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The Blow-Up of the Local Energy Solution to the Wave Equation with a Nontrivial Boundary Condition

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Abstract: In this study, we examine the wave equation with a nontrivial boundary condition. The main target of this study is to prove the local-in-time existence and the blow-up in finite time of the energy solution. Through the construction of an auxiliary function and the imposition of appropriate conditions on the initial data, we establish the both lower and upper bounds for the blow-up time of the solution. Meanwhile, based on these estimates, we obtain the result of the local-in-time existence and the blow-up of the energy solution. This approach enhances our understanding of the dynamics leading to blow-up in the considered condition.

Keywords: positive initial energy; boundary value problem; auxiliary function; lower and upper bounds

MSC: 35B35; 35L05; 35L20

1. Introduction

In this paper, we are concerned with the local-in-time existence of the energy solution to the following wave equation:

$$\begin{cases} u''(x,t) - \mu(t)\triangle u(x,t) + h(u(x,t)) = 0 & \text{in } \Omega \times (0,+\infty) \\ u(x,t) = 0 & \text{on } \Gamma_0 \times (0,+\infty) \\ \mu(t)\frac{\partial u}{\partial \nu} + g(u') = |u|^\gamma u & \text{on } \Gamma_1 \times (0,+\infty) \\ u(x,0) = u_0(x), u'(x,0) = u_1(x) & \text{in } \Omega \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^n (n = 1,2,3)$ with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ of class C^2. Here, $\Gamma_0 \neq \emptyset$, Γ_0 and Γ_1 are closed and disjoint. Let ν be the outward normal to Γ; $\triangle$ stands for the Laplace operator.

System (1) has been studied in [1]. When μ,γ and g satisfy appropriate assumptions, the solution of System (1) will blow up within a finite time. In this article, based on the solution blow-up, we will continue to study the upper and lower bounds for the blow-up time of System (1). Based on these estimates, we will obtain the result of the local-in-time existence of the energy solution. There is relatively little existing literature on the problem of calculating the upper and lower bounds of the blow-up time, but accurately calculating the blow-up time has significant practical significance in specific engineering problems. The authors of Ref. [2] study a nonlinear viscoelastic wave equation with damping and source terms. By using the concavity method, it shows a finite time blow-up result and obtains the upper bound for the blow-up time. Ref. [3] deals with a nonlinear viscoelastic wave equation with strong damping. By means of a first-order differential inequality technique, the estimate the lower bound for the blow-up time is obtained. Ref. [4] deals with the blow-up for a class of nonlinear viscoelastic wave equations. Based on a first-order differential inequality technique and some Sobolev-type inequality, a lower bound for blow-up time is obtained. However, each reference listed above has the Dirichlet's boundary condition. The previous studies that have been performed related to trivial boundary conditions. More importantly, the problem with nontrivial boundary conditions has extremely few

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results. In the references above, it always assumes that $u = 0$ on $\partial\Omega$, which greatly reduces the difficulty of estimating the blow-up time boundary. Unlike the previous literature, our article considers nontrivial boundary conditions, and these boundary conditions are nonlinear, increasing the difficulty of estimating the blow-up time. In addition, nontrivial boundary conditions can also cause difficulties in inequality estimation and auxiliary function construction. Therefore, our research can fill the gap in this area of study. There are still many other studies on handling blow-up time under trivial boundary conditions, for example, G.A. Philippin [5] explores the lower bounds for the blow-up time in the context of the wave equation with trivial boundary conditions. However, this study does not currently address the upper bounds for blow-up time. Future research endeavors may extend the investigation to include upper bounds and further enrich our understanding of the dynamics in this particular scenario. J. Zhou [6] considered the blow-up time with three different ranges of initial energy under the condition of atrivial boundary. Furthermore, considering positive initial energy and nonlinear boundary damping, T.G.Ha [1,7] established the blow-up of solutions for the semilinear wave equation. However, the specific determination of the blow-up time is not addressed within the current scope of the research. Investigating the blow-up time in this context could provide valuable insights into the temporal evolution of the solutions.

On the other hand, the blow-up behavior of solutions to the wave equation is not only related to the interaction between damping terms and source terms, but also to the sign of the system's initial energy. Generally speaking, negative initial energy is more likely to cause system solution blow-up, while positive initial energy requires stricter conditions for system solution blow-up. This article has already addressed the issue of system solution blow-up under positive initial energy, and we further estimate the upper and lower bounds of the solution blow-up time. Considerable progress has been made in demonstrating to the wave equation, especially in cases where the initial energy is negative, the conclusions about blow-up solution have been proved [8–15]. Meanwhile, many similar results also have been found when the initial energy is positive (see [16–19]). However, the problem of computing exact blow-up time T has not been considered. In instances where the solution of the wave equation experiences blow-up, the exact computation of the blow-up time T is often not feasible. So figuring out the bounds for T is valuable in practical applications. In recent years, there have been some advances in research on the bounds of blow-up time. However, a great deal of research work has focused on parabolic equations [20–25]. Very few researchers have focused their work on hyperbolic equations with nontrivial boundary conditions [5]. In addition, the above literature only obtained the bounds of blow-up time, but did not analyze the sharpness of blow-up time. In [26,27], not only the limit of blasting time is obtained, but also the sharpness of blasting time is analyzed.

Compared to existing literature results, this paper addresses a notable gap in the existing research, as minimal attention has been dedicated to investigating the lower and upper bounds for the blow-up time of the wave equation with weak boundary damping and source term. The primary focus of this work is to contribute to this specific aspect of the field.

This paper aims to investigate how the interaction between the damping term and source term influences the occurrence of blow-up in the solution. Specifically, the focus is on demonstrating that the blow-up and blow-up time are intricately controlled by the interplay of these two terms. Building upon the findings of [7], the objective is to extend and generalize the results by precisely computing both lower and upper bounds for the blow-up time T in the context of the wave equation with weak boundary damping and source term. Therefore, the motivation of this paper is to generalize the results under trivial boundary conditions, and further solve the problem of constructing new auxiliary functions to estimate the bounds of blow-up time under nontrivial boundary conditions.

This paper follows a structured organization. Section 2 provides a review of notation, hypotheses, and crucial preliminary steps. It also introduces the blow-up solution for Equation (1). Moving on to Section 3, the main result is presented, and the paper precisely computes both lower and upper bounds for the blow-up time T in the context of problem (1). This organization ensures a clear and systematic presentation of the research.

2. Preliminaries

Before delving into our principal discovery, it is crucial to take a moment to revisit the extant body of research pertaining to the local existence, uniqueness, and blow-up of the solution. This foundational understanding will provide a solid foundation for our forthcoming discussion and findings. We begin this part by outlining a few theories and some necessary results. To be more precise, we have the following hypotheses:

Hypothesis 1. *$\Omega \subset \mathbb{R}^n$ is a bounded domain, $n \geq 1$, where the boundary of Ω is $\Gamma = \Gamma_0 \cup \Gamma_1$ of class C^2.*

Here, $\Gamma_0 \neq \emptyset$, Γ_0 and Γ_1 are closed and disjoint, satisfying the following conditions:

$$m(x) \cdot \nu(x) \geq \sigma > 0 \text{ on } \Gamma_1, m(x) \cdot \nu(x) \leq 0 \text{ on } \Gamma_0,$$

$$m(x) = x - x^0 \ (x^0 \in \mathbb{R}^n) \text{ and } R = max_{x \in \overline{\Omega}} |m(x)|,$$

where ν represents the unit outward normal vector to Γ. We assume that

$$\mu(0) \frac{\partial u_0}{\partial \nu} + g(u_1) = |u_0|^\gamma u_0 \text{ on } \Gamma_1.$$

Hypothesis 2. *Assume $\mu \in W^{2,\infty}(0,T) \cap W^{2,1}(0,T)$, and $\mu(t) > 0$ is monotonic decreasing. Meanwhile, $h : \mathbb{R} \to \mathbb{R}$ is a continuous function and $h(s)s \geq 0$ for all $s \in \mathbb{R}$.*

Hypothesis 3. *Assume γ is a constant and satisfies requirements:*

$$\begin{cases} 0 \leq \gamma < 1, & \text{if } n = 3, \\ \gamma \geq 0, & \text{if } n = 1, 2. \end{cases}$$

Hypothesis 4. *Assume g is a monotone increasing function and satisfies $g(0) = 0$. There exist a non-negative constant m and a strictly increasing and odd function β of C^1 class on $[-1, 1]$ such that*

$$|\beta(s)| \leq |g(s)| \leq |\beta^{-1}(s), \text{ if } |s| \leq 1,$$
$$C_1 |s|^{m+1} \leq |g(s)| \leq C_2 |s|^{m+1}, \text{ if } |s| > 1,$$

where β^{-1} denotes the inverse function of β and C_1 and C_2 are positive constants.

2.1. Wellposedness Result

First of all, one can define the energy $E(t)$ associated with system (1).

$$E(t) = \frac{1}{2}\|u'\|_2^2 + \int_\Omega \Psi(u(x,t))dx + \frac{1}{2}\mu(t)\|\nabla u\|_2^2 - \frac{1}{\gamma+2}\|u\|_{\gamma+2,\Gamma_1}^{\gamma+2}, \tag{2}$$

where $\Psi(t) = \int_0^t h(s)dx$. By calculation, we can obtain

$$E'(t) = \mu'(t)\|\nabla u\| - \int_{\Gamma_1} g(u')u' d\Gamma, \tag{3}$$

where ∇ is the gradient operator. According to the Hypothesis 2, $E(t)$ is monotone decreasing function.

Remark 1. *The proof of the energy identity (2)–(3) will be proved in (7)–(8).*

With the notion we set, the following conclusion will be obtained [28].

Theorem 1. *Assume Hypotheses 1–4 hold and*

$$m \geq \gamma \text{ or } (u_0, u_1) \in \{(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega);\ \|\nabla u\| < \lambda_0,\ E(0) < d\}.$$

Then, the problem (1) has a unique local solution

$$u \in C^0(0, T; H^1_{\Gamma_0}(\Omega)) \cap C^1(0, T; L^2(\Omega)).$$

2.2. Blow-Up Solution

Next, we have the corresponding blow-up result.

Theorem 2. *Assume Hypotheses 1–4 hold and $m < \gamma$. Meanwhile, to System (1), we suppose that*

$$(u_0, u_1) \in \{(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega); \|\nabla u_0\|_2 > \lambda_0, E(0) < E_1 < d, E_1 \in \mathbb{R}\}$$

and

$$\beta^{-1} \leq \left(\frac{(\gamma+2)(\mu_0 \gamma \lambda_0 - 2(\gamma+2)E_1)^2}{8(\gamma+1) meas(\Gamma_1)(\mu_0 \lambda_0^2 - 2E_1)} \right)^{\frac{\gamma+1}{\gamma+2}}.$$

Consequently, the solution $u(t)$ blows up.

The result of Theorem 2 has been obtained in [7]. We can now report our primary finding in the next section.

3. The Bounds for *T*

In this section, we turn our attention to examining the lower bound of the blow-up time for the blow-up solution of Equation (1). Prior to presenting and demonstrating our primary result, we require the following lemma that plays a pivotal role in establishing the upper bounds for the blow-up time T.

Lemma 1. *In the case of the assumptions specified in Theorem 2, the solution to system (1) yields the following result*

$$\|\nabla u(t)\|_2 > \lambda_0.$$

Lemma 1 closely parallels Lemma 1 in [1]. Therefore, the proof will be omitted for brevity.

Theorem 3. *Assume Hypotheses 1–4 hold. Under the result in Theorem 2, the solution $u(x,t)$ to System (1) will blow up at a finite T, and blow-up time T satisfies*

$$\int_{F(0)}^{F(T^-)} \frac{dy}{y + 2\mathbf{k} + C_0 y^{\frac{2\gamma+3}{2\gamma+4}} (meas(\Gamma_1))^{\frac{1}{2\gamma+4}}} \leq T \leq \frac{1-\overline{\chi}}{C_7 [L(0)]^{\frac{\overline{\chi}}{1-\overline{\chi}}}},$$

where C_0, C_7 is a positive constant, $0 < \overline{\chi} < \chi < \frac{\gamma - m}{(m+2)(\gamma+2)}$ and

$$F(0) = \int_\Omega |u_0|^2 dx + \int_{\partial\Omega} |u_0|^{\gamma+2} d\Gamma + t^* \int_{\partial\Omega} |u_1|^{\gamma+2} d\Gamma,$$
$$\mathbf{k} := \frac{1}{2}\|u_1\|^2 + \int_\Omega \Psi(u_0) dx + \frac{1}{2}\mu(t)\|\nabla u_0\|_2^2 - \frac{1}{\gamma+2}\|u_0\|^{\gamma+2}_{\gamma+2,\Gamma_1}. \tag{4}$$

Proof. (i): In this section, we initiate the estimation of the upper bound of time *T* using auxiliary function that allows us to establish an upper bound

E_1 is a constant and satisfies $E(0) < E_1 < d$; then, we define $H(t)$ as follows:

$$H(t) = E_1 - E(t).$$

Subsequently, we obtain

$$H'(t) = 0 - E'(t) = -E'(t) \geq 0.$$

It is straightforward to derive that $H(t)$ is nondecreasing. Meanwhile, we have,

$$H(t) \geq H_0 := E_1 - E(0) \geq 0, \ for\ all\ t \geq 0.$$

According to Lemma 1 and Hypothesis 2, we obtain

$$\begin{aligned} H(t) &\leq E_1 - \frac{1}{2}\mu(0)\|\nabla u\|_2^2 + \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2} \\ &< d - \frac{1}{2}\mu(0)\lambda_0^2 + \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2} \\ &\leq \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2}. \end{aligned}$$

Afterward, we will examine:

$$I = \frac{d}{dt}\int_\Omega u' u dx = (\frac{\Phi'(t)}{2})'.$$

By similar calculation as in [1,7], we can see that the following estimate holds:

$$\begin{aligned} I &= \|u'\|_2^2 - \int_\Omega h(u)udx - \mu(t)\|\nabla u\|_2^2 + \|u\|_{\gamma+2,\Gamma_1}^{\gamma+2} - \int_{\Gamma_1} g(u')ud\Gamma + \theta E(t) - \theta E(t) \\ &\geq \left(1+\frac{\theta}{2}\right)\|u'\|_2^2 - \mu_0\left(\frac{\theta}{2}-1\right)\|\nabla u\|_2^2 - \theta E_1 + \left(1-\frac{\theta}{\gamma+1}\right)\|u\|_{\gamma+2,\Gamma_1}^{\gamma+2} \\ &+ \theta H(t) - \int_{\Gamma_1} g(u')ud\Gamma \\ &\geq C_5\left(\|u'\|_2^2 + \|u(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2} + H(t) - H'(t)H_0^{\overline{\chi}-\chi}H^{-\overline{\chi}(t)}\right), \end{aligned} \tag{5}$$

where $0 < \overline{\chi} < \chi < \frac{\gamma-m}{(m+2)(\gamma+2)}$ and $C_5 > 0$.

To derive the corresponding estimate, we construct an auxiliary function that allows us to establish the upper bound for T.

$$L(t) = H^{1-\overline{\chi}}(t) + \delta\Phi'(t),$$

where $\Phi(t) = \|u(t)\|_2^2$.

By taking the derivative of $L(t)$ and utilizing (5), we obtain:

$$\begin{aligned} L'(t) &= (1-\overline{\chi})H^{-\overline{\chi}}(t)H'(t) + \delta\Phi''(t) \\ &\geq \left(1-\overline{\chi}-2C_5\delta H^{\overline{\chi}-\chi}\right)H^{-\overline{\chi}}H'(t) + 2C_5\delta\left(\|u'\|_2^2 + \|u(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2} + H(t)\right). \end{aligned}$$

Choosing $0 < \overline{\chi} < min\{\frac{1}{2}, \chi\}$ and making δ sufficiently small, we establish

$$L'(t) \geq C_6\left(\|u'\|_2^2 + \|u(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2} + H(t)\right),$$

where C_6 is a positive number, so $L(t) > 0$ is an increasing function. Using the same reasoning as in [1], we establish:

$$L'(t) \geq C_7 L^{\frac{1}{1-\overline{\chi}}}(t), \qquad for\ all\ t \in [0, T], \tag{6}$$

where $C_7 > 0$ is constant and satisfies $1 < \frac{1}{1-\overline{\chi}} < 2$.

Then, a straightforward integration of (6) over $(0, T)$ produces

$$L^{\frac{\overline{\chi}}{1-\overline{\chi}}} \geq \frac{1}{L^{\frac{-\overline{\chi}}{1-\overline{\chi}}}(0) - C_7 T \frac{\overline{\chi}}{1-\overline{\chi}}},$$

therefore $L(t)$ blows up in time

$$T \leq \frac{1-\overline{\chi}}{C_7 \overline{\chi} [L(0)]^{\frac{\overline{\chi}}{1-\overline{\chi}}}}.$$

(ii): In this section, we initiate the estimation of the lower bound of time T using a series of energy mode estimation and inequality reduction techniques.

By multiplying both sides of the first equation by u' and integrating over the domain, we obtain the following energy mode estimate:

$$\int_\Omega (u''u' - \mu(t)\triangle u u' + h(u)u')dx = 0$$

by Green's formulas we have

$$\int_\Omega u''u'dx - \mu(t)(\int_{\partial\Omega} u' \frac{\partial u}{\partial \nu} d\Gamma - \int_\Omega \nabla u \nabla u' dx) + \int_\Omega h(u)u'dx = 0$$

Because of the bound condition of Γ in Equation (1), we obtain

$$\int_\Omega u''u'dx - \int_{\Gamma_1} |u|^{\gamma+1} u u' d\Gamma + \int_{\Gamma_1} u' g(u') d\Gamma + \mu(t) \int_\Omega \nabla u \nabla u' dx + \int_\Omega h(u)u'dx = 0.$$

It is straightforward to derive that

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\int_\Omega |u'|^2 dx - \frac{1}{\gamma+2}\int_{\Gamma_1} |u|^{\gamma+2} d\Gamma + \frac{1}{2}\mu(t)\int_\Omega |\nabla u|^2 dx + \int_\Omega \int_0^u h(s)dsdx\right) \\ =&\frac{1}{2}\mu'(t)\int_\Omega |\nabla u|^2 dx - \int_{\Gamma_1} g(u')u'd\Gamma. \end{aligned} \tag{7}$$

Through the above calculation, we can deduce the result (2), and

$$E'(t) = \frac{1}{2}\mu'(t)\int_\Omega |\nabla u|^2 dx - \int_{\Gamma_1} g(u')u'd\Gamma \leq 0. \tag{8}$$

So

$$E(t) \leq E(0) = \mathbf{k}.$$

It is straightforward to derive that

$$\begin{aligned} &2E(t) + \frac{2}{\gamma+2}\|u\|_{\gamma+2,\Gamma_1}^{\gamma+2} \\ &\leq 2\mathbf{k} + \frac{2}{\gamma+2}\|u\|_{\gamma+2,\Gamma_1}^{\gamma+2}. \end{aligned} \tag{9}$$

Next, we will define an auxiliary function as follows:

$$F(t) = \int_\Omega |u|^2 dx + \int_{\partial\Omega} |u|^{\gamma+2} d\Gamma + (t^* - t)\int_{\partial\Omega} |u_1|^{2\gamma+3} d\Gamma,$$

where $t^* > 0$ is a time large enough; furthermore, we can set $t^* = \frac{1-\overline{\chi}}{C_7\overline{\chi}[L(0)]^{\frac{\overline{\chi}}{1-\overline{\chi}}}}$, then

$$F'(t) = 2\int_{\Omega} uu'dx + (\gamma+2)\int_{\partial\Omega} |u|^{\gamma} uu' d\Gamma - \int_{\partial\Omega} |u_1|^{2\gamma+3} d\Gamma.$$

By using a series of inequality reduction techniques, we have

$$\begin{aligned}
F'(t) \le& \int_{\Omega} |u|^2 dx + \int_{\Omega} |u'|^2 dx + \varepsilon \int_{\partial\Omega} |u'|^{2\gamma+3} d\Gamma \\
&+ C_\varepsilon \int_{\partial\Omega} |u|^{(\gamma+1)\frac{2\gamma+3}{2\gamma+2}} d\Gamma - \int_{\partial\Omega} |u_1|^{2\gamma+3} d\Gamma \\
\le& \int_{\Omega} |u|^2 dx + \int_{\Omega} |u'|^2 dx + \varepsilon \int_{\partial\Omega} |u'|^{2\gamma+3} d\Gamma \\
&+ C_\varepsilon \left(\int_{\partial\Omega} |u|^{\gamma+2} d\Gamma \right)^{\frac{2\gamma+3}{2\gamma+4}} (meas(\Gamma))^{\frac{1}{2\gamma+4}} - \int_{\partial\Omega} |u_1|^{2\gamma+3} d\Gamma,
\end{aligned}$$

where C_ε is a constant depending on ε. Furthermore we choose $\varepsilon = \frac{\int_{\partial\Omega} |u_1|^{2\gamma+3} d\Gamma}{2\int_{\partial\Omega} |u'|^{2\gamma+3} d\Gamma}$ small enough so that

$$\varepsilon \int_{\partial\Omega} |u'|^{2\gamma+3} d\Gamma - \int_{\partial\Omega} |u_1|^{2\gamma+3} d\Gamma \le 0,$$

then, we have

$$F'(t) \le \int_{\Omega} |u|^2 dx + \int_{\Omega} |u'|^2 dx + C_\varepsilon \left(\int_{\partial\Omega} |u|^{\gamma+2} d\Gamma \right)^{\frac{2\gamma+3}{2\gamma+4}} (meas(\Gamma_1))^{\frac{1}{2\gamma+4}}.$$

Through (9) we obtain

$$\begin{aligned}
F'(t) \le& \int_{\Omega} |u|^2 dx + 2\mathbf{k} + \frac{2}{\gamma+2} \|u\|_{\gamma+2,\Gamma_1}^{\gamma+2} \\
&+ C_\varepsilon \left(\int_{\partial\Omega} |u|^{\gamma+2} d\Gamma \right)^{\frac{2\gamma+3}{2\gamma+4}} (meas(\Gamma_1))^{\frac{1}{2\gamma+4}} \\
\le& \int_{\Omega} |u|^2 dx + 2\mathbf{k} + \frac{2}{\gamma+2} \|u\|_{\gamma+2,\Gamma_1}^{\gamma+2} \\
&+ C_\varepsilon \left(\int_{\partial\Omega} |u|^{\gamma+2} d\Gamma \right)^{\frac{2\gamma+3}{2\gamma+4}} (meas(\Gamma_1))^{\frac{1}{2\gamma+4}} + (t^* - t) \int_{\partial\Omega} |u_1|^{\frac{\gamma+2}{2}} d\Gamma \\
\le& \int_{\Omega} |u|^2 dx + \|u\|_{\gamma+2,\Gamma_1}^{\gamma+2} + (t^* - t) \int_{\partial\Omega} |u_1|^{\frac{\gamma+2}{2}} d\Gamma + 2\mathbf{k} \\
&+ C_\varepsilon (F(t))^{\frac{2\gamma+3}{2\gamma+4}} (meas(\Gamma_1))^{\frac{1}{2\gamma+4}} \\
\le& F(t) + 2\mathbf{k} + C_0 (F(t))^{\frac{2\gamma+3}{2\gamma+4}} (meas(\Gamma_1))^{\frac{1}{2\gamma+4}},
\end{aligned} \tag{10}$$

where $C_0 = \text{Max}_{t\in(0,T)}\{C_\varepsilon\}$.

From Theorem 2, it is straightforward to derive that the solution of System (1) blows up; as a consequence, we concluded that

$$\lim_{t\to T^-} F(t) = \infty$$

and by (10), we have

$$\int_{F(0)}^{\infty} \frac{dy}{y + 2\mathbf{k} + C_0 y^{\frac{2\gamma+3}{2\gamma+4}} (meas(\Gamma_1))^{\frac{1}{2\gamma+4}}} \le T.$$

Finally, we complete the proof of Theorem 3. □

Remark 2. *Theorem 3 gives an upper and lower bound on the blow-up time, but does not analyze the sharpness of the blow-up time. According to (10), we have*

$$F'(t) \le F(t) + C_a (F(t))^{\frac{2\gamma+3}{2\gamma+4}}$$

with $p = \frac{2\gamma+3}{2\gamma+4}$*. This differential inequality may be reduced to a linear differential inequality by the process of solving the Bernoulli equation. Moreover, we can obtain*

$$(F(t))^{1-p} \ge (F(0))^{1-p} + C_a e^{(1-p)t} - C_a,$$

where C_a *is a positive constant. Hence,* $F(t)$ *bounded for* $t \in [0, T)$ *with*

$$\frac{1}{1-p}(log\{1 + \frac{1}{C_a}(F(0))^{1-p}\}) \le T$$

if we let

$$u(x,0) = \varepsilon_1 u_0(x), u'(x,0) = \varepsilon_2 u_1(x),$$

where ε_1 *and* ε_2 *are small parameters. Then, we can determine the sharpness of the lower bound*

$$C_m log(1 + \varepsilon_0^{2-2p}) \le T(\varepsilon_0)$$

where $\varepsilon_0 = max\{\varepsilon_1, \varepsilon_2\}$ *and* C_m *is a positive constant independent of* ε_0*.*

In the same way, we can obtain the sharpness of the upper bound

$$T(\varepsilon_0) \le C_M \varepsilon_0^{\gamma+2-\frac{\gamma-m}{m+2}}.$$

where C_M *is a positive constant independent of* ε_0*.*

4. Conclusions

The present paper substantially expands upon T.G. Ha's findings on the blow-up solution to the wave equation with damping and source terms, which were initially introduced in 2015. The main conclusions of this article are as follows: Firstly, through a series of energy mode estimations and auxiliary function techniques, the result of both upper and lower bounds for blow-up time is obtained. Based on these estimates, we obtain the result of the local-in-time existence of the energy solution. In comparison to previous studies, the most original contribution of this paper is the construction of a new auxiliary function under nontrivial boundary conditions, which solves the problem of estimating the bounds of blow-up time. Second, this work also delves deeper into how the source term and damping term impact blow-up time, revealing their effects more comprehensively. These insights provide valuable theoretical support and reference points for real-world engineering applications. Moving forward, we will study the effect of increasing the viscoelastic term and the time delay term on the blasting of the equation solution, and then estimate the upper and lower bounds of the blow-up time. Therefore, estimating the upper and lower bounds of the blow-up time under the influence of the viscoelastic term and the time delay term within the domain will be the focus of our next research.

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