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New Infinite Classes for Normal Trimagic Squares of Even Orders Using Row–Square Magic Rectangles

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Abstract: As matrix representations of magic labelings of related hypergraphs, magic squares and their various variants have been applied to many domains. Among various subclasses, trimagic squares have been investigated for over a hundred years. The existence problem of trimagic squares with singly even orders and orders $16n$ has been solved completely. However, very little is known about the existence of trimagic squares with other even orders, except for only three examples and three families. We constructed normal trimagic squares by using product constructions, row–square magic rectangles, and trimagic pairs of orthogonal diagonal Latin squares. We gave a new product construction: for positive integers p, q, and r having the same parity, other than $1, 2, 3$, or 6, if normal $p \times q$ and $r \times q$ row–square magic rectangles exist, then a normal trimagic square with order pqr exists. As its application, we constructed normal trimagic squares of orders $8q^3$ and $8pqr$ for all odd integers q not less than 7 and $p, r \in \{7, 11, 13, 17, 19, 23, 29, 31, 37\}$. Our construction can easily be extended to construct multimagic squares.

Keywords: trimagic square; row–square magic rectangle; trimagic pair; product construction; diagonal Latin square; hypergraph; magic labeling

MSC: 05B15; 05B30; 05C78

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1. Introduction

As the oldest known combinatorial designs, magic squares and their various variants have been applied to many fields, such as engineering technology, cryptography, physics, pedagogy, number theory, matrix algebra, and graph theory. Here are some specific examples. By using various magic squares, many researchers [1–4] proposed new physical repositioning techniques to maximize power output and enhance the solar photovoltaic array's performance. Kravchenko et al. [5] constructed synthesizing nonequidistant sparse antenna arrays based on magic squares providing a high degree of dilution and sufficiently small side radiation. Magic squares have been used to encrypt and decrypt images [6–8]. Hyodo and Kitabayashi [9] established the relationship between magic squares with the Dirac flavor neutrino mass matrix. In [10], to group students more appropriately to increase learning achievement, Peng et al. provided a new magic square-based heterogeneous grouping algorithm. In [11], by rearranging the rows of a special magic square of order $2n-3$, Sim and Wong showed that there exists an arrangement of m consecutive integers containing an $(n-1)$-term monotone arithmetic progression and avoiding an n-term monotone arithmetic progression. In [12], Nordgren investigated various matrix operations of special magic squares and obtained many new results. Magic squares and their many relatives can be viewed as matrix representations of magic labelings of related set systems or hypergraphs [13,14] and have been used to construct some labelings of graphs [15–19].

Let n and d be two positive integers, and let $I_n = \{0, 1, \ldots, n-1\}$. Let A be a d-dimensional matrix of side or order n consisting of integers with entries $a(x)$ or $a_{i_1,\ldots,i_d}$, where $x \in I_n^d$ or $i_1, \ldots, i_d \in I_n$. A row, line, or hyperedge of A is an n-tuple of positions or points $(i_1, \ldots, i_d)$ such that the number of identical coordinates for any two points is exactly $d-1$. A (space) diagonal is an n-tuple of points $\{(x, i_2, \ldots, i_d) \colon i_k = x \text{ or } n-1-x, 2 \le k \le d, x \in I_n\}$. Let $\mathbb{L}_n^d$ denote the set of dn^{d-1} rows and 2^{d-1} diagonals of any d-dimensional matrix of side n. Obviously, $(I_n^d, \mathbb{L}_n^d)$ is an n-uniform hypergraph or a set system. Similar to the definition for a magic labeling of a set system [13], we give the following definitions. A magic labeling of the hypergraph $(I_n^d, \mathbb{L}_n^d)$ is a labeling a of its points by integers such that every line has the same sum, that is, $\sum_{x \in L} a(x) = c$ for some constant c and $L \in \mathbb{L}_n^d$; we call the constant c the magic sum. Let $A = (a(x))$ for $x \in I_n^d$; then, the matrix A can be thought of as the label matrix of the above labeling a of the hypergraph $(I_n^d, \mathbb{L}_n^d)$. We call the matrix A a d-dimensional magic hypercube of side or order n. Other types of magic hypercubes can be defined similarly. A d-dimensional magic hypercube with side n is normal if it consists of n^d consecutive integers. Usually, a two-dimensional magic hypercube of order n is called a magic square of order n (MS(n)). The existence problem of normal magic hypercubes has been completely solved [20,21].

In this paper, we are interested in investigating multimagic squares. Let n, d, and t be positive integers. Let A and B be integer matrices with the same size and let $A \times B$ denote their Hadamard product, with (i, j) entry $a_{i,j} b_{i,j}$. Based on the reference [22], we write $A^{\times d} = (a_{i,j}^d)$; obviously, $A^{\times d}$ is the Hadamard product $A \times \cdots \times A$ (where the matrix A appears d times). Now, let A have order n; then, A is a t-multimagic square (MS(n, t)) if, for $d \in \{1, \ldots, t\}$, $A^{\times d}$ is a magic square. We denote by an NMS(n, t) a normal MS(n, t). Usually, we call an MS$(n, 2)$ a bimagic square and an MS$(n, 3)$ a trimagic square. Many researchers have done a lot of work on the existence and construction of normal multimagic squares [23–30].

We now review the research on normal trimagic squares with even orders. In 2007, Derksen et al. [22] proved that there is no NMS$(2n, 3)$ for every positive odd integer n. In 2023, Hu et al. [25] proved that an NMS$(16n, 3)$ exists for all positive integers n. However, very little is known about the existence of trimagic squares of doubly even orders not divisible by 16, except for the following three examples and three families. In 2017, based on known bimagic squares and trimagic squares of orders 12, 24, and 40, Li et al. [31] showed that there is an NMS$(mn, 3)$ for $m \in \{12, 24, 40\}$ and infinitely many odd positive integers n. Therefore, the existence problem of normal trimagic squares of even orders is at present far from being solved.

This paper aims to construct new infinite families for normal trimagic squares of even orders. Our construction tools include a known product construction (see Section 2, Lemma 2), two new product constructions (see Section 3, Theorem 1 and Lemma 9), trimagic pairs, and (quasi-normal) row–square magic rectangles (new notions, see Section 2 for the definitions).

The outline of the rest of this paper is as follows. Section 2 presents some related preliminaries, including notations, notions, and lemmas. In Section 3, we prove main results. We first show that one can construct an NMS$(n, 3)$ based on a trimagic pair of orthogonal diagonal Latin squares and construct an NMS$(pqr, 3)$ based on normal $p \times q$ and $r \times q$ row–square magic rectangles for appropriate positive integers p, q, and r. Furthermore, we prove several lemmas and theorems for the existence of quasi-normal even row–square magic rectangles and construct their several new infinite classes. Finally, we provide new families of normal trimagic squares of even orders. In Section 4, we give conclusions of the work.

2. Preliminaries

Let T be an n-set. For any $m \times n$ integer matrix A, let $|A| = (|a_{i,j}|)$, where $|a_{i,j}|$ denotes the absolute value of $a_{i,j}$; we denote by $\mathfrak{S}_A$ the entry-set of A and index the rows by I_m and columns by I_n. An $m \times m$ array A is a diagonal Latin square over the set T with order m (DLS(m)) if

$$\{a_{k,j} \colon j \in I_m\} = \{a_{i,k} \colon i \in I_m\} = \{a_{j,j} \colon j \in I_m\} = \{a_{j,n-1-j} \colon j \in I_m\} = T$$

for $k \in I_m$. Two DLS(m)s C and D are called orthogonal if $\{(c_{i,j}, d_{i,j})\colon i,j \in I_m\} = \mathfrak{S}_C \times \mathfrak{S}_D$. The following can be found in [32].

Lemma 1 ([32])**.** *Two orthogonal DLS(m)s exist if and only if $m \notin \{2,3,6\}$.*

Remark 1. *Let A be a DLS(m) over I_m; then, A is an MS(m,t) for every positive t. In fact, for $L \in \mathbb{L}_m^2$, by definition, we know that $\{a_{i,j}\colon (i,j) \in L\} = I_m$, thus, we obtain $\sum_{(i,j)\in L} a_{i,j}^t = \sum_{x\in I_m} x^t = \sum_{x=0}^{m-1} x^t$, which is independent of lines and, hence, is a constant.*

Let C and D be two orthogonal DLS(n)s over I_n. The pair (C,D) is a trimagic pair (TMP(n)) if $C \times D$, $C^{\times 2} \times D$, and $C \times D^{\times 2}$ are all magic squares.

Let A be an $m \times n$ integer matrix. We call A a row-magic rectangle if the row sums of A are all a constant. If A and $A^{\times 2}$ are both row-magic rectangles, then we call A a bimagic row-magic rectangle (BRMR(m,n)). A row-magic rectangle is called a magic rectangle if its transpose is also a row-magic rectangle. We call a BRMR(m,n) A a row–square magic rectangle (RSMR(m,n)) if A is a magic rectangle. Similarly, one can define a t-multimagic rectangle. An integer matrix is normal if it consists of consecutive integers. Similar to a bimagic square, we call a 2-multimagic rectangle a bimagic rectangle. A bimagic rectangle and its transpose are row–square magic rectangles, but not converse. For results of multimagic rectangles, the interested reader is referred to the references [33,34].

Let $J_{m\times n}$ be an $m \times n$ all ones matrix. For integers c and d with $c \le d$ and $c \equiv d \pmod 2$, let $[c,d]_2$ denote the set $\{c, c+2, \ldots, d-2, d\}$. A BRMR$(p,2q)$ A is odd-normal if $\mathfrak{S}_{|A|} = [1, 4pq-1]_2$. An $m \times n$ integer matrix A is quasi-normal if $\mathfrak{S}_A = [1-mn, mn-1]_2$. An odd-normal BRMR$(p,2q)$ A is balanced if A has an equal number of positive and negative numbers in each row and each row sum of $|A|$ is the same.

To construct new families of trimagic squares of even orders, we need the following lemmas.

Lemma 2 ([31])**.** *For positive integers m and t, if there exist an NMS$(2m, 2t+1)$ and an NMS$(n, 2t)$, then there exists an NMS$(2mn, 2t+1)$.*

Lemma 3 ([31])**.** *There exists an NMS$(n,3)$ if $n \in \{12, 24, 40\} \cup \{2^k\colon k > 3\}$.*

Lemma 4 ([29])**.** *There exists an NMS$(n,2)$ if $n \in \{2u\colon u \ge 4\} \cup \{pq\colon$ odd $p,q \ge 5\} \cup [9,63]_2$.*

Remark 2. *By Lemmas 2–4, there exists an NMS$(24q,3)$ for odd $q \ne 3$.*

3. Main Results

In this section, we firstly construct an NMS$(n,3)$ using a trimagic pair of orthogonal DLS(n)s, then, based on row–square magic rectangles, establish a new product construct theorem, finally, provide new families of trimagic squares of even orders by constructing new corresponding infinite classes of row–square magic rectangles.

3.1. Construction of an NMS$(m,3)$

In this section, we extend Construction 2.1 in [35] for a magic pair of orthogonal bimagic squares to a trimagic pair of orthogonal diagonal Latin squares. The following can be obtained by modifying the proof in [35].

Lemma 5. *If there exists a TMP(m), then there exists an NMS$(m,3)$.*

Proof. Let (C,D) be a TMP(m) and $F = mC + D$. By Remark 1, we know that C and D are MS$(m,3)$s. Next, we prove that F is an NMS$(m,3)$. By the proof of Construction 2.1 in [35], we know that F is an NMS$(m,2)$. Therefore, it suffices to show that $F^{\times 3}$ is an MS(m).

By hypothesis, $C^{\times 3}$, $D^{\times 3}$, $C^{\times 2} \times D$, and $C \times D^{\times 2}$ are MS(m)s. Let $S_{C^{\times 3}}$, $S_{D^{\times 3}}$, $S_{C^{\times 2}\times D}$, and $S_{C\times D^{\times 2}}$ be the magic sums of $C^{\times 3}$, $D^{\times 3}$, $C^{\times 2} \times D$, and $C \times D^{\times 2}$, respectively. For $L \in \mathbb{L}_m^2$, we obtain

$$\begin{aligned}\sum_{(u,v)\in L} f_{u,v}^3 &= \sum_{(u,v)\in L} (mc_{u,v} + d_{u,v})^3 \\ &= m^3 \sum_{(u,v)\in L} c_{u,v}^3 + 3m^2 \sum_{(u,v)\in L} c_{u,v}^2 d_{u,v} + 3m \sum_{(u,v)\in L} c_{u,v} d_{u,v}^2 + \sum_{(u,v)\in L} d_{u,v}^3 \\ &= m^3 S_{C^{\times 3}} + 3m^2 S_{C^{\times 2}\times D} + 3m S_{C\times D^{\times 2}} + S_{D^{\times 3}}.\end{aligned}$$

It follows that $F^{\times 3}$ is a magic square. □

3.2. A New Product Construction

Based on Lemma 5, to obtain an NMS$(pqr, 3)$, it suffices to find a TMP(pqr). In this section, we address this problem using row–square magic rectangles and orthogonal diagonal Latin squares.

Theorem 1. *Let p, q, and r be positive integers, other than $1, 2, 3$, or 6 having the same parity. If there exist a normal RSMR(p, q) and a normal RSMR(r, q), then there exists an NMS$(pqr, 3)$.*

Proof. Let $m = pqr$. By Lemma 1, we can suppose that A and $\bar{A}$ are a pair of orthogonal DLS(p)s over I_p, B and $\bar{B}$ are a pair of orthogonal DLS(q)s over I_q, and C and $\bar{C}$ are a pair of orthogonal DLS(r)s over I_r. We write $A = (a_{u,x})$, $\bar{A} = (\bar{a}_{u,x})$, $B = (b_{v,y})$, $\bar{B} = (\bar{b}_{v,y})$, $C = (c_{w,z})$, and $\bar{C} = (\bar{c}_{w,z})$, where $u, x \in I_p$, $v, y \in I_q$, and $w, z \in I_r$. Let F be an $r \times q$ normal row–square magic rectangle over I_{qr} and we write $F = (f_{k,l})$. Let H be a $p \times q$ normal row–square magic rectangle over I_{pq}, and we write $H = (h_{s,t})$. Let F_r, F_c, and $F_{r,2}$ denote the row-magic sum of F, column-magic sum of F, and row-magic sum of $F^{\times 2}$, respectively, and let H_r, H_c and $H_{r,2}$ denote the row-magic sum of H, column-magic sum of H, and row-magic sum of $H^{\times 2}$, respectively. Let S_a, $S_{a,2}$, S_c, and $S_{c,2}$ denote the magic sums of A, $A^{\times 2}$, $\bar{C}$, and $\bar{C}^{\times 2}$, respectively. Let

$$E = (e_{i,j}), \qquad G = (g_{i,j}),$$

where

$e_{i,j} = qra_{u,x} + f_{c_{w,z}, b_{v,y}}$,

$g_{i,j} = pq\bar{c}_{w,z} + h_{\bar{a}_{u,x}, \bar{b}_{v,y}}$,

$i = qru + rv + w, \quad j = qrx + ry + z, \quad u, x \in I_p, \quad v, y \in I_q, \quad w, z \in I_r, \quad i, j \in I_m.$

Next, we prove that (E, F) is a TMP(m).

(i) We prove that E and G are DLS(m)s. Noting that A, B, and C are all diagonal Latin squares, for $j \in I_m$, we have

$$\begin{aligned}\{e_{i,j} \colon i \in I_m\} &= \bigcup_{u\in I_p} \bigcup_{v\in I_q} \bigcup_{w\in I_r} (qra_{u,x} + f_{c_{w,z}, b_{v,y}}) = \bigcup_{u\in I_p} \bigcup_{v\in I_q} \bigcup_{w\in I_r} (qra_{u,x} + f_{w, b_{v,y}}) \\ &= \bigcup_{u\in I_p} \bigcup_{v\in I_q} \bigcup_{w\in I_r} (qra_{u,x} + f_{w,v}) = \bigcup_{u\in I_p} \bigcup_{v\in I_q} \bigcup_{w\in I_r} (qru + f_{w,v}) = I_m.\end{aligned}$$

Similarly, we obtain

$$\{e_{i,j} \colon j \in I_m\} = I_m, \quad i \in I_m, \quad \{e_{i,i} \colon i \in I_m\} = I_m, \{e_{i,m-1-i} \colon i \in I_m\} = I_m.$$

Therefore, E is a DLS(m). Similarly, one can prove that G is a DLS(m).

(ii) We prove that E and G are orthogonal. Let $i^* = qru^* + rv^* + w^*$ and $j^* = qrx^* + ry^* + z^*$, where $u^*, x^* \in I_p$, $v^*, y^* \in I_q$, $w^*, z^* \in I_r$, and $i^*, j^* \in I_m$. We suppose that $e_{i,j} = e_{i^*,j^*}$ and $g_{i,j} = g_{i^*,j^*}$. We have

$$qra_{u,x} + f_{c_{w,z},b_{v,y}} = qra_{u^*,x^*} + f_{c_{w^*,z^*},b_{v^*,y^*}},$$
$$pq\bar{c}_{w,z} + h_{\bar{a}_{u,x},\bar{b}_{v,y}} = pq\bar{c}_{w^*,z^*} + h_{\bar{a}_{u^*,x^*},\bar{b}_{v^*,y^*}}.$$

Noting that

$$f_{c_{w,z},b_{v,y}}, f_{c_{w^*,z^*},b_{v^*,y^*}} \in I_{qr}, \quad a_{u,x}, a_{u^*,x^*} \in I_p,$$
$$h_{\bar{a}_{u,x},\bar{b}_{v,y}}, h_{\bar{a}_{u^*,x^*},\bar{b}_{v^*,y^*}} \in I_{pq}, \quad \bar{c}_{w,z}, \bar{c}_{w^*,z^*} \in I_r,$$

we obtain

$$a_{u,x} = a_{u^*,x^*}, \quad f_{c_{w,z},b_{v,y}} = f_{c_{w^*,z^*},b_{v^*,y^*}},$$
$$\bar{c}_{w,z} = \bar{c}_{w^*,z^*}, \quad h_{\bar{a}_{u,x},\bar{b}_{v,y}} = h_{\bar{a}_{u^*,x^*},\bar{b}_{v^*,y^*}}.$$

Since F and H are both normal, we get

$$a_{u,x} = a_{u^*,x^*}, \quad c_{w,z} = c_{w^*,z^*}, \quad b_{v,y} = b_{v^*,y^*},$$
$$\bar{c}_{w,z} = \bar{c}_{w^*,z^*}, \quad \bar{a}_{u,x} = \bar{a}_{u^*,x^*}, \quad \bar{b}_{v,y} = \bar{b}_{v^*,y^*}.$$

Therefore, we have

$$(a_{u,x}, \bar{a}_{u,x}) = (a_{u^*,x^*}, \bar{a}_{u^*,x^*}),$$
$$(b_{v,y}, \bar{b}_{v,y}) = (b_{v^*,y^*}, \bar{b}_{v^*,y^*}),$$
$$(c_{w,z}, \bar{c}_{w,z}) = (c_{w^*,z^*}, \bar{c}_{w^*,z^*}).$$

Since A and $\bar{A}$ are orthogonal, B and $\bar{B}$ are orthogonal, and C and $\bar{C}$ are orthogonal, we obtain

$$u = u^*, \quad v = v^*, \quad w = w^*, \quad x = x^*, \quad y = y^*, \quad z = z^*,$$

which indicates that $i = i^*$ and $j = j^*$. It follows that E and G are orthogonal. By (i), E and G are orthogonal DLS(m)s over I_m.

(iii) We prove that $E \times G$ is an MS(m). For $j \in I_m$, we can write $j = qrx + ry + z$, where $x \in I_p$, $y \in I_q$, and $z \in I_r$. We get

$$\begin{aligned}
\sum_{i=0}^{m-1} e_{i,j}g_{i,j} &= \sum_{u=0}^{p-1}\sum_{v=0}^{q-1}\sum_{w=0}^{r-1}(qra_{u,x} + f_{c_{w,z},b_{v,y}})(pq\bar{c}_{w,z} + h_{\bar{a}_{u,x},\bar{b}_{v,y}}) \\
&= pq^2r\sum_{v=0}^{q-1}\sum_{u=0}^{p-1}\sum_{w=0}^{r-1} a_{u,x}\bar{c}_{w,z} + qr\sum_{w=0}^{r-1}\sum_{u=0}^{p-1}\sum_{v=0}^{q-1} a_{u,x}h_{\bar{a}_{u,x},\bar{b}_{v,y}} \\
&+ pq\sum_{u=0}^{p-1}\sum_{w=0}^{r-1}\sum_{v=0}^{q-1} \bar{c}_{w,z}f_{c_{w,z},b_{v,y}} + \sum_{u=0}^{p-1}\sum_{v=0}^{q-1}\sum_{w=0}^{r-1} f_{c_{w,z},b_{v,y}}h_{\bar{a}_{u,x},\bar{b}_{v,y}} \\
&= pq^3rS_aS_c + qr^2S_aH_r + p^2qS_cF_r + pF_cH_r.
\end{aligned}$$

Similarly, we have

$$\sum_{j=0}^{m-1} e_{i,j}g_{i,j} = S_{E\times G}, \text{for } i \in I_m, \sum_{i=0}^{m-1} e_{i,i}g_{i,i} = S_{E\times G}, \sum_{i=0}^{m-1} e_{i,m-1-i}g_{i,m-1-i} = S_{E\times G},$$

where $S_{E\times G} = pq^3rS_aS_c + qr^2S_aH_r + p^2qS_cF_r + pF_cH_r$. Thus, $E \times G$ is an MS(m).

(iv) We prove that $E^{\times 2} \times G$ is an MS(m). For $j \in I_m$, we can write $j = qrx + ry + z$, where $x \in I_p$, $y \in I_q$, and $z \in I_r$. We get

$$\begin{aligned}
\sum_{i=0}^{m-1} e_{i,j}^2 g_{i,j} &= \sum_{u=0}^{p-1}\sum_{v=0}^{q-1}\sum_{w=0}^{r-1} (qra_{u,x} + f_{c_{w,z},b_{v,y}})^2 (pq\bar{c}_{w,z} + h_{\bar{a}_{u,x},\bar{b}_{v,y}}) \\
&= pq^3r^2 \sum_{v=0}^{q-1}\sum_{u=0}^{p-1}\sum_{w=0}^{r-1} a_{u,x}^2 \bar{c}_{w,z} + q^2r^2 \sum_{w=0}^{r-1}\sum_{u=0}^{p-1}\sum_{v=0}^{q-1} a_{u,x}^2 h_{\bar{a}_{u,x},\bar{b}_{v,y}} \\
&+ 2pq^2r \sum_{u=0}^{p-1}\sum_{w=0}^{r-1}\sum_{v=0}^{q-1} a_{u,x}\bar{c}_{w,z} f_{c_{w,z},b_{v,y}} + 2qr \sum_{u=0}^{p-1}\sum_{v=0}^{q-1}\sum_{w=0}^{r-1} a_{u,x} f_{c_{w,z},b_{v,y}} h_{\bar{a}_{u,x},\bar{b}_{v,y}} \\
&+ pq \sum_{u=0}^{p-1}\sum_{w=0}^{r-1}\sum_{v=0}^{q-1} \bar{c}_{w,z} f_{c_{w,z},b_{v,y}}^2 + \sum_{w=0}^{r-1}\sum_{v=0}^{q-1}\sum_{u=0}^{p-1} f_{c_{w,z},b_{v,y}}^2 h_{\bar{a}_{u,x},\bar{b}_{v,y}} \\
&= pq^4r^2 S_{a,2} S_c + q^2r^3 S_{a,2} H_r + 2pq^2r S_a S_c F_r + 2qr S_a F_c H_r + p^2 q S_c F_{r,2} + r H_c F_{r,2}.
\end{aligned}$$

Similarly, we have

$$\sum_{j=0}^{m-1} e_{i,j}^2 g_{i,j} = S_{E^{\times 2}\times G}, \text{for } i \in I_m, \sum_{i=0}^{m-1} e_{i,i}^2 g_{i,i} = S_{E^{\times 2}\times G}, \sum_{i=0}^{m-1} e_{i,m-1-i}^2 g_{i,m-1-i} = S_{E^{\times 2}\times G},$$

where $S_{E^{\times 2}\times G} = pq^4r^2 S_{a,2} S_c + q^2r^3 S_{a,2} H_r + 2pq^2r S_a S_c F_r + 2qr S_a F_c H_r + p^2 q S_c F_{r,2} + r H_c F_{r,2}$. Thus, $E^{\times 2} \times G$ is an MS(m).

(v) We prove that $E \times G^{\times 2}$ is an MS(m). For $j \in I_m$, we can write $j = qrx + ry + z$, where $x \in I_p$, $y \in I_q$, and $z \in I_r$. We get

$$\begin{aligned}
\sum_{i=0}^{m-1} e_{i,j} g_{i,j}^2 &= \sum_{u=0}^{p-1}\sum_{v=0}^{q-1}\sum_{w=0}^{r-1} (qra_{u,x} + f_{c_{w,z},b_{v,y}}) (pq\bar{c}_{w,z} + h_{\bar{a}_{u,x},\bar{b}_{v,y}})^2 \\
&= p^2q^3r \sum_{v=0}^{q-1}\sum_{u=0}^{p-1}\sum_{w=0}^{r-1} a_{u,x} \bar{c}_{w,z}^2 + p^2q^2 \sum_{u=0}^{p-1}\sum_{w=0}^{r-1}\sum_{v=0}^{q-1} \bar{c}_{w,z}^2 f_{c_{w,z},b_{v,y}} \\
&+ 2pq^2r \sum_{u=0}^{p-1}\sum_{w=0}^{r-1}\sum_{v=0}^{q-1} a_{u,x}\bar{c}_{w,z} h_{\bar{a}_{u,x},\bar{b}_{v,y}} + 2pq \sum_{w=0}^{r-1}\sum_{v=0}^{q-1}\sum_{u=0}^{p-1} \bar{c}_{w,z} f_{c_{w,z},b_{v,y}} h_{\bar{a}_{u,x},\bar{b}_{v,y}} \\
&+ qr \sum_{w=0}^{r-1}\sum_{u=0}^{p-1}\sum_{v=0}^{q-1} a_{u,x} h_{\bar{a}_{u,x},\bar{b}_{v,y}}^2 + \sum_{u=0}^{p-1}\sum_{v=0}^{q-1}\sum_{w=0}^{r-1} f_{c_{w,z},b_{v,y}} h_{\bar{a}_{u,x},\bar{b}_{v,y}}^2 \\
&= p^2q^4r S_a S_{c,2} + p^3q^2 S_{c,2} F_r + 2pq^2r S_a S_c H_r + 2pq S_c F_r H_c + qr^2 S_a H_{r,2} + p F_c H_{r,2}.
\end{aligned}$$

Similarly, we have

$$\sum_{j=0}^{m-1} e_{i,j} g_{i,j}^2 = S_{E\times G^{\times 2}}, \text{for } i \in I_m, \sum_{i=0}^{m-1} e_{i,i} g_{i,i}^2 = S_{E\times G^{\times 2}}, \sum_{i=0}^{m-1} e_{i,m-1-i} g_{i,m-1-i}^2 = S_{E\times G^{\times 2}},$$

where $S_{E\times G^{\times 2}} = p^2q^4r S_a S_{c,2} + p^3q^2 S_{c,2} F_r + 2pq^2r S_a S_c H_r + 2pq S_c F_r H_c + qr^2 S_a H_{r,2} + p F_c H_{r,2}$. It follows that $E \times G^{\times 2}$ is an MS(m).

In summary, we know that (E, G) is a TMP(m). Based on Lemma 5, there exists an NMS$(m, 3)$, $mE + G$. □

3.3. Row–Square Magic Rectangles

In this section, our main purpose is to construct several infinite classes of row–square magic rectangles. For this purpose, we need the following lemmas.

Lemma 6. *If an odd-normal BRMR$(p, 2q)$ exists, then a quasi-normal RSMR$(2p, 2q)$ exists.*

Proof. Let A be an odd-normal BRMR$(p, 2q)$ and $B = \binom{A}{-A}$. Clearly, $-A$ is also an odd-normal BRMR$(p, 2q)$ and $\mathfrak{S}_{|-A|} = \mathfrak{S}_{|A|}$. Obviously, if $a \in \mathfrak{S}_A$, then $-a \in \mathfrak{S}_{-A}$. Therefore, we have $\mathfrak{S}_{-A} \cup \mathfrak{S}_A = \cup_{a \in \mathfrak{S}_{|A|}} \{-a, a\} = [1-4pq, -1]_2 \cup [1, 4pq-1]_2 = [1-4pq, 4pq-1]_2$. Obviously, each column sum of B is zero. In summary, we know at once that the conclusion holds. □

Based on Lemma 6, to obtain a quasi-normal RSMR$(2p, 2q)$, it is sufficient to find an odd-normal BRMR$(p, 2q)$.

Lemma 7. *If a quasi-normal RSMR$(2p, 2q)$ exists, then a normal RSMR$(2p, 2q)$ exists.*

Proof. If A is a quasi-normal RSMR$(2p, 2q)$, then $\frac{1}{2}(A + (4pq-1)J_{2p \times 2q})$ is a normal RSMR$(2p, 2q)$. □

Based on Lemmas 6 and 7, to obtain a normal RSMR$(2p, 2q)$, it suffices to find an odd-normal BRMR$(p, 2q)$.

Lemma 8. *Let p, q, and r be positive integers with $q \geq 2$. If there exist an odd-normal BRMR$(p, 2q)$ and balanced BRMR$(p, 4r)$, then there exists an odd-normal BRMR$(p, 2q+4r)$.*

Proof. Let A be an odd-normal BRMR$(p, 2q)$ and B an odd-normal balanced BRMR$(p, 4r)$. Let $k = 4pq$ and C be a $p \times (2q+4r)$ matrix defined by

$$\begin{cases} c_{i,j} = a_{i,j} & j \in I_{2q}, \\ c_{i,j+2q} = b_{i,j} + sign(b_{i,j})k & j \in I_{4r}, \end{cases} \quad i \in I_p, \tag{1}$$

where $\text{sign}(b_{i,j}) = b_{i,j}/|b_{i,j}|$. Clearly, $\text{sign}(b_{i,j})b_{i,j} = |b_{i,j}|$. Now, we prove that each row sum of C is zero. It should be noted that row sums of A and B are zero and B is balanced; for $i \in I_p$, we have

$$\begin{aligned} \sum_{j \in I_{2q+4r}} c_{i,j} &= \sum_{j \in I_{2q}} a_{i,j} + \sum_{j \in I_{4r}} (b_{i,j} + sign(b_{i,j})k) \\ &= \sum_{j \in I_{2q}} a_{i,j} + \sum_{j \in I_{4r}} b_{i,j} + k \sum_{j \in I_{4r}} sign(b_{i,j}) \\ &= 0 + 0 + k \times 0 \\ &= 0. \end{aligned}$$

It follows that each row sum of C is zero. Next, we show that C is odd-normal. It should be noted that $|b_{i,j} + sign(b_{i,j})k| = k + |b_{i,j}|$; we have

$$\begin{aligned} \{|c_{i,j}| \colon i \in I_p, j \in I_{2q+4r}\} &= \{|a_{i,j}| \colon i \in I_p, j \in I_{2q}\} \cup \{|b_{i,j} + sign(b_{i,j})k| \colon i \in I_p, j \in I_{4r}\} \\ &= [1, k-1]_2 \cup \{k + |b_{i,j}| \colon i \in I_p, j \in I_{4r}\} \\ &= [1, k-1]_2 \cup [k+1, k+8pr-1]_2 \\ &= [1, 2p(2q+4r)-1]_2, \end{aligned}$$

which indicates that C is odd-normal. Finally, we prove that $C^{\times 2}$ is a row-magic rectangle. Let $S_{|B|}$, $S_{A^{\times 2}}$, and $S_{B^{\times 2}}$ denote row sums of $|B|$, $A^{\times 2}$, and $B^{\times 2}$, respectively. For $i \in I_p$, we have

$$\begin{aligned} \sum_{j \in I_{2q+4r}} c_{i,j}^2 &= \sum_{j \in I_{2q}} a_{i,j}^2 + \sum_{j \in I_{4r}} (b_{i,j} + sign(b_{i,j})k)^2 \\ &= S_{A^{\times 2}} + \sum_{j \in I_{4r}} b_{i,j}^2 + 2k \sum_{j \in I_{4r}} |b_{i,j}| + \sum_{j \in I_{4r}} k^2 \\ &= S_{A^{\times 2}} + S_{B^{\times 2}} + 2kS_{|B|} + 4rk^2, \end{aligned}$$

which is independent of i and, hence, is a constant. Thus $C^{\times 2}$ is a row-magic rectangle. In summary, C is an odd-normal BRMR$(p, 2q+4r)$. $\square$

Lemma 9. *Let p, q, and r be positive integers with $q \geq 2$. If there exist an odd-normal balanced BRMR$(p, 4r)$ and $2r$ odd-normal BRMR$(p, 2q+2m)$ for $m \in I_{2r}$, then there exists an odd-normal BRMR$(p, 2n)$ for $n \geq q$.*

Proof. Let $v \in [2q, 2q+4r-2]_2$ and u be a nonnegative integer. Now, by induction, we prove the statement $\mathcal{P}_v(u)$: There exists an odd-normal BRMR$(p, 4ru+v)$ for $u \geq 0$. Firstly, $\mathcal{P}_v(0)$ is obviously true by hypothesis. Secondly, we suppose that $\mathcal{P}_v(w)$ is true for nonnegative integer w, that is, there exists an odd-normal BRMR$(p, 4rw+v)$. Finally, we prove that $\mathcal{P}_v(w+1)$ is correct, in other word, an odd-normal BRMR$(p, 4r(w+1)+v)$ exists. It should be noted that there exist an odd-normal balanced BRMR$(p, 4r)$ by hypothesis and an odd-normal BRMR$(p, 4rw+v)$ by our induction hypothesis; by Lemma 8 we see that there exists an odd-normal BRMR$(p, 4rw+v+4r)$, that is, BRMR$(p, 4r(w+1)+v)$. It follows that the statement $\mathcal{P}_v(w+1)$ is true. For $v \in [2q, 2q+4r-2]_2$, by induction, $\mathcal{P}_v(u)$ is true for $u \geq 0$. In summary, there exists an odd-normal BRMR$(p, 2n)$ for $n \geq q$. $\square$

Next, we construct several infinite classes of odd-normal bimagic row-magic rectangles.

Lemma 10. *There exists an odd-normal BRMR$(p, 2q)$ for $p \in \{7, 11, 13, 17, 19, 23, 29, 31, 37\}$ and $q \geq 3$.*

Proof. See Appendix A. $\square$

3.4. New Classes of Trimagic Squares of Even Orders

In the following, we provide two applications of Theorem 1.

Theorem 2. *There exists an NMS$(8pqr, 3)$ for $p, r \in \{7, 11, 13, 17, 19, 23, 29, 31, 37\}$ and odd q not less than 7.*

Proof. Let $\mathcal{S} = \{7, 11, 13, 17, 19, 23, 29, 31, 37\}$. By Lemma 10, there exists an odd-normal BRMR$(p, 2q)$ for $p \in \mathcal{S}$ and $q \geq 3$. By Lemma 7, there exists a normal BRMR$(2p, 2q)$ for $p \in \mathcal{S}$ and $q \geq 3$. Therefore, for $p, r \in \mathcal{S}$ and odd q not less than 7, there exist a normal BRMR$(2p, 2q)$ and a normal BRMR$(2r, 2q)$. Since $2p, 2q, 2r \notin \{1, 2, 3, 6\}$, by Lemma 1, we know that there is a pair of orthogonal DLS$(2n)$s over I_{2n} for $n \in \{p, q, r\}$. By Theorem 1, we prove that there exists an NMS$(2p \times 2r \times 2q, 3)$, that is, NMS$(8pqr, 3)$. $\square$

Theorem 3. *There exists an NMS$(8q^3, 3)$ for all positive integers q greater than 1.*

Proof. When $q = 2$, we have $8q^3 = 8 \times 2^3 = 64$. By Lemma 3, there is an NMS$(8 \times 2^3, 3)$. When $q = 3$, we have $8q^3 = 8 \times 3^3 = 24 \times 9$. Since there is an NMS$(24, 3)$ by Lemma 3 and there is an NMS$(9, 2)$ by Lemma 4, there is an NMS$(8 \times 3^3, 3)$ by Lemma 2. Based on Lemma 4, for every positive integer q greater than 3, there is a normal RSMR$(2q, 2q)$. Since $2q \notin \{1, 2, 3, 6\}$, by Lemma 1, we know that there is a pair of orthogonal DLS$(2q)$s over I_{2q}. By Theorem 1, we prove that there is an NMS$(2q \times 2q \times 2q, 3)$, that is, NMS$(8q^3, 3)$. $\square$

Let $\Omega = \{q^3 \colon q \text{ integer}, q \geq 2\} \cup \{pqr \colon p, r \in \{7, 11, 13, 17, 19, 23, 29, 31, 37\}, \text{odd } q \geq 7\}$. Then, by Lemma 2 and Theorems 2 and 3, we have

Corollary 1. *There is an NMS$(8mn, 3)$ for $m \in \Omega$ and $n \in \{pq \colon \text{odd } p, q \geq 5\} \cup [9, 63]_2$.*

4. Conclusions

In this paper, by giving a new product construction theorem, we reduced the problem of constructing one normal trimagic square of order pqr to the problem of constructing two normal $p \times q$ and $r \times q$ row–square magic rectangles; we especially reduced the problem of constructing normal $2p \times 2q$ row–square magic rectangles to the problem of constructing odd-normal $p \times 2q$ row–square magic rectangles. More precisely, we proved that there exist normal trimagic squares of orders $8u^3$ and $8pqr$ for all positive integers u with $u \geq 2$ and $p, r \in \{7, 11, 13, 17, 19, 23, 29, 31, 37\}$ and odd q not less than 7. These results are new to previous literature. Our new product construction method can easily be extended to construct multimagic squares.

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Appendix A

The Proof of Lemma 10.

Let

$$A_7 = \begin{pmatrix} -83 & -19 & 1 & 3 & 17 & 81 \\ -75 & -39 & -7 & 5 & 49 & 67 \\ -71 & -43 & -13 & 9 & 57 & 61 \\ -77 & -29 & -21 & 11 & 53 & 63 \\ -79 & -31 & -15 & 25 & 27 & 73 \\ -69 & -35 & -33 & 23 & 55 & 59 \\ -65 & -41 & -37 & 45 & 47 & 51 \end{pmatrix}, \qquad B_7 = \begin{pmatrix} -79 & -75 & -69 & -1 & 3 & 55 & 81 & 85 \\ -83 & -73 & -63 & -5 & 7 & 43 & 77 & 97 \\ -95 & -67 & -53 & -9 & 11 & 41 & 71 & 101 \\ -89 & -51 & -47 & -37 & 13 & 17 & 91 & 103 \\ -93 & -87 & -29 & -15 & 33 & 35 & 49 & 107 \\ -105 & -61 & -31 & -27 & 19 & 39 & 57 & 109 \\ -99 & -59 & -45 & -21 & 23 & 25 & 65 & 111 \end{pmatrix},$$

$$C_7 = \begin{pmatrix} -91 & -89 & -85 & -83 & -1 & 3 & 25 & 103 & 107 & 111 \\ -97 & -95 & -77 & -75 & -5 & 7 & 19 & 99 & 109 & 115 \\ -105 & -87 & -79 & -67 & -11 & 9 & 15 & 93 & 113 & 119 \\ -123 & -81 & -69 & -65 & -13 & 17 & 27 & 71 & 101 & 135 \\ -129 & -73 & -63 & -57 & -29 & 21 & 23 & 59 & 117 & 131 \\ -127 & -125 & -35 & -33 & -31 & 49 & 51 & 53 & 61 & 137 \\ -139 & -133 & -41 & -37 & 39 & 43 & 45 & 47 & 55 & 121 \end{pmatrix}, \qquad D_7 = \begin{pmatrix} -41 & -19 & -1 & 55 & 69 & 75 \\ -37 & -21 & -3 & 53 & 65 & 77 \\ -33 & -23 & -5 & 51 & 67 & 71 \\ -39 & -17 & -7 & 49 & 63 & 83 \\ -31 & -27 & -9 & 43 & 59 & 73 \\ -29 & -25 & -11 & 45 & 57 & 79 \\ -35 & -15 & -13 & 47 & 61 & 81 \end{pmatrix},$$

$$A_{11} = \begin{pmatrix} -127 & -35 & 1 & 3 & 29 & 129 \\ -125 & -47 & -7 & 5 & 57 & 117 \\ -131 & -41 & -9 & 11 & 59 & 111 \\ -121 & -61 & 13 & 15 & 31 & 123 \\ -107 & -93 & 17 & 25 & 53 & 105 \\ -101 & -89 & -19 & 39 & 71 & 99 \\ -115 & -75 & -21 & 49 & 77 & 85 \\ -113 & -65 & -27 & 23 & 87 & 95 \\ -109 & -73 & -33 & 45 & 79 & 91 \\ -103 & -63 & -51 & 37 & 83 & 97 \\ -119 & -55 & -43 & 67 & 69 & 81 \end{pmatrix}, \qquad B_{11} = \begin{pmatrix} -123 & -119 & -109 & -1 & 3 & 93 & 127 & 129 \\ -121 & -115 & -111 & -5 & 7 & 73 & 133 & 139 \\ -137 & -117 & -87 & -11 & 9 & 69 & 131 & 143 \\ -141 & -135 & -63 & -13 & 15 & 89 & 97 & 151 \\ -149 & -107 & -79 & -17 & 19 & 55 & 125 & 153 \\ -157 & -91 & -83 & -21 & 23 & 65 & 95 & 169 \\ -161 & -103 & -45 & -43 & 25 & 59 & 101 & 167 \\ -163 & -105 & -57 & -27 & 37 & 67 & 77 & 171 \\ -147 & -145 & -31 & -29 & 51 & 61 & 81 & 159 \\ -165 & -75 & -71 & -41 & 33 & 47 & 99 & 173 \\ -155 & -113 & -49 & -35 & 39 & 53 & 85 & 175 \end{pmatrix},$$

$$C_{11} = \begin{pmatrix} -179 & -151 & -109 & -87 & -23 & 1 & 55 & 131 & 161 & 201 \\ -181 & -153 & -105 & -85 & -25 & 3 & 53 & 129 & 165 & 199 \\ -207 & -183 & -159 & 5 & 27 & 57 & 77 & 107 & 127 & 149 \\ -171 & -143 & -101 & -83 & -51 & 7 & 29 & 125 & 185 & 203 \\ -177 & -147 & -123 & -103 & 9 & 31 & 49 & 81 & 175 & 205 \\ -191 & -167 & -121 & -59 & -11 & 33 & 71 & 95 & 141 & 209 \\ -163 & -133 & -119 & -73 & -63 & 13 & 35 & 97 & 189 & 217 \\ -219 & -193 & -139 & 15 & 39 & 65 & 69 & 93 & 113 & 157 \\ -197 & -145 & -111 & -61 & -37 & 17 & 75 & 89 & 155 & 215 \\ -169 & -135 & -115 & -91 & -41 & 19 & 47 & 79 & 195 & 211 \\ -187 & -137 & -117 & -67 & -43 & 21 & 45 & 99 & 173 & 213 \end{pmatrix}, \quad D_{11} = \begin{pmatrix} -63 & -31 & -1 & 87 & 109 & 119 \\ -65 & -29 & -3 & 83 & 107 & 123 \\ -55 & -33 & -5 & 85 & 105 & 125 \\ -53 & -35 & -7 & 81 & 101 & 129 \\ -61 & -27 & -9 & 79 & 103 & 131 \\ -51 & -39 & -11 & 73 & 97 & 115 \\ -47 & -41 & -13 & 69 & 99 & 121 \\ -49 & -43 & -15 & 67 & 91 & 111 \\ -59 & -25 & -17 & 75 & 95 & 117 \\ -57 & -23 & -19 & 77 & 93 & 127 \\ -45 & -37 & -21 & 71 & 89 & 113 \end{pmatrix},$$

$$A_{13} = \begin{pmatrix} -151 & -43 & -3 & 1 & 49 & 147 \\ -155 & -51 & -7 & 5 & 91 & 117 \\ -149 & -59 & -9 & 11 & 75 & 131 \\ -153 & -53 & -13 & 15 & 77 & 127 \\ -135 & -99 & 17 & 19 & 85 & 113 \\ -125 & -101 & 21 & 27 & 35 & 143 \\ -133 & -87 & -23 & 41 & 81 & 121 \\ -123 & -105 & -25 & 71 & 79 & 103 \\ -145 & -93 & 29 & 31 & 63 & 115 \\ -139 & -111 & 33 & 61 & 73 & 83 \\ -129 & -65 & -57 & 37 & 95 & 119 \\ -137 & -109 & 39 & 45 & 55 & 107 \\ -141 & -67 & -47 & 69 & 89 & 97 \end{pmatrix}, \quad B_{13} = \begin{pmatrix} -161 & -151 & -103 & -1 & 33 & 73 & 127 & 183 \\ -167 & -145 & -101 & -3 & 31 & 77 & 123 & 185 \\ -169 & -141 & -71 & -35 & 5 & 95 & 129 & 187 \\ -159 & -153 & -75 & -29 & 7 & 99 & 121 & 189 \\ -163 & -149 & -67 & -37 & 9 & 97 & 115 & 195 \\ -173 & -139 & -93 & -11 & 39 & 61 & 125 & 191 \\ -179 & -137 & -87 & -13 & 51 & 53 & 119 & 193 \\ -165 & -113 & -83 & -55 & 15 & 49 & 155 & 197 \\ -175 & -105 & -89 & -47 & 17 & 57 & 143 & 199 \\ -171 & -135 & -65 & -45 & 19 & 81 & 109 & 207 \\ -157 & -147 & -91 & -21 & 27 & 69 & 117 & 203 \\ -177 & -131 & -85 & -23 & 41 & 63 & 107 & 205 \\ -181 & -133 & -59 & -43 & 25 & 79 & 111 & 201 \end{pmatrix},$$

$$C_{13} = \begin{pmatrix} -211 & -195 & -177 & -65 & -1 & 27 & 103 & 129 & 155 & 235 \\ -237 & -213 & -199 & 3 & 29 & 67 & 99 & 125 & 151 & 175 \\ -209 & -181 & -153 & -101 & -5 & 31 & 61 & 121 & 197 & 239 \\ -193 & -179 & -147 & -123 & -7 & 33 & 63 & 95 & 217 & 241 \\ -215 & -203 & -127 & -69 & -35 & 9 & 81 & 149 & 167 & 243 \\ -247 & -173 & -145 & -85 & 11 & 37 & 71 & 119 & 189 & 223 \\ -219 & -207 & -137 & -73 & -13 & 39 & 79 & 115 & 171 & 245 \\ -221 & -201 & -139 & -75 & -15 & 41 & 83 & 117 & 157 & 253 \\ -231 & -183 & -133 & -87 & -17 & 43 & 77 & 109 & 165 & 257 \\ -187 & -163 & -143 & -111 & -47 & 19 & 57 & 93 & 227 & 255 \\ -185 & -169 & -141 & -105 & -51 & 21 & 55 & 91 & 233 & 251 \\ -229 & -161 & -131 & -107 & -23 & 49 & 59 & 89 & 205 & 249 \\ -191 & -159 & -135 & -113 & -53 & 25 & 45 & 97 & 225 & 259 \end{pmatrix}, D_{13} = \begin{pmatrix} -77 & -35 & -1 & 101 & 129 & 153 \\ -71 & -37 & -3 & 103 & 127 & 141 \\ -67 & -39 & -5 & 99 & 125 & 149 \\ -63 & -41 & -7 & 97 & 123 & 151 \\ -65 & -43 & -9 & 89 & 117 & 155 \\ -75 & -31 & -11 & 93 & 119 & 145 \\ -59 & -47 & -13 & 81 & 121 & 147 \\ -57 & -49 & -15 & 85 & 109 & 139 \\ -55 & -51 & -17 & 79 & 111 & 143 \\ -53 & -45 & -19 & 87 & 113 & 137 \\ -69 & -29 & -21 & 95 & 107 & 131 \\ -73 & -27 & -23 & 83 & 115 & 135 \\ -61 & -33 & -25 & 91 & 105 & 133 \end{pmatrix},$$

$$A_{17} = \begin{pmatrix} -199 & -43 & 1 & 3 & 37 & 201 \\ -203 & -39 & -7 & 5 & 49 & 195 \\ -197 & -67 & -9 & 11 & 79 & 183 \\ -185 & -93 & 13 & 15 & 59 & 191 \\ -193 & -85 & -17 & 19 & 135 & 141 \\ -189 & -83 & -21 & 23 & 95 & 175 \\ -159 & -151 & 25 & 27 & 113 & 145 \\ -173 & -107 & -31 & 29 & 129 & 153 \\ -181 & -101 & -33 & 47 & 111 & 157 \\ -177 & -73 & -65 & 35 & 109 & 171 \\ -169 & -115 & -41 & 75 & 87 & 163 \\ -167 & -123 & -45 & 89 & 121 & 125 \\ -179 & -103 & -51 & 97 & 99 & 137 \\ -149 & -133 & -53 & 69 & 119 & 147 \\ -165 & -161 & 55 & 63 & 77 & 131 \\ -187 & -139 & 57 & 61 & 91 & 117 \\ -155 & -105 & -81 & 71 & 127 & 143 \end{pmatrix}, \quad B_{17} = \begin{pmatrix} -259 & -231 & -53 & -1 & 131 & 135 & 137 & 141 \\ -257 & -233 & -51 & -3 & 123 & 133 & 139 & 149 \\ -251 & -239 & -49 & -5 & 117 & 129 & 147 & 151 \\ -269 & -213 & -55 & -7 & 105 & 119 & 153 & 167 \\ -263 & -225 & -47 & -9 & 113 & 127 & 145 & 159 \\ -267 & -209 & -57 & -11 & 89 & 115 & 169 & 171 \\ -271 & -217 & -43 & -13 & 121 & 125 & 143 & 155 \\ -243 & -241 & -45 & -15 & 103 & 107 & 161 & 173 \\ -261 & -207 & -59 & -17 & 83 & 97 & 181 & 183 \\ -265 & -199 & -61 & -19 & 85 & 91 & 175 & 193 \\ -223 & -219 & -81 & -21 & 67 & 71 & 185 & 221 \\ -215 & -197 & -109 & -23 & 27 & 93 & 195 & 229 \\ -245 & -163 & -111 & -25 & 35 & 95 & 177 & 237 \\ -227 & -189 & -99 & -29 & 31 & 101 & 165 & 247 \\ -255 & -187 & -69 & -33 & 73 & 79 & 157 & 235 \\ -253 & -191 & -63 & -37 & 65 & 75 & 201 & 203 \\ -249 & -179 & -77 & -39 & 41 & 87 & 205 & 211 \end{pmatrix},$$

$$C_{17} = \begin{pmatrix} -339 & -185 & -179 & -147 & 1 & 33 & 155 & 161 & 193 & 307 \\ -337 & -175 & -173 & -165 & 3 & 29 & 151 & 167 & 189 & 311 \\ -313 & -191 & -177 & -169 & 5 & 27 & 149 & 163 & 171 & 335 \\ -333 & -305 & -205 & -7 & 35 & 125 & 135 & 159 & 181 & 215 \\ -331 & -301 & -209 & -9 & 39 & 119 & 131 & 139 & 201 & 221 \\ -329 & -299 & -211 & -11 & 41 & 109 & 129 & 141 & 199 & 231 \\ -327 & -213 & -197 & -113 & 13 & 37 & 127 & 143 & 227 & 303 \\ -325 & -281 & -157 & -87 & 15 & 59 & 101 & 183 & 239 & 253 \\ -323 & -257 & -219 & -51 & 17 & 83 & 121 & 145 & 195 & 289 \\ -265 & -259 & -255 & -71 & 19 & 91 & 93 & 97 & 273 & 277 \\ -283 & -275 & -271 & -21 & 81 & 85 & 89 & 95 & 249 & 251 \\ -267 & -261 & -245 & -77 & 23 & 69 & 99 & 105 & 263 & 291 \\ -293 & -269 & -223 & -65 & 25 & 73 & 103 & 115 & 247 & 287 \\ -315 & -279 & -225 & -31 & 45 & 53 & 123 & 153 & 235 & 241 \\ -321 & -237 & -217 & -75 & 43 & 55 & 57 & 203 & 207 & 285 \\ -309 & -297 & -137 & -107 & 47 & 63 & 79 & 133 & 233 & 295 \\ -317 & -229 & -187 & -117 & 49 & 61 & 67 & 111 & 243 & 319 \end{pmatrix}, \; D_{17} = \begin{pmatrix} -101 & -45 & -1 & 135 & 167 & 203 \\ -97 & -47 & -3 & 133 & 163 & 201 \\ -93 & -49 & -5 & 131 & 169 & 177 \\ -99 & -43 & -7 & 127 & 165 & 199 \\ -89 & -51 & -9 & 125 & 157 & 193 \\ -79 & -53 & -11 & 129 & 161 & 195 \\ -81 & -55 & -13 & 119 & 159 & 191 \\ -83 & -57 & -15 & 115 & 151 & 181 \\ -95 & -41 & -17 & 123 & 149 & 189 \\ -75 & -61 & -19 & 109 & 153 & 187 \\ -77 & -63 & -21 & 113 & 137 & 173 \\ -69 & -65 & -23 & 111 & 139 & 197 \\ -73 & -67 & -25 & 103 & 141 & 171 \\ -87 & -35 & -27 & 121 & 155 & 185 \\ -71 & -59 & -29 & 105 & 147 & 175 \\ -85 & -39 & -31 & 117 & 143 & 179 \\ -91 & -37 & -33 & 107 & 145 & 183 \end{pmatrix},$$

$$A_{19} = \begin{pmatrix} -227 & -31 & 1 & 3 & 29 & 225 \\ -219 & -75 & -5 & 7 & 91 & 205 \\ -221 & -81 & -9 & 11 & 109 & 191 \\ -189 & -121 & 13 & 15 & 59 & 223 \\ -215 & -83 & -19 & 17 & 99 & 201 \\ -209 & -101 & -21 & 23 & 125 & 183 \\ -203 & -115 & 25 & 27 & 53 & 213 \\ -187 & -135 & 33 & 35 & 37 & 217 \\ -197 & -127 & -39 & 89 & 105 & 169 \\ -193 & -133 & -41 & 87 & 123 & 157 \\ -185 & -177 & 43 & 77 & 95 & 147 \\ -173 & -149 & -45 & 69 & 131 & 167 \\ -175 & -143 & -47 & 57 & 153 & 155 \\ -199 & -165 & 49 & 73 & 113 & 129 \\ -181 & -179 & 51 & 71 & 79 & 159 \\ -207 & -103 & -55 & 85 & 117 & 163 \\ -195 & -119 & -61 & 97 & 137 & 141 \\ -171 & -139 & -63 & 67 & 145 & 161 \\ -211 & -93 & -65 & 107 & 111 & 151 \end{pmatrix}, \quad B_{19} = \begin{pmatrix} -287 & -261 & -59 & -1 & 147 & 151 & 153 & 157 \\ -303 & -241 & -61 & -3 & 137 & 149 & 159 & 163 \\ -277 & -271 & -55 & -5 & 135 & 141 & 165 & 167 \\ -301 & -243 & -57 & -7 & 133 & 143 & 155 & 177 \\ -299 & -237 & -63 & -9 & 101 & 145 & 175 & 187 \\ -275 & -273 & -49 & -11 & 129 & 139 & 169 & 171 \\ -291 & -253 & -51 & -13 & 123 & 131 & 173 & 181 \\ -285 & -255 & -53 & -15 & 111 & 125 & 183 & 189 \\ -297 & -227 & -67 & -17 & 93 & 115 & 195 & 205 \\ -289 & -235 & -65 & -19 & 87 & 117 & 191 & 213 \\ -293 & -203 & -91 & -21 & 69 & 103 & 207 & 229 \\ -251 & -225 & -109 & -23 & 45 & 97 & 219 & 247 \\ -265 & -211 & -107 & -25 & 31 & 121 & 223 & 233 \\ -295 & -201 & -85 & -27 & 99 & 113 & 127 & 269 \\ -283 & -217 & -79 & -29 & 35 & 161 & 197 & 215 \\ -279 & -221 & -75 & -33 & 77 & 95 & 179 & 257 \\ -267 & -199 & -105 & -37 & 43 & 89 & 231 & 245 \\ -249 & -239 & -81 & -39 & 41 & 119 & 185 & 263 \\ -281 & -209 & -71 & -47 & 73 & 83 & 193 & 259 \end{pmatrix},$$

$$C_{19} = \begin{pmatrix} -379 & -207 & -183 & -181 & 1 & 35 & 173 & 197 & 199 & 345 \\ -377 & -225 & -179 & -169 & 3 & 37 & 155 & 201 & 211 & 343 \\ -375 & -195 & -193 & -187 & 5 & 31 & 165 & 185 & 215 & 349 \\ -373 & -213 & -189 & -175 & 7 & 29 & 167 & 191 & 205 & 351 \\ -371 & -255 & -163 & -161 & 9 & 43 & 125 & 217 & 219 & 337 \\ -369 & -347 & -223 & -11 & 33 & 145 & 157 & 171 & 209 & 235 \\ -367 & -241 & -203 & -139 & 13 & 45 & 119 & 177 & 261 & 335 \\ -365 & -331 & -137 & -117 & 15 & 49 & 151 & 229 & 243 & 263 \\ -363 & -325 & -159 & -103 & 17 & 55 & 131 & 221 & 249 & 277 \\ -361 & -267 & -265 & -57 & 19 & 113 & 115 & 149 & 231 & 323 \\ -359 & -289 & -237 & -65 & 21 & 91 & 127 & 143 & 253 & 315 \\ -317 & -311 & -299 & -23 & 95 & 97 & 99 & 101 & 273 & 285 \\ -303 & -283 & -275 & -89 & 25 & 87 & 105 & 109 & 305 & 319 \\ -307 & -287 & -281 & -75 & 27 & 85 & 107 & 121 & 301 & 309 \\ -333 & -327 & -251 & -39 & 79 & 83 & 93 & 129 & 269 & 297 \\ -291 & -279 & -245 & -135 & 41 & 53 & 73 & 133 & 295 & 355 \\ -353 & -293 & -233 & -71 & 47 & 59 & 111 & 153 & 259 & 321 \\ -341 & -313 & -227 & -69 & 51 & 61 & 63 & 247 & 257 & 271 \\ -357 & -329 & -141 & -123 & 67 & 77 & 81 & 147 & 239 & 339 \end{pmatrix}, \; D_{19} = \begin{pmatrix} -113 & -51 & -1 & 149 & 189 & 217 \\ -111 & -49 & -3 & 151 & 187 & 227 \\ -107 & -53 & -5 & 147 & 179 & 225 \\ -103 & -55 & -7 & 145 & 177 & 219 \\ -97 & -57 & -9 & 143 & 181 & 215 \\ -109 & -47 & -11 & 141 & 183 & 209 \\ -93 & -59 & -13 & 139 & 175 & 213 \\ -91 & -61 & -15 & 129 & 185 & 211 \\ -89 & -63 & -17 & 133 & 171 & 203 \\ -83 & -65 & -19 & 137 & 165 & 207 \\ -85 & -69 & -21 & 127 & 163 & 197 \\ -77 & -73 & -23 & 135 & 153 & 199 \\ -81 & -75 & -25 & 119 & 155 & 205 \\ -87 & -71 & -27 & 115 & 157 & 193 \\ -79 & -67 & -29 & 123 & 161 & 201 \\ -95 & -45 & -31 & 131 & 169 & 195 \\ -105 & -41 & -33 & 121 & 173 & 191 \\ -99 & -39 & -35 & 125 & 167 & 223 \\ -101 & -43 & -37 & 117 & 159 & 221 \end{pmatrix},$$

$$A_{23} = \begin{pmatrix}
-275 & -31 & -3 & 1 & 35 & 273 \\
-269 & -83 & -7 & 5 & 105 & 249 \\
-271 & -61 & -11 & 9 & 69 & 265 \\
-253 & -121 & 13 & 15 & 91 & 255 \\
-231 & -163 & 17 & 19 & 117 & 241 \\
-263 & -109 & -23 & 21 & 173 & 201 \\
-251 & -139 & 25 & 27 & 93 & 245 \\
-211 & -183 & 29 & 33 & 73 & 259 \\
-247 & -125 & -39 & 37 & 149 & 225 \\
-267 & -95 & -45 & 41 & 157 & 209 \\
-227 & -147 & -51 & 43 & 161 & 221 \\
-223 & -167 & -47 & 89 & 129 & 219 \\
-261 & -97 & -49 & 55 & 115 & 237 \\
-215 & -159 & -63 & 53 & 179 & 205 \\
-203 & -143 & -103 & 57 & 195 & 197 \\
-217 & -169 & -59 & 87 & 151 & 207 \\
-243 & -137 & -65 & 99 & 155 & 191 \\
-213 & -165 & -67 & 71 & 175 & 199 \\
-257 & -181 & 75 & 85 & 107 & 171 \\
-233 & -145 & -77 & 131 & 135 & 189 \\
-239 & -127 & -79 & 81 & 177 & 187 \\
-235 & -123 & -101 & 133 & 141 & 185 \\
-229 & -119 & -111 & 113 & 153 & 193
\end{pmatrix}, \quad
B_{23} = \begin{pmatrix}
-195 & -193 & -175 & -173 & 1 & 77 & 291 & 367 \\
-191 & -187 & -181 & -177 & 3 & 73 & 295 & 365 \\
-227 & -199 & -169 & -141 & 5 & 79 & 289 & 363 \\
-225 & -223 & -145 & -143 & 7 & 81 & 287 & 361 \\
-253 & -211 & -157 & -115 & 9 & 89 & 279 & 359 \\
-255 & -235 & -133 & -113 & 11 & 97 & 271 & 357 \\
-249 & -239 & -129 & -119 & 13 & 91 & 277 & 355 \\
-217 & -203 & -165 & -151 & 15 & 61 & 307 & 353 \\
-205 & -197 & -171 & -163 & 17 & 55 & 313 & 351 \\
-241 & -189 & -179 & -127 & 19 & 63 & 305 & 349 \\
-265 & -231 & -137 & -103 & 21 & 85 & 283 & 347 \\
-247 & -229 & -139 & -121 & 23 & 69 & 299 & 345 \\
-257 & -213 & -155 & -111 & 25 & 67 & 301 & 343 \\
-315 & -285 & -109 & -27 & 83 & 105 & 273 & 275 \\
-323 & -221 & -149 & -43 & 29 & 135 & 261 & 311 \\
-303 & -269 & -99 & -65 & 31 & 131 & 281 & 293 \\
-337 & -207 & -159 & -33 & 71 & 93 & 263 & 309 \\
-339 & -245 & -117 & -35 & 49 & 147 & 243 & 297 \\
-341 & -251 & -107 & -37 & 95 & 125 & 183 & 333 \\
-233 & -219 & -209 & -75 & 39 & 57 & 319 & 321 \\
-259 & -215 & -161 & -101 & 41 & 51 & 317 & 327 \\
-335 & -201 & -153 & -47 & 45 & 123 & 237 & 331 \\
-329 & -267 & -87 & -53 & 59 & 167 & 185 & 325
\end{pmatrix},$$

$$C_{23} = \begin{pmatrix}
-459 & -235 & -231 & -225 & 1 & 43 & 203 & 229 & 257 & 417 \\
-415 & -277 & -239 & -219 & 3 & 45 & 183 & 221 & 241 & 457 \\
-455 & -251 & -247 & -197 & 5 & 41 & 209 & 213 & 263 & 419 \\
-453 & -245 & -237 & -215 & 7 & 39 & 189 & 223 & 271 & 421 \\
-451 & -265 & -233 & -201 & 9 & 37 & 195 & 227 & 259 & 423 \\
-449 & -275 & -269 & -157 & 11 & 53 & 185 & 191 & 303 & 407 \\
-413 & -313 & -217 & -207 & 13 & 47 & 147 & 243 & 253 & 447 \\
-445 & -411 & -279 & -15 & 49 & 167 & 177 & 181 & 283 & 293 \\
-443 & -325 & -211 & -171 & 17 & 57 & 135 & 249 & 289 & 403 \\
-441 & -329 & -193 & -187 & 19 & 55 & 131 & 267 & 273 & 405 \\
-439 & -397 & -159 & -155 & 21 & 63 & 153 & 301 & 305 & 307 \\
-437 & -331 & -309 & -73 & 23 & 129 & 151 & 165 & 295 & 387 \\
-435 & -399 & -291 & -25 & 61 & 127 & 169 & 173 & 287 & 333 \\
-433 & -337 & -311 & -69 & 27 & 123 & 149 & 175 & 285 & 391 \\
-431 & -335 & -317 & -67 & 29 & 125 & 143 & 179 & 281 & 393 \\
-359 & -343 & -339 & -109 & 31 & 113 & 115 & 133 & 373 & 385 \\
-351 & -347 & -341 & -111 & 33 & 103 & 117 & 137 & 371 & 389 \\
-377 & -353 & -319 & -101 & 35 & 107 & 119 & 141 & 365 & 383 \\
-409 & -355 & -299 & -87 & 51 & 91 & 121 & 161 & 357 & 369 \\
-395 & -367 & -323 & -65 & 59 & 81 & 89 & 261 & 297 & 363 \\
-379 & -345 & -327 & -99 & 71 & 79 & 105 & 145 & 349 & 401 \\
-375 & -315 & -255 & -205 & 75 & 77 & 93 & 95 & 381 & 429 \\
-427 & -361 & -199 & -163 & 83 & 85 & 97 & 139 & 321 & 425
\end{pmatrix}, \quad
D_{23} = \begin{pmatrix}
-133 & -63 & -1 & 183 & 229 & 269 \\
-135 & -61 & -3 & 181 & 227 & 259 \\
-129 & -65 & -5 & 177 & 223 & 263 \\
-123 & -67 & -7 & 179 & 217 & 273 \\
-137 & -59 & -9 & 175 & 211 & 261 \\
-117 & -69 & -11 & 173 & 221 & 257 \\
-113 & -71 & -13 & 171 & 225 & 245 \\
-109 & -73 & -15 & 167 & 219 & 275 \\
-115 & -75 & -17 & 157 & 207 & 271 \\
-125 & -57 & -19 & 169 & 213 & 265 \\
-131 & -53 & -21 & 163 & 215 & 267 \\
-105 & -77 & -23 & 165 & 203 & 233 \\
-107 & -79 & -25 & 151 & 201 & 249 \\
-99 & -83 & -27 & 161 & 187 & 251 \\
-103 & -81 & -29 & 155 & 191 & 237 \\
-101 & -91 & -31 & 141 & 189 & 235 \\
-93 & -89 & -33 & 143 & 199 & 241 \\
-97 & -85 & -35 & 145 & 195 & 231 \\
-95 & -87 & -37 & 139 & 193 & 243 \\
-127 & -47 & -39 & 153 & 205 & 239 \\
-111 & -55 & -41 & 149 & 209 & 255 \\
-121 & -51 & -43 & 147 & 197 & 253 \\
-119 & -49 & -45 & 159 & 185 & 247
\end{pmatrix},$$

$$A_{29} = \begin{pmatrix}
-331 & -91 & 1 & 3 & 73 & 345 \\
-337 & -93 & -5 & 7 & 95 & 333 \\
-347 & -79 & -11 & 9 & 105 & 323 \\
-289 & -181 & 13 & 15 & 103 & 339 \\
-335 & -99 & 17 & 19 & 57 & 341 \\
-329 & -131 & -21 & 23 & 155 & 303 \\
-271 & -223 & 25 & 27 & 121 & 321 \\
-277 & -201 & 29 & 31 & 75 & 343 \\
-319 & -159 & 33 & 35 & 85 & 325 \\
-305 & -193 & 37 & 39 & 111 & 311 \\
-313 & -163 & -41 & 43 & 207 & 267 \\
-265 & -257 & 45 & 47 & 147 & 283 \\
-287 & -231 & 49 & 51 & 125 & 293 \\
-317 & -143 & -55 & 53 & 165 & 297 \\
-273 & -249 & 59 & 61 & 107 & 295 \\
-299 & -173 & -63 & 65 & 189 & 281 \\
-279 & -263 & 67 & 69 & 161 & 245 \\
-327 & -151 & -71 & 123 & 185 & 241 \\
-291 & -259 & 77 & 97 & 141 & 235 \\
-269 & -225 & -81 & 157 & 191 & 227 \\
-315 & -239 & 83 & 109 & 153 & 209 \\
-307 & -167 & -87 & 117 & 211 & 233 \\
-275 & -187 & -101 & 89 & 213 & 261 \\
-309 & -145 & -113 & 127 & 219 & 221 \\
-285 & -149 & -139 & 115 & 203 & 255 \\
-301 & -137 & -129 & 119 & 195 & 253 \\
-247 & -205 & -135 & 133 & 217 & 237 \\
-251 & -177 & -169 & 183 & 199 & 215 \\
-243 & -179 & -175 & 171 & 197 & 229
\end{pmatrix}, \quad
B_{29} = \begin{pmatrix}
-245 & -235 & -229 & -219 & 1 & 97 & 367 & 463 \\
-255 & -237 & -227 & -209 & 3 & 95 & 369 & 461 \\
-257 & -249 & -215 & -207 & 5 & 93 & 371 & 459 \\
-289 & -259 & -205 & -175 & 7 & 101 & 363 & 457 \\
-263 & -239 & -225 & -201 & 9 & 87 & 377 & 455 \\
-243 & -233 & -231 & -221 & 11 & 81 & 383 & 453 \\
-271 & -241 & -223 & -193 & 13 & 83 & 381 & 451 \\
-311 & -285 & -179 & -153 & 15 & 107 & 357 & 449 \\
-293 & -269 & -195 & -171 & 17 & 89 & 375 & 447 \\
-327 & -319 & -145 & -137 & 19 & 133 & 331 & 445 \\
-333 & -281 & -183 & -131 & 21 & 111 & 353 & 443 \\
-279 & -275 & -189 & -185 & 23 & 77 & 387 & 441 \\
-317 & -313 & -151 & -147 & 25 & 109 & 355 & 439 \\
-323 & -305 & -159 & -141 & 27 & 105 & 359 & 437 \\
-265 & -247 & -217 & -199 & 29 & 61 & 403 & 435 \\
-291 & -283 & -181 & -173 & 31 & 73 & 391 & 433 \\
-339 & -287 & -177 & -125 & 33 & 99 & 365 & 431 \\
-297 & -261 & -203 & -167 & 35 & 65 & 399 & 429 \\
-343 & -325 & -139 & -121 & 37 & 119 & 345 & 427 \\
-341 & -329 & -135 & -123 & 39 & 117 & 347 & 425 \\
-303 & -273 & -191 & -161 & 41 & 63 & 401 & 423 \\
-393 & -379 & -113 & -43 & 91 & 197 & 251 & 389 \\
-413 & -307 & -149 & -59 & 45 & 187 & 301 & 395 \\
-349 & -253 & -211 & -115 & 47 & 79 & 385 & 417 \\
-337 & -335 & -129 & -127 & 49 & 103 & 361 & 415 \\
-419 & -277 & -163 & -69 & 51 & 157 & 315 & 405 \\
-409 & -309 & -143 & -67 & 53 & 169 & 299 & 407 \\
-411 & -295 & -165 & -57 & 55 & 155 & 321 & 397 \\
-421 & -351 & -85 & -71 & 75 & 213 & 267 & 373
\end{pmatrix},$$

$$C_{29} = \begin{pmatrix}
-579 & -297 & -291 & -283 & 1 & 55 & 253 & 289 & 327 & 525 \\
-577 & -305 & -293 & -275 & 3 & 51 & 269 & 287 & 311 & 529 \\
-575 & -333 & -281 & -261 & 5 & 53 & 247 & 299 & 319 & 527 \\
-523 & -361 & -301 & -265 & 7 & 57 & 219 & 279 & 315 & 573 \\
-519 & -363 & -325 & -243 & 9 & 61 & 217 & 255 & 337 & 571 \\
-569 & -517 & -353 & -11 & 63 & 225 & 227 & 239 & 341 & 355 \\
-567 & -351 & -317 & -215 & 13 & 59 & 229 & 263 & 365 & 521 \\
-565 & -513 & -357 & -15 & 67 & 201 & 223 & 241 & 339 & 379 \\
-563 & -541 & -329 & -17 & 39 & 251 & 257 & 285 & 295 & 323 \\
-503 & -403 & -335 & -209 & 19 & 77 & 177 & 245 & 371 & 561 \\
-559 & -349 & -307 & -235 & 21 & 43 & 231 & 273 & 345 & 537 \\
-557 & -511 & -197 & -185 & 23 & 69 & 271 & 309 & 383 & 395 \\
-493 & -427 & -331 & -199 & 25 & 87 & 153 & 249 & 381 & 555 \\
-501 & -411 & -343 & -195 & 27 & 79 & 169 & 237 & 385 & 553 \\
-551 & -515 & -221 & -163 & 29 & 65 & 267 & 313 & 359 & 417 \\
-549 & -387 & -321 & -193 & 31 & 73 & 165 & 259 & 415 & 507 \\
-547 & -391 & -373 & -139 & 33 & 105 & 189 & 207 & 441 & 475 \\
-531 & -369 & -347 & -203 & 35 & 49 & 211 & 233 & 377 & 545 \\
-543 & -477 & -393 & -37 & 103 & 133 & 187 & 213 & 367 & 447 \\
-539 & -469 & -401 & -41 & 111 & 159 & 161 & 179 & 419 & 421 \\
-461 & -443 & -389 & -157 & 45 & 119 & 137 & 191 & 423 & 535 \\
-483 & -463 & -457 & -47 & 141 & 143 & 145 & 151 & 433 & 437 \\
-481 & -467 & -431 & -71 & 125 & 129 & 147 & 149 & 449 & 451 \\
-465 & -445 & -425 & -115 & 75 & 131 & 135 & 167 & 453 & 489 \\
-459 & -435 & -429 & -127 & 81 & 113 & 123 & 181 & 455 & 497 \\
-495 & -405 & -375 & -175 & 83 & 93 & 121 & 173 & 471 & 509 \\
-499 & -439 & -413 & -99 & 85 & 97 & 107 & 277 & 399 & 485 \\
-479 & -473 & -397 & -101 & 89 & 95 & 171 & 183 & 407 & 505 \\
-533 & -409 & -303 & -205 & 91 & 109 & 117 & 155 & 487 & 491
\end{pmatrix}, \quad
D_{29} = \begin{pmatrix}
-169 & -79 & -1 & 231 & 289 & 333 \\
-171 & -77 & -3 & 227 & 287 & 341 \\
-161 & -81 & -5 & 229 & 285 & 335 \\
-159 & -83 & -7 & 225 & 281 & 329 \\
-173 & -71 & -9 & 223 & 283 & 339 \\
-157 & -85 & -11 & 213 & 279 & 347 \\
-167 & -75 & -13 & 219 & 271 & 337 \\
-163 & -73 & -15 & 221 & 275 & 345 \\
-145 & -87 & -17 & 217 & 273 & 325 \\
-147 & -89 & -19 & 215 & 261 & 313 \\
-139 & -91 & -21 & 207 & 277 & 323 \\
-137 & -93 & -23 & 205 & 267 & 343 \\
-141 & -95 & -25 & 195 & 269 & 311 \\
-125 & -97 & -27 & 211 & 263 & 327 \\
-129 & -99 & -29 & 209 & 247 & 317 \\
-131 & -101 & -31 & 201 & 249 & 301 \\
-165 & -67 & -33 & 197 & 265 & 319 \\
-135 & -103 & -35 & 177 & 259 & 307 \\
-127 & -115 & -37 & 183 & 239 & 291 \\
-117 & -107 & -39 & 203 & 241 & 295 \\
-119 & -111 & -41 & 187 & 243 & 299 \\
-133 & -105 & -43 & 179 & 237 & 293 \\
-121 & -109 & -45 & 181 & 235 & 315 \\
-123 & -113 & -47 & 175 & 233 & 297 \\
-155 & -63 & -49 & 193 & 257 & 303 \\
-153 & -65 & -51 & 185 & 253 & 331 \\
-149 & -69 & -53 & 189 & 245 & 305 \\
-151 & -59 & -55 & 191 & 255 & 321 \\
-143 & -61 & -57 & 199 & 251 & 309
\end{pmatrix},$$

$$A_{31} = \begin{pmatrix}
-365 & -97 & -1 & 3 & 111 & 349 \\
-333 & -141 & 5 & 7 & 91 & 371 \\
-367 & -125 & -11 & 9 & 201 & 293 \\
-369 & -89 & -13 & 15 & 109 & 347 \\
-363 & -99 & 17 & 19 & 65 & 361 \\
-359 & -115 & -23 & 21 & 137 & 339 \\
-303 & -215 & 25 & 27 & 113 & 353 \\
-341 & -177 & 29 & 31 & 123 & 335 \\
-343 & -155 & 33 & 35 & 73 & 357 \\
-313 & -209 & 37 & 39 & 95 & 351 \\
-337 & -165 & -41 & 43 & 189 & 311 \\
-329 & -213 & 45 & 47 & 131 & 319 \\
-331 & -211 & 49 & 51 & 121 & 321 \\
-325 & -187 & -53 & 55 & 243 & 267 \\
-323 & -227 & 57 & 59 & 117 & 317 \\
-281 & -277 & 61 & 63 & 119 & 315 \\
-327 & -167 & -75 & 67 & 197 & 305 \\
-285 & -233 & -69 & 77 & 239 & 271 \\
-355 & -135 & -71 & 105 & 149 & 307 \\
-345 & -241 & 79 & 101 & 183 & 223 \\
-283 & -249 & -81 & 175 & 191 & 247 \\
-299 & -287 & 83 & 93 & 153 & 257 \\
-275 & -253 & -85 & 159 & 199 & 255 \\
-297 & -289 & 87 & 103 & 133 & 263 \\
-291 & -179 & -143 & 107 & 245 & 261 \\
-301 & -185 & -129 & 127 & 237 & 251 \\
-269 & -225 & -139 & 195 & 203 & 235 \\
-309 & -161 & -145 & 163 & 173 & 279 \\
-265 & -217 & -147 & 151 & 219 & 259 \\
-295 & -171 & -157 & 169 & 181 & 273 \\
-231 & -207 & -205 & 193 & 221 & 229
\end{pmatrix}, \quad
B_{31} = \begin{pmatrix}
-281 & -253 & -243 & -215 & 1 & 107 & 389 & 495 \\
-267 & -251 & -245 & -229 & 3 & 101 & 395 & 493 \\
-271 & -263 & -233 & -225 & 5 & 99 & 397 & 491 \\
-335 & -275 & -221 & -161 & 7 & 123 & 373 & 489 \\
-329 & -289 & -207 & -167 & 9 & 119 & 377 & 487 \\
-287 & -265 & -231 & -209 & 11 & 93 & 403 & 485 \\
-305 & -259 & -237 & -191 & 13 & 95 & 401 & 483 \\
-269 & -255 & -241 & -227 & 15 & 83 & 413 & 481 \\
-257 & -249 & -247 & -239 & 17 & 79 & 417 & 479 \\
-347 & -299 & -197 & -149 & 19 & 117 & 379 & 477 \\
-311 & -283 & -213 & -185 & 21 & 89 & 407 & 475 \\
-333 & -325 & -171 & -163 & 23 & 113 & 383 & 473 \\
-323 & -291 & -205 & -173 & 29 & 85 & 411 & 467 \\
-353 & -349 & -147 & -143 & 31 & 131 & 365 & 465 \\
-367 & -293 & -203 & -129 & 35 & 105 & 391 & 461 \\
-317 & -307 & -189 & -179 & 37 & 77 & 419 & 459 \\
-279 & -261 & -235 & -217 & 41 & 53 & 443 & 455 \\
-427 & -385 & -155 & -25 & 127 & 135 & 361 & 369 \\
-421 & -393 & -151 & -27 & 111 & 153 & 357 & 371 \\
-471 & -351 & -137 & -33 & 139 & 145 & 345 & 363 \\
-327 & -313 & -193 & -159 & 39 & 81 & 423 & 449 \\
-457 & -309 & -183 & -43 & 115 & 121 & 341 & 415 \\
-469 & -303 & -175 & -45 & 103 & 157 & 301 & 431 \\
-285 & -273 & -223 & -211 & 47 & 49 & 445 & 451 \\
-315 & -295 & -201 & -181 & 51 & 57 & 437 & 447 \\
-435 & -337 & -165 & -55 & 87 & 141 & 339 & 425 \\
-441 & -343 & -133 & -75 & 59 & 177 & 375 & 381 \\
-463 & -359 & -109 & -61 & 97 & 187 & 321 & 387 \\
-433 & -405 & -91 & -63 & 65 & 277 & 319 & 331 \\
-439 & -355 & -125 & -73 & 67 & 199 & 297 & 429 \\
-453 & -399 & -71 & -69 & 169 & 195 & 219 & 409
\end{pmatrix},$$

$$C_{31} = \begin{pmatrix}
-619 & -317 & -311 & -303 & 1 & 57 & 285 & 309 & 335 & 563 \\
-617 & -355 & -291 & -287 & 3 & 59 & 265 & 329 & 333 & 561 \\
-615 & -325 & -315 & -295 & 5 & 53 & 281 & 305 & 339 & 567 \\
-565 & -361 & -331 & -293 & 7 & 55 & 259 & 289 & 327 & 613 \\
-555 & -389 & -343 & -263 & 9 & 65 & 231 & 277 & 357 & 611 \\
-609 & -321 & -319 & -301 & 11 & 51 & 251 & 299 & 369 & 569 \\
-559 & -385 & -349 & -257 & 13 & 61 & 235 & 271 & 363 & 607 \\
-557 & -399 & -341 & -253 & 15 & 63 & 221 & 279 & 367 & 605 \\
-603 & -553 & -377 & -17 & 67 & 207 & 243 & 297 & 323 & 413 \\
-601 & -409 & -347 & -193 & 19 & 85 & 211 & 273 & 427 & 535 \\
-599 & -365 & -337 & -249 & 21 & 47 & 255 & 283 & 371 & 573 \\
-597 & -551 & -379 & -23 & 69 & 215 & 233 & 241 & 387 & 405 \\
-595 & -549 & -381 & -25 & 71 & 203 & 237 & 239 & 383 & 417 \\
-547 & -425 & -359 & -219 & 27 & 73 & 195 & 261 & 401 & 593 \\
-591 & -473 & -391 & -95 & 29 & 147 & 229 & 267 & 353 & 525 \\
-589 & -407 & -375 & -179 & 31 & 83 & 213 & 245 & 441 & 537 \\
-587 & -431 & -345 & -187 & 33 & 81 & 189 & 275 & 433 & 539 \\
-585 & -527 & -247 & -191 & 35 & 93 & 171 & 373 & 429 & 449 \\
-583 & -519 & -411 & -37 & 101 & 183 & 185 & 209 & 435 & 437 \\
-581 & -533 & -397 & -39 & 87 & 161 & 223 & 225 & 395 & 459 \\
-515 & -487 & -351 & -197 & 41 & 105 & 133 & 269 & 423 & 579 \\
-491 & -481 & -403 & -175 & 43 & 129 & 139 & 217 & 445 & 577 \\
-575 & -513 & -313 & -149 & 45 & 107 & 159 & 307 & 461 & 471 \\
-507 & -505 & -489 & -49 & 151 & 153 & 155 & 163 & 451 & 477 \\
-509 & -497 & -469 & -75 & 135 & 143 & 145 & 165 & 479 & 483 \\
-495 & -475 & -455 & -125 & 77 & 137 & 157 & 169 & 499 & 511 \\
-545 & -447 & -439 & -119 & 79 & 113 & 167 & 205 & 457 & 529 \\
-531 & -467 & -463 & -89 & 109 & 121 & 173 & 181 & 443 & 523 \\
-503 & -453 & -393 & -201 & 91 & 99 & 103 & 199 & 517 & 541 \\
-543 & -465 & -415 & -127 & 97 & 115 & 117 & 227 & 493 & 501 \\
-521 & -485 & -421 & -123 & 111 & 131 & 141 & 177 & 419 & 571
\end{pmatrix}, \;
D_{31} = \begin{pmatrix}
-183 & -83 & -1 & 247 & 309 & 355 \\
-179 & -85 & -3 & 243 & 307 & 361 \\
-181 & -81 & -5 & 245 & 303 & 367 \\
-185 & -79 & -7 & 237 & 305 & 369 \\
-169 & -87 & -9 & 241 & 301 & 353 \\
-167 & -89 & -11 & 239 & 291 & 363 \\
-165 & -91 & -13 & 229 & 297 & 371 \\
-177 & -77 & -15 & 235 & 299 & 359 \\
-161 & -93 & -17 & 225 & 293 & 349 \\
-153 & -95 & -19 & 233 & 283 & 357 \\
-149 & -97 & -21 & 231 & 281 & 365 \\
-147 & -99 & -23 & 223 & 295 & 331 \\
-175 & -75 & -25 & 227 & 287 & 339 \\
-145 & -101 & -27 & 221 & 285 & 323 \\
-141 & -103 & -29 & 213 & 289 & 337 \\
-151 & -105 & -31 & 199 & 273 & 347 \\
-133 & -107 & -33 & 215 & 275 & 351 \\
-171 & -73 & -35 & 219 & 277 & 341 \\
-135 & -109 & -37 & 207 & 269 & 335 \\
-143 & -111 & -39 & 197 & 263 & 319 \\
-139 & -123 & -41 & 191 & 249 & 317 \\
-131 & -121 & -43 & 189 & 257 & 345 \\
-127 & -119 & -45 & 203 & 251 & 321 \\
-129 & -113 & -47 & 193 & 265 & 333 \\
-137 & -115 & -49 & 187 & 255 & 315 \\
-173 & -67 & -51 & 209 & 261 & 325 \\
-125 & -117 & -53 & 195 & 253 & 313 \\
-155 & -69 & -55 & 205 & 279 & 343 \\
-157 & -71 & -57 & 211 & 259 & 327 \\
-159 & -63 & -59 & 217 & 271 & 311 \\
-163 & -65 & -61 & 201 & 267 & 329
\end{pmatrix},$$

$$A_{37} = \begin{pmatrix}
-443 & -43 & 1 & 3 & 41 & 441 \\
-435 & -97 & -5 & 7 & 99 & 431 \\
-429 & -125 & -9 & 11 & 131 & 421 \\
-439 & -93 & -15 & 13 & 109 & 425 \\
-411 & -179 & 17 & 19 & 137 & 417 \\
-437 & -141 & -21 & 23 & 197 & 379 \\
-415 & -159 & 25 & 27 & 89 & 433 \\
-389 & -217 & 29 & 31 & 123 & 423 \\
-419 & -185 & -33 & 35 & 279 & 323 \\
-381 & -229 & 37 & 39 & 107 & 427 \\
-403 & -189 & -47 & 45 & 209 & 385 \\
-407 & -207 & 49 & 51 & 101 & 413 \\
-399 & -203 & -55 & 53 & 249 & 355 \\
-375 & -235 & -59 & 57 & 259 & 353 \\
-373 & -289 & 61 & 63 & 171 & 367 \\
-347 & -319 & 65 & 67 & 165 & 369 \\
-409 & -257 & 69 & 71 & 183 & 343 \\
-341 & -277 & -73 & 75 & 291 & 325 \\
-405 & -237 & 77 & 79 & 95 & 391 \\
-397 & -201 & -81 & 83 & 247 & 349 \\
-333 & -293 & -85 & 117 & 287 & 307 \\
-335 & -301 & -87 & 181 & 225 & 317 \\
-387 & -313 & 91 & 135 & 175 & 299 \\
-395 & -195 & -103 & 153 & 169 & 371 \\
-363 & -191 & -157 & 105 & 245 & 361 \\
-383 & -205 & -111 & 121 & 213 & 365 \\
-345 & -281 & -113 & 215 & 261 & 263 \\
-359 & -251 & -115 & 147 & 273 & 305 \\
-351 & -255 & -119 & 155 & 233 & 337 \\
-377 & -211 & -129 & 127 & 275 & 315 \\
-401 & -177 & -133 & 173 & 199 & 339 \\
-393 & -163 & -161 & 139 & 267 & 311 \\
-329 & -219 & -193 & 143 & 271 & 327 \\
-357 & -223 & -151 & 145 & 265 & 321 \\
-331 & -243 & -167 & 149 & 283 & 309 \\
-303 & -269 & -187 & 221 & 241 & 297 \\
-295 & -239 & -231 & 227 & 253 & 285
\end{pmatrix}, \quad
B_{37} = \begin{pmatrix}
-319 & -301 & -291 & -273 & 1 & 125 & 467 & 591 \\
-311 & -307 & -285 & -281 & 3 & 121 & 471 & 589 \\
-323 & -305 & -287 & -269 & 5 & 119 & 473 & 587 \\
-329 & -309 & -283 & -263 & 7 & 117 & 475 & 585 \\
-327 & -297 & -295 & -265 & 9 & 113 & 479 & 583 \\
-343 & -321 & -271 & -249 & 11 & 115 & 477 & 581 \\
-375 & -347 & -245 & -217 & 13 & 129 & 463 & 579 \\
-333 & -299 & -293 & -259 & 15 & 105 & 487 & 577 \\
-337 & -303 & -289 & -255 & 17 & 103 & 489 & 575 \\
-325 & -317 & -275 & -267 & 19 & 99 & 493 & 573 \\
-349 & -315 & -277 & -243 & 21 & 101 & 491 & 571 \\
-345 & -313 & -279 & -247 & 23 & 97 & 495 & 569 \\
-379 & -335 & -257 & -213 & 25 & 109 & 483 & 567 \\
-341 & -331 & -261 & -251 & 27 & 93 & 499 & 565 \\
-377 & -353 & -239 & -215 & 29 & 107 & 485 & 563 \\
-389 & -357 & -235 & -203 & 31 & 111 & 481 & 561 \\
-361 & -351 & -241 & -231 & 33 & 95 & 497 & 559 \\
-445 & -339 & -253 & -147 & 35 & 139 & 453 & 557 \\
-423 & -397 & -195 & -169 & 37 & 143 & 449 & 555 \\
-417 & -383 & -209 & -175 & 39 & 127 & 465 & 553 \\
-427 & -399 & -193 & -165 & 41 & 141 & 451 & 551 \\
-501 & -469 & -171 & -43 & 145 & 161 & 431 & 447 \\
-519 & -457 & -163 & -45 & 155 & 159 & 433 & 437 \\
-521 & -443 & -173 & -47 & 137 & 167 & 419 & 461 \\
-415 & -411 & -205 & -153 & 49 & 131 & 455 & 549 \\
-545 & -407 & -181 & -51 & 91 & 223 & 429 & 441 \\
-541 & -371 & -219 & -53 & 135 & 151 & 381 & 517 \\
-539 & -405 & -185 & -55 & 123 & 183 & 373 & 505 \\
-523 & -413 & -191 & -57 & 81 & 229 & 367 & 507 \\
-543 & -425 & -157 & -59 & 133 & 177 & 435 & 439 \\
-513 & -403 & -201 & -67 & 61 & 227 & 387 & 509 \\
-529 & -385 & -187 & -83 & 63 & 237 & 359 & 525 \\
-527 & -395 & -197 & -65 & 71 & 233 & 365 & 515 \\
-535 & -391 & -189 & -69 & 89 & 199 & 393 & 503 \\
-537 & -363 & -211 & -73 & 75 & 221 & 355 & 533 \\
-531 & -369 & -207 & -77 & 85 & 179 & 409 & 511 \\
-547 & -401 & -149 & -87 & 79 & 225 & 421 & 459
\end{pmatrix},$$

$$C_{37} = \begin{pmatrix}
-739 & -375 & -371 & -365 & 1 & 67 & 357 & 369 & 383 & 673 \\
-737 & -379 & -373 & -361 & 3 & 65 & 351 & 367 & 389 & 675 \\
-669 & -421 & -417 & -343 & 5 & 71 & 319 & 323 & 397 & 735 \\
-733 & -381 & -377 & -359 & 7 & 69 & 295 & 363 & 445 & 671 \\
-731 & -453 & -335 & -331 & 9 & 73 & 287 & 405 & 409 & 667 \\
-729 & -677 & -433 & -11 & 63 & 307 & 341 & 355 & 385 & 399 \\
-727 & -403 & -387 & -333 & 13 & 57 & 337 & 353 & 407 & 683 \\
-661 & -463 & -429 & -297 & 15 & 79 & 277 & 311 & 443 & 725 \\
-723 & -481 & -345 & -301 & 17 & 77 & 259 & 395 & 439 & 663 \\
-721 & -419 & -393 & -317 & 19 & 55 & 321 & 347 & 423 & 685 \\
-719 & -447 & -391 & -293 & 21 & 85 & 233 & 349 & 507 & 655 \\
-717 & -649 & -461 & -23 & 91 & 265 & 267 & 279 & 473 & 475 \\
-715 & -459 & -449 & -227 & 25 & 97 & 281 & 291 & 513 & 643 \\
-713 & -647 & -261 & -229 & 27 & 93 & 325 & 415 & 479 & 511 \\
-711 & -477 & -427 & -235 & 29 & 89 & 263 & 313 & 505 & 651 \\
-687 & -457 & -401 & -305 & 31 & 53 & 283 & 339 & 435 & 709 \\
-707 & -625 & -329 & -189 & 33 & 115 & 247 & 411 & 493 & 551 \\
-705 & -637 & -299 & -209 & 35 & 103 & 253 & 441 & 487 & 531 \\
-703 & -465 & -451 & -231 & 37 & 83 & 275 & 289 & 509 & 657 \\
-701 & -613 & -497 & -39 & 127 & 169 & 243 & 315 & 425 & 571 \\
-699 & -495 & -469 & -187 & 41 & 123 & 245 & 271 & 553 & 617 \\
-697 & -503 & -413 & -237 & 43 & 87 & 223 & 327 & 517 & 653 \\
-695 & -577 & -533 & -45 & 163 & 179 & 207 & 241 & 499 & 561 \\
-693 & -595 & -515 & -47 & 145 & 175 & 225 & 251 & 489 & 565 \\
-593 & -583 & -455 & -219 & 49 & 147 & 157 & 285 & 521 & 691 \\
-689 & -587 & -523 & -51 & 153 & 195 & 211 & 217 & 529 & 545 \\
-569 & -541 & -491 & -249 & 59 & 151 & 171 & 199 & 589 & 681 \\
-679 & -483 & -431 & -257 & 61 & 125 & 133 & 309 & 607 & 615 \\
-665 & -537 & -467 & -181 & 75 & 109 & 203 & 273 & 559 & 631 \\
-605 & -591 & -573 & -81 & 161 & 173 & 185 & 193 & 557 & 581 \\
-579 & -563 & -543 & -165 & 95 & 159 & 183 & 191 & 603 & 619 \\
-627 & -585 & -539 & -99 & 137 & 155 & 201 & 205 & 555 & 597 \\
-575 & -535 & -501 & -239 & 101 & 105 & 117 & 269 & 599 & 659 \\
-645 & -567 & -527 & -111 & 107 & 141 & 197 & 255 & 549 & 601 \\
-639 & -519 & -471 & -221 & 113 & 135 & 139 & 213 & 621 & 629 \\
-641 & -623 & -437 & -149 & 119 & 129 & 143 & 303 & 547 & 609 \\
-633 & -611 & -485 & -121 & 131 & 167 & 177 & 215 & 525 & 635
\end{pmatrix}, \quad
D_{37} = \begin{pmatrix}
-221 & -99 & -1 & 293 & 365 & 429 \\
-217 & -97 & -3 & 295 & 369 & 439 \\
-219 & -101 & -5 & 279 & 367 & 433 \\
-211 & -103 & -7 & 287 & 363 & 405 \\
-215 & -95 & -9 & 291 & 359 & 431 \\
-201 & -105 & -11 & 283 & 361 & 443 \\
-197 & -107 & -13 & 285 & 355 & 417 \\
-187 & -109 & -15 & 289 & 357 & 425 \\
-213 & -93 & -17 & 281 & 353 & 421 \\
-191 & -111 & -19 & 277 & 351 & 397 \\
-185 & -113 & -21 & 273 & 347 & 437 \\
-189 & -115 & -23 & 271 & 339 & 391 \\
-177 & -117 & -25 & 269 & 349 & 415 \\
-205 & -91 & -27 & 275 & 345 & 427 \\
-181 & -119 & -29 & 255 & 337 & 435 \\
-171 & -121 & -31 & 259 & 341 & 441 \\
-169 & -123 & -33 & 263 & 343 & 383 \\
-165 & -125 & -35 & 265 & 333 & 389 \\
-209 & -87 & -37 & 267 & 331 & 401 \\
-159 & -127 & -39 & 257 & 335 & 403 \\
-155 & -129 & -41 & 261 & 329 & 395 \\
-163 & -131 & -43 & 245 & 327 & 387 \\
-167 & -133 & -45 & 231 & 319 & 419 \\
-179 & -135 & -47 & 225 & 307 & 379 \\
-157 & -145 & -49 & 229 & 309 & 399 \\
-173 & -137 & -51 & 223 & 305 & 381 \\
-149 & -143 & -53 & 233 & 315 & 393 \\
-151 & -147 & -55 & 235 & 303 & 371 \\
-161 & -141 & -57 & 227 & 299 & 377 \\
-153 & -139 & -59 & 237 & 301 & 373 \\
-195 & -83 & -61 & 253 & 311 & 411 \\
-193 & -81 & -63 & 247 & 325 & 407 \\
-207 & -75 & -65 & 241 & 317 & 413 \\
-199 & -77 & -67 & 249 & 323 & 375 \\
-203 & -79 & -69 & 251 & 297 & 385 \\
-175 & -89 & -71 & 243 & 321 & 409 \\
-183 & -85 & -73 & 239 & 313 & 423
\end{pmatrix}.$$

With the aid of a computer, for $p \in \{7, 11, 13, 17, 19, 23, 29, 31, 37\}$, one can prove that A_p is an odd-normal BRMR$(p, 6)$, B_p is an odd-normal balanced BRMR$(p, 8)$, and C_p is an odd-normal BRMR$(p, 10)$. Let $k = 24p$ and $D_p = (d_{i,j}^{(p)})$ and we define a $p \times 6$ matrix $E_p = (e_{i,j}^{(p)})$ by

$$e_{i,j}^{(p)} = sign(d_{i,5-j}^{(p)})(k - |d_{i,5-j}^{(p)}|), \quad i \in I_p,\ j \in I_6,$$

where $\mathrm{sign}(d_{i,5-j}^{(p)}) = |d_{i,5-j}^{(p)}|/d_{i,5-j}^{(p)}$. Let $F_p = (D_p\ E_p)$. Then, one can prove that F_p is an odd-normal BRMR$(p, 12)$ for $p \in \{7, 11, 13, 17, 19, 23, 29, 31, 37\}$. This completes the proof. $\square$

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