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Joins, Secant Varieties and Their Associated Grassmannians

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Abstract: We prove a strong theorem on the partial non-defectivity of secant varieties of embedded homogeneous varieties developing a general set-up for families of subvarieties of Grassmannians. We study this type of problem in the more general set-up of joins of embedded varieties. Joins are defined by taking a closure. We study the set obtained before making the closure (often called the open part of the join) and the set added after making the closure (called the boundary of the join). For a point q of the open part, we give conditions for the uniqueness of the set proving that q is in the open part.

Keywords: joins; secant variety; Grassmannian; defective secant variety; solution sets of joins; boundaries of joins

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1. Introduction

We start with an example. Suppose you are interested in complex tensors of format $(n_1+1) \times \cdots \times (n_k+1)$, $k \geq 3$, i.e., elements of the vector space $\mathcal{V} := V_1 \otimes \cdots \cdots \otimes V_k$ with $V_i \cong \mathbb{C}^{n_i+1}$. Take $T \in \mathcal{V}$, $T \neq 0$. A rank 1 tensor is a tensor of the form $v_1 \otimes \cdots v_k$ with $v_i \in V_i \setminus \{0\}$. The rank of T is the minimal number of rank 1 tensors whose sum is T. The rank of a tensor is important for real-life applications since a standard tool for signal processing is to approximate a tensor with a low rank tensor, i.e., to fix a positive integer a and try to approximate T with rank a tensors. Hence, it is natural to ask the following questions:

(1) What is the rank of a "general" tensor?
(2) For any fixed positive integer a how large is the set of all tensors of rank a (or of rank at most a)?

The answers for all formats of complex tensors is not known (and, perhaps, in this strong form will not be known in the near future), but there are "good enough" partial answers using general old results proved by B. Ådlandsvik (Prop. 2.1, Cor. 2.3, Th. 3.10 [1]), which are improved here (with a prompt from conversations with the authors of [2]). Questions (1) and (2) may be rephrased as the computation of the dimensions of the secant varieties of image of the Segre embedding ν of a complex multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. The group $GL(n_1+1, \mathbb{C}) \times \cdots \times GL(n_k+1, \mathbb{C})$ acts transitively on $X := \nu(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$. Thus, X is an embedded homogeneous space. This framework also applies to partially symmetric tensors (Segre-Veronese embeddings of multiprojective spaces), anti-symmetric tensors (Plücker embeddings of Grassmannians) and Schur embeddings.

For this type of embedding, ref. [1] gives a strong tool to see that the set of all rank a tensors (or partially symmetric tensors or anti-symmetric tensors and many other objects) has the "expected dimensions". This set always has at most the "expect dimension" and proving that the other inequality holds for a very specific a is often quite hard. To apply [1], one needs to check its assumptions in the case to be studied (there is a "no secant variety is a cone" assumption without which almost nothing can be said (Remark 3)). For homogeneous

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embeddings, this assumption is always satisfied ([2]). The same paper [1] gives a very good upper bound for the "general" rank. Hence its improvements, e.g., Theorem 1, give an even better upper bound for the general rank. Keep in mind that over the complex numbers we are talking about "general for the Zariski topology", i.e., open subsets for the euclidean topology which are dense in the euclidean topology (every tensor may be approximated with arbitrary precision by a tensor of generic rank) and its complement is "small", a finite union of smooth complex varieties of dimension less than the one of the space $\mathcal{V}$.

The a-secant variety of an embedded variety $\sigma_a(X)$ is defined by taking a closure in the Zariski topology and so it contains all objects of rank a, all objects of rank $< a$ and something else, something "unknown", the **boundary**. In Section 4, we consider the subsets of the a-secant variety formed by the points with rank at most a and its complement, the **boundary**. All these results are framed in the more general framework of joins $[X_1;\dots;X_s]$ of s embedded varieties.

We recall the definition and elementary properties of **joins** ([3]).

Let X, Y, X_i, $1 \le i \le s$, be integral subvarieties of $\mathbb{P}^r$. The *join* $[X;Y]$ of X and Y is defined in the following way. If $X = Y$ and Y is a single point, p, set $[\{p\};\{p\}] := \{p\}$. In all other cases let $[X;Y]$ denote the closure of the union of all lines of $\mathbb{P}^r$ spanned by a point of X and a different point of Y. If $s \ge 3$ define $[X_1;\dots;X_s]$ by induction on s using the formula $[X_1;\dots;X_s] := [[X_1;\dots;X_{s-1}];X_s]$. Set $\sigma_1(X) := X$ ([3]). For all integers $i \ge 2$ let $\sigma_i(X)$ denote the join of s copies of X ([3]). The variety $\sigma_i(X)$ is called the ***i*-secant variety** of X.

Let $c(X)$ be the maximal integer c such that $\sigma_c(X) = c(n+1) - 1$. Let $g(X)$ be the minimal integer such that $\sigma_{g(X)}(X) = \mathbb{P}^r$. For all integers $i > 0$ set $a_i(X) := \dim\sigma_{i+1}(X) - \dim\sigma_i(X)$.

This is one of the main results proved in this paper.

Theorem 1. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Set $n := \dim X$, $g := g(X)$, $c = c(X)$ and $k := g - c$. Let H be an algebraic group with $\mathbb{P}^r$ as an irreducible projective representation. Assume that X is H-invariant. Then:*

1. *We have $0 \le k \le n$.*
2. *We have $a_{i+1}(X) < a_i(X)$ for all $i = c-1,\dots,g-2$.*
3. *We have $c \ge \frac{r+1}{n+1} - \frac{nk}{n+1} + \frac{k(k-1)}{2n+2}$.*
4. *We have $g \le \frac{r+1}{n+1} + k - \frac{k(k-1)}{2n+2}$.*

A Roadmap of the Paper

(a) In Section 2, we give 3 remarks related to Theorem 1. We explain the main cases in which Theorem 1 may be applied (Remark 1). We explain its use over $\mathbb{R}$ (Remark 2). Remark 3 shows a key requirement to obtain part (2) of Theorem 1 and gives a motivation for some of the results proved in Section 6.

(b) Theorem 1 is a particular case of a result (Theorem 2) which considers more general objects, instead of joins of subvarieties, "joins" of subvarieties of Grassmann varieties. See Section 3 for the definitions and main properties. Section 3 introduces the Grassmannian joins and Grassmannian secant varieties. We hope that they will be interesting. The main reason to introduce them in this paper is that the proofs for these Grassmannian secant varieties, i.e., Theorem 2, are verbatim the ones used to prove Theorem 1. In this roadmap we only describe why the set-up of Section 3 helps to prove theorems for secant varieties and joins each time we use the Terracini Lemma (Cor. 1.10 [3]). Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate projective variety. Let $X_{\rm reg}$ denote the set of all smooth points of X. Set $n := \dim X$. For any $p \in X_{\rm reg}$ the tangent space T_pX of X is n-dimensional linear subspace of $\mathbb{P}^r$, i.e., an element of the Grassmannian $G(n+1, r+1)$. The Gauss mapping $X_{\rm reg} \to G(n+1, r+1)$ has as its image an irreducible quasi-projective variety. In Section 3, we consider the case of arbitrary quasi-projective subvarieties V of $G(n+1, r+1)$, not just images of Gauss mappings, i.e., we do not assume that they are "integrable". Thus our theorems are more general, while as far as possible we give examples coming from images of Gauss mappings. At the end of the section we raise an open question.

(c) Section 4 contains the definitions of the **open part** or the **open join** and of the **boundary** of a join and of a secant variety. For secant varieties the definitions are well-known to the experts. In the definition of joins there is a closure of a certain set. In Section 4, we discuss the set obtained before making the closure. For any set $S \subset \mathbb{P}^r$ let $\langle S \rangle$ denote its linear span. Assuming that all varieties have positive dimension we may define $[X_1; \dots; X_s]$ as the closure of the union, $[X_1; \dots; X_s]^\circ$, of all linear spaces $\langle \{p_1, \dots, p_s\} \rangle$, where $p_i \in X_i$ for all i. We say that $[X_1; \dots; X_s]^\circ$ is the **open part** or **open join** of $[X_1; \dots; X_s]$, although sometimes $[X_1; \dots; X_s]^\circ$ is not an open subset of $[X_1; \dots; X_s]$. The open join $[X_1; \dots; X_s]^\circ$ is a constructible subset of the join $[X_1; \dots; X_s]$ (Ex. II.3.18, Ex. II.3.19 [4]) and in particular it contains a non-empty Zariski open subset of $[X_1; \dots; X_s]$. We say that $\mathrm{bnd}(X_1; \dots, X_s) := [X_1; \dots; X_s] \setminus [X_1; \dots; X_s]^\circ$ is the **boundary** of the join $[X_1; \dots; X_s]$. Often, the boundary is not closed. We set $\sigma_s(X)^\circ := [X_1; \dots; X_s]^\circ$ when $X_i = X$ for all i and call $\mathrm{bnd}(X, S) := \sigma_s(X) \setminus \sigma_s(X)^\circ$ the boundary of $\sigma_s(X)$. See [5] for several examples with $s = 2$, $r = 3$ and X a smooth curve in which the boundary is neither closed nor irreducible.
There is a huge difference between the case of secant varieties and the case of joins of different varieties.
For joins of "very different" varieties the boundary is empty (Proposition 3, Theorem 5, Remark 16).
The boundaries of secant varieties often contain a hypersurface of the secant variety, i.e., their dimension is the maximal a priori possible. For instance, if $r \geq 3$ and X is a rational normal curve of $\mathbb{P}^r$, then the boundary of $\sigma_2(X)$ is a hypersurface of $\sigma_2(X)$. See [6] for the case of the 2-secant variety of a Veronese variety. At the end of Section 4 we recall that definition of the tangential variety $\tau(X)$ and explain why it often gives the explicit description of a part of the boundary of $\sigma_s(X)$ which have codimension 1 in $\sigma_s(X)$ and hence it is as big as a priori possible.

(d) Take a point q in the open part of a join or a secant variety. In Section 5, we study the subsets which certificate that q is in the open part. We study the uniqueness of such certificates. When uniqueness fails we discuss when q may be determined by its solution set. We consider the **non-uniqueness set** of q. On this topic we also give a conjecture and several open questions.

(e) In Section 6, we introduce the definition of **universally good** and **strongly universally good** embedding of a variety $X \subset \mathbb{P}^r$, i.e., an embedded variety which has joins with the expected dimension with respect to any other variety $Y \subset \mathbb{P}^r$ (with finer definitions if we also give an upper bound on $\dim Y$ and allow to take a secant variety of X instead of X).

(f) In Section 7, we consider the products of two (or more) embedded varieties, say $X \subset \mathbb{P}^r$ and $Y \subset \mathbb{P}^m$, with $X \times Y$ embedded in the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^m$.

(g) Section 8 briefly describes how the previous sections may be generalized to the case of families of embedded varieties.

(h) In the last section, Conclusions, we add four open questions.

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2. Applications of Theorem 1

Remark 1. *The paper [2] contains many very strong consequences of their main result (Theorem 1.1 in [2]) which in the set-up of Theorem 1 says that $c \geq \frac{r+1}{n+1} - n - 1$ and $g \leq \frac{r+1}{n+1} + n + 1$). Obviously, Theorem 1 gives a small improvement of these results. Sometimes, it gives no improvement at all, because the gap between Theorem 1 and Theorem 1.1 in [2] is covered by previous literature* *listed in [2]. There are many exceptional cases, even in the case of tensors ([7]), skew symmetric tensors ([2,8]), additive decompositions of forms ([9–11]) and partially symmetric tensors ([12–15]. For many other formats of partially symmetric tensors very few cases are fully understood and many others are research projects.*

Remark 2. *We work over the field of complex numbers. We describe the information that the complex case, say complex tensors of a prescribed format, give if we are interested in the case over the field of real numbers, say real tensors of the same format. Suppose both your variety and its embedding $X \subset \mathbb{P}^r$ are defined over $\mathbb{R}$. The real locus $X(\mathbb{R}) := X(\mathbb{C}) \cap \mathbb{P}^r(\mathbb{R})$ may be small, even empty. We assume (as in the case of multiprojective spaces and Grassmannians needed for the main applications) that this set is as large as possible, i.e., that it contains a differential manifold of dimension $\dim_{\mathbb{C}} X(\mathbb{C})$. There is an upper bound, $2g(X(\mathbb{C})$, for the $X(\mathbb{C})$-rank of each point of $\mathbb{P}^r(\mathbb{C})$ and the $X(\mathbb{R})$-rank of every point of $\mathbb{P}^r(\mathbb{R})$ ([16]). Thus Theorem 1 and/or similar statements immediately give an upper bound for all ranks, real or complex. Examples show that sometimes the upper bound (or the related upper bounds, like $2g(X)(\mathbb{C}) - 1$ or $2g(X)(\mathbb{C}) - 2$ given in [16] if a few other assumptions are satisfied) are sharp over $\mathbb{C}$ and over $\mathbb{R}$. Over $\mathbb{R}$ there is no notion of "generic rank", but only the notion of typical ranks ([16–19]). There are many typical ranks, the smallest one being the generic complex $X(\mathbb{C})$-rank of a point of $\mathbb{P}^r(\mathbb{C})$. Obviously, all typical ranks are at most the maximum of all real ranks and hence [16] applies also to typical ranks.*

Remark 3. *By Th. 3.10 in [1] there is a complete description of the varieties $X \subset \mathbb{P}^r$ such that $k = n$, i.e., $g(X) = c(X) + n$ (even not homogeneous varieties but with no cone as one of their proper secant varieties). If we drop the assumption that no proper secant variety of X is a cone, then all non-increasing finite sequences $\{a_i(X)\}_{i>0}$ of positive integers $a_i(X) \le n+1$ are realized by some r and some n-dimensional variety $X \subset \mathbb{P}^r$, with the only restriction that 1 occurs only once ([20]).*

3. Grassmannian Joins and Grassmannian Secant Varieties

Fix an integer $s \ge 1$ and integers $0 \le n_i < r$, $1 \le i \le s$. Let $V_i \subset G(n_i+1, r+1)$, $1 \le i \le s$, be integral quasi-projective subvarieties. Let $\overline{V}_i$ denote the closure of $V_i \subset G(n_i+1, r+1)$. Set $\gamma(V_1, \dots, V_s) := \dim\langle A_1 \cup \cdots \cup A_s\rangle$, where A_i is a general element of V_i (and hence a general element of $\overline{V}_i$). By the semicontinuity theorem for cohomology the integer $\gamma(V_1, \dots, V_s)$ is well-defined and we call it the *Gdimension of the Gjoin* of $V_1, \dots, V_s$ or of $\overline{V}_1, \dots, \overline{V}_s$. Let $\Gamma(V_1, \dots, V_s)$ denote the closure in $G(\gamma(V_1, \dots, V_s)+1, r+1)$ of the set of all $\langle A_1 \cup \cdots \cup A_s\rangle$ with $A_i \in V_i$ and $\dim\langle A_1 \cup \cdots \cup A_s\rangle = \gamma(V_1, \dots, V_s)$. We call $\Gamma(V_1, \dots, V_s)$ the Gjoin of $V_1, \dots, V_s$. If we have another variety V_{s+1}, then we have the associative rule for Gjoins and their dimensions similar to the classical associative rule for joins $[X_1; \dots; X_{s+1}] = [[X_1; \dots; X_s]; X_{s+1}]$. Obviously, $\gamma(V_1, \dots, V_s) = \gamma(V_{\tau(1)}, \dots, V_{\tau(s)})$ and $\Gamma(V_1, \dots, V_s) = \Gamma(V_{\tau(1)}, \dots, V_{\tau(s)})$ for any bijection $\tau : \{1, \dots, s\} \to \{1, \dots, s\}$. If $\overline{V}_i = \overline{V}_1$ for all i we say that the integer $\gamma(V_1, \dots, V_s)$ is the Gs-secant dimension $\gamma_s(V_1)$ of V_1 (or of $\overline{V}_1$) and that $\Gamma_s(V_1)$ is the associated **tangent variety**. When we use the join of i times a varieties we use the notation $A^{(i)}$ and $\gamma(A^{(i)}, Y) := \gamma(A, \dots, A, Y)$ where A is repeated i times. Let $V \subset G(m+1, r+1)$ be a quasi-projective irreducible variety. We say that $p \in \mathbb{P}^r$ is contained the vertex of V if $p \in \cap_{A \in V} A$. We say that V is a cone with vertex E if $\cap_{A \in V} A \neq \emptyset$ and $\cap_{A \in V} A = E$. Note that the vertex E is a non-empty linear subspace of $\mathbb{P}^r$ and that the vertex of V and $\overline{V}$ are the same. We say that $V \subset G(n+1, r+1)$ is **non-degenerate** if $\langle \cup_{A \in V} A\rangle = \mathbb{P}^r$. We say that $V_1, \dots, V_s$ has **Gdefect** or that $V_1, \dots, V_s$ are **Gdefective** if $\gamma(V_1, \dots, V_s) < \min\{r, s-1+n_1+\cdots+n_s\}$.

Our main results are on the integers $\gamma(V_1, \dots, V_s)$, not the Gjoins.

Remark 4. *We are aware of biduality ([21]). Since we work over an algebraically closed field with characteristic 0, e.g., the field of complex numbers, biduality says that any Grassmann data of hyperplanes is integrable. However, if we translate biduality in our set-up, in general we do not know in which Grassmannian lives the data whose dual is our given data.*

Remark 5. *By the semicontinuity theorem for cohomology we obtain the same integer $\gamma(V_1, \dots, V_s)$ if instead of $V_1, \dots, V_s$ we take any non-empty open Zariski subset U_i of V_i. Now assume that the algebraically closed base field is the field $\mathbb{C}$ of the complex numbers. By the semicontinuity theorem for cohomology we obtain the same integers $\gamma(V_1, \dots, V_s)$ if instead of V_i we take a non-empty open subset of V_i for the euclidean topology. If each V_i is defined over $\mathbb{R}$ and $V_i(\mathbb{R})$ contains a smooth point, p_i, of the complex projective space $V_i(\mathbb{C})$ instead of V_i we may take an open neighborhood of p_i in $V_i(\mathbb{R})$ for the euclidean topology.*

Lemma 1. *Fix an integer $i \geq 2$ and quasi-projective varieties $A \subset G(a+1, r+1)$, $B \subset G(b+1, r+1)$ and $C \subset G(c+1, r+1)$. Then:*

(i) $\gamma(A, B, C) - \gamma(A, B) \leq \gamma(A, C) - a - 1$.

(ii) *If $\gamma(A^{(i)}, C) \leq \gamma(A^{(i)}) + c$ and $\Gamma(A^{(i-1)})$ is not a cone, then $\gamma(A^{(i+1)}) - \gamma(A^{(i)}) < \gamma(A^{(i)}) - \gamma(A^{(i-1)})$.*

Proof. The proof is the one in Proposition 2.1 of [1]. Indeed, the first step of the proof of Proposition 2.1 in [1] is to reduce the proof of Proposition 2.1 in [1] to the proof of Lemma 1. The second step of the quoted proof is to prove Lemma 1. □

Fix an irreducible locally closed variety $V \subset G(n+1, r+1)$. For all $i \geq 1$ set $a_i(V) := \gamma(V^{(i+1)}) - \gamma(V^{(i)})$. Obviously, $0 \leq a_i(V) \leq n+1$ for all i. Let $c(V)$ (or just c) be the last positive integer i such that $\gamma(V^{(i)}) = i(n+1) - 1$. If V is non-degenerate let $g(V)$ be the minimal integer g such that $\gamma(V^{(g)}) = r$.

Corollary 1. *If $i > c(V)$ and $\Gamma(V^{(i-1)})$ is not a cone, then $a_{i+1}(V) < a_i(V)$. Moreover, $g \leq c + n$.*

Proof. Part (ii) of Lemma 1 gives the first assertion of the corollary. To prove the "Moreover" part it is sufficient to observe that any strictly decreasing list of integers between n and 1 has at most n entries. □

Remark 6. *Take quasi-projective varieties $V_i \subset G(n_i+1, r+1)$, $1 \leq i \leq s$, and set $m_i := \dim V_i$. Obviously, $\dim \Gamma(V_1, \dots, V_s) \leq m_1 + \cdots + m_s$. If V_i is integrable, i.e., if it is the image of a Gauss mapping, then $m_i \leq n_i$. For integrable V_i we often have $m_i = n_i$. For instance, $m_i = n_i$ if X_i is smooth by an important theorem of Zak ([22]). The case $m_i < n_i$ occurs for all cones, but also for varieties which are not cones. For surfaces, case $n_i = 2$, we have $m_i = 1$ if and only if either the surface is a cone or it is the tangent developable of an irreducible curve.*

Remark 7. *Let $V \subset G(n+1, r+1)$ be an integral and non-degenerate variety. Let H be an algebraic group acting on $\mathbb{P}^r$ as an irreducible projective representation. Assume that V is H-invariant. Then no proper Gsecant variety of V is a cone.*

Theorem 2. *Let $V \subset G(n, r+1)$ be an integral and non-degenerate variety. Set $g := g(V)$, $c := c(V)$ and $k := g - c$. Let H be an algebraic group with $\mathbb{P}^r$ as an irreducible projective representation. Assume that V is H-invariant. Then:*

1. *We have $0 \leq k \leq n$.*
2. *We have $a_{i+1}(V) < a_i(V)$ for all $i = c-1, \dots, g-2$.*
3. *We have $c \geq \frac{r+1}{n+1} - \frac{nk}{n+1} + \frac{k(k-1)}{2n+2}$.*
4. *We have $g \leq \frac{r+1}{n+1} + k - \frac{k(k-1)}{2n+2}$.*

Proof. Since $c = \lfloor (r+1)/(n+1) \rfloor$ and $g = \lceil (r+1)/(n+1) \rceil$ if V is not Gdefective, we may assume that V is Gdefective. This assumption implies $c \leq g - 2$ and that c is the first integer such that $a_c(X) \leq n$. Every Gsecant variety of V is H-invariant. Since $\mathbb{P}^r$ is an irreducible representation of H and the vertex of any cone is a linear subspace, no proper Gsecant variety of V is a cone (Remark 7). Thus $g \leq c + n$ and for $i = c, \dots, g-1$ we have $a_{i+1}(V) < a_i(V)$, concluding the proof of parts (1) and (2). By the definition of c we have $r+1 = c(n+1) + \sum_{i=c}^{g-1} a_i(V)$. Since $a_c(V) \le n$, we have $a_i(V) \le n+c-i$ for all $i = c, \dots, g-1$. Thus $\sum_{i=c}^{g-1} a_i(V) \le k(n+n-k+1)/2 = nk - k(k-1)/2$, concluding the proof of (3). Since $a_g(V) \ge 1$, part (2) implies $c(n+1) \le r+1-k(k-1)/2$. Since $g = c+k$, we obtain $g \le \frac{r+1}{n+1} - \frac{k(k-1)}{2n+2} + k$. □

Remark 8. *Let H be an algebraic group acting on $\mathbb{P}^r$ as an irreducible projective representation. Let $X_i \subset \mathbb{P}^r$, $1 \le i \le s$, be H-invariant integral subvarieties. Then each X_i is non-degenerate and the join $[X_1; \cdots; X_s]$ is not a cone, unless $[X_1; \cdots; X_s] = \mathbb{P}^r$.*

Proof of Theorem 1. The proof of Theorem 2 works just writing $a_i(X)$ instead of $a_i(V)$. □

Proposition 1. *Let W and Y be integral subvarieties. Then $\dim[W; g(Y)] = \min\{r, \dim W + \dim Y + 1\}$ for a general $g \in \mathrm{Aut}(\mathbb{P}^r)$.*

Proof. Fix $p \in W_{\mathrm{reg}}$ and $q \in Y_{\mathrm{reg}}$. By (part (1) of Cor. 1.10 [3]) it is sufficient to prove that $\dim\langle T_pW \cup g(T_qY)\rangle = \dim[W; g(Y)] = \min\{r, \dim W + \dim Y + 1\}$ for a general $g \in \mathrm{Aut}(\mathbb{P}^r)$. This is true, because $\mathrm{Aut}(\mathbb{P}^r)$ acts transitively on the Grassmannian $G(\dim Y + 1, r+1)$ of all subspaces of $\mathbb{P}^r$ of dimension $\dim Y$. □

By induction on the integer s Proposition 1 gives the following more general result.

Proposition 2. *Fix integral varieties $X_1, \dots, X_s \subset \mathbb{P}^r$, $s \ge 2$. Then*

$$\dim[X_1; g_2(X_2); \dots; g_s(X_s)] = \min\{r, s-1+\dim X_1 + \cdots + \dim X_s\}$$

for a general $(g_2, \dots, g_s) \in \mathrm{Aut}(\mathbb{P}^r)^{s-1}$.

Question 1. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate n-dimensional variety. Suppose $\dim \sigma_i(X) = i(n+1)-1 < r$ and that $\sigma_i(X)$ is a cone. How large may be the dimension of the vertex E of $\sigma_i(X)$?*

X may have a vertex of dimension $\le n-2$, because X is integral and non-degenerate and $n < r$. Thus for $i = 1$ we have $\dim E \le n-2$. Easy examples show that any integer between 0 and $n-2$ occurs as the dimension of a cone. Now assume $i \ge 2$. Is $\dim E \le n$? Note that in the set-up of Question 1 $\dim \sigma_{i-1}(X) = (i-1)(n+1)-1$ and $\sigma_{i-1}(X)$ is not a cone, because we are assuming $\dim \sigma_i(X) = i(n+1)-1$.

Joins also occur in the definition of Generalized Additive Decomposition ([23]).

Remark 9. *Among the joins of $a+b$ subvarieties of $\mathbb{P}^r$ there is $[\sigma_a(X); \sigma_b(Y)]$ which is the join of a-copies of X and b copies of Y. When $a \ge 2$ and $b \ge 2$ we may apply the ideas of [1] both to X and Y. It would be very nice to do it efficiently, taking as a model Theorem 1 (or Theorem 2) and its proof.*

4. The Open Part of a Join

From now on in this paper we use the following notation.

Notation: For any integer $r \ge 2$ and any $q \in \mathbb{P}^r$ let $\ell_q : \mathbb{P}^r \setminus \{q\} \to \mathbb{P}^{r-1}$ denote the linear projection from q. For any set T and any positive integer x let $S(T, x)$ denote the set of all $S \subseteq T$ such that $\#S = x$.

In the definition of the join of two varieties we use the closure of a certain union of lines. Thus inductively we obtain several closures in the definition of the joins of at least three varieties and of the secant varieties $\sigma_i(X)$, $i \ge 3$. When $\sigma_s(X) \ne \mathbb{P}^r$ and X is not secant defective, the boundary has often codimension 1 in the secant variety. For instance, this is the case for $\sigma_2(X)$ when X is a Veronese variety ([6]). In the case of joins of different varieties it seems that the situation is quite different and we discuss in detail the case $s = 2$.

Remark 10. *With our definition we have $X_1 \cup \cdots \cup X_s \subset [X_1; \dots ; X_s]^\circ$ and $\sigma_s(X)^\circ$ is the set of all $p \in \mathbb{P}^r$ with X-rank at most s. In the literature one sees the same notation to denote the set of all $p \in \mathbb{P}^r$ with X-rank exactly s.*

Remark 11. *Let $K_{2,r}$ denote the set of all degree 2 zero-dimensional schemes $v \subset \mathbb{P}^r$. The set $K_{2,r}$ is an integral projective variety. Take $v \in K_{2,r}$. Either v is the union of 2 different points or it is connected. In both cases $\langle v\rangle$ is a line. If $\{v_\alpha\}_{\alpha\in\Delta}$ is a family of elements of $K_{2,r}$ and this family has a limit, v, then the family of lines $\{\langle v_\alpha\rangle\}_{\alpha\in\Delta} \subset G(2, r+1)$ has the line $\langle v\rangle$ as its limit.*

Proposition 3. *If $X_1 \cap X_2 = \emptyset$, then $[X_1; X_2]^\circ = [X_1; X_2]$.*

Proof. Our definition gives $X_1 \cup X_2 \subset [X_1; X_2]^\circ$ (Remark 10). Fix $p \in \mathbb{P}^r \setminus (X_1 \cup X_2)$. Since $p \notin (X_1 \cup X_2)$, both $\ell_{p|X_1}$ and $\ell_{p|X_2}$ are morphisms and hence $\ell_p(X_1)$ and $\ell_p(X_2)$ are closed subvarieties of $\mathbb{P}^{r-1}$. Since $X_1 \cap X_2 = \emptyset$, $p \in [X_1; X_2]^\circ$ if and only if $\ell_p(X_1) \cap \ell_p(X_2) \neq \emptyset$. Since the latter is a closed condition on p and $[X_1; X_2]$ is the closure of $[X_1; X_2]^\circ$, we obtain $[X_1; X_2]^\circ = [X_1; X_2]$. $\square$

Remark 12. *Take $X_2 := \{p\}$ with $p \in X_1$. The boundary of $[X_1; X_2]$ is contained in the tangent cone, θ, of X_1 at p. There are r and X_1 with $\mathrm{bnd}(X_1, \{p\}) = \theta \setminus \{p\}$. Take for instance $r = 2$ and X_1 a smooth conic. Take $r = 2$ and let X_1 be a general plane curve of degree > 3. In this case $\mathrm{bnd}(X_1, \{p\}) = \emptyset$. Now take $r = 2$ and as X_1 a smooth plane cubic. We have $\mathrm{bnd}(X_1, \{p\}) = \theta \setminus \{p\}$ if p is a flex of X_1 and $\mathrm{bnd}(X_1, \{p\}) = \emptyset$ if p is not a flex of X_1.*

Theorem 3. *Take integral varieties $X_i \subset \mathbb{P}^r$, $i = 1, 2$, such that $X_1 \neq X_2$. The boundary of the join of X_1 and X_2 is contained in the union of all Zariski tangent spaces of $X_1 \cup X_2$ at the points of $X_1 \cap X_2$.*

Proof. Let q be any point in the boundary of the join. By the definition of join as a closure, there is an integral quasi-projective variety $\Gamma \subset G(2, r+1)$ (i.e., Γ is a family of lines $\{L_\gamma\}_{\gamma\in\Gamma}$), a line $L \in G(2, r+1)$ containing q and in the closure of Γ such that each L_γ contains a point $a_\gamma \in X_1$, a point $b_\gamma \in X_2$ such that $b_\gamma \neq a_\gamma$ and the family $\{a_\gamma, b_\gamma\}_{\gamma\in\Gamma}$ is an algebraic subset of $S(\mathbb{P}^r, 2)$. By the projectivity of the Hilbert scheme of degree 2 subschemes of $X_1 \cup X_2$, the family $\{a_\gamma, b_\gamma\}_{\gamma\in\Gamma}$ has a flat limit $v \subset X_1 \cup X_2$. Since $\{a_\gamma, b_\gamma\} \subset L_\gamma$ for all γ, $v \subset L$. Note that $L = \langle v\rangle$ and that $q \in L$ (Remark 11). First assume that v is formed by two distinct points. One of these points is the limit of the family $\{a_\gamma\}_{\gamma\in\Gamma}$ and hence it is contained in X_1, while the other one is the limit of the family $\{b_\gamma\}_{\gamma\in\Gamma}$ and hence it is contained in X_2. Since $q \in L$, q is not in the boundary, a contradiction. Now assume that that v is connected and set $\{o\} := v_{\mathrm{red}}$. The point o is a limit of the family $\{a_\gamma\}_{\gamma\in\Gamma}$ and of the family $\{b_\gamma\}_{\gamma\in\Gamma}$. Thus $o \in X_1 \cap X_2$. One of the definitions of Zariski tangent space says that L (as a linear span of v) is contained in the Zariski tangent space of $X_1 \cup X_2$ at o (Ex. II.2.8 [4], [24], Ch. 7 [25]). Hence q is contained in the Zariski tangent space of $X_1 \cup X_2$ at o. $\square$

Remark 13. *Let X, Y be integral space curves such that $\langle X \cup Y\rangle = \mathbb{P}^3$. Then $[X_1; X_2] = \mathbb{P}^3$. Note that this is true even if X and/or Y are degenerate, we only need $\langle X \cup Y\rangle = \mathbb{P}^3$.*

Proposition 4. *Let $X, Y \subset \mathbb{P}^3$ be integral planar curve spanning different planes, M and N. Set $E := X \cap Y$ (set-theoretic intersection) and $L := M \cap N$. Then:*

(i) $X \cup Y \cup (\mathbb{P}^3 \setminus (M \cup N)) \supseteq [X; Y]^\circ$.
(ii) *Fix $q \in M \setminus (L \cup Y)$. We have $q \in \mathrm{bnd}(X, Y)$ if and only if for each $o \in E$ the curve X and the line $\langle\{q, o\}\rangle$ have order of contact $\deg(X)$ at o.*
(iii) *Fix $q \in N \setminus (L \cup X)$. We have $q \in \mathrm{bnd}(X, Y)$ if and only if for each $o \in E$ the curve Y and the line $\langle\{q, o\}\rangle$ have order of contact $\deg(Y)$ at o.*
(iv) *If $\#E \neq 1$, then $L \subset [X; Y]^\circ$. If $\#E = 1$, then $L \setminus E \subseteq \mathrm{bnd}(X, Y)$.*

Proof. Set $a := \deg(X)$ and $b := \deg(Y)$. Note that $X \cap Y \subset L$.

Take $q \in \mathbb{P}^3 \setminus (M \cup N)$. Since $\ell_{q|M}$ and $\ell_{q|N}$ are isomorphisms, $\deg(\ell_q(X)) = a$, $\deg(\ell_q(Y)) = b$ and the schemes $\ell_q(X) \cap \ell_q(Y)$ are projectively isomorphic subschemes of the line $\ell_q(L)$ with degree at most $\min\{a, b\}$. If $\ell_q(X) = \ell_q(Y)$, then $q \in [X;Y]^\circ$, because $X \neq Y$. Now assume $\ell_q(X) \neq \ell_q(Y)$. Note that $\deg(\ell_q(X) \cap \ell_q(Y)) = ab$ by the theorem of Bezout. Since $ab > \min\{a, b\}$, we obtain $q \in [X;Y]^\circ$.

Take $q \in M \setminus (L \cup X)$. Every line containing q and a point of X is contained in M. Thus $q \in [X;Y]^\circ$ if and only if there is $o \in Y \cap L$ such that the multiplicity of $\langle\{q, o\}\rangle$ and X at o is at most $a - 1$.

In the same way we prove part (iii).

Now we prove part (iv). Take $q \in L \setminus E$. If R is a line containing a point of X and a different point of Y, then $R = L$ and $\#E > 1$. The converse is obvious. □

Remark 14. *It is easy to construct examples as in Proposition 4 with* $\mathrm{bnd}(X, Y) = (M \cup N) \setminus X \cup Y$, *other examples with* $\mathrm{bnd}(X, Y) = M \setminus X$, *and other examples with* $\mathrm{bnd}(X, Y) = N \setminus X$. *There are many examples with* $\#E = 1$.

Proposition 5. *Let $X \subset \mathbb{P}^3$ be an integral and non-degenerate. Let $H \subset \mathbb{P}^3$ be a plane and $Y \subset H$ an integral curve of degree $b > 1$. Set $E := X \cap Y$ (set-theoretic intersection). For any $q \in H \setminus H \cap X$ let $C_q(Y)$ be the cone with vertex q and Y as a base. If $X \nsubseteq C_q(Y)$ for any $o \in E$ let $e_{q,o}$ denote the degree of the connected component of the zero-dimensional scheme $C_q(Y) \cap X$ with o as its reduction; set $\hat{e}_q := \sum_{o \in E} e_{q,o}$. If $E \neq \emptyset$ and $X \nsubseteq C_q(Y)$ we have $\hat{e}_q \le ab$.*

(i) *Fix $q \in \mathbb{P}^3 \setminus (H \cup X)$. We have $q \in [X;Y]^\circ$ if and only if either $E = \emptyset$ or $X \subset C_q(Y)$ or $\hat{e}_q < ab$*

(ii) *Fix $q \in H \setminus H \cap X$. We have $q \in [X;Y]^\circ$ if and only if there is $o \in E$ such that Y has order of contact $< b$ with the line $\langle\{q, o\}\rangle$.*

Proof. If $E = \emptyset$, then $\mathrm{bnd}(X, Y) = \emptyset$ (Proposition 3). Assume $E \neq \emptyset$.

(i) Fix $q \in \mathbb{P}^3 \setminus (X \cup H)$. Since $q \notin H$, $C_q(Y)$ is a degree b cone. First assume $X \subset C_q(Y)$. Since we are in characteristic 0, not all tangent lines of X at its smooth points contain q. Thus a general line of the cone $C_q(Y)$ gives $q \in [X;Y]^\circ$.
Now assume $X \nsubseteq C_q(Y)$. Since $Y \subset H$, $q \in [X;Y]^\circ$ if and only if the scheme $C_q(Y) \cap X$ contains a point $a \notin E$. By the theorem of Bezout the scheme $C_q(Y) \cap X$ has degree ab and $\hat{e}_q$ is the sum of the degrees of the connected components of the scheme $C_q(Y) \cap X$ with a point of E as their reduction. Thus we obtain part (i).

(ii) Fix $q \in H \setminus (Y \cup (X \cap H))$. Any line containing q and a point of Y is contained in H. The theorem of Bezout gives part (ii).

□

In the set-up of Proposition 5 there are obvious examples with $H \nsubseteq [X;Y]^\circ$.

Remark 15. *Take the assumptions of Theorem 3 and let q be a boundary point of the join of X_1 and X_2. The proof of Theorem 3 shows the existence of $o \in X_1 \cap X_2$ and a connected degree 2 zero-dimensional scheme $v \subset X_1 \cup X_2$ such that q is contained in the line $\langle v \rangle$.*

Theorem 4. *Let $X_i \subset \mathbb{P}^3$, $i = 1, 2$, be integral and non-degenerate curves such that X_1 and X_2 are smooth at each point of $E := X_1 \cap X_2$, at each $o \in E$ the tangent line T_oX_2 of X_2 at o is not contained in the osculating space M_o of X_1 at o and the tangent line T_oX_1 of X_1 at o is not contained in the osculating space N_o of X_2 at o. Moreover, if $\#E > 1$ assume that the tangent lines of X_i, $i = 1, 2$, at any 2 distinct points of E are disjoint. Then the join of X_1 and X_2 has no boundary point.*

Proof. Set $a := \deg(X_1)$, $b := \deg(X_2)$ and $e := \#E$. Our assumptions imply that $X_1 \cup X_2$ is nodal at each point of E. Thus E is the scheme-theoretic intersection of X_1 and X_2.

Assume, by contradiction, the existence of a boundary point, q. By Theorem 3 and Remark 15 there are $o \in E$ and a connected zero-dimensional scheme $v \subset X_1 \cup X_2$ such that $q \in L := \langle v \rangle$. Since $X_1 \cup X_2 \subset [X_1; X_2]^\circ$, $q \notin X_1 \cup X_2$ and hence $\ell_{q|X_i} : X_i \to \mathbb{P}^2$ is well-defined. If $\ell_{q|E}$ is not injective, then $q \in [X_1; X_2]^\circ$, a contradiction. Thus we may assume $\#\ell_q(E) = \#E$, i.e., $L \cap E = \{o\}$, and that no point of $X_1 \cup X_2 \setminus E$ is mapped by ℓ_q to a point of $\ell_q(E)$. To obtain a contradiction it is sufficient to prove that the plane curves $\ell_q(X_1)$ and $\ell_q(X_2)$ meet at a point not in $\ell_q(E)$.

Claim 1. *$\ell_{q|X_1}$ and $\ell_{q|X_2}$ are birational onto their images.*

Proof of Claim 1. Assume for instance that $\ell_{q|X_1} : X_1 \to \mathbb{P}^2$ is not birational onto its image. Thus $c := \deg(\ell_{q|X_1}) > 1$. Note that c divides $a - \deg(L \cap X_1)$ and that $a - \deg(L \cap X_1) = c \deg(\ell_q(X_1))$. Since X_1 is non-degenerate, $\ell_q(X_1)$ has degree > 1. If $L \cap X_1$ contains a point $u \neq o$, then $q \in [X_1; X_2]^\circ$, because $o \in X_2$, a contradiction. Thus $v \subset X_1$. Since X_1 is smooth at o, $L = T_oX$. First assume $e = 1$. Since X_1 and X_2 have different tangents at o by one of our assumptions and L meets X_2 only at o by the definition of boundary, $\ell_q(X_1)$ has a unibranch singularity at $\ell_q(o)$ and $\ell_q(X_2)$ is a degree b plane curve which is smooth at $\ell_q(o)$. Since $T_oX_2 \nsubseteq M_o$, the tangent line of $\ell_q(X_2)$ at $\ell_q(o)$ is not in the tangent cone of $\ell_q(X_1)$ at $\ell_q(o)$. We understand that the intersection number of $\ell_q(X_1)$ and $\ell_q(X)$ at $\ell_q(o)$ is the multiplicity μ of $\ell_q(X_1)$ at $\ell_q(o)$. We have $\mu \leq (a - \deg(L \cap X_1))/c$, while the scheme-theoretic intersection of $\ell_q(X_1)$ and $\ell_q(X_2)$ has degree $bc/(a - \deg(L \cap X_1)) > \mu$. Thus $\ell_q(X_1) \cap \ell_q(X_2)$ contains a point not in $\ell_q(E)$. Thus q is not in the boundary, a contradiction. Now assume $e > 1$. Fix any $u \in E \setminus \{o\}$. Since $\ell_{q|X_1}$ has degree > 1 and $u \in X_1 \cap X_2$, the line $\langle \{q, u\} \rangle$ is the tangent line of X_1 at u, contradicting one of our assumptions. □

Claim 2. $e \leq (a-1)(b-1) + 1$

Proof of Claim 2. If both X_1 and X_2 are smooth, then Claim 1 is true by (Th. 1 [26]). The proofs in [26] only use that X_1 and X_2 are smooth at each point of E.

By Claim 1 $\ell_q(X_1)$ and $\ell_q(X_2)$ are plane curves of degree a and b, respectively. To contradict the assumption that q is in the boundary it is sufficient to prove that $\ell_q(X_1) \cap \ell_q(X_2)$ contains a point not in $\ell_q(E)$. Call $\hat{e}$ the sum of the multiplicities of intersection of $\ell_q(X_1)$ and $\ell_q(X_2)$ at the points of $\ell_q(X)$.

(a) First assume that L is tangent to one of the curves X_1 or X_2, say to X_1, i.e., $L = T_oX_1$. Call μ the multiplicity of $\ell_q(X_1)$ at $\ell_q(o)$. Since $T_oX_2 \nsubseteq M_o$, $\ell_q(X_2)$ is smooth at $\ell_q(o)$ and with intersection multiplicity μ. We have $\mu \leq a$. Since $\mu \leq a$ and $b > 2$, Claim 2 gives $\hat{e} \leq \mu + e - 1 \leq a + (a-1)(b-1) < ab$, concluding the proof in this case.

(b) Assume that L is not tangent to X_1 or X_2. If q is contained in T_uX_i for some $u \in E \setminus \{o\}$, then we apply step (a) to u instead of o. Thus we may assume that $\ell_q(X_1)$ and $\ell_q(X_2)$ are smooth at $\ell_q(o)$. Since $T_oX_1 \nsubseteq N_o$ and $T_oX_2 \nsubseteq M_o$, $\ell_q(X_1)$ and $\ell_q(X_2)$ have intersection multiplicity 2 at $\ell_q(o)$. Thus $\hat{e} = 2 + (e-1) < ab$.

□

□

The assumptions of Theorem 4 are often satisfied, but at least some of them are needed. We give the following example.

Example 1. *Take a smooth quadric Q and X_1 and X_2 non-degenerate space curves contained in Q. Fix $o \in Q \setminus (X_1 \cup X_2)$. By the theorem of Bezout any line containing o and at least 2 different points of $X_1 \cup X_2$ is contained in Q. We may obtain smooth curves $X_1, X_2 \subset Q$ with $o \in \mathrm{bnd}(X_1, X_2)$ in the following way. Fix integers $a_1 \geq 2$ and $a_2 \geq 1$. Let $L_1 \in |\mathcal{O}_Q(1,0)|$ and $L_2 \in |\mathcal{O}_Q(0,1)|$ be the two lines of Q containing o. Fix $u_1 \in L_1 \setminus \{o\}$ and $u_2 \in L_2 \setminus \{o\}$. Let $Z_i \subset L_i$, $i = 1, 2$, be the only connected degree a_i zero-dimensional scheme with u_i as its reduction. Since $h^0(Q, \mathcal{O}_Q(a_2, a_1)) = (a_2+1)(a_1+1) \geq a_1 + a_2 + 2$, we have $\dim |\mathcal{I}_{Z_1 \cup Z_2}(a_2, a_1)| > 0$ and hence $X \neq Y$ for a general $X, Y \in |\mathcal{I}_{Z_1 \cup Z_2}(a_2, a_1)|$.*

Claim 3. *A general $W \in |\mathcal{I}_{Z_1 \cup Z_2}(a_2, a_1)|$ is smooth and irreducible.*

Proof of Claim 3. Note that $L_1 \cup L_2 \cup D \in |\mathcal{I}_{Z_1 \cup Z_2}(a_2, a_1)|$ for every $D \in |\mathcal{O}_Q(a_2 - 1, a_1 - 1)|$. Since $|\mathcal{O}_Q(a_2 - 1, a_1 - 1)|$ has no base points, the theorem of Bertini gives that X and Y are smooth outside $L_1 \cup L_2$ (III.10.9 [4], 6.3 [27]). We have $h^0(\mathcal{O}_Q(u, v)) = (u + 1)(v + 1)$ for all $(u, v) \in \mathbb{N}^2$. Thus $h^0(\mathcal{O}_Q(a_2, a_1)) - h^0(\mathcal{O}_Q(a_2 - 1, a_1)) > \deg(Z_1)$ and $h^0(\mathcal{O}_Q(a_2, a_1)) - h^0(\mathcal{O}_Q(a_2, a_1)) > \deg(Z_2)$. Thus neither L_1 nor L_2 is contained in the base locus of $|\mathcal{I}_{Z_1 \cup Z_2}(a_2, a_1)|$. Hence neither L_1 nor L_2 is an irreducible component of W. Since a_1 (resp. a_2) is the intersection number of $\mathcal{O}_Q(a_2, a_1)$ and $\mathcal{O}_Q(1, 0)$ (resp. $\mathcal{O}_Q(0, 1)$) we understand that Z_i is the scheme-theoretic intersection of W and L_i. Thus $\{u_1, u_2\}$ is the base locus of $|\mathcal{I}_{Z_1 \cup Z_2}(a_2, a_1)|$. Hence W is smooth outside $\{u_1, u_2\}$ by the theorem of Bertini. Take $D \in |\mathcal{O}_Q(a_2 - 1, a_1 - 1)|$ such that $D \cap \{u_1, u_2\} = \emptyset$. Since $L_1 \cup L_2 \cup D$ is smooth at each point of $\{u_1, u_2\}$ and smoothness is an open condition, W is smooth. Since a_2 and a_1 are positive, the intersection theory of Q gives that the smooth curve W is connected.

Since W is smooth and irreducible and $\deg(W \cap L_i) = a_i$, Z_i is the scheme-theoretic intersection of W and L_i. At the beginning of the example we explained why $o \in \operatorname{bnd}(X_1, X_2)$. $\square$

Proposition 6. *Let $X_i \subset \mathbb{P}^3$, $i = 1, 2$, be integral and non-degenerate curves. Assume that X_1 and X_2 are smooth and with a different tangent line at each point of $E := X_1 \cap X_2$. Assume that at each $o \in E$ the curves X_1 and X_2 have different osculating planes. Then* $\operatorname{bnd}(X_1, X_2) \subset \cup_{o \in E}(T_o X_1 \cup T_o X_2)$.

Proof. Fix $q \in \operatorname{bnd}(X_1, X_2)$ (if any). In order to obtain a contradiction we assume $q \notin \cup_{o \in E}(T_o X_1 \cup T_o X_2)$. Set $e := \#E$ and $a_i := \deg(X_i)$. Since X_i is non-degenerate, $a_i \geq 3$. Claim 2 of the proof of Theorem 4 gives $e \leq (a_1 - 1)(a_2 - 1) + 1$.

Since $E \subseteq X_1 \cup X_2$, if there is $o \in E$ and $u \in X_1 \cup X_2 \setminus \{o\}$ such that $\ell_q(o) = \ell_q(u)$, then $q \in [X_1; X_2]^\circ$, a contradiction. Thus we may assume $\#\ell_q(E) = e$ and $\ell_q(E) \cap \ell_q((X_1 \cup X_2) \setminus E) = \emptyset$. By Theorem 3 there is $o \in E$ such that q is contained in the Zariski tangent space of $X_1 \cup X_2$ at o. Since $q \notin \cup_{u \in E}(T_u X_1 \cup T_u X_2)$, each $\ell_q(X_i)$ is smooth at each point of $\ell_q(E)$. The assumption on the osculating planes of X_1 and X_2 at the points of E implies that $\ell_q(X_1) \cap \ell_q(X_1)$ are transversal at each point of $\ell_q(E)$. Recall that we assume $q \notin \cup_{o \in E}(T_o X_1 \cup T_o X_2)$. With this assumption the proof of Claim 1 of the proof of Theorem 4 gives that each $\ell_{q|X_i}$ is birational onto its image, i.e., $\deg(\ell_q(X_i)) = a_i$. Since $a_1 a_2 > e$, $q \in [X_1; X_2]^\circ$, a contradiction. $\square$

The following result shows that sometimes Zariski open subsets of an osculating plane are contained in the boundary.

Proposition 7. *Let $X \subset \mathbb{P}^3$ be an integral and non-degenerate curve. Fix a smooth point $o \in X_1$ and let M the osculating plane of X at o. Fix a line $L \subset M$ such that $L \cap X_1 = \{o\}$ (e.g., take a general line of M containing o).*

(i) If M contains another point of X, then $M \subset [X; L]^\circ$.
(ii) If M contains no other point of X, then $\operatorname{bnd}(X, L) = M \setminus L$.

Proof. Let $\ell_L : \mathbb{P}^3 \setminus L \to \mathbb{P}^1$ denote the linear projection. Note that $\ell_L(M \setminus L)$ is a point, u.

Fix $q \in M \setminus L$. Any line R containing q and meeting L is contained in M. Conversely, any line of M meets L. We see that $q \in [X; L]^\circ$ if and only if M contains another point of X.

Fix $q \in \mathbb{P}^3 \setminus M$. If $a \in \mathbb{P}^1 \setminus \{u\}$, then $\ell_L^{-1}(a) = N \setminus L$, where N is a plane containing L and $N \neq M$. Since $N \neq M$, M is the osculating space of X at o and X is smooth at o, the connected component of $X \cap N$ containing o has degree $< \deg(X)$. Thus there is $v \in X \cap (N \setminus L)$. Hence $q \in [X; L]^\circ$. $\square$

The following example shows the existence of X as in Proposition 7 for all integers $\deg(X) \geq 3$.

Example 2. *In the set-up of Example 1 take $a_1 = 1$. The smooth curve X_1 has degree $a_2 + 1$ and its tangent line at u_1 has order of contact $\deg(X_1) - 1$ with X_1 at u_1, the osculating plane has order of contact $\geq \deg(X_1)$ and hence it has order of contact $\deg(X_1)$.*

Theorem 5. *Fix an integer $s \geq 2$ and integral varieties $X_i \subset \mathbb{P}^r$, $1 \leq i \leq s$. Set $T_1 := X_1$. For $i = 2, \dots, s-1$ set $T_i := [X_1; \dots ; X_i]$. Assume $T_i \cap X_{i+1} = \emptyset$ for all i. Then the join $[X_1; \dots ; X_s]$ has no boundary.*

Proof. The case $s = 2$ is true by Proposition 3. The case $s > 2$ is true using induction on s and applying Proposition 3 to the varieties T_{s-1} and X_s. □

Remark 16. *Fix integers $s \geq 2$ and $n_i \geq 0$, $1 \leq i \leq s$. Take linear subspaces $L_i \subset \mathbb{P}^r$, $1 \leq i \leq s$, such that $\dim L_i = n_i$. Take a general $(g_1, \dots, g_{s-1}) \in \mathrm{Aut}(\mathbb{P}^r)^{s-1}$. Since $\mathrm{Aut}(\mathbb{P}^r)$ acts transitively on each Grassmannian, we have*

$$\dim\langle g_1(L_1) \cup \cdots \cup g_{s-1}(L_{s-1}) \cup L_s\rangle = \min\{r, s-1+n_1+\cdots+n_s\}.$$

Remark 17. *Fix irreducible varieties $T, X \subset \mathbb{P}^r$ and set $t := \dim T$ and $n := \dim X$. Fix a general $g \in \mathrm{Aut}(\mathbb{P}^r)$. We have $T \cap g(X) = \emptyset$ if $t + n < r$, while $T \cap X \neq \emptyset$ and $\dim T \cap X = t + n - r$ with $T \cap X$ equidimensional if $t + n \geq r$ (III.10.8 [4]). Now fix integral varieties $X_i \subset \mathbb{P}^r$, $i \leq i \leq s$. Set $n_i := \dim X_i$. Take a general $(g_1, \dots, g_{s-1}) \in \mathrm{Aut}(\mathbb{P}^r)^{s-1}$. We have*

$$\dim[g_1(X_1); \dots ; g_{s-1}(X_{s-1}); X_s] = \min\{r, s-1+n_1+\cdots+n_s\}$$

by the Terracini Lemma and Remark 16.

Fix integral varieties $X \subset \mathbb{P}^r$, $Y \subset \mathbb{P}^r$ and positive integers a and b. Let $\tau(X) \subseteq \mathbb{P}^r$ denote the tangential variety of X, i.e., the closure in $\mathbb{P}^r$ of the union of all T_pX, $p \in X_{\mathrm{reg}}$. The closed set $\tau(X)$ is an irreducible variety. We have $X \subseteq \tau(X) \subseteq \sigma_2(X)$, $\dim \tau(X) \leq 2\dim X$, and $\dim \sigma_2(X) \leq 2\dim X + 1$. Hence $\sigma_{a+1}(X) \subseteq [\sigma_a(X), \tau(X)] \subseteq \sigma_{a+2}(X)$ for all $a \geq 0$. Often, $[\sigma_a(X)^\circ, \tau(X)]^\circ \cap \mathrm{bnd}(\sigma_{a+2}(X))$ is a hypersurface of $\sigma_{a+2}(X)$. Now assume X is smooth. We often have $\tau(X) \setminus X = \mathrm{bnd}(\sigma_2(X))$. For instance, this is the case for Veronese varieties ([6]). If X is a Veronese variety, then the join of a copies of X and b copies of $\tau(X)$ is related to a certain generalized additive decomposition ([28]).

5. Solution Sets, Generic Uniqueness and the Reconstruction from the Solution Set

Let $[X_1; \dots ; X_s]^{\circ\circ}$ denote the set of all $q \in [X_1; \dots ; X_s]^\circ$ such that there is no $k < s$ and $i_1 < \cdots < i_k \leq s$ with $q \in [X_{i_1}; \dots ; X_{i_k}]^\circ$. We say that s is the join-rank of $X_1, \dots, X_s$ and write $r_{[X_{i_1};\dots;X_{i_k}]}(q) = s$. Assume that q has join-rank s. Let $\mathcal{S}(X_1, \dots, X_s; q)$ denote the set of all s-ples $(p_1, \dots, p_s)$ with $p_i \in X_i$ for all i and $p_i \neq p_j$ for all $i \neq j$. The set $\mathcal{S}(X_1, \dots, X_s; q)$ is called the **solution set** of the join. We have $\mathcal{S}(X_1, \dots, X_s; q) \neq \emptyset$ for all $q \in [X_1; \dots ; X_s]^{\circ\circ}$. Abusing notation, we say that a subset $S \subset X_1 \cup \cdots \cup X_s$ is a solution of q if it has an ordering $(p_1, \dots, p_s)$ of its points with $(p_1, \dots, p_s) \in \mathcal{S}(X_1, \dots, X_s; q)$.

Example 3. *Take $r = 3$, two distinct planes M_1 and M_2 of $\mathbb{P}^3$, and a point $o \in M_1 \cap M_2$. Let $X_i \subset M_i$, $i = 1, 2$, be a general smooth conic containing o. All $q \in M_i \setminus X_i$ have join-rank 2, but any 2-ple in $\mathcal{S}(X_1, X_2; q)$ contains $o \in X_1 \cap X_2$.*

Remark 18. *Assume $\dim[X_1; \dots ; X_s] = s - 1 + \sum_{i=1}^{s} \dim X_i$. This assumption implies that for a general $q \in [X_1; \dots ; X_s]$ its solution set is finite. It also implies that $[X_1; \dots ; X_s]$ strictly contains each join of $s-1$ entries of the s-ple $(X_1, \dots, X_s)$ and the singular locus of the other entry. Thus there is a non-empty open subset $\mathcal{V}$ of $[X_1; \dots ; X_s]^{\circ\circ}$ such that for each $q \in \mathcal{V}$ its solution set is finite, for any $\{p_1, \dots, p_s\} \in \mathcal{S}(X_1, \dots, X_s; q)$ we have $\{p_1, \dots, p_s\} \cap \mathrm{Sing}(X_i) = \emptyset$ for all i and $p_i \notin X_j$ for all $i \neq j$. Now assume $r \geq s + \dim X_1 + \cdots + \dim X_s$. Fix any $q \in \mathcal{V}$ and any $(p_1, \dots, p_s) \in \mathcal{S}(X_1, \dots, X_s)$. Since $p_i \notin \mathrm{Sing}(X_i)$ for all i, it is well-defined the tangent* *locus introduced in [29]. Take a solution set $(p_1,\dots,p_s)$ with $p_i \in (X_i)_{\mathrm{reg}}$ for all i. By the Terracini Lemma $\langle T_{p_1}X_1 \cup \cdots \cup T_{p_s}X_s\rangle$ is contained in the tangent space of $[X_1;\dots;X_s]$ and equality holds for a general $q \in [X_1;\dots;X_s]$ and all $S \in \mathcal{S}(X_1,\dots,X_s;s)$. Our assumption on r implies $\langle T_{p_1}X_1 \cup \cdots \cup T_{p_s}X_s\rangle \neq \mathbb{P}^r$. Thus there is a non-empty Zariski open subset $\mathcal{G}$ of $\mathcal{V}$ such that $T_q[X_1;\dots;X_s] = \langle T_{p_1}X_1 \cup \cdots \cup T_{p_s}X_s\rangle$ for all $q \in \mathcal{G}$ and all solution sets $S = \{p_1,\dots,p_s\}$ of q. The* **tangential contact locus** *$\mathbf{t}(S)$ of S is the union of the irreducible components of the contact locus of $X_1 \cup \cdots \cup X_s$ containing at least one point of $\{p_1,\dots,p_s\}$. Note that $\mathbf{t}(S) \supseteq S$ and equality holds if and only if $\dim \mathbf{t}(S) = 0$. As in [30–33] we say that $[X_1;\dots;X_s]$ is* **tangential defective** *if $\dim \mathbf{t}(S) > 0$ for a general $q \in [X_1,\dots,X_s]$ and some $S = \{p_1,\dots,p_s\} \in \mathcal{S}(X_1,\dots,X_s;q)$. A dimensional count shows that if $[X_1;\dots;X_s]$ is tangential defective, then $\dim \mathbf{t}(S) > 0$ for a general $q \in [X_1;\dots;X_s]$ and all $S \in \mathcal{S}(X_1,\dots,X_s;q)$ and that the dimension and the number of the irreducible components of $\mathbf{t}(S)$ are the same for a general $q \in [X_1;\dots;X_s]$ and all $S \in \mathcal{S}(X_1,\dots,X_s;q)$.*

Proposition 8. *Assume $r \geq s + \dim X_1 + \cdots + \dim X_s$ with each X_i non-degenerate. Take a join $[X_1;\dots;X_s]$ with $\dim[X_1;\dots;X_s] = s - 1 + \sum_{i=1}^{s} \dim X_i$ and not tangential defective. Then $\#\mathcal{S}(X_1,\dots,X_s;q) = 1$ for a general $q \in [X_1;\dots;X_s]$.*

Proof. The assumption that $[X_1;\dots;X_s]$ is not defective and the assumption on r allow us the definition of contact locus. With the definition given in Remark 18 the proof given in (Prop. 14 [31]) works with no modification. □

Theorem 6. *Assume $r \geq 2s$. Let $X_i \subset \mathbb{P}^r$, $1 \leq i \leq s$, be integral and non-degenerate curves. Then $\#\mathcal{S}(X_1,\dots,X_s;q) = 1$ for a general $q \in [X_1;\dots;X_s]$.*

Proof. The join $[X_1;\dots;X_s]$ has dimension $2s - 1$ (Cor. 1.5 [3]). Any hyperplane is tangent to a non-degenerate curve only at finitely many points. Hence the theorem is a corollary of Proposition 8. □

Now we consider a general $q \in \mathbb{P}^r$. Hence we are in the case $[X_1;\dots;X_s] = \mathbb{P}^r$. We also assume $r = s - 1 + \dim X_1 + \cdots + \dim X_s$. These assumptions imply that $\mathcal{S}(X_1,\dots,X_s;q)$ is finite. In the case of secant varieties, i.e., $X_i = X_j$ for all $i \neq j$, seldom $\#\mathcal{S}(X_1,\dots,X_s;q) = 1$ for a general $q \in \mathbb{P}^r$. For the secant varieties of curves this is true if and only if r is odd and X_1 is a rational normal curve (Th. 3.1 [30]). If $\#\mathcal{S}(X_1,\dots,X_s;q) = 1$ for a general $q \in .\mathbb{P}^r$ we say that **generic uniqueness** holds.

Theorem 7. *Let $X \subset \mathbb{P}^3$ be an integral, smooth and non-degenerate curve. Let $Y \subset \mathbb{P}^3$ be an integral and smooth curve such that $Y \neq X$. Set $d := \deg(X)$. Generic uniqueness holds for the join of X and Y if and only if X is rational, Y is a line and $\deg(Y \cap X) = d - 1$.*

Proof. Set $a := \deg(Y)$, $E := X \cap Y$ (set-theoretic intersection) and $e := \#E$. For each $o \in E$ let m_o denote the order of contact of X and Y at o, i.e., let m_o be the degree of the connected component of $X \cap Y$ (scheme-theoretic intersection) containing o. Set $\hat{e} := \sum_{o\in E} m_o$.

Since X is non-degenerate, $d \geq 3$. Let $g(X)$ and $g(Y)$ denote the genus of the smooth curve X and Y, respectively. Take a general $q \in \mathbb{P}^3$. Since q is general and E is finite, $\#\ell_q(E) = \#E$, $\ell_q(X)$ is a degree d nodal curve with exactly $(d-1)(d-2)/2 - g(X)$ nodes, $\ell_q(Y)$ is a degree a nodal curve with exactly $(a-1)(a-2)/2 - g(Y)$ nodes and $\ell_q(X) \cap \mathrm{Sing}(\ell_q(Y)) = \ell_q(Y) \cap \mathrm{Sing}(\ell_q(X)) = \emptyset$. Moreover, for a general q we may also determine that $\ell_q(X)$ and $\ell_q(Y)$ intersect transversally outside $\ell_q(E)$ and that $\ell_q(u) \notin \ell_q((X \cup Y) \setminus \{u\})$ for all $u \in E$. Set $Z := \ell_q(X) \cap \ell_q(Y)$ (scheme-theoretic intersection) and set $F := Z_{\mathrm{red}}$ and $f := \#F$, i.e., let f the number of common points of $\ell_q(X)$ and $\ell_q(Y)$. We have $\deg(Z) = ad$ by the theorem of Bezout. Note that $F \supseteq \ell_q(E)$ and that $f - e = \#(\mathcal{S}(X,Y;q))$. Since for each $o \in E$ the integer m_o is the intersection multiplicity of $\ell_q(X)$ and $\ell_q(Y)$ at $\ell_q(o)$, we have $f - e = da - \hat{e}$.

(a) Assume that Y is non-degenerate. Thus $a \geq 3$. In this case changing if necessary X and Y we may assume $a \leq d$. Assume for the moment $a \geq 5$. By the smoothness of Y and Theorem 1 and Remark 14 in [26] we have $\hat{e} \leq (a-1)(d-1)+1$ and hence $ad - \hat{e} > 1$. In this step to use Remark 14 in [26] it is essential the assumption that Y is smooth. Now assume $a \in \{3,4\}$. In this case Y is contained in a quadric Q. If $X \nsubseteq Q$, then $\hat{e} \leq 2d$ (and hence $da - \hat{e} > 1$) by the theorem of Bezout. If $X \subset Q$ we obtain the inequality $da - \hat{e} > 1$ by the classification of smooth curves contained in integral quadric surfaces (see Ex. III.5.6 in [4] for smooth quadrics and Ex. V.2.9 in [4] for quadric cones).

(b) Assume that Y is a plane curve, but not a line. Thus $a \geq 2$ and $M := \langle Y \rangle$ intersects X in a degree d scheme. Thus $\hat{e} \leq d$. We have $da - \hat{e} \geq d > 1$.

(c) Assume that Y is a line. We obtain $f - e = 1$ if and only if $\sum_{o \in E} m_o = d - 1$. To conclude the "only if" part of the proof of the theorem it is sufficient to prove that if Y is a line and $\deg(Y \cap X) = 1$, then X is rational. Call $\ell : \mathbb{P}^3 \setminus Y \to \mathbb{P}^1$ the linear projection from the line Y. Since X is smooth, the rational map $\ell_{|X \setminus E} : X \setminus E \to \mathbb{P}^1$ extends to a morphism $\mu : X \to \mathbb{P}^1$. We have $\deg(\mu) = d - \deg(Y \cap X)$ and $\deg(\mu) = \#\mathcal{S}(X, y; q)$.

The last part of step (c) also proves the "if" part of the theorem. □

Theorem 8. *Fix an integer $s \geq 2$ and set $r := 2s - 1$. Let $X \subset \mathbb{P}^r$ be an integral, smooth and non-degenerate curve of degree d. Let $L \subset \mathbb{P}^r$ be a line. Set $X_i := X$ for $i = 1, \dots, s-1$ and $X_s := L$. We have $[X_1; \dots; X_s] = \mathbb{P}^r$. Generic uniqueness holds for this join if and only if X is a rational curve and L is a line with* $\deg(L \cap X) = d + 2 - r$.

Proof. To prove the "only if" part we use induction on the integer s, the case $s = 2$ being true by Theorem 7. Assume $s > 2$. Let $\ell_L : \mathbb{P}^r \setminus L \to \mathbb{P}^{r-1}$ denote the linear projection from L. Since $X \neq L$ and X is smooth, the rational map $\ell_{L|X \setminus X \cap L} : X \setminus X \cap L \to \mathbb{P}^{r-2}$ extends to a morphism $\mu : X \to \mathbb{P}^{r-2}$. Note that $d - \deg(L \cap X) = \deg(\mu)\deg(\mu(X))$. Since X is non-degenerate, $\mu(X)$ is non-degenerate. Generic uniqueness for the join of $X_1, \dots, X_s$ holds if and only if $\deg(\mu) = 1$ and generic uniqueness holds for the $(s-1)$-th secant variety of $\mu(X)$ in $\mathbb{P}^{r-2}$. The latter condition holds if and only if $\mu(X)$ is a rational normal curve of $\mathbb{P}^{r-2}$ (Th. 3.1 [30]). Assume that $\mu(X)$ is a rational normal curve of $\mathbb{P}^{r-2}$. Since X is smooth, we have $\deg(\mu) = 1$ if and only if X is a rational curve and $d = r - 2 - \deg(L \cap X)$.

Now we prove the "if part". Fix a general $q \in [X; \dots; X; L]$. Since q is general, $q \notin L$ and hence $\ell_L(q)$ is a well-defined point of $\mathbb{P}^{r-2}$. Since q is general, $\ell_L(q)$ is a general point of $\mathbb{P}^{r-2}$. By the easy part of (Th. 3.1 [30]) there is a unique $S \subset X$ such that $\ell_L(q) \in \langle \ell_L(S) \rangle$. Since L is a line, the set $L \cap \langle S \cup \{q\} \rangle$ is a unique point, o. The set $S \cup \{o\}$ is the unique element of $\mathcal{S}(X, L; q)$. □

Conjecture 1. *Take an integer $s \geq 2$ and s smooth non-degenerate curves $C_i \subset \mathbb{P}^r$, $r = 2s - 1$, such that $C_1 \neq C_2$. Is generic uniqueness always false for the join of $C_1, \dots, C_s$?*

Question 2. *Is Conjecture 1 true if we allowed the curves C_i to be singular?*

Remark 19. *If Question 2 or Conjecture 1 are false we expect that the counterexamples form a short list and that each of them has very interesting geometric properties.*

Now we consider solutions sets for subvarieties of Grassmannians.

We consider the space of all solutions, but in this definition we need to distinguish between V_i and $\overline{V}_i$. In algebraic geometry a constructible subset of an algebraic variety Y is a finite union of locally closed subsets of Y (Ex. II.3.18 and II.3.19 [4]). To be as general as possible we fix a Zariski dense constructible subset U_i of $\overline{V}_i$ and say that we look at solutions coming from $U_1, \dots, U_s$. Since U_i is constructible and $\overline{U}_i = \overline{V}_i$, U_i contains a non-empty Zariski open subset of $\overline{V}_i$ (Ex. II.3.18 and II.3.19 [4]). Fix $q \in \mathbb{P}^r$. We say that q has **$(U_1, \dots, U_s)$-rank s** (or just **Grank s with respect to $U_1, \dots, U_s$**) if there are

$A_i \in U_i$, $1 \le i \le s$, such that $q \in \langle A_1 \cup \cdots \cup A_s \rangle$, $\dim\langle A_1 \cup \cdots \cup A_s \rangle = \gamma(V_1, \dots, V_s)$ and there is no integer $1 \le m < s$, $1 \le i_1 < \cdots < i_m \le s$ and $B_{i_j} \in U_{i_j}$, $1 \le j \le m$, such that $q \in \langle B_{i_1} \cup \cdots \cup \rangle$. If $\gamma(V_1, \dots, V_s) = s - 1 + n_1 + \cdots + n_s$ there are points of Grank s. For each $q \in \mathbb{P}^r$ of Grank s with respect to $U_1, \dots, U_q$ let $\mathcal{S}(U_1, \dots, U_s; q)$ denote the set of all $(A_1, \dots, A_s)$ such that $A_i \in U_i$, $1 \le i \le s$, such that $q \in \langle A_1 \cup \cdots \cup A_s \rangle$ and $\dim\langle A_1 \cup \cdots \cup A_s \rangle = \gamma(V_1, \dots, V_s)$. The non-empty set $\mathcal{S}(U_1, \dots, U_s; q)$ is called the **solution set**.

In many important cases the solution set is not a singleton, i.e., uniqueness does not hold. We describe how to use the solution set, if large, to reconstruct the element $q \in \mathbb{P}^r$ from it solution set. We state it in the case of secant varieties, the case of joins requiring only notational modifications. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. For each $q \in \mathbb{P}^r$ let $r_X(q)$ denote the X-rank of q, i.e., the first integer a such that there is $S \subset X$ such that $q \in \langle S \rangle$. Let $\mathcal{S}(X, q)$ (the solution set of q with respect to X) denote the set of all $S \subset X$ such that $q \in \langle S \rangle$ and $\#S = r_X(q)$. Set $W(X)_q := \cap_{S \in \mathcal{S}(X,q)} \langle S \rangle$. The set $W(X)_q$ is a linear subspace, the **non-uniqueness set** of q with respect to X. The point q is uniquely reconstructed by its solution set if $W(X)_q = \{q\}$. Set $w(X)_q := \dim W(X)_q$. Since $W(X)_q$ is a linear space, $w(X)_q = 0$ if and only if $W(X)_q = \{q\}$. Obviously, $W(X)_q = \{q\}$ if $q \in X$, i.e., if $r_X(q) = 1$. By the semicontinuity theorem and standard results for constructible sets over the complex numbers (or over any algebraically closed field), the integer $w(X)_q$ is the same for all q in a non-empty Zariski open subset of $\mathbb{P}^r$ (Ex. II.3.18 and II.3.19 [4]). We call this, integer, $w(X)$, the **generic indeterminacy number of X**. Now assume that X is defined over $\mathbb{R}$, that $X_{\mathrm{reg}}(\mathbb{R}) \neq \emptyset$ and that $q \in \mathbb{P}^r(\mathbb{R})$. For any $q \in \mathbb{P}^r$ with real rank $r_{X(\mathbb{R})}(q)$ let $\mathcal{S}_{\mathbb{R}}(X, q)$ denote the set of all $S \subset X(\mathbb{R})$ such that $\#S = r_{X(\mathbb{R})}(q)$ and $q \in \langle S \rangle$. Set $W(X, \mathbb{R})_q := \mathbb{P}^r(\mathcal{R}) \cap (\cap_{S \in \mathcal{S}_{\mathbb{R}}(X,q)} \langle S \rangle)$. The set $W(X, \mathbb{R})_q$ is a real vector space and we call $w(X, \mathbb{R})_q$ its dimension. By results on real semi-algebraic geometry (fully explained in the papers describing the typical ranks ([16–19]), there are finitely many euclidean open subset U_i, $1 \le i \le e$, such that $r_{X(\mathbb{R})}(q)$ and $w(X, \mathbb{R})_q$, are the same for all $q \in U_i$. These integers $w(X, \mathbb{R})_q$ are called the **typical non-uniqueness numbers**.

Question 3. *Are the typical non-uniqueness numbers a connected set of integers, i.e., do they contain all integers between the minimum and the maximum typical?*

Question 4. *Assume that X is real. What is its position of $w(X)$ with respect to the list of typical non-uniqueness numbers?*

The set $\mathcal{S}(X, q)$ may be generalized in the following way. Fix $q \in \mathbb{P}^r$. For each integer $b \ge r_X(q)$ let $\mathcal{S}_1(X, q; b)$ denote the set of all $S \subset X$ such that $\#S = b$ and $q \in \langle S \rangle$. Set $W_1(X, b)_q := \cap_{S \in \mathcal{S}_1(X,q;b)} \langle S \rangle$. Let $\mathcal{S}(X, q; b)$ denote the set of all $S \in \mathcal{S}_1(X, q; b)$ such that $q \notin \langle S' \rangle$ for each $S' \subsetneq S$. If $\mathcal{S}(X, q; b) \neq \emptyset$ set $W(X, b)_q := \cap_{S \in \mathcal{S}(X,q;b)} \langle S \rangle$. Set $w_1(X, b)_q := \dim W_1(X, b)_q$ and $w(X, b)_q := \dim W(X, b)_q$ with the convention $w(X, b)_q = -\infty$ if $W(X, b)_q = \emptyset$.

Question 5. *Give upper and lower bounds for the first integer b such that $w_1(X, b)_q = \{q\}$ for a general $q \in \mathbb{P}^r$ and for the first integers b_1 such that $W_1(X, b_1)_q = \{q\}$ for all $q \in \mathbb{P}^r$.*

Remark 20. *It is easy to check that $r - \dim X + 1$ is an upper bound for the integer b_1 in Question 5.*

Suggestion: Assume that X is real, take $q \in \mathbb{P}^r(\mathbb{R})$ and fix an integer $b > r_{X(\mathrm{R}}(q)$. Define the set $\mathcal{S}_{\mathbb{R}}(X, q; b)$, the real linear space $W_{\mathbb{R}}(X, b)_q$ and set $w_{\mathrm{R}}(X, b)_q := \dim_{\mathrm{R}} W_{\mathbb{R}}(X, b)_q$. Rephrase Question 5 for the euclidean open subsets of $\mathbb{P}^r(\mathbb{R})$ corresponding to typical ranks.

6. Good X with Respect to Joins by an Arbitrary Y

Definition 1. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Set $n := \dim X$. Fix positive integers i and m such that $i(n+1) + m \le r$. We say that X is (i, m)-universal (resp.*

strongly (i,m)-universal) nondefective or just (i,m)-universal (resp. strongly (i,m)-universal) if $\dim[\sigma_i(X);Y] = i(n+1) + \dim Y$ *for all integral and non-degenerate (resp. integral) varieties Y of dimension $\le m$. We say that X is universal (resp. strongly universal) secant nondefective if it is (i,m)-universal (resp. strongly (i,m)-universal) for all positive integers i and m such that $i(n+1)+m \le r$. We say that X is m-universal or strongly m-universal if it is $(1,m)$-universal or strongly $(1,m)$-universal.*

Remark 21. *Note that if X is (i,m)-universal, then* $\dim \sigma_i(X) = i(n+1) - 1$*. By (Cor. 1.5 [3]) X is $(i,1)$-universal if and only if* $\dim \sigma_i(X) = i(n+1) - 1 < r$*.*

Remark 22. *Let $X \subset \mathbb{P}^r$ be an integral n-dimensional variety. If $i \ge 2$ and* $\dim \sigma_i(X) = i(n+1) - 1$*, then $\sigma_{i-1}(X)$ is not a cone.*

Proposition 9. *Let $X \subset \mathbb{P}^r$, $r \ge 2$, be an integral and non-degenerate curve defined over any algebraically closed field. Then X is not secant defective, $g(X) = \lceil (r+1)/(n+1) \rceil$ and no proper secant variety of X is a cone.*

Proof. X is not secant defective (Cor. 1.5 [3]) and hence $g(X) = \lceil (r+1)/(n+1) \rceil$. Remark 22 proves the case r odd. If r is even and $i \le g(X) - 2$, we apply Remark 22. Assume that r is even and $\sigma_{g(X)-1}(X)$ is a cone. Take a point o in the vertex of $\sigma_{g(X)-1}(X)$. By (1.3 and 1.4 [3]) and induction on i we obtain $\dim[\sigma_i(X); \{o\}] = 2i$ for all $i \le g(X) - 1$. The contradiction arises for $i = g(X) - 1$. □

Proposition 10. *In arbitrary characteristic all secant varieties of a non-degenerate curve are strongly universal.*

Proof. Let $Y \subset \mathbb{P}^r$ be any integral variety. Let $C \subset \mathbb{P}^r$ be an integral and non-degenerate curve. Fix an integer $i > 0$. We have $\dim \sigma_i(C) = \min\{r, 2i-1\}$ (Cor 1.5 [3]). We have $\dim[\sigma_i(C); Y] = \min\{r, 2i + \dim Y\}$ (use Prop. 1.3 [3] and induction on i). □

By the Terracini Lemma the definition of strong $(1,m)$-universality may be rephrased in the following way.

Remark 23. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Set $n := \dim X$ and fix a positive integer $m < r - n$. The following conditions are equivalent:*

1. *X is strongly $(1,m)$-universal;*
2. *for every $M \in G(m+1, r+1)$ there is $o \in X_{\mathrm{reg}}$ such that $M \cap T_oX = \emptyset$;*
3. *for every $M \in G(m+1, r+1)$, we have $\dim \ell_M(X \setminus X \cap M) = n$, where $\ell_M : \mathbb{P}^r \setminus M \to \mathbb{P}^{r-m-1}$ denote the linear projection from M.*

Proposition 11. *Fix integers $n > t \ge 2$. Then there is a non-defective n-dimensional smooth projective variety $X \subset \mathbb{P}^r$ such that no $\sigma_i(X)$, $i < g(X)$, is a cone.*

Proof. Set $g := \lceil (r+1)/(n+1) \rceil$. Note that $g = g(X)$ for every non-defective n-dimensional variety $X \subset \mathbb{P}^r$ and that X is not secant defective if and only if $g = g(X)$ and $\dim \sigma_{g-1}(X) = (g-1)(n+1) - 1$. Moreover, no proper secant variety of X is a cone if and only if $\sigma_{g-1}(X)$ is a cone. For any $p \in \mathbb{P}^r$ let $G(n+1, r+1)_p$ denote the sets of all n-dimensional linear subspaces of $\mathbb{P}^r$ containing p. Let Γ be denote the subset of $(\mathbb{P}^r)^g \times G(n+1, r+1)^g$ formed by all $(p_1, \dots, p_g, V_1, \dots, V_g)$ such that $p_i \ne p_j$ for all $i \ne j$ and $V_i \in G(n+1, r+1)_{p_i}$ for all $i = 1, \dots, g$. The set Γ is an irreducible quasi-projective variety of dimension $g(r + r(r-n+1)) = gr(r-n)$. Fix a general $A = (p_1, \dots, p_g, V_1, \dots, V_g)$.

Claim 4. *There is a smooth n dimensional variety $X \subset \mathbb{P}^r$ such that $p_i \in X$ and $T_{p_i}(X) = V_i$ for all i.*

Proof of Claim 4. Fix an integer $d \geq 4(r-n)g$. Our aim is to prove that we may take as X the general complete intersection of $r-g$ degree d hypersurfaces. For $i = 1, \dots, g$ and $j = 1, \dots, r-n$ let $H_i(j)$ be a general hyperplane containing V_i. Note that $\cap_{j=1}^{r-n} H_i(j) = V_i$. We first find degree d smooth hypersurfaces X_j containing $\{p_1, \dots, p_g\}$ and with $T_{p_i}X_j = H_i(j)$ for all i. If we omit the requirement that X_j is smooth outside the set $\{p_1, \dots, p_g\}$, then is an easy interpolation problem which is true if $d \geq g$ and we use the inequality $d \geq 2g$ and the theorem of Bertini to obtain the smoothness of X_j outside $\{p_1, \dots, p_g\}$. For $j = 1, \dots, r-n$ set $X[j] := \cap_{h=1}^{j} X_h$. Since $d \geq 4(r-n)g$, by induction on j and using the theorem of Bertini we recognize that each $X[j]$ is smooth. Note that $X := X[r-n]$ is a solution of Claim 4. □

Take X as in Claim 4. For each $I \subseteq \{1, \dots, g\}$ let A_I be the linear span of $\cup_{i \in I} V_i$. Since A is general, we have $A_I = \mathbb{P}^r$ if $I = \{1, \dots, g\}$ and $\dim A_I = \#I(n+1) - 1$ if $I \neq \{1, \dots, g\}$. Thus X is not secant defective by the Terracini Lemma. Assume that $\sigma_{g-1}(X)$ is a cone and take a point o in the vertex of the cone. Let T be the set of all $S \subset \{p_1, \dots, p_g\}$ such that $\#S = g-1$. For any $S \in T$ set $W_S := \langle \cup_{p \in S} V_p \rangle$. The Terracini Lemma gives $o \in W_S$ for all $S \in T$. Thus $o \in \cap_{S \in T} W_S$. The generality of A and the definition of g gives $\cap_{S \in T} W_S = \emptyset$, a contradiction. □

Proposition 12. *Fix integers $r > n > 0$ such that $r+1 \equiv 0 \pmod{n+1}$. Set $g := (r+1)/(n+1)$. Take an integral, non degenerate and not secant defective variety $X \subset \mathbb{P}^r$. Then $g = g(X) \geq 2$ and no secant variety $\sigma_i(X)$, $i < g$, is a cone.*

Proof. Since X is not secant defective and $r = g(n+1) - 1$, $g = g(X)$ and $\dim \sigma_i(X) = \dim \sigma_{i+1}(X) - n - 1$. Assume that $\sigma_i(X)$ is a cone and let $i \in \{1, \dots, g-1\}$ be the first integer such that $\sigma_i(X)$ is a cone. Take a point o in the vertex of $\sigma_i(X)$. Take a general $S \subset X_{\mathrm{reg}}$ such that $\#S = i+1$. For any $I \subset S$ set $V_I := \langle \cup_{p \in I} T_pX \rangle$. Take $I \subset S$ such that $\#I = i-1$ and set $\{p, q\} := S \setminus I$, $E := I \cup \{p\}$ and $F := I \cup \{q\}$. By assumption and the Terracini Lemma we have $o \in V_E \cap V_F$ and $o \notin V_i$. Since $\dim V_S = \dim V_I + 2n + 1$, we obtain a contradiction. □

Proposition 13. *Fix integers $r > n > 0$ and set $i := \lfloor (r+1)/(n+1) \rfloor$ and $a := r+1-(n+1)i$. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety such that $\dim \sigma_i(X) = r - a$. Assume $a \in \{1, 2\}$.*

(a) X is (i, ∞)-universal.
(b) Assume $a = 1$. X is strongly (i, ∞)-universal if and only $\sigma_i(X)$ is not a cone.

Proof. By assumption $\sigma_i(X)$ has codimension a. Let $Y \subset \mathbb{P}^r$ be an integral and non-degenerate variety. To prove part (a) it is sufficient to prove that $[\sigma_i(X); Y] = \mathbb{P}^r$. Since Y is non-degenerate, there is a non-degenerate curve $C \subseteq Y$. Thus it is sufficient to quote (Prop. 1.3 [3]).

Assume $a = 1$. If $\sigma_i(X)$ is not a cone, then $[\sigma_i(X); \{p\}] = \mathbb{P}^r$ for every $p \in \mathbb{P}^r$, while if $\sigma_i(X)$ is a cone with vertex V, then $[\sigma_i(X); \{p\}] = \sigma_i(X)$ for every $p \in V$. □

Theorem 9. *Fix positive integers n, m and r_0 and an integral n-dimensional variety X. Then there is a very ample line bundle L on X such that $h^0(L) > r_0$ and the embedding of X by the complete linear system $|L|$ is strongly m-universal.*

Proof. Fix an arbitrary embedding $X \subset \mathbb{P}^N$ and take a positive integer d and set $\mathcal{O}_X(1) := \mathcal{O}_{\mathbb{P}^N}(1)_{|X}$. Fix an integer $d \geq n+m$ such that $h^0(\mathcal{O}_X(d)) > r_0$. and set $L := \mathcal{O}_X(d)$. The line bundle L is very ample. Let $X_1 \subset \mathbb{P}^r$, $r := h^0(L) - 1$, denote the image of X by the embedding j of X induced by the complete linear system $|L|$. Fix an integral m-dimensional variety Y. By the Terracini Lemma to prove that $\dim[X_1; Y] = n+m+1$ it is sufficient to prove that $\dim[X_1; W] = n+m+1$ for a general tangent space W of Y_{reg}. Thus it is sufficient to prove that $\dim[X_1; V] = n+m+1$ for every m-dimensional linear space V.

Let $\ell_V : \mathbb{P}^r \setminus V \to \mathbb{P}^{r-m-1}$ the linear projection from V. Since $X_1 \nsubseteq V$, to conclude the proof it is sufficient to prove that the quasi-projective variety $\ell_V(X_1 \setminus X_1 \cap V)$ has dimension n (use the Terracini Lemma). Let $v_{N,d} : \mathbb{P}^N \to \mathbb{P}^M$, $M := \binom{N+M}{N} - 1$, denote the order d Veronese embedding of $\mathbb{P}^N$. Note that $L \cong \mathcal{O}_{\mathbb{P}^N}(1)_{|v_{N,d}(\mathbb{P}^N)}$. Let $C \subset X$ be any integral curve. Note that $v_{N,d}(C)$ spans a linear space of dimension at least d and hence $j(C)$ spans a linear subspace of dimension at least d. Since $d > m$, $V \cap X_1$ is a finite set and hence $j(C) \cap V$ is finite. Since $d \geq m+2$, no $(m+1)$-dimensional linear subspace of $\mathbb{P}^r$ contains C. Thus $\ell_V(j(C) \setminus j(C) \cap V)$ is a curve. Thus $\ell_V(X_1 \setminus X_1 \cap V)$ has dimension n. □

In Theorem 9 we cannot require that the embedding of X is given by a complete linear system. Indeed, in general there is no integer $r_0(n,m)$ such that for all $r \geq r_0(n,m)$ we have a non-degenerate embedding of X in $\mathbb{P}^r$ which is universally m-join, because often, e.g., for $\mathbb{P}^n$, $n \geq 2$, not all large integers r are of the form $h^0(L) - 1$ for some line bundle L on X.

Theorem 10. *Fix a positive integer m. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Set $n := \dim X$ and assume $\dim \sigma_{m+2}(X) = (m+2)(n+1) - 1$. Then X is strongly m-universal.*

Proof. By the Terracini Lemma it is sufficient to prove that $\dim[X;T] = n+m+1$ for any m-dimensional linear subspace $T \subset \mathbb{P}^r$. Fix T. By the Terracini Lemma it is sufficient to prove that $T_pX \cap T = \emptyset$ for a general $p \in X_{\mathrm{reg}}$. Assume $T_pX \cap T \neq \emptyset$ for a general $p \in X_{\mathrm{reg}}$. Fix a general $S \subset X_{\mathrm{reg}}$ such that $\#S = m+2$, say $S = \{p_1, \dots, p_{m+2}\}$. By assumption there is $o_i \in T \cap T_{p_i}X$, $i = 1, \dots, m+2$. Since $\dim \sigma_{m+2}(X) = (m+2)(n+1) - 1$, the Terracini Lemma implies that $o_1, \dots, o_{m+2}$ are linearly independent. Thus $\dim T > m$, a contradiction. □

Note that if $m \geq 2$ and X satisfies the assumptions of Theorem 10 for the integer m, then it satisfies the assumptions of Theorem 10 for all positive integers $< m$.

Theorem 11. *Fix positive integers n, m and i and integral n-dimensional projective variety X. Let N be the local embedding dimension of X, i.e., the maximum of the embedding dimension of all points of X. Fix an integer $r \geq \max\{n+N, (i+m+3)(n+1)-1\}$. Then there is a non-degenerate embedding $j : X \to \mathbb{P}^r$ such that $j(X)$ is not secant defective and strongly (i,m)-universal.*

Proof. Fix any very ample line bundle L on X. By [34] there is an integer d_0 such that for all $d \geq d_0$ the embedding j_1 of X by the complete linear system $|L^{\otimes d}|$ is not secant defective. Fix any $d \geq d_0$ such that $h^0 L^{\otimes d}) > r$ and let $j_1 : X \to \mathbb{P}^x$, $x = h^0(L^{\otimes d}) - 1$ denote the embedding of X induced by the complete linear system $|L^{\otimes d}|$. Let $V \subset \mathbb{P}^x$ be a general $(r-x-1)$-dimensional linear subspace, with the convention $V = \emptyset$ if $x = r$. Let $\ell_V : \mathbb{P}^x \setminus V \to \mathbb{P}^r$ denote the linear projection for V. Since $x - r < x - n$ and V is general, $V \cap j_1(X) = \emptyset$ and hence $\ell_{V|j_1(X)} : j_1(X) \to \mathbb{P}^r$ is a morphism. Set $j := \ell_V \circ j_1$. Since j_1 is an embedding, $r \geq \max\{n+N+2n+1\}$ and V is general, j is an embedding.

Claim 5. *$j(X)$ is not secant defective.*

Proof of Claim 5. Set $g := \lceil (r+1)/(n+1) \rceil$. To prove Claim 5 it is sufficient to prove that $\dim \sigma_{g-1}(j(X)) = (g-1)(n+1) - 1$ and $\sigma_g(X) = \mathbb{P}^r$. Since $j_1(X)$ is not secant defective, $\dim \sigma_{g-1}(j_1(X)) = (g-1)(n+1) - 1$. By the definition of $x - r - 1$ and the generality of V we have $V \cap \sigma_{g-1}(X) = \emptyset$. Thus $\ell_V(\sigma_{g-1}(j_1(X))) = \sigma_{g-1}(j(X))$ and $\dim \sigma_{g-1}(j(X)) = (g-1)(n+1) - 1$. Fix a general $S \subset X_{\mathrm{reg}}$ such that $\#S = g$. By the Terracini Lemma and the nondefectivity of $j_1(X)$, the vector space $W := \langle \cup_{p \in j_1(S)} T_p j_1(X) \rangle$ has dimension $(n+1)g - 1 \geq r$. Since V is general, we have $V \cap W = \emptyset$ and $\ell_V(W) = \mathbb{P}^r$. Thus $\langle \cup_{p \in j(S)} T_p j(X) \rangle = \mathbb{P}^r$. The Terracini Lemma gives $\sigma_g(X) = \mathbb{P}^r$. □

Note that $g - 3 \geq i + m$ (resp. $g - 2 \geq i + m$). Since $\dim \sigma_{g-1}(j(X)) = \dim \sigma_{g-2}(j(X)) + n + 1$, $\sigma_{g-2}(j(X))$ is not a cone. Part (i) of (Cor. 2.3 [1]) gives $\dim[\sigma_i(j(X)); Y] = \dim \sigma_i(j(X)) + n + 1$. Thus to prove Theorem 11 it is sufficient to quote part (i) of (Cor. 2.3 [1]). □

Note that $N = n$ in Theorem 11 if X is smooth and that if X is contained in an M-dimensional projective space, then $N \le M$.

7. Tensors and Products

In this section, we fix an integer $k \ge 2$ and non-zero vector spaces V_h, $1 \le h \le k$. We set $V := V_1 \otimes \cdots \otimes V_k$. Not that V is the vector spaces of all k-order tensors with format $(\dim V_1, \dots, \dim V_k)$. We take the set-up of the other sections of the papers for very particular subvarieties X and Y of $\mathbb{P}V$: we assume that both X and Y are product of k varieties X_h and Y_h. We prove the following result.

Theorem 12. *Fix integral and non-degenerate varieties $X_h \subset \mathbb{P}V_h$, $1 \le h \le k$, and integral varieties $Y_h \subset \mathbb{P}V_h$, $1 \le h \le k$. Set $n_h := \dim X_h$, $m_h := \dim Y_h$, $X := X_1 \times \cdots \times X_k$, and $Y := Y_1 \times \cdots \times Y_k$. See both X and Y as subvarieties of $\mathbb{P}V$ using the Segre embedding of the multiprojective space $\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k$ into $\mathbb{P}V$. Fix a positive integer i and assume $\dim[\sigma_i(X_h); Y_h] = m_h + i(n_h + 1)$ for all $h = 1, \dots, k$. Then $\dim[\sigma_i(X); Y] = i(\dim X + 1) + \dim Y$.*

Proof. Fix a general $S \subset S(X_{\mathrm{reg}}, i)$ and a general $p \in Y_{\mathrm{reg}}$. Set $W := \langle \cup_{q \in S} T_q X \cup T_p Y \rangle$ and $W_h := \langle \cup_{o \in S} T_{o_h} X_h \cup T_{p_h} Y_h \rangle$. By the Terracini Lemma it is sufficient to prove that $\dim W = i(\dim X + 1) + \dim Y$. We have $\dim X = n_1 + \cdots + n_k$ and $\dim Y = m_1 + \cdots + m_k$. Write $p = (p_1, \dots, p_k)$. For all $o \in S$ write $o = (o_1, \dots, o_k)$. Note that $[\sigma_i(X) : Y] \subseteq \prod_{h=1}^{k} [\sigma_i(X_h); Y_h]$. Let $\mathcal{A} \subset X \times Y$ denote the set of all products of the coordinates of the $i+1$ elements of $S \cup \{p\}$. We have $\#\mathcal{A} = k(i+1)$. The Terracini Lemma and the assumption $\dim[\sigma_i(X_h); Y_h] = m_h + i(n_h + 1)$ gives $\dim W_h = m_h + i(n_h + 1)$. Hence the definition of tensor product gives $\dim \otimes_h W_h = \prod_h (m_h + i(n_h + 1))$. We understand that the tangent spaces to X and Y appearing in the definition of W are linearly independent, i.e., W has the claimed dimension. $\square$

8. Family of Joins

All questions considered in this paper may be considered in the following more general set-up. Let $X_i \subset \mathbb{P}^r$, $1 \le i \le s$, be integral projective varieties. Set $n_i := \dim X_i$. Suppose that there is a family $\mathcal{F}_i$, $1 \le i \le s$, of subvarieties of $\mathbb{P}^n$, i.e., algebraic constructible subsets of the Hilbert scheme of $\mathbb{P}^r$, with $X_i \in \mathcal{F}_i$ for all i. Consider the family of all irreducible varieties $[Y_1; \dots; Y_s]$, $Y_i \in \mathcal{F}_i$, for all i. If each $\mathcal{F}_i$ is irreducible, we obtain an irreducible family of joins parametrized (not one-to-one and may be not even finite-to-one) by the irreducible variety $\mathcal{F}_1 \times \cdots \times \mathcal{F}_s$. In this case it makes sense to consider the "generic case", i.e., the generic uniqueness problem or the dimension of the boundary, for a general $(T_1, \dots, T_s) \in \mathcal{F}_1 \times \cdots \times \mathcal{F}_s$. For instance, we may consider a small neighborhood of X_i in the Hilbert scheme of $\mathbb{P}^r$ assuming (as in many important examples) that the Hilbert scheme of $\mathbb{P}^r$ is "nice" (e.g., irreducible) at $[Y_i]$. This choice (for a sufficiently small neighborhood of X_i) has the advantage that if X_i is non-degenerate then all nearby $Y_i \in \mathcal{F}_i$ are non-degenerate and that if X_i is smooth, then all nearby $Y_i \in \mathcal{F}_i$ are smooth. We give some examples. Take for instance $s = 2$ and assume $n_1 + n_2 < r$. Just using $\mathcal{F}_1 := X_1$, i.e., not moving X_1 and taking as $\mathcal{F}_2$ the family of all $g(X_2)$, $g \in \mathrm{Aut}(\mathbb{P}^r)$, we have $T_1 \cap T_2 = \emptyset$ and hence the boundary of $[T_1; T_2]$ is empty by Proposition 1. Similarly, if $n_1 + \cdots + n_s + s \le r$ and a general $(g_1, \dots, g_{s-1}) \in \mathrm{Aut}(\mathbb{P}^r)s - 1$ we understand that the boundary of $X_1; g_1(X_2); \dots; g_{s-1}(X_s)]$ is empty (Proposition 2). In the case of two space curves X, Y with a unique common point it is each to obtain $[X; g(Y)]^\circ = \mathbb{P}^3$ for a general $g \in \mathrm{Aut}(\mathbb{P}^3)$ such that $g(o) = o$. We think that similar results may be proved for the general element of families with prescribed, but small, intersection.

9. Conclusions

We prove a strong theorem on partial non-defectivity of secant varieties of embedded homogeneous varieties developing a general set-up for families of subvarieties of Grassmannians. We study these type of problems in the more general set-up of joins of embedded varieties. Joins are defined by taking a closure. We study the set obtained before the closure (often called the **open part of the join**) and the set added taking the closure is called the **boundary**. For a point q of the open part we give conditions for the uniqueness or non-uniqueness of the set proving that q is in the open part. On this topics we give a conjecture and several open questions. For the applications a very promising area is the reconstruction of an object, say a tensor T, from a small family of rank 1 decompositions, not (or at least not known) having the minimal number of addenda. The space of possible tensors is a linear space and reconstruction holds if this linear subspace has dimension 1, i.e., it it spanned by T. Note that the answer may be YES without knowing T.

We state the following four additional questions framed in the set-up of tensors:

(1) Is a general tensor of a given format uniquely determined by the sets of its solutions? May it be reconstructed in a computational efficient way?

(2) Take a format for tensors and a positive integer a such that the set of all rank a tensors (of the given format) has not the expected dimension. Is a general rank a tensor uniquely determined by its set of solutions?

(3) Take a rank a tensor T of tensor rank a and a given format. Describe the integer $b > a$ such that T may be uniquely reconstructed by the set of all its decompositions as a sum of b rank 1 tensors.

These questions make sense even for real tensors and real decompositions of real tensors.

(4) For any variety $X \subset \mathbb{P}^r$ and all positive integer a and b compute the dimension of the join of a copies of X and b copies of the tangential variety $\tau(X)$ of X. The interested reader may work on these questions for different objects, e.g., for partially symmetric tensors or skew tensors.

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