrm{d}u\mathrm{d}v > 1, \tag{33}$$

hold, then (1) is oscillatory.

Proof. Suppose for the sake of contradiction that x is a nonoscillatory solution of (1) on $[s_0, \infty)$. Without the loss of generality, we may take $s_1 \ge s_0$ such that $x(s) > 0$ and $x(\tau_i(s)) > 0$ for $s \ge s_1$. By Lemma 1 , four possible cases may occur for $s \ge s_1$.

The proof of (C_1), (C_3) and (C_4) proceeds in the same manner as that in Theorem 2 (Theorems 4 and 5).

Now suppose that (C_2) holds. Integrating (4) from u to s $(> u)$ and using the fact the monotonicity of x/π , we have

$$\begin{aligned}
L_2x(u) &\ge L_2x(u) - L_2x(s) \ge \int_{u}^{s} Q(\theta)x(\tilde{\tau}(\theta))\mathrm{d}\theta \\
&\ge \int_{u}^{s} Q(\theta)\frac{x(\tilde{\tau}(\theta))}{\pi(\tilde{\tau}(\theta))}\pi(\tilde{\tau}(\theta))\mathrm{d}\theta \\
&\ge \frac{x(\tilde{\tau}(s))}{\pi(\tilde{\tau}(s))} \int_{u}^{s} Q(\theta)\pi(\tilde{\tau}(\theta))\mathrm{d}\theta
\end{aligned}$$

that is,

$$(L_1x)'(u) \ge \frac{x(\tilde{\tau}(s))}{\pi(\tilde{\tau}(s))} \frac{1}{r_2(s)} \int_{u}^{s} Q(\theta)\pi(\tilde{\tau}(\theta))\mathrm{d}\theta.$$

Repeating the above process of integration from u to s $(> u)$ twice, we obtain

$$x(u) \geq \frac{x(\widetilde{\tau}(s))}{\pi(\widetilde{\tau}(s))} \int_u^s \frac{1}{r_1(v)} \int_v^s \frac{1}{r_2(\varkappa)} \int_\varkappa^s Q(\theta)\pi(\widetilde{\tau}(\theta))\mathrm{d}\theta\mathrm{d}\varkappa\mathrm{d}v. \tag{34}$$

Setting $u = \widetilde{\tau}(s)$ in (23), we obtain a contradiction with (33). The proof is complete. □

Example 1. *Let us consider the third-order delay differential equation of Euler type*

$$\left(s^b\left(s^a x'(s)\right)'\right)' + \sum_{i=1}^n q_0 s^{a+b-3} x(\tau_i s) = 0,\ s \geq 1, \tag{35}$$

where $a > 1$, $b > 1$, $q_0 > 0$, and $\tau_i \in (0,1]$, $i = 1,2,\ldots,n$. By comparing (1) and (35), we observe that $r_1(s) = s^a$, $r_2(s) = s^b$, $q(s) = q_0 s^{a+b-3}$, $\tau_i(s) = \tau_i s$. It is straightforward to find that

$$\pi_1(s) = \frac{1}{(a-1)s^{a-1}},\ \pi_2(s) = \frac{1}{(b-1)s^{b-1}},\ \pi(s) = \frac{1}{(b-1)(a+b-2)s^{a+b-2}},$$

$$\tau(s) = \max\{\tau_i(s),\ i = 1,2,...,n\} = \tau s,$$

$$\widetilde{\tau}(s) = \min\{\tau_i(s),\ i = 1,2,...,n\} = \widetilde{\tau} s,$$

and

$$Q(s) = nq_0 s^{a+b-3}.$$

Now, condition (5) leads to

$$\begin{aligned}
\int_{s_0}^\infty \frac{1}{r_1(v)} \int_{s_0}^v \frac{1}{r_2(u)} \int_{s_0}^u Q(\theta)\mathrm{d}\theta\mathrm{d}u\mathrm{d}v &= \int_{s_0}^\infty \frac{1}{v^a} \int_{s_0}^v \frac{1}{u^b} \int_{s_0}^u nq_0\theta^{a+b-3}\mathrm{d}\theta\mathrm{d}u\mathrm{d}v \\
&= \frac{nq_0}{a+b-2} \int_{s_0}^\infty \frac{1}{v^a} \int_{s_0}^v u^{a-2}\mathrm{d}u\mathrm{d}v \\
&= \frac{nq_0}{(a+b-2)(a-1)} \int_{s_0}^\infty v^{-1}\mathrm{d}v \\
&= \frac{nq_0}{(a+b-2)(a-1)} \lim_{s\to\infty} \ln s = \infty.
\end{aligned}$$

Then by by Theorem 1, we conclude that (35) has property A.

On the other hand, condition (18) leads to

$$\begin{aligned}
&\liminf_{s\to\infty} \int_{\widetilde{\tau}(s)}^s \frac{1}{r_1(v)} \int_{s_0}^v \frac{1}{r_2(u)} \int_{s_0}^u Q(\theta)\mathrm{d}\theta\mathrm{d}u\mathrm{d}v \\
= &\liminf_{s\to\infty} \int_{\widetilde{\tau}s}^s \frac{1}{v^a} \int_{s_0}^v \frac{1}{u^b} \int_{s_0}^u nq_0\theta^{a+b-3}\mathrm{d}\theta\mathrm{d}u\mathrm{d}v \\
= &\frac{nq_0}{(a+b-2)(a-1)} \ln\frac{1}{\widetilde{\tau}},
\end{aligned}$$

which is satisfied when

$$q_0 > \frac{(a+b-2)(a-1)}{n \ln\frac{1}{\widetilde{\tau}}\mathrm{e}} 1, \tag{36}$$

and condition (19) leads to

$$\begin{aligned}
&\limsup_{s\to\infty}\int_{\tilde{\tau}(s)}^{s}\frac{1}{r_1(v)}\int_{v}^{s}\frac{1}{r_2(u)}\int_{u}^{s}Q(\theta)\mathrm{d}\theta\mathrm{d}u\mathrm{d}v \\
=\;&\limsup_{s\to\infty}\int_{\tilde{\tau}s}^{s}\frac{1}{v^a}\int_{v}^{s}\frac{1}{u^b}\int_{u}^{s}nq_0\theta^{a+b-3}\mathrm{d}\theta\mathrm{d}u\mathrm{d}v \\
=\;&\frac{nq_0}{a+b-2}\limsup_{s\to\infty}\int_{\tilde{\tau}s}^{s}\frac{1}{v^a}\int_{v}^{s}\frac{1}{u^b}\left(s^{a+b-2}-u^{a+b-2}\right)\mathrm{d}u\mathrm{d}v \\
=\;&\frac{nq_0}{a+b-2}\limsup_{s\to\infty}\int_{\tilde{\tau}s}^{s}\frac{1}{v^a}\int_{v}^{s}\left(\frac{s^{a+b-2}}{u^b}-u^{a-2}\right)\mathrm{d}u\mathrm{d}v \\
=\;&\frac{nq_0}{a+b-2}\limsup_{s\to\infty}\int_{\tilde{\tau}s}^{s}\frac{1}{v^a}\left(\left(\frac{1}{1-b}\left(s^{a-1}-\frac{s^{a+b-2}}{v^{b-1}}\right)\right)-\frac{1}{a-1}\left(s^{a-1}-v^{a-1}\right)\right)\mathrm{d}v \\
=\;&\frac{nq_0}{a+b-2}\limsup_{s\to\infty}\int_{\tilde{\tau}s}^{s}\left(\left(\frac{1}{1-b}\left(\frac{s^{a-1}}{v^a}-\frac{s^{a+b-2}}{v^{a+b-1}}\right)\right)-\frac{1}{a-1}\left(\frac{s^{a-1}}{v^a}-\frac{1}{v}\right)\right)\mathrm{d}v \\
=\;&\frac{nq_0}{(a+b-2)(a-1)}\left[\left(\frac{1}{a-1}+\frac{1}{b-1}\right)\left(1-\frac{1}{\tilde{\tau}^{a-1}}\right)+\ln\frac{1}{\tilde{\tau}}\right] \\
&+\frac{nq_0}{(1-b)(a+b-2)^2}\left(1-\frac{1}{\tilde{\tau}^{a+b-2}}\right)
\end{aligned}$$

which is satisfied when

$$\begin{aligned}
&\frac{nq_0}{(a-1)}\left[\left(\frac{1}{a-1}+\frac{1}{b-1}\right)\left(1-\frac{1}{\tilde{\tau}^{a-1}}\right)+\ln\frac{1}{\tilde{\tau}}\right] \\
&+\frac{nq_0}{(1-b)(a+b-2)}\left(1-\frac{1}{\tilde{\tau}^{a+b-2}}\right)>(a+b-2).
\end{aligned} \tag{37}$$

By Theorem 2, Equation (35) is oscillatory if both (36) and (37) hold.

Example 2. *Consider the specific instance of Equation (35), given by*

$$\left(s^2\left(s^2x'(s)\right)'\right)'+q_0s\left(x\left(\frac{1}{2}s\right)+x\left(\frac{1}{3}s\right)+x\left(\frac{1}{4}s\right)\right)=0,\ s\geq 1, \tag{38}$$

By comparing (1) and (38), we see that $r_1(s)=s^2$, $r_2(s)=s^2$, $q(s)=q_0s$. It is straightforward to find that

$$\pi_1(s)=\frac{1}{s},\ \pi_2(s)=\frac{1}{s},\ \pi(s)=\frac{1}{2s^2},$$
$$\tau(s)=\frac{1}{2}s,\ \tilde{\tau}(s)=\frac{1}{4}s,$$

and

$$Q(s)=3q_0s.$$

Now, condition (5) leads to

$$\begin{aligned}
\int_{s_0}^{\infty}\frac{1}{r_1(v)}\int_{s_0}^{v}\frac{1}{r_2(u)}\int_{s_0}^{u}Q(\theta)\mathrm{d}\theta\mathrm{d}u\mathrm{d}v &= \int_{s_0}^{\infty}\frac{1}{v}\int_{s_0}^{v}\frac{1}{u}\int_{s_0}^{u}3q_0\theta\mathrm{d}\theta\mathrm{d}u\mathrm{d}v \\
&= \frac{3q_0}{2}\int_{s_0}^{\infty}\frac{1}{v^2}\int_{s_0}^{v}\mathrm{d}u\mathrm{d}v \\
&= \frac{3q_0}{2}\int_{s_0}^{\infty}v^{-1}\mathrm{d}v \\
&= \frac{3q_0}{2}\lim_{s\to\infty}\ln s=\infty.
\end{aligned}$$

Then by by Theorem 1, we conclude that (38) has property A.

On the other hand, condition (18) leads to

$$\begin{aligned} & \liminf_{s\to\infty} \int_{\widetilde{\tau}(s)}^{s} \frac{1}{r_1(v)} \int_{s_0}^{v} \frac{1}{r_2(u)} \int_{s_0}^{u} Q(\theta) \mathrm{d}\theta \mathrm{d}u \mathrm{d}v \\ = & \liminf_{s\to\infty} \int_{\frac{1}{4}s}^{s} \frac{1}{v^2} \int_{s_0}^{v} \frac{1}{u^2} \int_{s_0}^{u} 3q_0 \theta \mathrm{d}\theta \mathrm{d}u \mathrm{d}v \\ = & \frac{3q_0}{2} \ln 4, \end{aligned}$$

which is satisfied when

$$q_0 > 0.176\,91, \tag{39}$$

and condition (19) is satisfied when

$$q_0 > 0.07018. \tag{40}$$

By Theorem 2, Equation (38) is oscillatory if both (39) and (40) hold.

3. Conclusions

This paper has presented a comprehensive investigation into the asymptotic and oscillatory properties of a certain type of third-order linear differential equation with multiple delays, in a noncanonical case. By applying the comparative method and the Riccati method, we have established new and stringent criteria to determine whether the solutions of the studied equation exhibit oscillatory behavior or approach zero. Our results not only enhance the understanding of this particular differential equation but also contribute to the broader literature on delay differential equations.

Furthermore, in light of future research directions, we recognize the potential for extending the scope of this study. Specifically, we propose exploring the application of the same techniques to establish criteria for determining the oscillatory or convergent behavior of solutions for half-linear neutral differential equations of the form

$$\left(r_2(s)\left(r_1(s)\left((x(s)+p(s)x(\sigma(s)))'\right)^{\alpha}\right)'\right)' + \sum_{i=1}^{n} q_i(s) x^{\alpha}(\tau_i(s)) = 0.$$

Such an expansion of our study could significantly enhance the applicability of the methods employed in this paper to a wider range of differential equations, thereby fostering continued advancement in the field.

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