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Contravariant Curvatures of Doubly Warped Product Poisson Manifolds

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Abstract: In this paper, we investigate the sectional contravariant curvature of a doubly warped product manifold $({}_fB \times_b F, \tilde{g}, \Pi = \Pi_B + \Pi_F)$ equipped with a product Poisson structure Π, using warping functions and sectional curvatures of its factor manifolds $(B, \tilde{g}_B, \Pi_B)$ and $(F, \tilde{g}_F, \Pi_F)$. Qualar and null sectional contravariant curvatures of $({}_fB \times_b F, \tilde{g}, \Pi)$ are also given. As an example, we construct a four-dimensional Lorentzian doubly warped product Poisson manifold where qualar and sectional curvatures are obtained.

Keywords: doubly warped product; Poisson geometry; sectional contravariant curvature; qualar; curvature

MSC: 53C20; 53D17

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1. Introduction

The concept of warped products was proposed by Bishop and O'Neill [1] to investigate Riemannian manifolds of negative sectional curvatures. Such products, which are generalizations of direct product Riemannian manifolds, play an important role in differential geometry and also have many applications in physics, particularly in the theory of relativity. In fact, many exact solutions to Einstein's field equation can be expressed in terms of Lorentzian warped products [2]. The last theory demands a larger class of manifolds, and then the idea of doubly warped products was introduced as a generalization of warped product manifolds. For two given pseudo-Riemannian manifolds, $(B, \tilde{g}_B)$ and $(F, \tilde{g}_F)$, and for two positive smooth functions, b and f, on B and F, respectively, the doubly warped product $({}_fB \times_b F, \tilde{g})$ is the product manifold $B \times F$ equipped with the metric $\tilde{g} = f^2\tilde{g}_B + b^2\tilde{g}_F$. The smooth functions b and f are called warping functions, B is called base manifold, and F is called fiber manifold.

The concept of Poisson structure first appeared with Poisson [3] in order to obtain new integrals of motions in Hamiltonian mechanics. Later, Lichnerowicz [4] introduced the notion of the Poisson manifold as a smooth manifold equipped with a Poisson structure. In [5], Vaisman introduced the idea of a contravariant derivative on the Poisson manifold. Then, Fernandes discussed many properties of the Poisson manifold with contravariant connections [6]. It is worth noting that Poisson structures play an important role in Hamiltonian mechanics and also have interaction with the theory of relativity. Recently, [7,8] studied contravariant gravity on Poisson manifolds equipped with a Riemannian metric using Levi–Civita contravariant connections. The compatibility between a Poisson structure and a pseudo-Riemannian metric was investigated on different classes of smooth manifolds by many authors [9–12]. In [13], Nasri and Mustapha studied some of the geometric properties of the product Riemannian Poisson manifold. After that, [14] introduced the concept of the warped product Poisson manifold as a generalization of the product Riemannian Poisson manifold. The contravariant curvatures of the warped product Poisson manifold were studied in [15].

Inspired by these studies, we introduce the notion of the doubly warped product Poisson manifold $({}_f B \times_b F, \tilde{g}, \Pi = \Pi_B + \Pi_F)$, associated with components $(B, \tilde{g}_B, \Pi_B)$ and $(F, \tilde{g}_F, \Pi_F)$, where Π_B and Π_F are Poisson tensors on B and F, respectively, and we investigate some geometric structures on $({}_f B \times_b F, \tilde{g}, \Pi)$, such as the Levi–Civita contravariant connection, curvature tensor, sectional curvature, qualar, and null sectional curvatures. Some interesting consequences of the sectional curvature of a doubly warped product Poisson manifold are given.

This paper is organized as follows: Section 2 represents some Poisson-tensor-, contravariant-connection-, and curvature-related formulas on a Poisson manifold equipped with a pseudo-Riemannian metric. In Section 3, we derive the Levi–Civita contravariant connection, curvature tensor, and sectional contravariant curvature of a doubly warped product Poisson manifold $({}_f B \times_b F, \tilde{g}, \Pi)$ in terms of Levi–Civita connections, curvatures, and sectional curvatures of components $(B, \tilde{g}_B, \Pi_B)$ and $(F, \tilde{g}_F, \Pi_F)$. Then, using warping functions f, b, and sectional curvatures of $(B, \tilde{g}_B, \Pi_B)$ and $(F, \tilde{g}_F, \Pi_F)$, we investigate the sectional contravariant curvature of $({}_f B \times_b F, \tilde{g}, \Pi)$. In Section 4, we compute the qualar and null sectional contravariant curvatures of a degenerate plane at a point in $B \times F$. In the final section, we give an example of a four-dimensional Lorentzian doubly warped product Poisson manifold, where Levi–Civita connection, qualar, and sectional curvatures are obtained.

2. Preliminaries

2.1. Poisson Structure

A Poisson bracket $\{,\}$ on a smooth manifold M is a Lie bracket $\{,\}$ on the space of a real-valued smooth function $\mathcal{C}^\infty(M)$ on M,

$$\{,\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M), \quad (\phi, \psi) \mapsto \{\phi, \psi\},$$

which satisfies the following Leibniz identity:

$$\{\phi, \psi\varphi\} = \{\phi, \psi\}\varphi + \psi\{\phi, \varphi\}, \quad \forall\ \phi, \psi, \varphi \in \mathcal{C}^\infty(M).$$

From the Leibniz identity, for any smooth function ϕ on a Poisson manifold $(M, \{,\})$, the map $\psi \mapsto \{\phi, \psi\}$ is a derivation, and thus, there exists a unique vector field X_ϕ on M such that:

$$X_\phi(\psi) = \{\phi, \psi\}, \quad \forall\ \psi \in \mathcal{C}^\infty(M).$$

This vector X_ϕ is called a Hamiltonian vector field of ϕ and if $X_\phi \equiv 0$, and the smooth function ϕ is called a Casimir function on M.

It also follows from the Leibniz identity that, for each Poisson structure $\{,\}$ on M, there exists a bivector field $\Pi \in \Gamma(\Lambda^2 TM)$, called a Poisson tensor, on M such that for any $\phi, \psi \in \mathcal{C}^\infty(M)$,

$$\{\phi, \psi\} = \Pi(d\phi, d\psi).$$

A Poisson manifold (M, Π) is a smooth manifold M equipped with a Poisson tensor Π.

2.2. Contravariant Connection and Curvature Tensor

The notion of contravariant connection defined on the Poisson manifold is similar to the notion of usual covariant connection. This connection was introduced by Vaismann in [5] and analyzed in detail by Fernandes [6].

Let (M, Π) be a Poisson manifold. For each Poisson tensor Π, we associate the anchor map $\sharp_\Pi : T^*M \to TM$ defined by $\gamma(\sharp_\Pi(\beta)) = \Pi(\beta, \gamma)$ and the Koszul bracket $[,]_\Pi$ defined on the space of differential 1-forms $\Omega^1(M)$ by,

$$[\beta, \gamma]_\Pi = \mathcal{L}_{\sharp_\Pi(\beta)}\gamma - \mathcal{L}_{\sharp_\Pi(\gamma)}\beta - d(\Pi(\beta, \gamma)), \quad \beta, \gamma \in \Omega^1(M).$$

A contravariant connection $\mathcal{D}$ on M with respect to the Poisson tensor Π is a $\mathbb{R}-$bilinear map

$$\mathcal{D} : \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M), \quad (\beta, \gamma) \mapsto \mathcal{D}_\beta \gamma,$$

satisfying the following properties:

(i) The mapping $\beta \mapsto \mathcal{D}_\beta \gamma$ is $\mathcal{C}^\infty(M)$-linear, i.e.,

$$\mathcal{D}_{\psi\beta}\gamma = \psi \mathcal{D}_\beta \gamma;$$

(ii) The mapping $\gamma \mapsto \mathcal{D}_\beta \gamma$ is a derivation in the following sense:

$$\mathcal{D}_\beta(\psi\gamma) = \psi \mathcal{D}_\beta \gamma + \sharp_\Pi(\beta)(\psi)\gamma, \quad \forall \psi \in \mathcal{C}^\infty(M).$$

The torsion $\mathcal{T}$ and the curvature $\mathcal{R}$ of a contravariant connection $\mathcal{D}$ are $(2,1)$- and $(3,1)$-type tensor fields, defined, respectively, by:

$$\mathcal{T}(\beta, \gamma) = \mathcal{D}_\beta \gamma - \mathcal{D}_\gamma \beta - [\beta, \gamma]_\Pi,$$

$$\mathcal{R}(\beta, \gamma)\eta = \mathcal{D}_\beta \mathcal{D}_\gamma \eta - \mathcal{D}_\gamma \mathcal{D}_\beta \eta - \mathcal{D}_{[\beta,\gamma]_\Pi}\eta, \tag{1}$$

where $\beta, \gamma, \eta \in \Omega^1(M)$. When $\mathcal{T} \equiv 0$ (respectively, $\mathcal{R} \equiv 0$,) the connection $\mathcal{D}$ is called torsion-free (respectively, flat).

Now, let $(M, \tilde{g})$ be a smooth manifold equipped with a covariant pseudo-Riemannian metric $\tilde{g}$. Using the metric $\tilde{g}$, we can define the isomorphisms,

$$\flat_{\tilde{g}} : TM \to T^*M; X \mapsto \tilde{g}(X, .)$$

and its inverse $\natural_{\tilde{g}}$, called musical isomorphisms. The contravariant metric g associated with $\tilde{g}$ is defined for any 1-forms $\beta, \gamma \in \Omega^1(M)$ by

$$g(\beta, \gamma) = \tilde{g}(\natural_{\tilde{g}}(\beta), \natural_{\tilde{g}}(\gamma)). \tag{2}$$

For a Poisson tensor Π on M, there exists a unique contravariant connection $\mathcal{D}$ associated with (g, Π), called the Levi–Civita contravariant connection, such that the metric g is parallel with respect to $\mathcal{D}$, i.e.,

$$\sharp_\Pi(\beta) g(\gamma, \eta) = g(\mathcal{D}_\beta \gamma, \eta) + g(\gamma, \mathcal{D}_\beta \eta),$$

and that $\mathcal{D}$ is torsion-free, i.e.,

$$\mathcal{D}_\beta \gamma - \mathcal{D}_\gamma \beta = [\beta, \gamma]_\Pi.$$

The contravariant connection $\mathcal{D}$ is the analog of the usual covariant Levi–Civita connection and can be expressed by the Koszul formula:

$$\begin{aligned} 2g(\mathcal{D}_\beta \gamma, \eta) \quad = \quad & \sharp_\Pi(\beta) g(\gamma, \eta) + \sharp_\Pi(\gamma) g(\beta, \eta) - \sharp_\Pi(\eta) g(\beta, \gamma) \\ + \quad & g([\beta, \gamma]_\Pi, \eta) + g([\eta, \beta]_\Pi, \gamma) + g([\eta, \gamma]_\Pi, \beta). \end{aligned} \tag{3}$$

For any smooth function $\psi \in \mathcal{C}^\infty(M)$, we can write $\mathcal{D}\psi = d\psi \circ \sharp_\Pi$, and for any 1-form $\beta \in \Omega^1(M)$, we have:

$$(\mathcal{D}\psi)(\beta) = \mathcal{D}_\beta \psi = \sharp_\Pi(\beta)(\psi) = d\psi(\sharp_\Pi(\beta)).$$

The relation between the Poisson tensor Π and the metric g is defined by the field endomorphism $J : T^*M \to T^*M$ as follows:

$$\Pi(\beta, \gamma) = g(J\beta, \gamma) = -g(\beta, J\gamma), \qquad \beta, \gamma \in \Omega^1(M).$$

The contravariant Laplacian operator Δ^D of any tensor field T on an $m-$dimensional manifold M, associated with the Levi–Civita connection $\mathcal{D}$, is defined by [16]:

$$\begin{aligned} \Delta^{\mathcal{D}}(T) &= -\sum_{i=1}^{m} \mathcal{D}^2_{\beta_i,\beta_i} T \\ &= \sum_{i=1}^{m} -\mathcal{D}_{\beta_i}\mathcal{D}_{\beta_i} T + \mathcal{D}_{\mathcal{D}_{\beta_i}\beta_i} T, \end{aligned} \tag{4}$$

where $\{\beta_1, \ldots, \beta_m\}$ is a local $g-$orthonormal coframe field of M.

For any smooth function ψ on M, the contravariant Hessian $\mathcal{H}^{\psi}_{\Pi}$ of ψ associated with $\mathcal{D}$ is given by [14]:

$$\begin{aligned} \mathcal{H}^{\psi}_{\Pi}(\beta,\gamma) &= -\sharp_{\Pi}(\mathcal{D}_{\beta}\gamma)(\psi) + \sharp_{\Pi}(\beta)(\sharp_{\Pi}(\gamma)(\psi)) \\ &= -g(\mathcal{D}_{\beta} J d\psi, \gamma). \end{aligned} \tag{5}$$

From Equations (4) and (5), for any $\psi \in \mathcal{C}^{\infty}(M)$, the contravariant Laplacian of ψ is given by:

$$\begin{aligned} \Delta^{\mathcal{D}}(\psi) &= -\sum_{i=1}^{m} \mathcal{H}^{\psi}_{\Pi}(\beta_i,\beta_i) \\ &= \sum_{i=1}^{m} g(\mathcal{D}_{\beta_i} J d\psi, \beta_i). \end{aligned}$$

If $\mathcal{R}$ is the curvature of the connection $\mathcal{D}$ and $\pi = \operatorname{span}(\beta_p, \gamma_p)$ is a non-degenerate plane at the point $p \in M$, spanned by two non-parallel cotangent vectors β_p and γ_p, the sectional contravariant curvature of (M, Π, g) is the number given by:

$$\mathcal{K}_p(\beta_p,\gamma_p) = \frac{g(\mathcal{R}^M(\beta_p,\gamma_p)\gamma_p,\beta_p)}{Q(\beta_p,\gamma_p)}, \tag{6}$$

where $Q(\beta_p,\gamma_p) = g(\beta_p,\beta_p)g(\gamma_p,\gamma_p) - g^2(\beta_p,\gamma_p)$. Note that, the connection $\mathcal{D}$ is flat if and only if the sectional contravariant curvature is identically zero.

2.3. Qualar and Null Sectional Contravariant Curvatures

The sectional covariant curvature function is defined only on non-degenerate planes. Then, a new type of curvature is needed for degenerate planes. Therefore, Harris [17] proposed what is called the null sectional curvature of a degenerate plane in order to study Lorentzian manifold. The geometrical interpretation of this curvature was studied by Albujer and Haesen in [18]. Some relations between qualar and null sectional curvatures are given by Gülbahar in [19].

By analogy with the covariant case, we define the qualar and null sectional contravariant curvatures on a Poisson manifold equipped with a pseudo-Riemannian metric.

Firstly, we need to define the notions of timelike, null, and spacelike one forms. Using Equation (2), a one form $\beta \in \Omega^1(M)$ is said to be:

(i) Timelike, if $g(\beta,\beta) = \tilde{g}(\sharp_{\tilde{g}}(\beta), \sharp_{\tilde{g}}(\beta)) < 0$.
(ii) Null, if $g(\beta,\beta) = \tilde{g}(\sharp_{\tilde{g}}(\beta), \sharp_{\tilde{g}}(\beta)) = 0$ and $\beta \neq 0$.
(iii) Spacelike, if $g(\beta,\beta) = \tilde{g}(\sharp_{\tilde{g}}(\beta), \sharp_{\tilde{g}}(\beta)) > 0$.

Let (M, Π, g) be an m-dimensional Poisson manifold equipped with a pseudo-Riemannian metric g of index s, and let $(dx_1, \ldots, dx_m)$ be a local $g-$orthonormal basis, such that:

$$g(dx_i, dx_i) = -1, \forall i \in \{1, \ldots, s\} \quad \text{and} \quad g(dx_k, dx_k) = 1, \forall k \in \{s+1, \ldots, m\}.$$

The qualar contravariant curvature at a point $p \in M$ is defined by

$$\operatorname{qual}(p) = 2\sum_{i=1}^{s}\sum_{k=s+1}^{m} \mathcal{K}(dx_i, dx_k). \tag{7}$$

Using timelike and spacelike one forms, we can define null one forms as follows:

$$\eta_k^i = \frac{1}{\sqrt{2}}(dx_i + dx_k),$$

where $i \in \{1, \ldots, s\}$ and $k \in \{s+1, \ldots, m\}$.

For any $l \in \{1, \ldots, m\}$, $l \neq i, l \neq k$, the plane spanned by η_k^i and dx_l is a degenerate plane, and its null sectional contravariant curvature $\widetilde{\mathcal{K}}$ is defined by:

$$\begin{aligned} \widetilde{\mathcal{K}}(\eta_k^i, dx_l) &= \frac{g(\mathcal{R}(\eta_k^i, dx_l)dx_l, \eta_k^i)}{g(dx_l, dx_l)} \\ &= -\frac{1}{2}\mathcal{K}(dx_i, dx_l) + \frac{1}{2}\mathcal{K}(dx_k, dx_l) + \xi_l g(\mathcal{R}(dx_i, dx_l)dx_l, dx_k), \end{aligned} \tag{8}$$

where $\xi_l = g(dx_l, dx_l) = \pm 1$.

2.4. Vertical and Horizontal Lifts

In this subsection, we recall the notions of vertical and horizontal lifts of tensor fields on the product manifold (see [14,20]).

Let B and F be two smooth manifolds. We denote by $\mathfrak{X}(B)$ and $\mathfrak{X}(F)$ the spaces of vector fields on B and F, respectively, and by $\sigma_1 : B \times F \to B$ and $\sigma_2 : B \times F \to F$ the first and the second projection of $B \times F$ on B and F, respectively.

Let $f \in \mathcal{C}^\infty(F)$ be a smooth function on F. The vertical lift of f to $B \times F$ is the smooth function $f^v = f \circ \sigma_2$ on $B \times F$.

Let $q \in F$ and $X_q \in T_q F$. For any $p \in B$, the vertical lift of X_q to (p, q) is the unique tangent vector field $X^v_{(p,q)}$ in $T_{(p,q)}(B \times F)$ such that

$$\begin{cases} d_{(p,q)}\sigma_1(X^v_{(p,q)}) = 0 \\ d_{(p,q)}\sigma_2(X^v_{(p,q)}) = X_q. \end{cases}$$

We can define similarly the horizontal lift b^h of a function $b \in \mathcal{C}^\infty(B)$ and the horizontal lift X^h of a vector field $X \in \mathfrak{X}(B)$ on B to $B \times F$ by using the first projection σ_1.

Now, let $\alpha_2 \in \Omega^1(F)$ be a smooth 1-form on F; then, its pullback $\sigma_2^*(\alpha_2) = \alpha_2^v$ by the second projection σ_2, is a smooth 1-form α_2^v on $B \times F$, called the vertical lift of α_2 to $B \times F$, such that for any $u \in T_{(p,q)}(B \times F))$, we have,

$$(\alpha_2^v)_{(p,q)}(u) = (\alpha_2)_q(d_{(p,q)}\sigma_2(u)).$$

Similarly, we can define the horizontal lift α_1^h of a smooth 1-form $\alpha_1 \in \Omega^1(B)$ by using the first projection σ_1.

Lemma 1 ([21]). *For any smooth functions $b \in \mathcal{C}^\infty(B)$, $f \in \mathcal{C}^\infty(F)$ and for any vector fields $X_1 \in \mathfrak{X}(B)$ and $X_2 \in \mathfrak{X}(F)$, we have:*

$$\begin{aligned} X_1^h(b^h) &= (X_1(b))^h, \quad X_1^h(f^v) = 0, \quad X_2^v(br^h) = 0, \\ X_2^v(f^v) &= (X_2(f))^v, \quad (bX_1)^h = b^h X_1^h \ \textit{and} \ (fX_2)^v = f^v X_2^v. \end{aligned}$$

2.5. Doubly Warped Product Poisson Manifold

Let Π_B and Π_F be Poisson tensors on B and F, respectively. The product Poisson structure on the product manifold $B \times F$ is the unique Poisson structure $\Pi = \Pi_B + \Pi_F$ such that for any $\beta_1, \gamma_1 \in \Omega^1(B)$ and $\beta_2, \gamma_2 \in \Omega^1(F)$, we have [14]:

$$\begin{cases} \Pi(\beta_1^h, \gamma_1^h) &= \Pi_B(\beta_1, \gamma_1)^h \\ \Pi(\beta_2^v, \gamma_2^v) &= \Pi_F(\beta_2, \gamma_2)^v \\ \Pi(\beta_1^h, \gamma_2^v) &= \Pi(\beta_2^v, \gamma_1^h) = 0. \end{cases}$$

Proposition 1 ([14]). *Let $\beta_1, \gamma_1 \in \Omega^1(B)$, $\beta_2, \gamma_2 \in \Omega^1(F)$, $\beta = \beta_1^h + \beta_2^v$ and $\gamma = \gamma_1^h + \gamma_2^v$. Then, we have:*

1. $\sharp_\Pi(\beta) = [\sharp_{\Pi_B}(\beta_1)]^h + [\sharp_{\Pi_F}(\beta_2)]^v$,
2. $\mathcal{L}_{\sharp_\Pi(\beta)}\gamma = [\mathcal{L}_{\sharp_{\Pi_B}(\beta_1)}\gamma_1]^h + [\mathcal{L}_{\sharp_{\Pi_F}(\beta_2)}\gamma_2]^v$,
3. $[\beta, \gamma]_\Pi = [\beta_1, \gamma_1]_{\Pi_B}^h + [\beta_2, \gamma_2]_{\Pi_F}^v$

Now, let $\tilde{g}_B$ and $\tilde{g}_F$ be pseudo-Riemannian metrics on B and F, respectively. The doubly warped product $({}_fB \times_b F, \tilde{g})$ is the product manifold $B \times F$ equipped with the metric,

$$\tilde{g} = (f^v)^2 \sigma_1^*(\tilde{g}_B) + (b^h)^2 \sigma_2^*(\tilde{g}_F),$$

where $f : F \to (0, \infty)$ and $b : B \to (0, \infty)$ are smooth positive functions on F and B, respectively, called warping functions.

The doubly warped metric is defined explicitly, for any $X_1, Y_1 \in \mathfrak{X}(B)$ and $X_2, Y_2 \in \mathfrak{X}(F)$, by:

$$\begin{cases} \tilde{g}(X_1^h, Y_1^h) &= (f^v)^2 \tilde{g}_B(X_1, Y_1)^h \\ \tilde{g}(X_2^v, Y_2^v) &= (b^h)^2 \tilde{g}_F(X_2, Y_2)^v \\ \tilde{g}(X_1^h, Y_2^v) &= g(X_2^v, Y_1^h) = 0. \end{cases}$$

The contravariant pseudo-Riemannian metric g associated with $\tilde{g}$ is given explicitly, for any $\beta_1, \gamma_1 \in \Omega^1(B)$ and $\beta_2, \gamma_2 \in \Omega^1(F)$, by [20]:

$$\begin{cases} g(\beta_1^h, \gamma_1^h) &= \frac{1}{(f^v)^2} g_B(\beta_1, \gamma_1)^h \\ g(\beta_2^v, \gamma_2^v) &= \frac{1}{(b^h)^2} g_F(\beta_2, \gamma_2)^v \\ g(\beta_1^h, \gamma_2^v) &= g(\beta_2^v, \gamma_1^h) = 0. \end{cases} \tag{9}$$

Definition 1. *Let (B, g_B, Π_B) and (F, g_F, Π_F) be two pseudo-Riemannian manifolds equipped with Poisson tensor Π_B and Π_F, respectively, and let b and f be two smooth positive functions on B and F, respectively. A doubly warped product manifold $({}_fB \times_b F, g, \Pi)$ equipped with the product Poisson structure $\Pi = \Pi_B + \Pi_F$ will be called doubly warped product Poisson manifold of (B, g_B, Π_B) and (F, g_F, Π_F).*

3. Sectional Curvature of Doubly Warped Product Poisson Manifolds

In this section, we calculate the Levi–Civita connection, curvature tensor and sectional contravariant curvature of a doubly warped product Poisson manifold $({}_fB \times_b F, g, \Pi)$. Then, using sectional contravariant curvatures and warping functions of components (B, g_B, Π_B) and (F, g_F, Π_F), we discuss the sectional curvature of $({}_fB \times_b F, g, \Pi)$.

Proposition 2. *Let $\mathcal{D}^B$, $\mathcal{D}^F$ and $\mathcal{D}$ be the Levi–Civita contravariant connections associated, respectively, with (g_B, Π_B), (g_F, Π_F) and (g, Π). Then, for any $\beta_1, \gamma_1 \in \Omega^1(B)$ and $\beta_2, \gamma_2 \in \Omega^1(F)$, we have:*

1. $\mathcal{D}_{\beta_1^h}\gamma_1^h = (\mathcal{D}^B_{\beta_1}\gamma_1)^h - \frac{(b^h)^2}{(f^v)^3} g_B(\beta_1, \gamma_1)^h (J_F df)^v$.
2. $\mathcal{D}_{\beta_1^h}\gamma_2^v = \mathcal{D}_{\gamma_2^v}\beta_1^h = \frac{1}{f^v} g_F(\gamma_2, J_F df)^v \beta_1^h + \frac{1}{b^h} g_B(\beta_1, J_B db)^h \gamma_2^v$
3. $\mathcal{D}_{\beta_2^v}\gamma_2^v = (\mathcal{D}^F_{\beta_2}\gamma_2)^v - \frac{(f^v)^2}{(b^h)^3} g_F(\beta_2, \gamma_2)^v (J_B db)^h$.

Proof. 1. Using Lemma 1, Proposition 1, Equation (9) and taking $\beta=\beta_1^h, \gamma=\gamma_1^h, \eta=\eta_1^h$ in the Koszul Formula (3), we obtain:

$$\begin{aligned} 2g(\mathcal{D}_{\beta_1^h}\gamma_1^h,\eta_1^h) &= \sharp_\Pi(\beta_1^h)g(\gamma_1^h,\eta_1^h)+\sharp_\Pi(\gamma_1^h)g(\beta_1^h,\eta_1^h)-\sharp_\Pi(\eta_1^h)g(\beta_1^h,\gamma_1^h)\\ &+ g([\beta_1^h,\gamma_1^h]_\Pi,\eta_1^h)+g([\eta_1^h,\beta_1^h]_\Pi,\gamma_1^h)+g([\eta_1^h,\gamma_1^h]_\Pi,\beta_1^h)\\ &= (\sharp_{\Pi_B}(\beta_1))^h(\tfrac{1}{(f^v)^2}g_B(\gamma_1,\eta_1)^h)+(\sharp_{\Pi_B}(\gamma_1))^h(\tfrac{1}{(f^v)^2}g_B(\beta_1,\eta_1)^h)\\ &- (\sharp_{\Pi_B}(\eta_1))^h(\tfrac{1}{(f^v)^2}g_B(\beta_1,\gamma_1)^h)+\tfrac{1}{(f^v)^2}g_B([\beta_1,\gamma_1]_{\Pi_B},\eta_1)^h\\ &+ \tfrac{1}{(f^v)^2}g_B([\eta_1,\beta_1]_{\Pi_B},\gamma_1)^h+\tfrac{1}{(f^v)^2}g_B([\eta_1,\gamma_1]_{\Pi_B},\beta_1)^h\\ &= \tfrac{2}{(f^v)^2}g_B(\mathcal{D}^B_{\beta_1}\gamma_1,\eta_1)^h\\ &= 2g((\mathcal{D}^B_{\beta_1}\gamma_1)^h,\eta_1^h). \end{aligned}$$

Similarly, taking $\beta=\beta_1^h, \gamma=\gamma_1^h, \eta=\eta_2^v$ in Equation (3), we obtain:

$$\begin{aligned} 2g(\mathcal{D}_{\beta_1^h}\gamma_1^h,\eta_2^v) &= \sharp_\Pi(\beta_1^h)g(\gamma_1^h,\eta_2^v)+\sharp_\Pi(\gamma_1^h)g(\beta_1^h,\eta_2^v)-\sharp_\Pi(\eta_2^v)g(\beta_1^h,\gamma_1^h)\\ &+ g([\beta_1^h,\gamma_1^h]_\Pi,\eta_2^v)+g([\eta_2^v,\beta_1^h]_\Pi,\gamma_1^h)+g([\eta_2^v,\gamma_1^h]_\Pi,\beta_1^h)\\ &= -\sharp_\Pi(\eta_2^v)g(\beta_1^h,\gamma_1^h)\\ &= -(\sharp_{\Pi_F}(\eta_2))^v(\tfrac{1}{(f^v)^2}g_B(\beta_1,\gamma_1)^h)\\ &= \tfrac{2}{(f^v)^3}(\sharp_{\Pi_F}(\eta_2)(f))^v g_B(\beta_1,\gamma_1)^h\\ &= \tfrac{2}{(f^v)^3}\Pi_F(\eta_2,df)^v g_B(\beta_1,\gamma_1)^h\\ &= -\tfrac{2}{(f^v)^3}g_F(\eta_2,J_Fdf)^v g_B(\beta_1,\gamma_1)^h\\ &= 2g_F(-\tfrac{(b^h)^2}{(f^v)^3}g_B(\beta_1,\gamma_1)^h(J_Fdf)^v,\eta_2^v). \end{aligned}$$

Therefore,

$$\mathcal{D}_{\beta_1^h}\gamma_1^h=(\mathcal{D}^B_{\beta_1}\gamma_1)^h-\frac{(b^h)^2}{(f^v)^3}g_B(\beta_1,\gamma_1)^h(J_Fdf)^v.$$

2. Using Lemma 1, Proposition 1, Equation (9) and taking $\beta=\beta_1^h, \gamma=\gamma_2^v, \eta=\eta_1^h$ in (3), we obtain:

$$\begin{aligned} g(\mathcal{D}_{\beta_1^h}\gamma_2^v,\eta_1^h) &= \tfrac{1}{2}\sharp_\Pi(\gamma_2^v)g(\beta_1^h,\eta_1^h)\\ &= \tfrac{1}{2}(\sharp_{\Pi_F}(\gamma_2))^v(\tfrac{1}{(f^v)^2}g_B(\beta_1,\eta_1)^h)\\ &= -\tfrac{1}{(f^v)^3}(\sharp_{\Pi_F}(\gamma_2)(f))^v g_B(\beta_1,\eta_1)^h\\ &= \tfrac{1}{(f^v)^3}g_F(\gamma_2,J_Fdf)^v g_B(\beta_1,\eta_1)^h\\ &= g(\tfrac{1}{f^v}g_F(\gamma_2,J_Fdf)^v\beta_1^h,\eta_1^h). \end{aligned}$$

Similarly, taking $\beta=\beta_1^h, \gamma=\gamma_2^v, \eta=\eta_2^v$ in (3), we obtain:

$$\begin{aligned} g(\mathcal{D}_{\beta_1^h}\gamma_2^v,\eta_2^v) &= \tfrac{1}{2}\sharp_\Pi(\beta_1^h)g(\gamma_2^v,\eta_2^v)\\ &= \tfrac{1}{2}(\sharp_{\Pi_B}(\beta_1))^h(\tfrac{1}{(b^h)^2}g_F(\gamma_2,\eta_2)^v)\\ &= -\tfrac{1}{(b^h)^3}(\sharp_{\Pi_B}(\beta_1)(b))^h g_F(\gamma_2,\eta_2)^v\\ &= \tfrac{1}{(b^h)^3}g_B(\beta_1,J_Bdb)^h g_F(\gamma_2,\eta_2)^v\\ &= g(\tfrac{1}{b^h}g_B(\beta_1,J_Bdb)^h\gamma_2^v,\eta_2^v). \end{aligned}$$

Then, the first part of 2 follows.

Moreover, since $\mathcal{D}$ is torsion-free and $[\beta_1^h,\gamma_2^v]_\Pi=0$, then $\mathcal{D}_{\beta_1^h}\gamma_2^v=\mathcal{D}^B_{\gamma_2^v}\beta_1^h$.

3. This is analogous to the proofs of 1.

□

Lemma 2. *Let $\mathcal{R}^B$, $\mathcal{R}^F$ and $\mathcal{R}$ be the curvatures tensors of $\mathcal{D}^B$, $\mathcal{D}^F$ and $\mathcal{D}$, respectively. Then, for any $\beta_1,\gamma_1,\eta_1\in\Omega^1(B)$ and $\beta_2,\gamma_2,\eta_2\in\Omega^1(F)$, we have :*

1.

$$\begin{aligned}\mathcal{R}(\beta_1^h,\gamma_1^h)\eta_2^v &= \left[\tfrac{g_B(\gamma_1,J_Bdb)}{b}\right]^h\left[\tfrac{g_F(\eta_2,J_Fdf)}{f}\right]^v\beta_1^h-\left[\tfrac{g_B(\beta_1,J_Bdb)}{b}\right]^h\left[\tfrac{g_F(\eta_2,J_Fdf)}{f}\right]^v\gamma_1^h\\ &+ \left[\tfrac{1}{b}\{g_B(\gamma_1,\mathcal{D}^B_{\beta_1}J_Bdb)-g_B(\beta_1,\mathcal{D}^B_{\gamma_1}J_Bdb)\}\right]^h\eta_2^v,\end{aligned}$$

2.

$$\begin{aligned}\mathcal{R}(\beta_1^h,\gamma_1^h)\eta_1^h &= \left[\mathcal{R}^B(\beta_1,\gamma_1)\eta_1\right]^h+(\tfrac{\|J_Fdf\|_F^2}{f^4})^v\left[b^2\{g_B(\beta_1,\eta_1)\gamma_1-g_B(\gamma_1,\eta_1)\beta_1\}\right]^h\\ &+ (\tfrac{1}{f^3})^v\left[b\{g_B(\beta_1,J_Bdb)g_B(\gamma_1,\eta_1)-g_B(\gamma_1,J_Bdb)g_B(\beta_1,\eta_1)\}\right]^h(J_Fdf)^v,\end{aligned}$$

3.

$$\begin{aligned}\mathcal{R}(\beta_1^h,\gamma_2^v)\eta_1^h &= 2\left[b^2g_B(\beta_1,\eta_1)\right]^h\left[\tfrac{g_F(\gamma_2,J_Fdf)}{f^4}\right]^v(J_Fdf)^v\\ &- \left[\tfrac{g_B(\beta_1,\eta_1)}{b}\right]^h\left[\tfrac{g_F(\gamma_2,J_Fdf)}{f}\right]^v(J_Bdb)^h+\left[\tfrac{g_B(\eta_1,J_Bdb)}{b}\right]^h\left[\tfrac{g_F(\gamma_2,J_Fdf)}{f}\right]^v\beta_1^h\\ &+ 2\left[\tfrac{1}{b^2}\{g_B(\eta_1,J_Bdb)g_B(\beta_1,J_Bdb)+\tfrac{b}{2}g_B(\eta_1,\mathcal{D}^B_{\beta_1}J_Bdb)\}\right]^h\gamma_2^v\\ &+ \tfrac{1}{(f^v)^3}\left[b^2g_B(\beta_1,\eta_1)\right]^h(\mathcal{D}^B_{\gamma_2}J_Fdf)^v,\end{aligned}$$

4.

$$\begin{aligned}\mathcal{R}(\beta_1^h,\gamma_2^v)\eta_2^v &= \left[\tfrac{g_F(\gamma_2,\eta_2)}{f}\right]^v\left[\tfrac{g_B(\beta_1,J_Bdb)}{b}\right]^h(J_Fdf)^v\\ &- 2\left[\tfrac{g_B(\beta_1,J_Bdb)}{b^4}\right]^h\left[f^2g_F(\gamma_2,\eta_2)\right]^v(J_Bdb)^h\\ &- \left[\tfrac{1}{f^2}\{2g_F(\gamma_2,J_Fdf)g_F(\eta_2,J_Fdf)+fg_F(\eta_2,\mathcal{D}^F_{\gamma_2}J_Fdf)\}\right]^v\beta_1^h\\ &- \left[\tfrac{g_F(\eta_2,J_Fdf)}{f}\right]^v\left[\tfrac{g_B(\beta_1,J_Bdb)}{b}\right]^h\gamma_2^v-\tfrac{1}{(b^h)^3}\left[f^2g_F(\gamma_2,\eta_2)\right]^v(\mathcal{D}^B_{\beta_1}J_Bdb)^h,\end{aligned}$$

5.

$$\begin{aligned}\mathcal{R}(\beta_2^v,\gamma_2^v)\eta_2^v &= \left[\mathcal{R}^F(\beta_2,\gamma_2)\eta_2\right]^v+(\tfrac{\|J_Bdb\|_B^2}{b^4})^h\left[f^2\{g_F(\beta_2,\eta_2)\gamma_2-g_F(\gamma_2,\eta_2)\beta_2\}\right]^v\\ &+ (\tfrac{1}{b^3})^h\left[f\{g_F(\beta_2,J_Fdf)g_F(\gamma_2,\eta_2)-g_F(\gamma_2,J_Fdf)g_F(\beta_2,\eta_2)\}\right]^v(J_Bdb)^h,\end{aligned}$$

6.

$$\begin{aligned}\mathcal{R}(\beta_2^v,\gamma_2^v)\eta_1^h &= -\left[\tfrac{g_F(\beta_2,J_Fdf)}{f}\right]^v\left[\tfrac{g_B(\eta_1,J_Bdb)}{b}\right]^h\gamma_2^v+\left[\tfrac{g_F(\gamma_2,J_Fdf)}{f}\right]^v\left[\tfrac{g_B(\eta_1,J_Bdb)}{b}\right]^h\beta_2^v\\ &+ \left[\tfrac{1}{f}\{g_F(\gamma_2,\mathcal{D}^F_{\beta_2}J_Fdf)-g_F(\beta_2,\mathcal{D}^F_{\gamma_2}J_Fdf)\}\right]^v\eta_1^h.\end{aligned}$$

Proof. 1. Taking $\beta=\beta_1^h,\gamma=\gamma_1^h$ and $\eta=\eta_2^v$ in Equation (1), we obtain,

$$\mathcal{R}(\beta_1^h,\gamma_1^h)\eta_2^v=\mathcal{D}_{\beta_1^h}\mathcal{D}_{\gamma_1^h}\eta_2^v-\mathcal{D}_{\gamma_1^h}\mathcal{D}_{\beta_1^h}\eta_2^v-\mathcal{D}_{[\beta_1^h,\gamma_1^h]_\Pi}\eta_2^v. \tag{10}$$

Using Lemma 1 and Proposition 2 in the first term $T_1=\mathcal{D}_{\beta_1^h}\mathcal{D}_{\gamma_1^h}\eta_2^v$ of (10), we obtain:

$$\begin{aligned}T_1 &= \mathcal{D}_{\beta_1^h}\mathcal{D}_{\gamma_1^h}\eta_2^v\\ &= \mathcal{D}_{\beta_1^h}(\tfrac{1}{f^v}g_F(\eta_2,J_Fdf)^v\gamma_1^h+\tfrac{1}{b^h}g_B(\gamma_1,J_Bdb)^h\eta_2^v)\\ &= \tfrac{1}{f^v}g_F(\eta_2,J_Fdf)^v\mathcal{D}_{\beta_1^h}\gamma_1^h-\tfrac{1}{(b^h)^2}\Pi_B(\beta_1,db)^hg_B(\gamma_1,J_Bdb)^h\eta_2^v+\tfrac{1}{b^h}\mathcal{D}_{\beta_1^h}(g_B(\gamma_1,J_Bdb)^h)\eta_2^v\\ &+ \tfrac{1}{b^h}g_B(\gamma_1,J_Bdb)^h\mathcal{D}_{\beta_1^h}\eta_2^v\\ &= \tfrac{1}{f^v}g_F(\eta_2,J_Fdf)^v\Big((\mathcal{D}^B_{\beta_1}\gamma_1)^h-\tfrac{(b^h)^2}{(f^v)^3}g_B(\beta_1,\gamma_1)^h(J_Fdf)^v\Big)+\tfrac{1}{(b^h)^2}g_B(\beta_1,J_Bdb)^hg_B(\gamma_1,J_Bdb)^h\eta_2^v\\ &+ \tfrac{1}{b^h}\sharp\Pi_B(\beta_1)^hg_B(\gamma_1,J_Bdb)^h\eta_2^v+\tfrac{1}{b^h}g_B(\gamma_1,J_Bdb)^h\Big(\tfrac{1}{f^v}g_F(\eta_2,J_Fdf)^v\beta_1^h+\tfrac{1}{b^h}g_B(\beta_1,J_Bdb)^h\eta_2^v\Big).\end{aligned}$$

Interchanging β_1 and γ_1 in the previous equation, the second term T_2 of (10) is given by:

$$\begin{aligned}
T_2 &= \mathcal{D}_{\gamma_1^h}\mathcal{D}_{\beta_1^h}\eta_2^v \\
&= \tfrac{1}{f^v} g_F(\eta_2, J_F df)^v \Big((\mathcal{D}^B_{\gamma_1}\beta_1)^h - \tfrac{(b^h)^2}{(f^v)^3} g_B(\gamma_1,\beta_1)^h (J_F df)^v \Big) + \tfrac{1}{(b^h)^2} g_B(\gamma_1, J_B db)^h g_B(\beta_1, J_B db)^h \eta_2^v \\
&+ \tfrac{1}{b^h} \sharp_{\Pi_B}(\gamma_1)^h g_B(\beta_1, J_B db)^h \eta_2^v + \tfrac{1}{b^h} g_B(\beta_1, J_B db)^h \Big(\tfrac{1}{f^v} g_F(\eta_2, J_F df)^v \gamma_1^h + \tfrac{1}{b^h} g_B(\gamma_1, J_B db)^h \eta_2^v \Big).
\end{aligned}$$

Using Proposition 2, the third term T_3 of (10) is given by:

$$\begin{aligned}
T_3 &= \mathcal{D}_{[\beta_1^h,\gamma_1^h]_\Pi}\eta_2^v \\
&= \mathcal{D}_{[\beta_1,\gamma_1]^h_{\Pi_B}}\eta_2^v \\
&= \tfrac{1}{f^v} g_F(\eta_2, J_F df)^v [\beta_1,\gamma_1]^h_{\Pi_B} + \tfrac{1}{b^h} g_B([\beta_1,\gamma_1]_{\Pi_B}, J_B db)^h \eta_2^v \\
&= \tfrac{1}{f^v} g_F(\eta_2, J_F df)^v ((\mathcal{D}^B_{\beta_1}\gamma_1)^h - (\mathcal{D}^B_{\gamma_1}\beta_1)^h) + \tfrac{1}{b^h}(g_B(\mathcal{D}^B_{\beta_1}\gamma_1, J_B db) - g_B(\mathcal{D}^B_{\gamma_1}\beta_1, J_B db))^h \eta_2^v.
\end{aligned}$$

Using the above terms in Equation (10), we obtain:

$$\begin{aligned}
\mathcal{R}(\beta_1^h,\gamma_1^h)\eta_2^v &= T_1 - T_2 - T_3 \\
&= \tfrac{1}{b^h} g_B(\gamma_1, J_B db)^h \Big(\tfrac{1}{f^v} g_F(\eta_2, J_F df)^v \beta_1^h \Big) - \tfrac{1}{b^h} g_B(\beta_1, J_B db)^h \Big(\tfrac{1}{f^v} g_F(\eta_2, J_F df)^v \gamma_1^h \Big) \\
&+ \tfrac{1}{b^h} \Big(\sharp_{\Pi_B}(\beta_1) g_B(\gamma_1, J_B db) - g_B(\mathcal{D}^B_{\beta_1}\gamma_1, J_B db) \Big)^h \eta_2^v \\
&- \tfrac{1}{b^h} \Big(\sharp_{\Pi_B}(\gamma_1) g_B(\beta_1, J_B db) - g_B(\mathcal{D}^B_{\gamma_1}\beta_1, J_B db) \Big)^h \eta_2^v \\
&= \Big[\tfrac{g_B(\gamma_1, J_B db)}{b} \Big]^h \Big[\tfrac{g_F(\eta_2, J_F df)}{f} \Big]^v \beta_1^h - \Big[\tfrac{g_B(\beta_1, J_B db)}{b} \Big]^h \Big[\tfrac{g_F(\eta_2, J_F df)}{f} \Big]^v \gamma_1^h \\
&+ \Big[\tfrac{1}{b} \{ g_B(\gamma_1, \mathcal{D}^B_{\beta_1} J_B db) - g_B(\beta_1, \mathcal{D}^B_{\gamma_1} J_B db) \} \Big]^h \eta_2^v,
\end{aligned}$$

and the first part of the lemma follows.

2. Taking $\beta = \beta_1^h, \gamma = \gamma_1^h$ and $\eta = \eta_1^h$ in (1), we obtain,

$$\mathcal{R}(\beta_1^h,\gamma_1^h)\eta_1^h = \mathcal{D}_{\beta_1^h}\mathcal{D}_{\gamma_1^h}\eta_1^h - \mathcal{D}_{\gamma_1^h}\mathcal{D}_{\beta_1^h}\eta_1^h - \mathcal{D}_{[\beta_1,\gamma_1]^h_{\Pi_B}}\eta_1^h. \tag{11}$$

Applying Proposition 2 in the first term T_1 of (11), we obtain:

$$\begin{aligned}
T_1 &= \mathcal{D}_{\beta_1^h}\mathcal{D}_{\gamma_1^h}\eta_1^h \\
&= \mathcal{D}_{\beta_1^h}\Big((\mathcal{D}^B_{\gamma_1}\eta_1)^h - \tfrac{(b^h)^2}{(f^v)^3} g_B(\gamma_1,\eta_1)^h (J_F df)^v \Big) \\
&= (\mathcal{D}^B_{\beta_1}\mathcal{D}^B_{\gamma_1}\eta_1)^h - \tfrac{(b^h)^2}{(f^v)^3} g_B(\beta_1, \mathcal{D}^B_{\gamma_1}\eta_1)^h (J_F df)^v - \tfrac{(b^h)^2}{(f^v)^3} g_B(\gamma_1,\eta_1)^h \mathcal{D}_{\beta_1^h}(J_F df)^v \\
&+ 2\tfrac{b^h}{(f^v)^3} g_B(\beta_1, J_B db)^h g_B(\gamma_1,\eta_1)^h (J_F df)^v - \tfrac{(b^h)^2}{(f^v)^3} \mathcal{D}_{\beta_1^h}(g_B(\gamma_1,\eta_1)^h)(J_F df)^v \\
&= (\mathcal{D}^B_{\beta_1}\mathcal{D}^B_{\gamma_1}\eta_1)^h - \tfrac{(b^h)^2}{(f^v)^3} g_B(\beta_1, \mathcal{D}^B_{\gamma_1}\eta_1)^h (J_F df)^v \\
&- \tfrac{(b^h)^2}{(f^v)^3} g_B(\gamma_1,\eta_1)^h \Big(\big(\tfrac{\|J_F df\|_F^2}{f}\big)^v \beta_1^h + \tfrac{1}{b^h} g_B(\beta_1, J_B db)^h (J_F df)^v \Big) \\
&+ 2\tfrac{b^h}{(f^v)^3} g_B(\beta_1, J_B db)^h g_B(\gamma_1,\eta_1)^h (J_F df)^v - \tfrac{(b^h)^2}{(f^v)^3} \sharp_{\Pi_B}(\beta_1)^h g_B(\gamma_1,\eta_1)^h (J_F df)^v.
\end{aligned}$$

Interchanging β_1 and γ_1 in the previous equation, the second term T_2 of (11) is given by:

$$\begin{aligned}
T_2 &= \mathcal{D}_{\gamma_1^h}\mathcal{D}_{\beta_1^h}\eta_1^h \\
&= (\mathcal{D}^B_{\gamma_1}\mathcal{D}^B_{\beta_1}\eta_1)^h - \tfrac{(b^h)^2}{(f^v)^3} g_B(\gamma_1, \mathcal{D}^B_{\beta_1}\eta_1)^h (J_F df)^v \\
&- \tfrac{(b^h)^2}{(f^v)^3} g_B(\beta_1,\eta_1)^h \Big(\big(\tfrac{\|J_F df\|_F^2}{f}\big)^v \gamma_1^h + \tfrac{1}{b^h} g_B(\gamma_1, J_B db)^h (J_F df)^v \Big) \\
&+ 2\tfrac{b^h}{(f^v)^3} g_B(\gamma_1, J_B db)^h g_B(\beta_1,\eta_1)^h (J_F df)^v - \tfrac{(b^h)^2}{(f^v)^3} \sharp_{\Pi_B}(\gamma_1)^h g_B(\beta_1,\eta_1)^h (J_F df)^v.
\end{aligned}$$

Applying Proposition 2 in the third term T_3 of (11), we obtain:

$$\begin{aligned} T_3 &= \mathcal{D}_{[\beta_1,\gamma_1]^h_{\Pi_B}}\eta_1^h \\ &= (\mathcal{D}^B_{[\beta_1,\gamma_1]_{\Pi_B}}\eta_1)^h - \frac{(b^h)^2}{(f^v)^3}g_B([\beta_1,\gamma_1]_{\Pi_B},\eta_1)^h(J_F df)^v. \end{aligned}$$

Replacing the above terms in (11) and after some calculations the result follows.

3. Taking $\beta=\beta_1^h, \gamma=\gamma_2^v$ and $\eta=\eta_1^h$ in (1), we obtain,

$$\begin{aligned} \mathcal{R}(\beta_1^h,\gamma_2^v)\eta_1^h &= \mathcal{D}_{\beta_1^h}\mathcal{D}_{\gamma_2^v}\eta_1^h - \mathcal{D}_{\gamma_2^v}\mathcal{D}_{\beta_1^h}\eta_1^h - \mathcal{D}_{[\beta_1^h,\gamma_2^v]_\Pi}\eta_1^h \\ &= \mathcal{D}_{\beta_1^h}\mathcal{D}_{\gamma_2^v}\eta_1^h - \mathcal{D}_{\gamma_2^v}\mathcal{D}_{\beta_1^h}\eta_1^h. \end{aligned} \tag{12}$$

Applying Proposition 2 in the first term T_1 of (12), we obtain:

$$\begin{aligned} T_1 &= \mathcal{D}_{\beta_1^h}\mathcal{D}_{\gamma_2^v}\eta_1^h \\ &= \mathcal{D}_{\beta_1^h}\left(\frac{1}{f^v}g_F(\gamma_2,J_F df)^v\eta_1^h + \frac{1}{b^h}g_B(\eta_1,J_B db)^h\gamma_2^v\right) \\ &= \frac{1}{f^v}g_F(\gamma_2,J_F df)^v\mathcal{D}_{\beta_1^h}\eta_1^h + \frac{1}{b^h}g_B(\eta_1,J_B db)^h\mathcal{D}_{\beta_1^h}\gamma_2^v + \frac{1}{(b^h)^2}g_B(\beta_1,J_B db)^h g_B(\eta_1,J_B db)^h\gamma_2^v \\ &+ \frac{1}{b^h}\sharp_{\Pi_B}(\beta_1)^h g_B(\eta_1,J_B db)^h\gamma_2^v \\ &= \frac{1}{f^v}g_F(\gamma_2,J_F df)^v\left((\mathcal{D}^B_{\beta_1}\eta_1)^h - \frac{(b^h)^2}{(f^v)^3}g_B(\beta_1,\eta_1)^h(J_F df)^v\right) \\ &+ \frac{1}{b^h}g_B(\eta_1,J_B db)^h\left(\frac{1}{f^v}g_F(\gamma_2,J_F df)^v\beta_1^h + \frac{1}{b^h}g_B(\beta_1,J_B db)^h\gamma_2^v\right) \\ &+ \frac{1}{(b^h)^2}g_B(\beta_1,J_B db)^h g_B(\eta_1,J_B db)^h\gamma_2^v + \frac{1}{b^h}\sharp_{\Pi_B}(\beta_1)^h g_B(\eta_1,J_B db)^h\gamma_2^v. \end{aligned}$$

Applying Proposition 2 in the second term T_2 of (12), we obtain:

$$\begin{aligned} T_2 &= \mathcal{D}_{\gamma_2^v}\mathcal{D}_{\beta_1^h}\eta_1^h \\ &= \mathcal{D}_{\gamma_2^v}\left((\mathcal{D}^B_{\beta_1}\eta_1)^h - \frac{(b^h)^2}{(f^v)^3}g_B(\beta_1,\eta_1)^h(J_F df)^v\right) \\ &= \frac{1}{f^v}g_F(\gamma_2,J_F df)^v(\mathcal{D}^B_{\beta_1}\eta_1)^h + \frac{1}{b^h}g_B(\mathcal{D}^B_{\beta_1}\eta_1,J_B db)^h\gamma_2^v \\ &- 3\frac{(b^h)^2}{(f^v)^4}g_B(\beta_1,\eta_1)^h g_F(\gamma_2,J_F df)^v(J_F df)^v - \frac{(b^h)^2}{(f^v)^3}g_B(\beta_1,\eta_1)^h\mathcal{D}_{\gamma_2^v}(J_F df)^v \\ &= \frac{1}{f^v}g_F(\gamma_2,J_F df)^v(\mathcal{D}^B_{\beta_1}\eta_1)^h + \frac{1}{b^h}g_B(\mathcal{D}^B_{\beta_1}\eta_1,J_B db)^h\gamma_2^v \\ &- 3\frac{(b^h)^2}{(f^v)^4}g_B(\beta_1,\eta_1)^h g_F(\gamma_2,J_F df)^v(J_F df)^v - \frac{(b^h)^2}{(f^v)^3}g_B(\beta_1,\eta_1)^h(\mathcal{D}^F_{\gamma_2}J_F df)^v \\ &+ \frac{1}{f^v b^h}g_B(\beta_1,\eta_1)^h g_F(\gamma_2,J_F df)^v(J_B db)^h \end{aligned}$$

Using the above terms in (12), we obtain:

$$\begin{aligned} \mathcal{R}(\beta_1^h,\gamma_2^v)\eta_1^h &= T_1 - T_2 \\ &= 2\frac{(b^h)^2}{(f^v)^4}g_B(\beta_1,\eta_1)^h g_F(\gamma_2,J_F df)^v(J_F df)^v - \frac{1}{f^v b^h}g_B(\beta_1,\eta_1)^h g_F(\gamma_2,J_F df)^v(J_B db)^h \\ &+ \frac{1}{b^h f^v}g_B(\eta_1,J_B db)^h g_F(\gamma_2,J_F df)^v\beta_1^h + \frac{(b^h)^2}{(f^v)^3}g_B(\beta_1,\eta_1)^h(\mathcal{D}^F_{\gamma_2}J_F df)^v \\ &+ 2\left[\frac{1}{b^2}\{g_B(\eta_1,J_B db)g_B(\beta_1,J_B db) + \frac{b}{2}(\sharp_{\Pi_B}(\beta_1)g_B(\eta_1,J_B db) - g_B(\mathcal{D}^B_{\beta_1}\eta_1,J_B db))\}\right]^h\gamma_2^v \\ &= 2\left[b^2 g_B(\beta_1,\eta_1)\right]^h\left[\frac{g_F(\gamma_2,J_F df)}{f^4}\right]^v(J_F df)^v \\ &- \left[\frac{g_B(\beta_1,\eta_1)}{b}\right]^h\left[\frac{g_F(\gamma_2,J_F df)}{f}\right]^v(J_B db)^h + \left[\frac{g_B(\eta_1,J_B db)}{b}\right]^h\left[\frac{g_F(\gamma_2,J_F df)}{f}\right]^v\beta_1^h \\ &+ 2\left[\frac{1}{b^2}\{g_B(\eta_1,J_B db)g_B(\beta_1,J_B db) + \frac{b}{2}g_B(\eta_1,\mathcal{D}^B_{\beta_1}J_B db)\}\right]^h\gamma_2^v \\ &+ \frac{1}{(f^v)^3}\left[b^2 g_B(\beta_1,\eta_1)\right]^h(\mathcal{D}^B_{\gamma_2}J_F df)^v, \end{aligned}$$

and the third part of the lemma follows.

The proofs of Parts 4, 5, and 6 are analogous to Proofs 3, 2 and 1, respectively.

□

Theorem 1. *Let $\mathcal{K}^B$, $\mathcal{K}^F$ and $\mathcal{K}$ be the sectional contravariant curvatures of (B, g_B, Π_B), (F, g_F, Π_F) and $({}_fB \times_b F, g, \Pi)$, respectively. Then, for any $\beta_1, \gamma_1 \in \Omega^1(B)$ and $\beta_2, \gamma_2 \in \Omega^1(F)$, we have:*

1. $\mathcal{K}(\beta_1^h, \gamma_1^h) = (f^2)^v \mathcal{K}^B(\beta_1, \gamma_1)^h - (b^2)^h \left(\frac{\|J_F df\|_F^2}{f^2}\right)^v$.
2. $\mathcal{K}(\beta_1^h, \gamma_2^v) = -(f^2)^v \left[2\frac{g_B^2(\beta_1, J_B db)}{b^2 g_B(\beta_1,\beta_1)} + \frac{g_B(\mathcal{D}^B_{\beta_1} J_B db, \beta_1)}{b g_B(\beta_1,\beta_1)}\right]^h - (b^2)^h \left[2\frac{g_F^2(\gamma_2, J_F df)}{f^2 g_F(\gamma_2,\gamma_2)} + \frac{g_F(\mathcal{D}^F_{\gamma_2} J_F df, \gamma_2)}{f g_F(\gamma_2,\gamma_2)}\right]^v$.
3. $\mathcal{K}(\beta_2^v, \gamma_2^v) = (b^2)^h \mathcal{K}^F(\beta_2, \gamma_2)^v - (f^2)^v \left(\frac{\|J_B db\|_B^2}{b^2}\right)^h$.

Proof. 1. Using Lemma 2 and taking $\beta = \beta_1^h$, $\gamma = \gamma_1^h$ in Equation (6), we obtain:

$$
\begin{aligned}
g(\mathcal{R}(\beta_1^h, \gamma_1^h)\gamma_1^h, \beta_1^h) &= g([\mathcal{R}^B(\beta_1, \gamma_1)\gamma_1]^h + (\tfrac{\|J_F df\|_F^2}{f^4})^v [b^2\{g_B(\beta_1, \gamma_1)\gamma_1 - g_B(\gamma_1, \gamma_1)\beta_1\}]^h, \beta_1^h) \\
&= \frac{1}{(f^v)^2} g_B(\mathcal{R}^B(\beta_1, \gamma_1)\gamma_1, \beta_1)^h + \left(\frac{\|J_F df\|_F^2}{f^6}\right)^v [b^2\{g_B^2(\beta_1, \gamma_1) \\
&- g_B(\beta_1, \beta_1) g_B(\gamma_1, \gamma_1)\}]^h
\end{aligned}
$$

and

$$
Q(\beta_1^h, \gamma_1^h) = g(\beta_1^h, \beta_1^h) g(\gamma_1^h, \gamma_1^h) - g^2(\beta_1^h, \gamma_1^h) = \frac{1}{(f^v)^4}[g_B(\beta_1, \beta_1) g_B(\gamma_1, \gamma_1) - g_B^2(\beta_1, \gamma_1)]^h.
$$

Hence,

$$
\mathcal{K}(\beta_1^h, \gamma_1^h) = \frac{g(\mathcal{R}(\beta_1^h, \gamma_1^h)\gamma_1^h, \beta_1^h)}{Q(\beta_1^h, \gamma_1^h)} = (f^2)^v \mathcal{K}^B(\beta_1, \gamma_1)^h - (b^2)^h \left(\frac{\| J_F df \|_F^2}{f^2}\right)^v.
$$

2. Similarly, using Lemma 2 and taking $\beta = \beta_1^h$, $\gamma = \gamma_2^v$ in (6), we obtain:

$$
\begin{aligned}
g(\mathcal{R}(\beta_1^h, \gamma_2^v)\gamma_2^v, \beta_1^h) &= -2\left[g_F(\gamma_2, \gamma_2)^v \left(\frac{g_B^2(\beta_1, J_B db)}{b^4}\right)^h + g_B(\beta_1, \beta_1)^h \left(\frac{g_F^2(\gamma_2, J_F df)}{f^4}\right)^v\right] \\
&- \left[g_F(\gamma_2, \gamma_2)^v \left(\frac{g_B(\mathcal{D}^B_{\beta_1} J_B db, \beta_1)}{b^3}\right)^h + g_B(\beta_1, \beta_1)^h \left(\frac{g_F(\mathcal{D}^F_{\gamma_2} J_F df, \gamma_2)}{f^3}\right)^v\right]
\end{aligned}
$$

and

$$
Q(\beta_1^h, \gamma_2^v) = g(\beta_1^h, \beta_1^h) g(\gamma_2^v, \gamma_2^v) - g^2(\beta_1^h, \gamma_2^v) = \frac{1}{(f^v)^2 (b^h)^2} g_B(\beta_1, \beta_1)^h g_F(\gamma_2, \gamma_2)^v.
$$

Hence,

$$
\begin{aligned}
\mathcal{K}(\beta_1^h, \gamma_2^v) &= \frac{g(\mathcal{R}(\beta_1^h, \gamma_2^v)\gamma_2^v, \beta_1^h)}{Q(\beta_1^h, \gamma_2^v)} \\
&= -(f^2)^v \left[2\frac{g_B^2(\beta_1, J_B db)}{b^2 g_B(\beta_1,\beta_1)} + \frac{g_B(\mathcal{D}^B_{\beta_1} J_B db, \beta_1)}{b g_B(\beta_1,\beta_1)}\right]^h - (b^2)^h \left[2\frac{g_F^2(\gamma_2, J_F df)}{f^2 g_F(\gamma_2,\gamma_2)} + \frac{g_F(\mathcal{D}^F_{\gamma_2} J_F df, \gamma_2)}{f g_F(\gamma_2,\gamma_2)}\right]^v.
\end{aligned}
$$

3. This is analogous to the proofs of 1.
□

Corollary 1. *If $\mathcal{D}^B$ and $\mathcal{D}^F$ are flat then, $\mathcal{D}$ is flat if and only if b and f are Casimir functions.*

Proof. First, note that b (respectively, f) is a Casimir function if and only if $J_B db = 0$ (respectively, $J_F df = 0$). Now, if $\mathcal{D}^B$ and $\mathcal{D}^F$ are flat, then from Theorem 1, the sectional curvature of $({}_fB \times_b F, g, \Pi)$ becomes:

$$
\begin{cases}
\mathcal{K}(\beta_1^h, \gamma_1^h) = -(b^2)^h \left(\frac{\|J_F df\|_F^2}{f^2}\right)^v, \\
\mathcal{K}(\beta_1^h, \gamma_2^v) = -(f^2)^v \left[2\frac{g_B^2(\beta_1, J_B db)}{b^2 g_B(\beta_1,\beta_1)} + \frac{g_B(\mathcal{D}^B_{\beta_1} J_B db, \beta_1)}{b g_B(\beta_1,\beta_1)}\right]^h - (b^2)^h \left[2\frac{g_F^2(\gamma_2, J_F df)}{f^2 g_F(\gamma_2,\gamma_2)} + \frac{g_F(\mathcal{D}^F_{\gamma_2} J_F df, \gamma_2)}{f g_F(\gamma_2,\gamma_2)}\right]^v, \\
\mathcal{K}(\beta_2^v, \gamma_2^v) = -(f^2)^v \left(\frac{\|J_B db\|_B^2}{b^2}\right)^h.
\end{cases}
\tag{13}
$$

$\Rightarrow$ Assume that $\mathcal{D}$ is flat. Using the first and third equations of (13), we obtain:

$$\|J_B db\|_B = \|J_F df\|_F = 0 \Rightarrow J_B db = J_F df = 0.$$

Thus b and f are Casimir functions.

$\Leftarrow$ Assume that b and f are Casimir functions. Taking $J_B db = J_F df = 0$ in Equation (13), we obtain $\mathcal{K} = 0$, and then $\mathcal{D}$ is flat. $\square$

Corollary 2. *If (B, g_B) and (F, g_F) are Riemannian manifolds, then $\mathcal{D}$ is flat if and only if $\mathcal{D}^B$, $\mathcal{D}^F$ are flat and b, f are Casimir functions.*

Proof. If $\mathcal{D}$ is flat, using the fact that B and F are Riemannian manifolds in Part 2 of Theorem 1, we obtain:

$$g_B(\beta_1, J_B db) = g_F(\gamma_2, J_F df) = 0, \quad \forall \beta_1 \in \Omega^1(B), \forall \gamma_2 \in \Omega^1(F).$$

Then,

$$J_B db = J_F df = 0. \tag{14}$$

Now, using Equation (14) in Parts 1 and 3 of Theorem 1, we obtain:

$$\mathcal{K}^B(\beta_1, \gamma_1) = \mathcal{K}^F(\beta_2, \gamma_2) = 0, \quad \forall \beta_1, \gamma_1 \in \Omega^1(B), \quad \forall \beta_2, \gamma_2 \in \Omega^1(F). \tag{15}$$

Hence, from (14) and (15), we deduce that b and f are Casimir functions and $\mathcal{D}^B$ and $\mathcal{D}^F$ are flat.

Converse of this corollary directly follows from Corollary 1. $\square$

Corollary 3. *If $({}_f B \times_b F, g, \Pi)$ has a positive sectional curvature, then (B, Π_B, g_B) and (F, g_B, Π_B) have positive sectional curvatures.*

Proof. If $\mathcal{K} > 0$, then from Theorem 1, for any $\beta_1, \gamma_1 \in \Omega^1(B)$ and $\beta_2, \gamma_2 \in \Omega^1(F)$, we have:

$$\begin{array}{rcl} \mathcal{K}^B(\beta_1, \gamma_1)^h & = & \frac{1}{(f^2)^v}\left(\mathcal{K}(\beta_1^h, \gamma_1^h) + (b^2)^h (\frac{\|J_F df\|_F^2}{f^2})^v\right) > 0 \\ \mathcal{K}^F(\beta_2, \gamma_2)^v & = & \frac{1}{(b^2)^h}\left(\mathcal{K}(\beta_2^v, \gamma_2^v) + (f^2)^v (\frac{\|J_B db\|_B^2}{b^2})^h\right) > 0, \end{array}$$

and the corollary follows. $\square$

Corollary 4. *If b and f are Casimir functions on B and F, respectively, then the triplet $({}_f B \times_b F, g, \Pi)$ has non-negative (respectively, non-positive) sectional curvature if and only if (B, g_B, Π_B) and (F, g_F, Π_F) have non-negative (respectively, non-positive) sectional curvatures.*

Proof. If f and g are Casimir functions, then from Theorem 1, the sectional contravariant curvature $\mathcal{K}$ of $({}_f B \times_b F, g, \Pi)$ becomes:

$$\left\{ \begin{array}{rcl} \mathcal{K}(\beta_1^h, \gamma_1^h) & = & (f^2)^v \mathcal{K}^B(\beta_1, \gamma_1)^h, \\ \mathcal{K}(\beta_1^h, \gamma_2^v) & = & 0, \\ \mathcal{K}(\beta_2^v, \gamma_2^v) & = & (b^2)^h \mathcal{K}^F(\beta_2, \gamma_2)^v, \end{array} \right. \tag{16}$$

and the corollary follows. $\square$

Corollary 5. *If (B, g_B) and (F, g_F) are Riemannian manifolds, then $({}_f B \times_b F, g, \Pi)$ has non-negative sectional curvature if and only if (B, g_B, Π_B), (F, g_F, Π_F) have non-negative sectional curvatures and b, f are Casimir functions.*

Proof. If $\mathcal{K} \geq 0$, B and F are Riemannian manifolds, then from Part 2 of Theorem 1, we obtain:

$$J_B db = J_F df = 0. \tag{17}$$

Using Equation (17) in Parts 1 and 3 of Theorem 1, for any $\beta_1, \gamma_1 \in \Omega^1(B)$ and $\beta_2, \gamma_2 \in \Omega^1(F)$, we obtain:

$$\mathcal{K}^B(\beta_1, \gamma_1)^h = \frac{1}{(f^2)^v}\mathcal{K}(\beta_1^h, \gamma_1^h) \text{ and } \mathcal{K}^F(\beta_2, \gamma_2)^v = \frac{1}{(b^2)^h}\mathcal{K}(\beta_2^v, \gamma_2^v) \tag{18}$$

Hence, from (17) and (18), we deduce that b and f are Casimir functions and $\mathcal{K}^B \geq 0$, $\mathcal{K}^F \geq 0$.

Converse of this corollary follows from Corollary 4. □

Corollary 6. *If $({}_fB \times_b F, g, \Pi)$ has a nonzero constant sectional curvature c and f is a Casimir function on F then, both (B, g_B, Π_B) and (F, g_F, Π_F) have constant sectional curvatures given, respectively, by:*

$$\mathcal{K}^B = \frac{c}{f^2} \quad \text{and} \quad \mathcal{K}^F = \frac{c}{b^2}(1 + \frac{\|J_B db\|_B^2}{b^2\mathcal{K}^B}),$$

Furthermore, if b is a Casimir function, then $\mathcal{K}^F = \frac{c}{b^2}$.

Proof. By Part 1 of Theorem 1, if f is a Caismir function and $\mathcal{K} = c$, then for any 1-forms $\alpha_1, \beta_1 \in \Omega^1(B)$, we have $\mathcal{K}^B(\beta_1, \gamma_1)^h = (\frac{c}{f^2})^v$. Hence,

$$\mathcal{K}^B = \frac{c}{f^2}. \tag{19}$$

By Part 3 of Theorem 1 and Equation (19), for any $\beta_2, \gamma_2 \in \Omega^1(F)$, we have $\mathcal{K}^F(\beta_2, \gamma_2)^v = \left[\frac{c}{b^2}(1 + \frac{\|J_B db\|_B^2}{b^2\mathcal{K}^B})\right]^h$, hence

$$\mathcal{K}^F = \frac{c}{b^2}(1 + \frac{\|J_B db\|_B^2}{b^2\mathcal{K}^B}).$$

□

4. Qualar and Null Sectional Contravariant Curvatures of Doubly Warped Product Poisson Manifold

Let $\{f^v dx_1^h, \ldots, f^v dx_{m_1}^h, b^h dy_1^v, \ldots, b^h dy_{m_2}^v\}$ be a local $g-$orthonormal basis on a doubly warped product Poisson manifold $({}_fB \times_b F, g, \Pi)$, where $\{dx_1, \ldots, dx_{m_1}\}$ is a g_B-orthonormal basis on B of index s_1 and $\{dy_1, \ldots, dy_{m_2}\}$ is a g_F-orthonormal basis on F of index s_2.

From Equations (7) and (8), the qualar contravariant curvature at point $(p_1, p_2) \in B \times F$ and the null sectional contravariant curvature of $({}_fB \times_b F, g, \Pi)$ are defined, respectively, by:

$$\begin{aligned} \text{qual}(p_1, p_2) &= 2\Big[\sum_{i=1}^{s_1}\sum_{k=s_1+1}^{m_1}\mathcal{K}(f^v dx_i^h, f^v dx_k^h) + \sum_{i=1}^{s_1}\sum_{k=s_2+1}^{m_2}\mathcal{K}(f^v dx_i^h, b^h dy_k^v) \\ &+ \sum_{i=1}^{s_2}\sum_{k=s_1+1}^{m_1}\mathcal{K}(b^h dy_i^v, f^v dx_k^h) + \sum_{i=1}^{s_2}\sum_{k=s_2+1}^{m_2}\mathcal{K}(b^h dy_i^v, b^h dy_k^v),\Big] \end{aligned} \tag{20}$$

and

$$\widetilde{\mathcal{K}}(\eta_k^i, \alpha_l) = \frac{g(\mathcal{R}(\eta_k^i, \alpha_l)\alpha_l, \eta_k^i)}{g(\alpha_l, \alpha_l)}, \tag{21}$$

where $\alpha_l \in \{f^v dx_1^h, \dots, f^v dx_{m_1}^h, b^h dy_1^v, \dots, b^h dy_{m_2}^v\}$, $l \neq i$, $l \neq k$ and $\eta_k^i \in \{\eta_{k,h}^{i,h}, \eta_{k,v}^{i,h}, \eta_{k,h}^{i,v}, \eta_{k,v}^{i,v}\}$ such that:

$$\begin{cases} \eta_{k,h}^{i,h} &= \frac{f^v}{\sqrt{2}}(dx_i^h + dx_k^h), & \forall i \in \{1, \dots, s_1\}, k \in \{s_1+1, \dots, m_1\}, \\ \eta_{k,v}^{i,h} &= \frac{1}{\sqrt{2}}(f^v dx_i^h + b^h dy_k^v), & \forall i \in \{1, \dots, s_1\}, k \in \{s_2+1, \dots, m_2\}, \\ \eta_{k,h}^{i,v} &= \frac{1}{\sqrt{2}}(b^h dy_i^v + f^v dx_k^h), & \forall i \in \{1, \dots, s_2\}, k \in \{s_1+1, \dots, m_1\}, \\ \eta_{k,v}^{i,v} &= \frac{b^h}{\sqrt{2}}(dy_i^v + dy_k^v), & \forall i \in \{1, \dots, s_2\}, k \in \{s_2+1, \dots, m_2\}. \end{cases}$$

Theorem 2. *Let $({}_f B \times_b F, g, \Pi)$ be a doubly warped product Poisson manifold of (B, g_B, Π_B) and (F, g_F, Π_F), and let π be a degenerate plane spanned by a null 1-form $\eta_k^i \in \{\eta_{k,h}^{i,h}, \eta_{k,v}^{i,h}, \eta_{k,h}^{i,v}, \eta_{k,v}^{i,v}\}$ and a unit 1-form $\alpha_l \in \{f^v dx_1^h, \dots, f^v dx_{m_1}^h, b^h dy_1^v, \dots, b^h dy_{m_2}^v\}$. Then, the null sectional contravariant curvature $\widetilde{\mathcal{K}}$ of π is given by:*

1.

$$\begin{cases} \widetilde{\mathcal{K}}(\eta_{k,h}^{i,h}, f^v dx_l^h) &= (f^v)^2\Big[\frac{-1}{2}\mathcal{K}^B(dx_i, dx_l) + \frac{1}{2}\mathcal{K}^B(dx_k, dx_l) + \varepsilon_{l_B} g(\mathcal{R}^B(dx_i, dx_l)dx_l, dx_k)\Big]^h \\ \widetilde{\mathcal{K}}(\eta_{k,h}^{i,h}, b^h dy_l^v) &= -\frac{(f^2)^v}{(b^2)^h} g_B^2(dx_i + dx_k, J_B db)^h + \frac{(f^2)^v}{2b^h}\mathcal{H}_{\Pi_B}^b(dx_i + dx_k, dx_i + dx_k)^h \end{cases}$$

for any $i \in \{1, \dots, s_1\}, k \in \{s_1+1, \dots, m_1\}$, $i \neq l$, and $k \neq l$, where $\varepsilon_{l_B} = g_B(dx_l, dx_l)$.

2.

$$\begin{cases} \widetilde{\mathcal{K}}(\eta_{k,v}^{i,h}, f^v dx_l^h) &= -\frac{(f^v)^2}{2}\Big[\mathcal{K}^B(dx_i, dx_l) + \frac{2\varepsilon_{l_B}}{b^2} g_B^2(dx_l, J_B db) - \frac{\varepsilon_{l_B}}{b}\mathcal{H}_{\Pi_B}^b(dx_l, dx_l)\Big]^h \\ &+ \frac{(b^2)^h}{2}\Big[\frac{1}{f^2}\{\|J_F df\|_F^2 - 2g_F^2(dy_k, J_F df)\} + \frac{1}{f}\mathcal{H}_{\Pi_F}^f(dy_k, dy_k)\Big]^v \\ &+ g_B(dx_i, J_B db)^h g_F(dy_k, J_F df)^v \\ \widetilde{\mathcal{K}}(\eta_{k,v}^{i,h}, b^h dy_l^v) &= \frac{(b^h)^2}{2}\Big[\mathcal{K}^F(dy_k, dy_l) + \frac{2\varepsilon_{l_F}}{f^2} g_F^2(dy_l, J_F df) - \frac{\varepsilon_{l_F}}{f}\mathcal{H}_{\Pi_F}^f(dy_l, dy_l)\Big]^v \\ &- \frac{(f^2)^v}{2}\Big[\frac{1}{b^2}\{\|J_B db\|_B^2 + 2g_B^2(dx_i, J_B db)\} - \frac{1}{b}\mathcal{H}_{\Pi_B}^b(dx_i, dx_i)\Big]^h \\ &+ g_B(dx_i, J_B db)^h g_F(dy_k, J_F df)^v \end{cases}$$

for any $i \in \{1, \dots, s_1\}, k \in \{s_2+1, \dots, m_2\}$, $i \neq l$, and $k \neq l$, where $\varepsilon_{l_F} = g_F(dy_l, dy_l)$.

3.

$$\begin{cases} \widetilde{\mathcal{K}}(\eta_{k,h}^{i,v}, f^v dx_l^h) &= \frac{(f^v)^2}{2}\Big[\mathcal{K}^B(dx_k, dx_l) + \frac{2\varepsilon_{l_B}}{b^2} g_B^2(dx_l, J_B db) - \frac{\varepsilon_{l_B}}{b}\mathcal{H}_{\Pi_B}^b(dx_l, dx_l)\Big]^h \\ &- \frac{(b^2)^h}{2}\Big[\frac{1}{f^2}\{\|J_F df\|_F^2 + 2g_F^2(dy_i, J_F df)\} - \frac{1}{f}\mathcal{H}_{\Pi_F}^f(dy_i, dy_i)\Big]^v \\ &+ g_B(dx_k, J_B db)^h g_F(dy_i, J_F df)^v \\ \widetilde{\mathcal{K}}(\eta_{k,h}^{i,v}, b^h dy_l^v) &= -\frac{(b^h)^2}{2}\Big[\mathcal{K}^F(dy_i, dy_l) + \frac{2\varepsilon_{l_F}}{f^2} g_F^2(dy_l, J_F df) - \frac{\varepsilon_{l_F}}{f}\mathcal{H}_{\Pi_F}^f(dy_l, dy_l)\Big]^v \\ &+ \frac{(f^2)^v}{2}\Big[\frac{1}{b^2}\{\|J_B db\|_B^2 - 2g_B^2(dx_k, J_B db)\} + \frac{1}{b}\mathcal{H}_{\Pi_B}^b(dx_k, dx_k)\Big]^h \\ &+ g_B(dx_k, J_B db)^h g_F(dy_i, J_F df)^v \end{cases}$$

for any $i \in \{1, \dots, s_2\}, k \in \{s_1+1, \dots, m_1\}$, $i \neq l$, and $k \neq l$.

4.

$$\begin{cases} \widetilde{\mathcal{K}}(\eta_{k,v}^{i,v}, f^v dx_l^h) &= -\frac{(b^2)^h}{(f^2)^v} g_F^2(dy_i + dy_k, J_F df)^v + \frac{(b^2)^h}{2f^v}\mathcal{H}_{\Pi_F}^f(dy_i + dy_k, dy_i + dy_k)^v \\ \widetilde{\mathcal{K}}(\eta_{k,v}^{i,v}, b^h dy_l^v) &= (b^h)^2\Big[\frac{-1}{2}\mathcal{K}^F(dy_i, dy_l) + \frac{1}{2}\mathcal{K}^F(dy_k, dy_l) + \varepsilon_{l_F} g(\mathcal{R}^F(dy_i, dy_l)dy_l, dy_k)\Big]^v \end{cases}$$

for any $i \in \{1, \dots, s_2\}, k \in \{s_2+1, \dots, m_2\}$, $i \neq l$, and $k \neq l$.

Proof. Using Lemma 2, Theorem 1 and Equation (21), we obtain this theorem. For example, consider Part 2 of the theorem for all $i \in \{1, \dots, s_1\}, k \in \{s_2+1, \dots, m_2\}$. We obtain:

$$
\begin{aligned}
\widetilde{\mathcal{K}}(\eta^{i,h}_{k,v}, f^v dx^h_l) &= \frac{g(\mathcal{R}(\eta^{i,h}_{k,v}, f^v dx^h_l)f^v dx^h_l, \eta^{i,h}_{k,v})}{g(f^v dx^h_l, f^v dx^h_l)} \\
&= \frac{1}{2g(f^v dx^h_l, f^v dx^h_l)}\Big[g(\mathcal{R}(f^v dx^h_i, f^v dx^h_l)f^v dx^h_l, f^v dx^h_i) + g(\mathcal{R}(b^h dy^v_k, f^v dx^h_l)f^v dx^h_l, b^h dy^v_k) \\
&+ 2g(\mathcal{R}(f^v dx^h_i, f^v dx^h_l)f^v dx^h_l, b^h dy^v_k)\Big] \\
&= -\tfrac{1}{2}\mathcal{K}(dx^h_i, dx^h_l) + \tfrac{1}{2}\mathcal{K}(dy^v_k, dx^h_l) + (f^v)^3(b^h)\varepsilon_{l_B} g(\mathcal{R}(dx^h_i, dx^h_l)dx^h_l, dy^v_k) \\
&= -\tfrac{1}{2}\Big[(f^2)^v \mathcal{K}^B(dx_i, dx_l)^h - (b^2)^h(\tfrac{\|J_F df\|^2_F}{f^2})^v\Big] \\
&+ \tfrac{1}{2}\Big[-(f^2)^v \varepsilon_{l_B}\Big\{2\frac{g^2_B(dx_l, J_B db)}{b^2} + \frac{g_B(\mathcal{D}^B_{dx_l} J_B db, dx_l)}{b}\Big\}^h \\
&- (b^2)^h\Big\{\frac{2g^2_F(dy_k, J_F df)}{f^2} + \frac{g_F(dy_k, \mathcal{D}^F_{dy_k} J_F df)}{f}\Big\}^v\Big] \\
&+ (f^v)^3 b^h \varepsilon_{l_B} g(\tfrac{1}{(f^3)^v}[b\{\varepsilon_{l_B} g_B(dx_i, J_B db) - g_B(dx_l, J_B db) g_B(dx_i, dx_l)\}]^h (J_F df)^v, dy^v_k) \\
&= -\frac{(f^v)^2}{2}\Big[\mathcal{K}^B(dx_i, dx_l) + \frac{2\varepsilon_{l_B}}{b^2} g^2_B(dx_l, J_B db) - \frac{\varepsilon_{l_B}}{b}\mathcal{H}^b_{\Pi_B}(dx_l, dx_l)\Big]^h \\
&+ \frac{(b^2)^h}{2}\Big[\frac{1}{f^2}(\|J_F df\|^2_F - 2g^2_F(dy_k, J_F df)) + \frac{1}{f}\mathcal{H}^f_{\Pi_F}(dy_k, dy_k)\Big]^v \\
&+ g_B(dx_i, J_B db)^h g_F(dy_k, J_F df)^v,
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{\mathcal{K}}(\eta^{i,h}_{k,v}, b^h dy^v_l) &= \frac{g(\mathcal{R}(\eta^{i,h}_{k,v}, b^h dy^v_l)b^h dy^v_l, \eta^{i,h}_{k,v})}{g(b^h dy^v_l, b^h dy^v_l)} \\
&= \frac{1}{2g(b^h dy^v_l, b^h dy^v_l)}\Big[g(\mathcal{R}(f^v dx^h_i, b^h dy^v_l)b^h dy^v_l, f^v dx^h_i) + g(\mathcal{R}(b^h dy^v_k, b^h dy^v_l)b^h dy^v_l, b^h dy^v_k) \\
&+ 2g(\mathcal{R}(f^v dx^h_i, b^h dy^v_l)b^h dy^v_l, b^h dy^v_k)\Big] \\
&= -\tfrac{1}{2}\mathcal{K}(dx^h_i, dy^v_l) + \tfrac{1}{2}\mathcal{K}(dy^v_k, dy^v_l) + (b^h)^3 f^v \varepsilon_{l_F} g(\mathcal{R}(dx^h_i, dy^v_l)dy^v_l, dy^v_k) \\
&= \tfrac{1}{2}\Big[(b^2)^h \mathcal{K}^F(dy_k, dy_l)^v - (f^2)^v(\tfrac{\|J_B db\|^2_B}{b^2})^h\Big] \\
&- \tfrac{1}{2}\Big[-(f^2)^v\Big\{-2\frac{g^2_B(dx_i, J_B db)}{b^2} - \frac{g_B(\mathcal{D}^B_{dx_i} J_B db, dx_i)}{b}\Big\}^h \\
&- \varepsilon_{l_F}(b^2)^h\Big\{\frac{2g^2_F(dy_l, J_F df)}{f^2} + \frac{g_F(dy_l, \mathcal{D}^F_{dy_l} J_F df)}{f}\Big\}^v\Big] \\
&+ \varepsilon_{l_F}(b^2)^h g(g_F(dy_l, dy_l)^v g_B(dx_i, J_B db)^h (J_F df)^v, dy^v_k) \\
&= \frac{(b^h)^2}{2}\Big[\mathcal{K}^F(dy_k, dy_l) + \frac{2\varepsilon_{l_F}}{f^2} g^2_F(dy_l, J_F df) - \frac{\varepsilon_{l_F}}{f}\mathcal{H}^f_{\Pi_F}(dy_l, dy_l)\Big]^v \\
&- \frac{(f^2)^v}{2}\Big[\frac{1}{b^2}(\|J_B db\|^2_B + 2g^2_B(dx_i, J_B db)) - \frac{1}{b}\mathcal{H}^b_{\Pi_B}(dx_i, dx_i)\Big]^h \\
&+ g_B(dx_i, J_B db)^h g_F(dy_k, J_F df)^v,
\end{aligned}
$$

In the same way, we can prove the remaining parts of the theorem. □

Theorem 3. *Let $({}_f B \times_b F, g, \Pi)$ be a doubly warped product Poisson manifold of (B, g_B, Π_B) and (F, g_F, Π_F). Then, the qualar curvature at a point $(p_1, p_2) \in B \times F$ is given by:*

$$
\begin{aligned}
\mathrm{qual}(p_1, p_2) &= (f^2)^v \mathrm{qual}(p_1)^h + (b^2)^h \mathrm{qual}(p_2)^v - 2\Big[s_1(2 + m_1 - s_1)\frac{(b^2)^h}{(f^2)^v}(\|J_F df\|^2_F)^v \\
&- \frac{2m_2(f^2)^v}{(b^2)^h}\sum_{i=1}^{s_1} g^2_B(J_B db, dx_i)^h - \frac{m_2(f^2)^v}{b^h}\sum_{i=1}^{s_1} g_B(\mathcal{D}^B_{dx_i} J_B db, dx_i)^h + \frac{s_2(f^2)^v}{b^h}(\Delta^{\mathcal{D}^B}(b))^h \\
&+ s_2(2 + m_2 - s_2)\frac{(f^2)^v}{(b^2)^h}(\|J_B db\|^2_B)^h - \frac{2m_1(b^2)^h}{(f^2)^v}\sum_{i=1}^{s_2} g^2_F(J_F df, dy_i)^v \\
&- \frac{m_1(b^2)^h}{f^v}\sum_{i=1}^{s_2} g_F(\mathcal{D}^F_{dy_i} J_F df, dy_i)^v + \frac{s_1(b^2)^h}{f^v}(\Delta^{\mathcal{D}^F}(f))^v\Big]
\end{aligned}
$$

Proof. Using Theorem 1 and Equation (20), for any $(p_1, p_2) \in B \times F$, we obtain:

$$\begin{aligned}
\mathrm{qual}(p_1,p_2) &= 2\sum_{i=1}^{s_1}\sum_{k=s_1+1}^{m_1}\Big[(f^2)^v\mathcal{K}^B(dx_i,dx_k)^h-(b^2)^h(\tfrac{\|J_Fdf\|_F^2}{f^2})^v\Big]\\
&+\ 2\sum_{i=1}^{s_1}\sum_{k=s_2+1}^{m_2}\Big[-(f^2)^v\{\tfrac{-2g_B^2(dx_i,J_Bdb)}{b^2}-\tfrac{g_B(\mathcal{D}^B_{dx_i}J_Bdb,dx_i)}{b}\}^h\\
&-\ (b^2)^h\{\tfrac{2g_F^2(dy_k,J_Fdf)}{f^2}+\tfrac{g_F(\mathcal{D}^F_{dy_k}J_Fdf,dy_k)}{f}\}^v\Big]\\
&+\ 2\sum_{i=1}^{s_2}\sum_{k=s_1+1}^{m_1}\Big[-(f^2)^v\{\tfrac{2g_B^2(dx_k,J_Bdb)}{b^2}+\tfrac{g_B(\mathcal{D}^B_{dx_k}J_Bdb,dx_k)}{b}\}^h\\
&-\ (b^2)^h\{\tfrac{-2g_F^2(dy_i,J_Fdf)}{f^2}-\tfrac{g_F(\mathcal{D}^F_{dy_i}J_Fdf,dy_i)}{f}\}^v\Big]+2\sum_{i=1}^{s_2}\sum_{k=s_2+1}^{m_2}\Big[(b^2)^h\mathcal{K}^F(dy_i,dy_k)^v\\
&-\ (f^2)^v(\tfrac{\|J_Bdb\|_B^2}{b^2})^h\Big]\\
&=\ 2(f^2)^v\sum_{i=1}^{s_1}\sum_{k=s_1+1}^{m_1}\mathcal{K}^B(dx_i,dx_k)^h-2s_1(m_1-s_1)\tfrac{(b^2)^h}{(f^2)^v}(\|J_Fdf\|_F^2)^v\\
&+\ \tfrac{4(m_2-s_2)(f^2)^v}{(b^2)^h}\sum_{i=1}^{s_1}g_B^2(dx_i,J_Bdb)^h+\tfrac{2(m_2-s_2)(f^2)^v}{b^h}\sum_{i=1}^{s_1}g_B(\mathcal{D}^B_{dx_i}J_Bdb,dx_i)^h\\
&-\ \tfrac{4s_1(b^2)^h}{(f^2)^v}\sum_{k=s_2+1}^{m_2}g_F^2(dy_k,J_Fdf)^v-\tfrac{2s_1(b^2)^h}{f^v}\sum_{k=s_2+1}^{m_2}g_F(\mathcal{D}^F_{dy_k}J_Fdf,dy_k)^v\\
&-\ \tfrac{4s_2(f^2)^v}{(b^2)^h}\sum_{k=s_1+1}^{m_1}g_B^2(dx_k,J_Bdb)^h-\tfrac{2s_2(f^2)^v}{b^h}\sum_{k=s_1+1}^{m_1}g_B(\mathcal{D}^B_{dx_k}J_Bdb,dx_k)^h\\
&+\ \tfrac{4(m_1-s_1)(b^2)^h}{(f^2)^v}\sum_{i=1}^{s_2}g_F^2(dy_i,J_Fdf)^v+\tfrac{2(m_1-s_1)(b^2)^h}{f^v}\sum_{i=1}^{s_2}g_F(\mathcal{D}^F_{dy_i}J_Fdf,dy_i)^v\\
&+\ 2(b^2)^h\sum_{i=1}^{s_2}\sum_{k=s_2+1}^{m_1}\mathcal{K}^F(dy_i,dy_k)^h-2s_2(m_2-s_2)\tfrac{(f^2)^v}{(b^2)^h}(\|J_Bdb\|_B^2)^h,
\end{aligned}$$

and the theorem follows. □

Corollary 7. *Let $({}_fB\times_b F,g,\Pi)$ be a doubly warped product manifold such that the fiber F is a Riemannian manifold. Then, the qualar curvature at point $(p_1,p_2)\in B\times F$ becomes:*

$$\begin{aligned}
\mathrm{qual}(p_1,p_2) &= (f^2)^v\mathrm{qual}(p_1)^h-2\Big[s_1(2+m_1-s_1)\tfrac{(b^2)^h}{(f^2)^v}(\|J_Fdf\|_F^2)^v\\
&-\ \tfrac{2m_2(f^2)^v}{(b^2)^h}\sum_{i=1}^{s_1}g_B^2(J_Bdb,dx_i)^h-\tfrac{m_2(f^2)^v}{b^h}\sum_{i=1}^{s_1}g_B(\mathcal{D}^B_{dx_i}J_Bdb,dx_i)^h+\tfrac{s_1(b^2)^h}{f^v}(\Delta^{\mathcal{D}^F}(f))^v\Big].
\end{aligned}\tag{22}$$

Proof. Taking $s_2=0$ in the Theorem 3, the corollary follows directly. □

5. Example

Let (M,g,Π) be a pseudo-Riemannian manifold equipped with a Poisson tensor Π, and let $\mathcal{D}$ be the Levi–Civita connection associated with (g,Π). If $(x_1,\dots,x_m)$ is a local coordinates on a neighborhood $\mathcal{U}$ in M, then we can define the Christoffel symbols Γ_k^{ij} by [6]:

$$\mathcal{D}_{dx_i}dx_j=\sum_{k=1}^{m}\Gamma_k^{ij}dx_k.\tag{23}$$

Since $\sharp_\Pi(dx_i)=\sum_{l=1}^{m}\Pi^{il}\frac{\partial}{\partial x_l}$ and $[dx_i,dx_j]_\Pi=\sum_{l=1}^{m}\frac{\partial\Pi^{ij}}{\partial x_l}dx_l$, then from (3) and (23), we have:

$$\begin{aligned}
2g(\mathcal{D}_{dx_i}dx_j,dx_n) &= 2\sum_{k=1}^{m}\Gamma_k^{ij}g^{kn}\\
&= \sum_{l=1}^{m}\Big(\Pi^{il}\frac{\partial g^{jn}}{\partial x_l}+\Pi^{jl}\frac{\partial g^{in}}{\partial x_l}+\Pi^{ln}\frac{\partial g^{ij}}{\partial x_l}+g^{ln}\frac{\partial\Pi^{ij}}{\partial x_l}+g^{lj}\frac{\partial\Pi^{ni}}{\partial x_l}+g^{li}\frac{\partial\Pi^{nj}}{\partial x_l}\Big).
\end{aligned}$$

Multiplying both sides of the previous equation by $\sum_{n=1}^{m} g_{kn}$ we obtain:

$$\Gamma_k^{ij} = \frac{1}{2}\sum_{l=1}^{m}\sum_{n=1}^{m} g_{kn}\Big(\Pi^{il}\frac{\partial g^{jn}}{\partial x_l} + \Pi^{jl}\frac{\partial g^{in}}{\partial x_l} + \Pi^{ln}\frac{\partial g^{ij}}{\partial x_l} + g^{ln}\frac{\partial \Pi^{ij}}{\partial x_l} + g^{lj}\frac{\partial \Pi^{ni}}{\partial x_l} + g^{li}\frac{\partial \Pi^{nj}}{\partial x_l}\Big), \quad (24)$$

where g_{kn} is the inverse matrix of g^{kn}.

Now, we take $B = (c,d) \times \mathbb{R}$ with the Lorentzian metric $\tilde{g}_B = -dx_1^2 + dx_2^2$, where (c,d) is an open interval and $F = \mathbb{R}^2$ with the metric $\tilde{g}_F = dx_3^2 + dx_4^2$, and let $b : (c,d) \times \mathbb{R} \to (0,\infty)$ and $f : \mathbb{R}^2 \to (0,\infty)$ be positive smooth functions on $(c,d) \times \mathbb{R}$ and $\mathbb{R}^2$, respectively. Then, the product manifold $B \times F = (c,d) \times \mathbb{R}^3$ equipped with the following metric:

$$\tilde{g} = (f^v)^2(-dx_1^2 + dx_2^2) + (b^h)^2(dx_3^2 + dx_4^2)$$

is a 4-dimensional Lorentzian doubly warped product manifold, where (x_1), (x_2) and (x_3, x_4) are the coordinates on (c,d), $\mathbb{R}$, and $\mathbb{R}^2$, respectively. Furthermore, if g is the contravariant metric of $\tilde{g}$, then its local components are given by:

$$\begin{cases} g^{11} = g(dx_1^h, dx_1^h) = \frac{1}{(f^v)^2} g_B(dx_1, dx_1)^h & = & -\frac{1}{(f^v)^2} \\ g^{22} = g(dx_2^h, dx_2^h) = \frac{1}{(f^v)^2} g_B(dx_2, dx_2)^h & = & \frac{1}{(f^v)^2} \\ g^{33} = g(dx_3^v, dx_3^v) = \frac{1}{(b^h)^2} g_F(dx_3, dx_3)^h & = & \frac{1}{(b^h)^2} \\ g^{44} = g(dx_4^v, dx_4^v) = \frac{1}{(b^h)^2} g_F(dx_4, dx_4)^h & = & \frac{1}{(b^h)^2} \\ g^{12} = g^{13} = g^{14} = g^{23} = g^{24} = g^{34} & = & 0. \end{cases}$$

Let $\Pi_B = \Pi_B^{12}\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$ and $\Pi_F = \Pi_F^{34}\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}$ be Poisson tensors on B and F, respectively, then $\Pi = \Pi_F + \Pi_B$ is a Poisson tensor on ${}_f B \times_b F$ and $({}_f B \times_b F, g, \Pi)$ is a 4-dimensional Lorentzian doubly warped product Poisson manifold.

Using (24), the Christoffel symbols of (B, g_B, Π_B) and (F, g_F, Π_F) are given, respectively, by:

$$\Gamma_1^{11} = 0, \qquad \Gamma_1^{12} = -\frac{\partial \Pi_B^{21}}{\partial x_1}, \qquad \Gamma_2^{11} = \frac{\partial \Pi_B^{12}}{\partial x_1}, \qquad \Gamma_2^{12} = 0$$

$$\Gamma_1^{21} = 0, \qquad \Gamma_2^{21} = \frac{\partial \Pi_B^{21}}{\partial x_2}, \qquad \Gamma_1^{22} = \frac{\partial \Pi_B^{21}}{\partial x_2}, \qquad \Gamma_2^{22} = 0$$

and

$$\Gamma_3^{33} = 0, \qquad \Gamma_3^{34} = \frac{\partial \Pi_F^{34}}{\partial x_3}, \qquad \Gamma_4^{33} = -\frac{\partial \Pi_F^{34}}{\partial x_3}, \qquad \Gamma_4^{34} = 0$$

$$\Gamma_3^{43} = 0, \qquad \Gamma_4^{43} = \frac{\partial \Pi_F^{43}}{\partial x_4}, \qquad \Gamma_3^{44} = \frac{\partial \Pi_F^{34}}{\partial x_4}, \qquad \Gamma_4^{44} = 0.$$

Hence, from (23) and above equations of Christoffel symbols, we obtain:

$$\mathcal{D}_{dx_1}^B dx_1 = \frac{\partial \Pi_B^{12}}{\partial x_1} dx_2, \qquad \mathcal{D}_{dx_1}^B dx_2 = -\frac{\partial \Pi_B^{21}}{\partial x_1} dx_1$$

$$\mathcal{D}_{dx_2}^B dx_1 = \frac{\partial \Pi_B^{21}}{\partial x_2} dx_2, \qquad \mathcal{D}_{dx_2}^B dx_2 = \frac{\partial \Pi_B^{21}}{\partial x_2} dx_1$$

and

$$\mathcal{D}_{dx_3}^F dx_3 = -\frac{\partial \Pi_F^{34}}{\partial x_3} dx_4, \qquad \mathcal{D}_{dx_3}^F dx_4 = \frac{\partial \Pi_F^{34}}{\partial x_3} dx_3$$

$$\mathcal{D}_{dx_4}^F dx_3 = \frac{\partial \Pi_F^{43}}{\partial x_4} dx_4, \qquad \mathcal{D}_{dx_4}^F dx_4 = \frac{\partial \Pi_F^{34}}{\partial x_4} dx_3.$$

Using Equation (6), the sectional contravariant curvatures of the non-degenerated planes $\{dx_1, dx_2\}$ and $\{dx_3, dx_4\}$ are given, respectively, by:

$$\begin{aligned}
\mathcal{K}^B(dx_1, dx_2) &= (\frac{\partial \Pi_B^{12}}{\partial x_2})^2 - (\frac{\partial \Pi_B^{12}}{\partial x_1})^2 + \Pi_B^{21}[\frac{\partial^2 \Pi_B^{21}}{\partial x_1^2} - \frac{\partial^2 \Pi_B^{21}}{\partial x_2^2}], \\
\mathcal{K}^F(dx_3, dx_4) &= \Pi_F^{34}[\frac{\partial^2 \Pi_F^{34}}{\partial x_3^2} + \frac{\partial^2 \Pi_F^{34}}{\partial x_4^2}] - [(\frac{\partial \Pi_F^{34}}{\partial x_3})^2 + (\frac{\partial \Pi_F^{34}}{\partial x_4})^2].
\end{aligned} \tag{25}$$

Using the fact that $\{dx_1, dx_2\}$ and $\{dx_3, dx_4\}$ are, respectively, g_B- and g_F-orthonormal bases, we obtain the following identities:

$$\begin{cases}
J_B dx_1 = \Pi_B^{12} dx_2, \qquad J_B dx_2 = \Pi_B^{12} dx_1, \qquad J_F dx_3 = \Pi_F^{34} dx_4, \qquad J_F dx_4 = -\Pi_F^{34} dx_3, \\
J_B db = \Pi_B^{12}[\frac{\partial b}{\partial x_1} dx_2 + \frac{\partial b}{\partial x_2} dx_1], \qquad J_F df = \Pi_F^{34}[\frac{\partial f}{\partial x_3} dx_4 - \frac{\partial f}{\partial x_4} dx_3], \\
\|J_B db\|_B^2 =|\ (\Pi_B^{12})^2[(\frac{\partial b}{\partial x_1})^2 - (\frac{\partial b}{\partial x_2})^2]\ |, \qquad \|J_F df\|_F^2 = (\Pi_F^{34})^2[(\frac{\partial f}{\partial x_3})^2 + (\frac{\partial f}{\partial x_4})^2], \\
g_B(J_B db, dx_1) = -\Pi_B^{12} \frac{\partial b}{\partial x_2}, \qquad g_B(J_B db, dx_2) = \Pi_B^{12} \frac{\partial b}{\partial x_1}, \\
g_F(J_F df, dx_3) = -\Pi_F^{34} \frac{\partial f}{\partial x_4}, \qquad g_F(J_F db, dx_4) = \Pi_F^{34} \frac{\partial f}{\partial x_3}, \\
g_B(\mathcal{D}^B_{dx_1} J_B db, dx_1) = \Pi_B^{12}[\frac{\partial b}{\partial x_1} \frac{\partial \Pi_B^{21}}{\partial x_1} - \frac{\partial b}{\partial x_2} \frac{\partial \Pi_B^{12}}{\partial x_2} - \Pi_B^{12} \frac{\partial^2 b}{\partial x_2^2}], \\
g_B(\mathcal{D}^B_{dx_2} J_B db, dx_2) = \Pi_B^{21}[\frac{\partial b}{\partial x_1} \frac{\partial \Pi_B^{12}}{\partial x_1} + \frac{\partial b}{\partial x_2} \frac{\partial \Pi_B^{12}}{\partial x_2} + \Pi_B^{12} \frac{\partial^2 b}{\partial x_1^2}], \\
g_F(\mathcal{D}^F_{dx_3} J_B db, dx_3) = \Pi_F^{34}[\frac{\partial f}{\partial x_3} \frac{\partial \Pi_F^{34}}{\partial x_3} - \frac{\partial f}{\partial x_4} \frac{\partial \Pi_F^{34}}{\partial x_4} - \Pi_F^{34} \frac{\partial^2 f}{\partial x_4^2}], \\
g_F(\mathcal{D}^F_{dx_4} J_F df, dx_4) = \Pi_F^{43}[\frac{\partial f}{\partial x_3} \frac{\partial \Pi_F^{34}}{\partial x_3} + \frac{\partial f}{\partial x_4} \frac{\partial \Pi_F^{43}}{\partial x_4} + \Pi_F^{34} \frac{\partial^2 f}{\partial x_3^2}].
\end{cases}$$

Therefore, according to Theorem 1, the sectional curvature of the non-degenerate plane $\{dx_1^h + dx_3^v, dx_2^h + dx_4^v\}$ is given by:

$$\begin{cases}
\mathcal{K}(dx_1^h, dx_2^h) &= & (f^2)^v[(\frac{\partial \Pi_B^{12}}{\partial x_2})^2 - (\frac{\partial \Pi_B^{12}}{\partial x_1})^2 + \Pi_B^{21}(\frac{\partial^2 \Pi_B^{21}}{\partial x_1^2} - \frac{\partial^2 \Pi_B^{21}}{\partial x_2^2})]^h - \frac{(b^2)^h}{(f^2)^v}[(\Pi_F^{34})^2((\frac{\partial f}{\partial x_3})^2 + (\frac{\partial f}{\partial x_4})^2)]^v, \\
\mathcal{K}(dx_3^v, dx_4^v) &= & (b^2)^h[\Pi_F^{34}(\frac{\partial^2 \Pi_F^{34}}{\partial x_3^2} + \frac{\partial^2 \Pi_F^{34}}{\partial x_4^2}) - ((\frac{\partial \Pi_F^{34}}{\partial x_3})^2 + (\frac{\partial \Pi_F^{34}}{\partial x_4})^2)]^v - \frac{(f^2)^v}{(b^2)^h}[|\ (\Pi_B^{12})^2((\frac{\partial b}{\partial x_1})^2 - (\frac{\partial b}{\partial x_2})^2)\ |]^h, \\
\mathcal{K}(dx_1^h, dx_3^v) &= & (f^2)^v[\frac{2}{b^2}(\Pi_B^{12} \frac{\partial b}{\partial x_2})^2 + \frac{\Pi_B^{12}}{b}(\frac{\partial b}{\partial x_1} \frac{\partial \Pi_B^{21}}{\partial x_1} - \frac{\partial b}{\partial x_2} \frac{\partial \Pi_B^{12}}{\partial x_2} - \frac{\partial^2 b}{\partial x_2^2})]^h \\
&- & (b^2)^h[\frac{2}{f^2}(\Pi_F^{34} \frac{\partial f}{\partial x_4})^2 + \frac{\Pi_F^{34}}{f}(\frac{\partial f}{\partial x_3} \frac{\partial \Pi_F^{34}}{\partial x_3} - \frac{\partial f}{\partial x_4} \frac{\partial \Pi_F^{34}}{\partial x_4} - \Pi_F^{34} \frac{\partial^2 f}{\partial x_4^2})]^v, \\
\mathcal{K}(dx_1^h, dx_4^v) &= & (f^2)^v[\frac{2}{b^2}(\Pi_B^{12} \frac{\partial b}{\partial x_2})^2 + \frac{\Pi_B^{12}}{b}(\frac{\partial b}{\partial x_1} \frac{\partial \Pi_B^{21}}{\partial x_1} - \frac{\partial b}{\partial x_2} \frac{\partial \Pi_B^{12}}{\partial x_2} - \frac{\partial^2 b}{\partial x_2^2})]^h \\
&- & (b^2)^h[\frac{2}{f^2}(\Pi_F^{34} \frac{\partial f}{\partial x_3})^2 + \frac{\Pi_F^{43}}{f}(\frac{\partial f}{\partial x_3} \frac{\partial \Pi_F^{34}}{\partial x_3} + \frac{\partial f}{\partial x_4} \frac{\partial \Pi_F^{43}}{\partial x_4} + \Pi_F^{34} \frac{\partial^2 f}{\partial x_3^2})]^v, \\
\mathcal{K}(dx_2^h, dx_3^v) &= & -(f^2)^v[\frac{2}{b^2}(\Pi_B^{12} \frac{\partial b}{\partial x_1})^2 + \frac{\Pi_B^{12}}{b}(\frac{\partial b}{\partial x_1} \frac{\partial \Pi_B^{12}}{\partial x_1} + \frac{\partial b}{\partial x_2} \frac{\partial \Pi_B^{12}}{\partial x_2} + \Pi_B^{12} \frac{\partial^2 b}{\partial x_1^2})]^h \\
&- & (b^2)^h[\frac{2}{f^2}(\Pi_F^{34} \frac{\partial f}{\partial x_4})^2 + \frac{\Pi_F^{34}}{f}(\frac{\partial f}{\partial x_3} \frac{\partial \Pi_F^{34}}{\partial x_3} - \frac{\partial f}{\partial x_4} \frac{\partial \Pi_F^{34}}{\partial x_4} - \Pi_F^{34} \frac{\partial^2 f}{\partial x_4^2})]^v, \\
\mathcal{K}(dx_2^h, dx_4^v) &= & -(f^2)^v[\frac{2}{b^2}(\Pi_B^{12} \frac{\partial b}{\partial x_1})^2 + \frac{\Pi_B^{12}}{b}(\frac{\partial b}{\partial x_1} \frac{\partial \Pi_B^{12}}{\partial x_1} + \frac{\partial b}{\partial x_2} \frac{\partial \Pi_B^{12}}{\partial x_2} + \Pi_B^{12} \frac{\partial^2 b}{\partial x_1^2})]^h \\
&- & (b^2)^h[\frac{2}{f^2}(\Pi_F^{34} \frac{\partial f}{\partial x_3})^2 + \frac{\Pi_F^{43}}{f}(\frac{\partial f}{\partial x_3} \frac{\partial \Pi_F^{34}}{\partial x_3} + \frac{\partial f}{\partial x_4} \frac{\partial \Pi_F^{43}}{\partial x_4} + \Pi_F^{34} \frac{\partial^2 f}{\partial x_3^2})]^v.
\end{cases}$$

Using the above equations in (22), the qualar curvature of $({}_f B \times_b F, g, \Pi)$ at point (p_1, p_2) is given by:

$$\begin{aligned}\mathrm{qual}(p_1,p_2) &= 2(f^2)^v[(\frac{\partial\Pi_B^{12}}{\partial x_2})^2-(\frac{\partial\Pi_B^{12}}{\partial x_1})^2+\Pi_B^{21}(\frac{\partial^2\Pi_B^{21}}{\partial x_1^2}-\frac{\partial^2\Pi_B^{21}}{\partial x_2^2})]^h \\ &- 6\frac{(b^2)^h}{(f^2)^v}[(\Pi_F^{34})^2\{(\frac{\partial f}{\partial x_3})^2+(\frac{\partial f}{\partial x_4})^2\}]^v+8\frac{(f^2)^v}{(b^2)^h}[(\Pi_B^{12}\frac{\partial b}{\partial x_2})^2]^h \\ &+ 4\frac{(f^2)^v}{b^h}[\Pi_B^{12}(\frac{\partial b}{\partial x_1}\frac{\partial\Pi_B^{21}}{\partial x_1}-\frac{\partial b}{\partial x_2}\frac{\partial\Pi_B^{12}}{\partial x_2}-\Pi_B^{12}\frac{\partial^2 b}{\partial x_2^2})]^h+2\frac{(b^2)^h}{f^v}[(\Pi_F^{34})^2(\frac{\partial^2 f}{\partial x_3^2}+\frac{\partial^2 f}{\partial x_4^2})]^v.\end{aligned} \tag{26}$$

Remark 1. *If $\Pi_B^{12}=\lambda x_1$ and $\Pi_F^{34}=\lambda x_3$, $\forall\lambda\in\mathbb{R}$, then Equations (25) and (26) reduced to $\mathcal{K}^B(dx_1,dx_2)=\mathcal{K}^F(dx_3,dx_4)=-\lambda^2$ and*

$$\begin{aligned}\mathrm{qual}(p_1,p_2) &= -2\lambda^2\Big[(f^2)^v+3(b^2)^h\Big((\tfrac{x_3}{f})^2\{(\tfrac{\partial f}{\partial x_3})^2+(\tfrac{\partial f}{\partial x_4})^2\}\Big)^v-4(f^2)^v\Big((\tfrac{x_1}{b^2})^2(\tfrac{\partial b}{\partial x_2})^2\Big)^h \\ &+ 2(f^2)^v\Big(\tfrac{x_1}{b}(\tfrac{\partial b}{\partial x_1}+x_1\tfrac{\partial^2 b}{\partial x_2^2})\Big)^h-(b^2)^h\Big(\tfrac{x_3^2}{f}(\tfrac{\partial^2 f}{\partial x_3^2}+\tfrac{\partial^2 f}{\partial x_4^2})\Big)^v\Big].\end{aligned}$$

Furthermore, if b and f are positive constants with $f=c_1$, then $\mathrm{qual}(p_1,p_2)=-2(\lambda c_1)^2\leq 0$.

6. Conclusions

This research paper delved into the investigation of the sectional contravariant curvature of doubly warped product manifolds equipped with a product Poisson structure, which we called doubly warped product Poisson manifolds. Through our comprehensive investigation, we have uncovered several important results and made significant contributions to the understanding of some geometric structures, such as the Levi–Civita contravariant connection, curvature tensor, sectional curvature, qualar, and null sectional curvatures of this class of manifolds.

One of the main achievements of this paper is the establishment of various key findings regarding the properties of sectional contravariant curvature of doubly warped product Poisson manifolds. After expressing the Levi–Civita contravariant connection and the curvature tensor, we have established a relationship between the sectional contravariant curvature of a doubly warped product Poisson manifold and those of its factor manifolds. Using this relationship, we have explored some of its geometric properties, such as the determination of a necessary and sufficient condition for a doubly warped product Poisson manifold to be flat and the discussion of its sign in terms of signs of sectional curvatures of basic manifolds.

Additionally, we have calculated the qualar and null sectional contravariant curvatures of this class of manifolds. Moreover, we enhanced our results by providing an example of a four-dimensional Lorentzian doubly warped product Poisson manifold, in which Levi–Civita connection, qualar, and sectional curvatures are obtained.

The findings presented in this paper have implications for various areas of differential geometry and mathematical physics, contributing to the understanding of warped product manifolds, which are crucial for theoretical physics, especially in the theory of relativity. It fills a specific gap by extending the study of sectional contravariant curvatures to doubly warped product Poisson manifolds.

In conclusion, this paper expands our knowledge about some geometric structures on doubly warped product Poisson manifolds. The insights gained from this research, along with the expressions of the Levi–Civita contravariant connection and the curvature tensor, provide a solid foundation for further investigations and applications in related fields. In future research, we will explore how our results can be used to study the Ricci and scalar curvatures, Einstein's manifolds, contravariant Einstein's equation, and the cosmological constant on this class of manifolds.

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