mathematics

Article

Essential Norm of t-Generalized Composition Operators from $F(p,q,s)$ to Iterated Weighted-Type Banach Space

Shams Alyusof [1] and Nacir Hmidouch [2,*]

[1] Department of Mathematics and Statistics, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11564, Saudi Arabia; syyusof@imamu.edu.sa
[2] Department of Mathematical Sciences, Talladega College, Talladega, AL 35160, USA
* Correspondence: nhmidouch@talladega.edu

Abstract: In this work, we characterize the boundedness of t-generalized composition operators from $F(p,q,s)$ spaces to iterated weighted-type Banach space. We also provide estimates of the norm and the essential norm of t-generalized composition operators from $F(p,q,s)$ spaces to iterated weighted-type Banach space. As corollaries, we obtain approximations of the essential norm of integral operators and generalized composition operators from $F(p,q,s)$ spaces to iterated weighted-type Banach space. Moreover, we conclude our work by discussing the t-generalized composition operators and the special cases of $F(p,q,s)$.

Keywords: *F(p,q,s)*; iterated weighted-type Banach space; *t*-generalized composition operators; essential norm

MSC: 30H30; 31A05

check for updates

Citation: Alyusof, S.; Hmidouch, N. Essential Norm of *t*-Generalized Composition Operators from $F(p,q,s)$ to Iterated Weighted-Type Banach Space. *Mathematics* **2024**, *12*, 1320. https://doi.org/10.3390/math12091320

Academic Editor: Luigi Rodino

Received: 16 March 2024
Revised: 15 April 2024
Accepted: 20 April 2024
Published: 26 April 2024

1. Introduction

Let $H(\mathbb{D})$ denote the set of analytic functions on the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$, and let $S(\mathbb{D})$ represent the set of analytic self-maps of $\mathbb{D}$.

For $\varphi \in S(\mathbb{D})$, the composition operator C_φ acting on $H(\mathbb{D})$ is defined as follows:

$$C_\varphi f = f \circ \varphi. \tag{1}$$

In recent years, a growing focus has emerged on examining composition operators and their actions across various spaces of analytic functions. Particularly, significant attention has been devoted to exploring the intricate connections between C_φ and the properties of φ. This area of research has been extensively investigated and discussed in works such as [1–8], along with the references cited therein.

Given $g \in H(\mathbb{D})$, the integral operator I_g is defined as

$$(I_g f)(z) = \int_0^z f'(w)g(w)dw. \tag{2}$$

Assuming that $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, a linear operator is defined as follows:

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(w))g(w)dw. \tag{3}$$

This operator is referred to as the generalized composition operator. If $\varphi(z) = z$, C_φ^g reduces to the integral operator I_g. In the case where $g = \varphi'$, it is observed that the operator C_φ^g becomes a composition operator since $C_\varphi^{\varphi'} - C_\varphi$ is constant. Thus, C_φ^g serves as a generalization of the composition operator introduced in [9].

The study of the boundedness and compactness of generalized composition operators on Bloch-type spaces and Zygmund spaces has been explored in [9]. In [10], a new characterization of the generalized composition operator on Zygmund spaces was presented. Additional insights into the generalized composition operator on various spaces can be found in related works such as [11–14].

Consider $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, and $t \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Building upon the motivation provided by (1)–(3), Kamal, Abd-Elhafeez, and Eissa [15] introduced a new operator known as the t-generalized composition operator, defined as

$$(C_\varphi^{g,t} f)(z) = \int_0^z f'(\varphi(w)) g^{(t)}(w) dw.$$

This operator is an extension of the generalized composition operator. Specifically, when $t = 0$, $C_\varphi^{g,0}$ coincides with C_φ^g. Unlike the generalized composition operator, the t-generalized composition operator accommodates varying degrees of differentiability, governed by the parameter t. This parameterization opens up new avenues for analyzing the interplay between operator properties and function space characteristics.

Let μ be a positive continuous function on $\mathbb{D}$, which we refer to as a *weight*, and $k \in \mathbb{N}_0$. In [16], Stević introduced the iterated weighted-type Banach space $\mathcal{V}_{\mu,k}$ as follows:

$$\mathcal{V}_{\mu,k} = \left\{ f \in H(\mathbb{D}) : \sup_{z \in \mathbb{D}} \mu(z) |f^{(k)}(z)| < \infty \right\},$$

with the norm

$$\|f\|_{\mathcal{V}_{\mu,k}} := \sum_{m=0}^{k-1} |f^{(m)}(0)| + \sup_{z \in \mathbb{D}} \mu(z) |f^{(k)}(z)|.$$

The little iterated weighted-type space $\mathcal{V}_{\mu,k}^0$ is the closed subspace of $\mathcal{V}_{\mu,k}$ such that

$$\lim_{|z| \to 1} \mu(z) |f^{(k)}(z)| = 0.$$

For $k = 0, 1, 2$, the space $\mathcal{V}_{\mu,k}$ is the weighted-type space H_μ^∞, the weighted Bloch-type space B_μ, and the weighted Zygmund-type space Z_μ, respectively.

Consider $\alpha > 0$ and $\mu(z) = (1 - |z|^2)^\alpha$. When $n = 1, 2$, $\mathcal{V}_{\mu,k}$ coincides with the Bloch-type space B_α and the Zygmund-type space Z_α, respectively. In particular, for $\alpha = 1$, we obtain the classical Bloch space B and the Zygmund space Z, respectively. Moreover, when $\mu(z) = 1 - |z|^2$, as proven in Theorem 1 of [17], $\mathcal{V}_{\mu,k}$ serves as the dual of the Hardy space $H^{\frac{1}{k}}$ for all $k \geq 2$. For further details on these spaces, please refer to [18,19].

The iterated weighted-type Banach spaces have a significant role in the field of approximation theory and numerical analysis. They are particularly useful for measuring the precision of different numerical methods used to approximate functions with nth-order derivatives, like finite difference and finite element methods. Additionally, these spaces can be employed to determine the rates at which various approximation schemes converge and to calculate error limits for numerical solutions of differential equations. Additionally, they have applications in machine learning, where they are used to model complex data structures and make predictions based on them. More details can be found in [20–22].

Let $p > 0$, $s \geq 0$, and $q > -2$ such that $q + s > -1$. The general family space $F(p, q, s)$ is the set of all analytic functions that satisfy

$$\|f\|_{F(p,q,s)} := |f(0)| + \left(\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(w)|^p (1 - |w|^2)^q (1 - |\alpha_a(w)|^2)^s dm(w) \right)^{1/p} < \infty,$$

where dm denotes the Lebesgue area measure such that $m(\mathbb{D}) = 1$, and

$$\alpha_a(z) = \frac{a - z}{1 - \bar{a}z}.$$

The little space $F_0(p, q, s)$ is the closed subspace of $F(p, q, s)$ such that

$$\lim_{|a| \to 1} \int_{\mathbb{D}} |f'(w)|^p (1 - |w|^2)^q (1 - |\alpha_a(w)|^2)^s dm(w) = 0.$$

These spaces were introduced by Zhao [23]. Equipped with the above norm, the general family space $F(p, q, s)$ becomes a Banach space. It is well known in [24] that there is a positve constant C such that

$$(1 - |z|^2)^{m-1+\frac{q+2}{p}} |f^{(m)}(z)| \le C\|f\|_{F(p,q,s)} \quad \forall m \in \mathbb{N},\ f \in F(p, q, s). \tag{4}$$

Previous research efforts have made significant strides in characterizing the boundedness and compactness properties of operators across a variety of function spaces, ranging from $F(p, q, s)$ to several iterated weighted-type Banach spaces. For instance, Yang, as detailed in [25], provided a characterization of the boundedness and compactness of weighted differentiation composition operators from the $F(p, q, s)$ space to B_α. Similarly, Ye, in [26], examined the boundedness and compactness of the weighted composition operator from the general family space $F(p, q, s)$ to the logarithmic Bloch space $\mathcal{B}_{\log}$. Another contribution by Yang, discussed in [24], focused on investigating the boundedness and compactness of composition operators from the general family space $F(p, q, s)$ space to $\mathcal{V}_{\mu,k}$. Zhou and Chen, in their work [27], conducted a study on the weighted composition operator from the $F(p, q, s)$ space to B_α on the unit ball. Additionally, in [28,29], Stević engaged in discussions concerning the boundedness and compactness of integral operators between $F(p, q, s)$ spaces and Bloch-type spaces within the unit ball. These investigations contribute significantly to our understanding of the behavior of various operators on different function spaces, shedding light on the intricate interplay between operator-theoretic properties and function-space characteristics.

Expanding upon this existing body of literature, our research introduces a novel operator, the t-generalized composition operator. This operator extends the concept of generalized composition operators to a new level of generality and flexibility, offering insights into previously unexplored areas of operator theory. What sets t-generalized composition operators apart is their ability to capture and manipulate higher-order derivative information, providing a richer framework for analyzing the composition of functions. By incorporating tth-order derivatives of the function g into the composition process, t-generalized composition operators offer a more nuanced understanding of how compositions interact with the underlying function spaces. This additional degree of control over the composition process enables us to explore a broader range of phenomena and derive more refined results. In particular, our study investigates the boundedness and essential norm of t-generalized composition operators as they operate from $F(p, q, s)$ spaces to iterated type spaces, providing valuable contributions to the understanding of these operators' behaviors in diverse function-space settings. Furthermore, we discuss the special cases of $F(p, q, s)$ and the operator $C_\varphi^{g,t}$.

In this work, we will consistently use the symbol C to represent a positive constant that remains independent of the variables or parameters involved, although its value may vary with each instance. The notation $A \preceq B$ indicates that there exists a positive constant c such that $cA \le B$. Furthermore, we employ the notation $A \asymp B$ to signify that there exist positive constants c_1 and c_2, with $c_1 \le c_2$, such that $c_1 A \le B \le c_2 A$.

2. Boundedness

The main goal of this section is to characterize the boundedness of t-generalized compostion operators from $F(p,q,s)$ spaces to iterated weighted-type Banach spaces.

Lemma 1 (Lemma 4, [16]). *Given $f, g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, for $n \in \mathbb{N}$ and $z \in \mathbb{D}$,*

$$(g(f\circ\varphi))^{(n)}(z) = \sum_{\ell=0}^{n} f^{(\ell)}(\varphi(z)) \sum_{j=\ell}^{n} \binom{n}{j} g^{(n-j)}(z) A_{j,\ell}(\varphi'(z),\ldots,\varphi^{(j-\ell+1)}(z)),$$

where

$$A_{j,\ell}(\varphi'(z),\ldots,\varphi^{(j-\ell+1)}(z)) := \sum_{\ell_1,\ell_2,\ldots,\ell_j} \frac{j!}{\ell_1!\ell_2!\cdots\ell_j!} \prod_{m=1}^{j} \left(\frac{\varphi^{(m)}(z)}{m!}\right)^{\ell_m},$$

and the sum is taken over all nonnegative integers $\ell_1,\ldots,\ell_j$ such that $\ell = \ell_1+\cdots+\ell_j$, and $\ell_1+2\ell_2+\cdots+j\ell_j = j$.

Then, for the t-generalized compostion operator case, we have

$$\begin{aligned}
&((C_\varphi^{g,t})f)^{(n)}(z)\\
&= (g^{(t)}(f'\circ\varphi))^{(n-1)}(z)\\
&= \sum_{\ell=1}^{n} f^{(\ell)}(\varphi(z)) \sum_{j=\ell-1}^{n-1} \binom{n-1}{j} g^{(t+n-1-j)}(z) A_{j,\ell-1}(\varphi'(z),\ldots,\varphi^{(j-\ell+2)}(z)).
\end{aligned} \tag{5}$$

We set $k \in \mathbb{N}$ and $t \in \mathbb{N}_0$, as well as functions g and φ. For $z \in \mathbb{D}$, $\ell \in \{1,\ldots,k\}$, we define

$$\mathbf{N}_\ell{}^t(z) := \left|\sum_{j=\ell-1}^{k-1} \binom{k-1}{j} g^{(t+k-1-j)}(z) A_{j,\ell-1}(\varphi'(z),\ldots,\varphi^{(j-\ell+2)}(z))\right|.$$

Theorem 1. *We set $k \in \mathbb{N}$ and $t \in \mathbb{N}_0$ and let $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then, the following statements are equivalent.*

(a) $C_\varphi^{g,t} : F(p,q,s) \to \mathcal{V}_{\mu,k}$ *is bounded.*

(b) $M{:=}\sup_{z\in\mathbb{D}} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+\ell-1}} < \infty.$

Moreover, if $C_\varphi^{g,t}$ is bounded, then

$$\|C_\varphi^{g,t}\| \asymp \sup_{z\in\mathbb{D}} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^{\ell-1+\frac{q+2}{p}}}.$$

Proof. (b) $\Longrightarrow$ (a) Let $f \in F(p,q,s)$ such that $\|f\|_{F(p,q,s)} \le 1$ and $z \in \mathbb{D}$. By (4) and (5), we have

$$\begin{aligned}
&\mu(z)|(C_\varphi^{g,t}f)^{(k)}(z)|\\
&\le \mu(z)\sum_{\ell=1}^{k} |f^{(\ell)}(\varphi(z))| \left|\sum_{j=\ell-1}^{k-1} \binom{k-1}{j} g^{(t+k-1-j)}(z) A_{j,\ell-1}(\varphi'(z),\ldots,\varphi^{(j-\ell+2)}(z))\right| \\
&\preceq \mu(z)\sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^{\ell-1+\frac{q+2}{p}}}.
\end{aligned} \tag{6}$$

Taking the supremum over all z in $\mathbb{D}$, we obtain

$$\sup_{z\in\mathbb{D}} \mu(z)|(C_\varphi^{g,t} f)^{(k)}(z)| \preceq \sup_{z\in\mathbb{D}} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^{\ell-1+\frac{q+2}{p}}}. \tag{7}$$

Noting $(C_\varphi^{g,t} f)(0) = 0$ and again by (4), for each $m \in \{1,\dots,k-1\}$, we have

$$\begin{aligned}
&|(C_\varphi^{g,t} f)^{(m)}(0)| \\
&\leq \sum_{\ell=1}^{m} |f^{(\ell)}(\varphi(0))| \left| \sum_{j=\ell-1}^{m-1} \binom{m-1}{j} g^{(t+m-1-j)}(0) A_{j,\ell-1}(\varphi'(0),\dots,\varphi^{(j-\ell+2)}(0)) \right| \\
&\preceq \sum_{\ell=1}^{m} \frac{\mathbf{N}_\ell{}^t(0)}{(1-|\varphi(0)|^2)^{\ell-1+\frac{q+2}{p}}} \\
&\preceq \sup_{z\in\mathbb{D}} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^{\ell-1+\frac{q+2}{p}}}.
\end{aligned} \tag{8}$$

Combining (7) and (8), we obtain

$$\|C_\varphi^{g,t} f\|_{\mathcal{V}_{\mu,k}} \preceq \sup_{z\in\mathbb{D}} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^{\ell-1+\frac{q+2}{p}}}$$

which proves that $C_\varphi^{g,t}$ is bounded.

By taking the supremum over all f in the unit ball of $F(p,q,s)$, we obtain the upper estimate.

(a) $\Longrightarrow$ (b) Let $k \in \mathbb{N}$ and $w, a \in \mathbb{D}$. By [30] and Lemma 3 in [16], for each $l \in \{0,\dots,k\}$, there exist unique real numbers $c_0,\dots,c_k$ such that

$$f_a(z) := \sum_{j=0}^{k} \frac{c_j(1-|a|^2)^{j+1}}{(1-\bar{a}z)^{j+\frac{q+2}{p}}}, \quad z \in \mathbb{D}, \tag{9}$$

which satisfies the conditions

$$\begin{aligned}
f_a^{(l)}(a) &= \frac{\bar{a}^l}{(1-|a|^2)^{l-1+\frac{q+2}{p}}} \sum_{j=0}^{k} c_j \prod_{r=0}^{l-1} (j+r+\frac{q+2}{p}) = \frac{\bar{a}^l}{(1-|a|^2)^{l-1+\frac{q+2}{p}}}, \\
f_a^{(t)}(a) &= 0, \quad \text{for } t \in \{0,\dots,k\} \setminus \{l\}.
\end{aligned}$$

Moreover, $L := \sup_{a\in\mathbb{D}} \|f_a\|_{F(p,q,s)} < \infty$.

Since $C_\varphi^{g,t}$ is bounded, then by (5), we obtain

$$\begin{aligned}
&\mu(w) \left| \sum_{\ell=1}^{k} f^{(\ell)}(\varphi(w)) \sum_{j=\ell-1}^{k-1} \binom{k-1}{j} g^{(t+k-1-j)}(w) A_{j,\ell-1}(\varphi'(w),\dots,\varphi^{(j-\ell+2)}(w)) \right| \\
&= \mu(w)|(C_\varphi^{g,t} f_{\varphi(w)})^{(k)}(w)| \\
&\leq L\|C_\varphi^{g,t}\|
\end{aligned} \tag{10}$$

where for fixed $\ell = 0,\dots,k$ and $m = 0,\dots,k$,

$$|f_{\varphi(w)}^{(m)}(\varphi(w))| = \begin{cases} \frac{|\varphi(w)|^\ell}{(1-|\varphi(w)|^2)^{\ell-1+\frac{q+2}{p}}} & \text{for } m = \ell \\ 0 & \text{for } m \neq \ell. \end{cases} \tag{11}$$

Hence, by (10), we obtain

$$\frac{\mu(w)|\varphi(w)|^{\ell}\mathbf{N}_{\ell}{}^{t}(w)}{(1-|\varphi(w)|^2)^{\ell-1+\frac{q+2}{p}}} \leq L\|C_{\varphi}^{g,t}\|.$$

Therefore, if $|\varphi(w)| > 1/2$, then

$$\frac{\mu(w)\mathbf{N}_{\ell}{}^{t}(w)}{(1-|\varphi(w)|^2)^{\ell-1+\frac{q+2}{p}}} \leq \frac{L}{|\varphi(w)|^{\ell}}\|C_{\varphi}^{g,t}\| \leq 2^{\ell}L\|C_{\varphi}^{g,t}\|. \tag{12}$$

On the other hand, when $|\varphi(w)| \leq 1/2$, it follows that for each $\ell \in \{1,\ldots,k\}$, we have

$$\frac{\mathbf{N}_{\ell}{}^{t}(w)}{(1-|\varphi(w)|^2)^{\ell-1+\frac{q+2}{p}}} \leq \left(\frac{4}{3}\right)^{\ell-1+\frac{q+2}{p}}\mathbf{N}_{\ell}{}^{t}(w). \tag{13}$$

Combining (12) and (13), it follows that to prove that

$$\frac{\mu(w)\mathbf{N}_{\ell}{}^{t}(w)}{(1-|\varphi(w)|^2)^{\ell-1+\frac{q+2}{p}}} \leq C\|C_{\varphi}^{g,t}\|, \tag{14}$$

and it suffices to show that

$$\mu(w)\mathbf{N}_{\ell}{}^{t}(w) \leq C\|C_{\varphi}^{g,t}\|. \tag{15}$$

For a non-negative integer n, let $p_n(z) = z^n$. By Proposition 2.13 in [23], $p_n \in F(p,q,s)$. Moreover, for all $n \in \{0,\ldots,k\}$, $\|p_n\|_{F(p,q,s)}$ is bounded by a constant C.

We establish (15) using an induction proof on $\ell \in \{1,\ldots,k\}$. For $\ell = 1$, we have

$$\begin{aligned}
&\mu(w)|((C_{\varphi}^{g,t})p_1^{(\ell)}\varphi(w))^{(k)}(w)| \\
&= \mu(w)\left|\sum_{\ell=1}^{k} p_1^{(\ell)}(\varphi(w))\sum_{j=\ell-1}^{k-1}\binom{k-1}{j}g^{(t+k-1-j)}(w)A_{j,\ell-1}(\varphi'(w),\ldots,\varphi^{(j-\ell+2)}(w))\right| \\
&= \mu(w)\mathbf{N}_1^t(w) \\
&\leq C\|C_{\varphi}^{g,t}\|.
\end{aligned}$$

Therefore, we have

$$\mu(w)\mathbf{N}_1^t(w) \leq C\|C_{\varphi}^{g,t}\|.$$

Assume that for $n \in \{1,\ldots,\ell-1\}$, we have

$$\mu(w)\mathbf{N}_n^t(w) \leq C\|C_{\varphi}^{g,t}\|.$$

Observe that

$$p_{\ell}^{(j)}(z) = \begin{cases} \ell\cdots(\ell-j+1)z^{\ell-j} & \text{for } j = 0,\cdots\ell \\ 0 & \text{for } j = \ell+1,\cdots,k. \end{cases}$$

Therefore, we have

$$
\begin{aligned}
&C\|C_\varphi^{g,t}\| \\
&\geq \mu(w)|(C_\varphi^{g,t} p_\ell)^{(k)}(w)| \\
&= \mu(w)\left|\sum_{n=1}^{k} p_\ell^{(n)}(\varphi(w)) \sum_{j=n-1}^{k-1} \binom{k-1}{j} g^{(t+k-1-j)}(z) A_{j,n-1}(\varphi'(w),\dots,\varphi^{(j-n+2)}(w))\right| \\
&= \mu(w)\left|\sum_{n=1}^{\ell} p_\ell^{(n)}(\varphi(w)) \sum_{j=n-1}^{k-1} \binom{k-1}{j} g^{(t+k-1-j)}(z) A_{j,n-1}(\varphi'(w),\dots,\varphi^{(j-n+2)}(w))\right. \\
&\quad \left. + \sum_{n=\ell+1}^{k} p_\ell^{(n)}(\varphi(w)) \sum_{j=n-1}^{k-1} \binom{k-1}{j} g^{(t+k-1-j)}(z) A_{j,n-1}(\varphi'(w),\dots,\varphi^{(j-n+2)}(w))\right| \\
&= \mu(w)\left|\sum_{n=1}^{\ell-1} \ell\cdots(\ell-n+1)(\varphi(w))^{\ell-n}\right. \\
&\qquad \times \sum_{j=n-1}^{k-1} \binom{k-1}{j} g^{(t+k-1-j)}(z) A_{j,n-1}(\varphi'(w),\dots,\varphi^{(j-n+2)}(w)) \\
&\quad \left. + \ell! \sum_{j=\ell-1}^{k-1} \binom{k-1}{j} g^{(t+k-1-j)}(z) A_{j,n-1}(\varphi'(w),\dots,\varphi^{(j-\ell+2)}(w))\right|.
\end{aligned}
$$

Therefore, we have

$$\ell!\mu(w)\mathbf{N}_\ell{}^t \leq C\|C_\varphi^{g,t}\| + \mu(w)\sum_{n=1}^{\ell-1} \ell\cdots(\ell-n+1)\mathbf{N}_n \preceq \|C_\varphi^{g,t}\|.$$

By (12) and (14), for each $w \in \mathbb{D}$, we obtain

$$\frac{\mu(w)\mathbf{N}_\ell{}^t(w)}{(1-|\varphi(w)|^2)^{\ell-1+\frac{q+2}{p}}} \preceq \|C_\varphi^{g,t}\|. \tag{16}$$

By summing over all $\ell \in \{1,\dots,k\}$ and taking the supremum over all w in $\mathbb{D}$, we obtain

$$\sup_{w\in\mathbb{D}} \mu(w) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(w)}{(1-|\varphi(w)|^2)^{\ell-1+\frac{q+2}{p}}} \preceq \|C_\varphi^{g,t}\|,$$

which completes our proof. □

Focusing on the component operators C_φ^g and I_g, we derive the following two results.

Corollary 1. *Let $k \in \mathbb{N}$, $g \in H(\mathbb{D})$, and $\varphi \in S(\mathbb{D})$. Then, the following statements are equivalent.*
(a) $C_\varphi^g : F(p,q,s) \to \mathcal{V}_{\mu,k}$ *is bounded.*
(b) $\sup_{z\in\mathbb{D}} \mu(z) \sum_{\ell=1}^{k} \frac{|\sum_{j=\ell-1}^{k-1} \binom{k-1}{j} g^{(k-1-j)}(z) A_{j,\ell-1}(\varphi'(z),\dots,\varphi^{(j-\ell+2)}(z))|}{(1-|\varphi(z)|^2)^{\ell-1+\frac{q+2}{p}}} < \infty.$
Moreover, if C_φ^g is bounded, then

$$\|C_\varphi^g\| \asymp \sup_{z\in\mathbb{D}} \mu(z) \sum_{\ell=1}^{k} \frac{|\sum_{j=\ell-1}^{k-1} \binom{k-1}{j} g^{(k-1-j)}(z) A_{j,\ell-1}(\varphi'(z),\dots,\varphi^{(j-\ell+2)}(z))|}{(1-|\varphi(z)|^2)^{\ell-1+\frac{q+2}{p}}}.$$

Corollary 2. *Let $k \in \mathbb{N}$, and let $g \in H(\mathbb{D})$. Then, the following statements are equivalent.*

(a) $I_g : F(p,q,s) \to \mathcal{V}_{\mu,k}$ *is bounded.*

(b) $\sup\limits_{z\in\mathbb{D}} \mu(z) \sum_{\ell=1}^{k} \frac{|g^{(k-\ell-2)}(z)|}{(1-|\varphi(z)|^2)^{\ell-1+\frac{q+2}{p}}} < \infty$.

Moreover, if I_g is bounded, then

$$\|I_g\| \asymp \sup_{z\in\mathbb{D}} \mu(z) \sum_{\ell=1}^{k} \frac{|g^{(k-\ell)}(z)|}{(1-|z|^2)^{\ell-1+\frac{q+2}{p}}}.$$

3. Essential Norm

The result presented in [7] is crucial for characterizing the compactness of the operators under investigation in this study.

Lemma 2 ([7], Lemma 3.7)**.** *Let X, Y be Banach spaces of analytic functions on $\mathbb{D}$, and let $T : X \to Y$ be a bounded linear operator. Suppose the following:*

(i) *The point evaluation functionals on X are continuous;*

(ii) *The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets;*

(iii) *T is continuous when X and Y are given the topology of uniform convergence on compact sets.*

Then, T is a compact operator if and only if for any bounded sequence $\{f_n\}$ in X such that f_n converges uniformly to zero on compact sets, the sequence $\{Tf_n\}$ converges to zero in the norm of Y.

Recall that the essential norm of a bounded linear operator $W : X \to Y$, where X and Y are Banach spaces, is given by

$$\|W\|_e := \inf\big\{\|W - T\| \,/\, T : X \to Y \text{ compact}\big\}.$$

Therefore, a bounded linear operator W is compact if and only $\|W\|_e = 0$.

The following lemma will be used to prove the main result of this section, and the proof is similar to the one in Lemma 3.1 in [31].

Lemma 3. *Let $k \in \mathbb{N}$, and let $0 \le r < 1$. For $f \in F(p,q,s)$, the dilation function W_r in $F(p,q,s)$ is defined by $W_r f(z) := f(rz)$ for all $z \in \mathbb{D}$. Then, W_r is compact on $F(p,q,s)$ and*

$$\tau := \sup_{0\le r<1} \|W_r\| < \infty. \tag{17}$$

Moreover, for $\varepsilon > 0$ and $a \in (0,1)$, there exists $r \in (0,1)$ such that

$$\sup_{\|f\|_{F(p,q,s)}=1} \sup_{|z|\le a} \left|((I - W_r)f))^{(j)}(z)\right| < \varepsilon, \quad \textit{for all } j = 1,\ldots,k. \tag{18}$$

Now, we are ready to state the main result of this section.

Theorem 2. *Let $k \in \mathbb{N}$, $t \in \mathbb{N}_0$, $g \in H(\mathbb{D})$, and $\varphi \in S(\mathbb{D})$. If $C_\varphi^{g,t} : F(p,q,s) \to \mathcal{V}_{\mu,k}$ is bounded, then*

$$\|C_\varphi^{g,t}\|_e \asymp \lim_{a\to1} \sup_{|\varphi(z)|>a} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^{\ell-1+\frac{q+2}{p}}}.$$

Proof. To prove the upper estimate, let $a \in (0,1)$, $\varepsilon > 0$, and $0 \le r < 1$. $C_\varphi^{g,t} W_r$ is compact, since W_r is compact and $C_\varphi^{g,t}$ is bounded. Then, by (4), (6), (17), and (18), we have the following:

$$
\begin{aligned}
\|C_\varphi^{g,t}\|_e &\le \|C_\varphi^{g,t} - C_\varphi^{g,t} W_r\| \\
&= \sup_{\|f\|_{F(p,q,s)}=1} \|(C_\varphi^{g,t}(I - W_r))f\|_{\mathcal{V}_{\mu,k}} \\
&= \sup_{\|f\|_{F(p,q,s)}=1} \Big(\sum_{j=1}^{k-1} \big|(C_\varphi^{g,t}(I - W_r)f)^{(j)}(0)\big| + \sup_{z\in\mathbb{D}} \mu(z)\big|(C_\varphi^{g,t}(I - W_r)f)^{(k)}(z)\big| \Big) \\
&\le (k-1)\varepsilon + \sup_{\|f\|_{F(p,q,s)}=1} \sup_{|\varphi(z)|\le a} \mu(z)\big|(C_\varphi^{g,t}(I - W_r)f)^{(k)}(z)\big| \\
&\quad + \sup_{\|f\|_{F(p,q,s)}=1} \sup_{a<|\varphi(z)|<1} \mu(z)\big|(C_\varphi^{g,t}(I - W_r)f)^{(k)}(z)\big| \\
&\le (k-1)\varepsilon + \sup_{\|f\|_{F(p,q,s)}=1} \sup_{|\varphi(z)|\le a} \mu(z) \sum_{\ell=1}^{k} |((I - W_r)f)^{(\ell)}(\varphi(z))| \mathbf{N}_\ell{}^t(z) \\
&\quad + \sup_{\|f\|_{F(p,q,s)}=1} \sup_{a<|\varphi(z)|<1} \mu(z) \sum_{\ell=1}^{k} |((I - W_r)f)^{(\ell)}(\varphi(z))| \mathbf{N}_\ell{}^t(z). \\
&\le (k-1)\varepsilon + \varepsilon \sup_{|\varphi(z)|\le a} \mu(z) \sum_{\ell=1}^{k} \mathbf{N}_\ell{}^t(z) \\
&\qquad + C \sup_{\|f\|_{F(p,q,s)}=1} \sup_{a<|\varphi(z)|<1} \mu(z) \Big[\|f\|_{F(p,q,s)} + \|W_r f\|_{F(p,q,s)}\Big] \\
&\qquad\qquad \times \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1 - |\varphi(z)|^2)^{\ell-1+\frac{q+2}{p}}} \\
&\le (k - 1 + M)\varepsilon + C(1+\tau) \sup_{a<|\varphi(z)|<1} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1 - |\varphi(z)|^2)^{\ell-1+\frac{q+2}{p}}}.
\end{aligned}
$$

For sufficiently small ε, we obtain

$$
\|C_\varphi^{g,t}\|_e \preceq \lim_{a\to 1} \sup_{|\varphi(z)|>a} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1 - |\varphi(z)|^2)^{\ell-1+\frac{q+2}{p}}}.
$$

To prove the lower estimate, let $\{w_n\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(w_n)| \to 1$ and let $\ell \in \{1, \ldots, k\}$. Then, the sequence $f_n := f_{\varphi(w_n)}$ defined in the proof of Theorem 1 converges to 0 uniformly on compact subsets. Moreover, $G := \sup_{n\in\mathbb{N}} \|f_n\|_{F(p,q,s)} < \infty$.

Let $W : F(p,q,s) \to \mathcal{V}_{\mu,k}$ be a compact operator. Then, by Lemma 2, $\lim_{n\to\infty} \|W f_n\|_{\mathcal{V}_{\mu,k}} = 0$. Hence, by (5) and (11), we have

$$
\begin{aligned}
G\|C_\varphi^{g,t} - W\| &\ge \limsup_{n\to\infty} \|(C_\varphi^{g,t} - W) f_n\|_{\mathcal{V}_{\mu,k}} \\
&\ge \limsup_{n\to\infty} \mu(w_n) |(C_\varphi^{g,t} f_n)^{(k)}(w_n)| \\
&= \limsup_{n\to\infty} \frac{\mu(w_n)\mathbf{N}_\ell{}^t(w_n)}{(1 - |\varphi(w_n)|^2)^{\ell-1+\frac{q+2}{p}}}.
\end{aligned}
$$

Summing over all $\ell \in \{1, \ldots, k\}$ and taking the infimum over all compact operators $W : F(p,q,s) \to \mathcal{V}_{\mu,k}$, we obtain

$$\limsup_{n\to\infty} \mu(w_n) \sum_{\ell=1}^{k} \frac{\mathbf{N}_{\ell}{}^{t}(w_n)}{(1-|\varphi(w_n)|^2)^{\ell-1+\frac{q+2}{p}}} \preceq \|C_{\varphi}^{g,t}\|_e.$$

□

Focusing on the component operators C_{φ}^{g} and I_g, we derive the following results.

Corollary 3. *Let $k \in \mathbb{N}$, $g \in H(\mathbb{D})$, and $\varphi \in S(\mathbb{D})$. If $C_{\varphi}^{g} : F(p,q,s) \to \mathcal{V}_{\mu,k}$ is bounded, then*

$$\|C_{\varphi}^{g}\|_e \asymp \lim_{a\to 1} \sup_{|\varphi(z)|>a} \mu(z) \sum_{\ell=1}^{k} \frac{\left| \sum_{j=\ell-1}^{k-1} \binom{k-1}{j} g^{(k-1-j)}(z) A_{j,\ell-1}(\varphi'(z), \ldots, \varphi^{(j-\ell+2)}(z)) \right|}{(1-|\varphi(z)|^2)^{\ell-1+\frac{q+2}{p}}}.$$

Corollary 4. *Let $k \in \mathbb{N}$ and $\varphi \in S(\mathbb{D})$. If $I_g : F(p,q,s) \to \mathcal{V}_{\mu,k}$ is bounded, then*

$$\|I_g\|_e \asymp \lim_{a\to 1} \sup_{|z|>a} \mu(z) \sum_{\ell=1}^{k} \frac{|g^{(k-\ell)}(z)|}{(1-|z|^2)^{\ell-1+\frac{q+2}{p}}}.$$

4. The Special Cases of the Space of $F(p,q,s)$ and the Operators $C_{\varphi}^{g,t}$

We conclude this paper by exploring several special cases of $F(p,q,s)$ and $C_{\varphi}^{g,t}$. To accomplish this, we begin by stating some fundamental definitions.

The space BMOA of analytic functions of bounded mean oscillation, defined as the space of analytic functions on unit disk such that

$$\|f\|_* = \sup_{a\in\mathbb{D}} \|f \circ \alpha_a - f(a)\|_{H^2},$$

where H^2 is the Hilbert Hardy space. With the norm

$$\|f\|_{BMOA} := |f(0)| + \|f\|_*,$$

BMOA is a Banach space.

For $q > -1$, the weighted Dirichlet D_q is the collection of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{D}$ such that

$$\sum_{n=0}^{\infty} n^{1-q} |a_n|^2 < \infty.$$

For $p \geq 1$, the Bergman space L_a^p is defined as the space of all functions $f \in H(\mathbb{D})$ such that

$$\int_{\mathbb{D}} |f(z)|^p \, dA(z) < \infty.$$

L_a^p is a Banach space with the norm

$$\|f\|_{L_a^p} := \left(\int_{\mathbb{D}} |f(z)|^p \, dA(z) \right)^{1/p} < \infty.$$

For $p > 1$, an analytic function f on $\mathbb{D}$ belongs to Besov space B^p if

$$\|f\|_{B^p} = |f(0)| + \left(\int_{\mathbb{D}} |f'(z)|^2 \left(1 - |z|^2\right)^{p-2} dA(z)\right)^{\frac{1}{p}} < \infty.$$

In [23], Zhao proved that the above spaces coincide with $F(p,q,s)$ as follows:

- $F(p,q,s) = B_{\frac{q+2}{p}}$ for $s > 1$;
- $F(p,p-2,s) = B$ for $s > 1$;
- $F(2,1,0) = H^2$;
- $F(2,0,1) = BMOA$;
- $F(p,p,0) = L_a^p$ for $p \geq 1$;
- $F(p,p-2,0) = B^p$ for $p > 1$;
- $F(2,q,0) = D_q$ for $q > -1$.

Therefore, using Theroems 1 and 2, we deduce the following:

Corollary 5. *Let $k \in \mathbb{N}$, $t \in \mathbb{N}_0$, $p > 1$, $g \in H(\mathbb{D})$, and $\varphi \in S(\mathbb{D})$. Then, the following statements are equivalent.*

(a) $C_\varphi^{g,t} : B^p \to \mathcal{V}_{\mu,k}$ *is bounded.*

(b) $C_\varphi^{g,t} : BMOA \to \mathcal{V}_{\mu,k}$ *is bounded.*

(c) $C_\varphi^{g,t} : B \to \mathcal{V}_{\mu,k}$ *is bounded.*

(d) $\sup_{z\in\mathbb{D}} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^\ell} < \infty.$

Moreover, if $C_\varphi^{g,t}$ is bounded, then

$$\|C_\varphi^{g,t}\| \asymp \sup_{z\in\mathbb{D}} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^\ell},$$

$$\|C_\varphi^{g,t}\|_e \asymp \lim_{a\to 1} \sup_{|\varphi(z)|>a} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^\ell}.$$

Corollary 6. *Let $k \in \mathbb{N}$, $t \in \mathbb{N}_0$, $p > 0$, and $q > -2$. Let $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then, the following statements are equivalent.*

(a) $C_\varphi^{g,t} : B_{\frac{q+2}{p}} \to \mathcal{V}_{\mu,k}$ *is bounded.*

(b) $\sup_{z\in\mathbb{D}} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^{\ell-1+\frac{q+2}{p}}} < \infty.$

Moreover, if $C_\varphi^{g,t}$ is bounded, then

$$\|C_\varphi^{g,t}\| \asymp \sup_{z\in\mathbb{D}} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^{\ell-1+\frac{q+2}{p}}},$$

$$\|C_\varphi^{g,t}\|_e \asymp \lim_{a\to 1} \sup_{|\varphi(z)|>a} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^{\ell-1+\frac{q+2}{p}}}.$$

Corollary 7. *Let $k \in \mathbb{N}$, $t \in \mathbb{N}_0$, $g \in H(\mathbb{D})$, and $\varphi \in S(\mathbb{D})$. Then, the following statements are equivalent.*

(a) $C_\varphi^{g,t} : H^2 \to \mathcal{V}_{\mu,k}$ *is bounded.*

(b) $\sup_{z\in\mathbb{D}} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^{\ell+\frac{1}{2}}} < \infty.$

Moreover, if $C_\varphi^{g,t}$ is bounded, then

$$\|C_\varphi^{g,t}\| \asymp \sup_{z\in\mathbb{D}} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^{\ell+\frac{1}{2}}},$$
$$\|C_\varphi^{g,t}\|_e \asymp \lim_{a\to 1} \sup_{|\varphi(z)|>a} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^{\ell+\frac{1}{2}}}.$$

Corollary 8. *Let $k \in \mathbb{N}$, $t \in \mathbb{N}_0$, and $q > -1$. Let $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then, the following statements are equivalent.*

(a) $C_\varphi^{g,t} : D_q \to \mathcal{V}_{\mu,k}$ *is bounded.*

(b) $\sup_{z\in\mathbb{D}} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^{\ell+\frac{q}{2}}} < \infty.$

Moreover, if $C_\varphi^{g,t}$ is bounded, then

$$\|C_\varphi^{g,t}\| \asymp \sup_{z\in\mathbb{D}} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^{\ell+\frac{q}{2}}},$$
$$\|C_\varphi^{g,t}\|_e \asymp \lim_{a\to 1} \sup_{|\varphi(z)|>a} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^{\ell+\frac{q}{2}}}.$$

Corollary 9. *Let $k \in \mathbb{N}$, $t \in \mathbb{N}_0$, and $p \geq 1$. Let $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then, the following statements are equivalent.*

(a) $C_\varphi^{g,t} : L_a^p \to \mathcal{V}_{\mu,k}$ *is bounded.*

(b) $\sup_{z\in\mathbb{D}} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^{\ell+\frac{2}{p}}} < \infty.$

Moreover, if $C_\varphi^{g,t}$ is bounded, then

$$\|C_\varphi^{g,t}\| \asymp \sup_{z\in\mathbb{D}} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^{\ell+\frac{2}{p}}},$$
$$\|C_\varphi^{g,t}\|_e \asymp \lim_{a\to 1} \sup_{|\varphi(z)|>a} \mu(z) \sum_{\ell=1}^{k} \frac{\mathbf{N}_\ell{}^t(z)}{(1-|\varphi(z)|^2)^{\ell+\frac{2}{p}}}.$$

Author Contributions: Conceptualization, S.A. and N.H.; Writing—original draft, S.A. and N.H.; Writing—review & editing, S.A. and N.H. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-RPP2023066).

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Cowen, C., Jr.; MacCluer, B. *Composition Operators on Spaces of Analytic Functions*; CRC Press: New York, NY, USA, 1995; Volume 20.
2. El-Sayed Ahmed, A.; Bakhit, M.A. Composition Operators on Some Holomorphic Banach Function Spaces. *Math. Scand.* **2009**, *104*, 275–295. [CrossRef]
3. Rashwan, R.A.; El-Sayed Ahmed, A.; Bakhit, M.A. Composition operators on some general families of function spaces. *J. Appl. Funct. Anal.* **2015**, *10*, 164–175.
4. Shapiro, J. *Composition Operators and Classical Function Theory*; Springer: New York, NY, USA, 1993.
5. Stević, S. Composition operators from the Hardy space to the nth weighted-type space on the unit disk and the half-plane. *Appl. Math. Comput.* **2010**, *215*, 3950–3955. [CrossRef]

6. Tjani, M. Compact Composition Operators on Some Möbius Invariant Banach Spaces. Doctoral Dissertation, Michigan State University, East Lansing, MI, USA, 1996.
7. Tjani, M. Compact composition operators on Besov spaces. *Trans. Am. Math. Soc.* **2003**, *355*, 4683–4698. [CrossRef]
8. Xiaonan, L.; González, P.; Rättyä, J. Composition Operators in Hyperbolic Q-Classes. *Ann. Acad. Sci. Math. Diss.* **2006**, *31*, 391–404.
9. Li, S.; Stević, S. Generalized composition operators on Zygmund spaces and Bloch type spaces. *J. Math. Anal. Appl.* **2008**, *338*, 1282–1295. [CrossRef]
10. Lindström, M.; Sanatpour, A. Derivative-free characterization of compact generalized composition operators between Zygmund type spaces. *Bull. Austral. Math. Soc.* **2010**, *81*, 398–408. [CrossRef]
11. Li, S. On an integral-type operator from the Bloch space into the $\mathcal{Q}_K(p,q)$ space. *Filomat* **2012**, *26*, 125–133. [CrossRef]
12. Pan, C. On an integral-type operator from $\mathcal{Q}_K(p,q)$ spaces to α-Bloch space. *Filomat* **2011**, *25*, 163–173. [CrossRef]
13. Stević, S. Generalized composition operators between mixed norm space and some weighted spaces. *Numer. Funct. Anal. Opt.* **2008**, *29*, 959–978. [CrossRef]
14. Zhang, F.; Liu, Y. Generalized compositions operators from Bloch type spaces to Q_K type spaces. *J. Funct. Spaces Appl.* **2010**, *8*, 55–66. [CrossRef]
15. Kamal, A.; Abd-Elhafeez, S.A.; Eissa, M.H. The s-Generalized Composition Operators from $\mathcal{B}_g^{(m,n)}$ to $\mathcal{Q}_P$ Spaces. *Appl. Math. Inf. Sci.* **2023**, *17*, 21–26.
16. Stević, S. Weighted differentiation composition operators from H^∞ and Bloch spaces to nth weighted-type spaces on the unit disk. *Appl. Math. Comput.* **2010**, *216*, 3634–3641. [CrossRef]
17. Duren, P.L.; Romberg, B.W.; Shields, A.L. Linear functionals on H^p spaces with $0 < p < 1$. *J. Reine Angew. Math.* **1969**, *238*, 32–60.
18. Colonna, F.; Hmidouch, N. Weighted composition operators on iterated weighted-type Banach spaces of analytic functions. *Complex Anal. Oper. Theory* **2019**, *13*, 1989–2016. [CrossRef]
19. Hmidouch, N. Weighted Composition Operators Acting on Some Classes of Banach Spaces of Analytic Functions. Doctoral Dissertation, George Mason University, Fairfax, VA, USA, 2017.
20. Alexandre, E.; Jean-Luc, E.G. *Finite Elements I: Approximation and Interpolation*; Springer: Berlin/Heidelberg, Germany, 2021.
21. Allen, R.L.; Mills, D. *Signal Analysis: Time, Frequency, Scale, and Structure*; Wiley-IEEE Press: Hoboken, NJ, USA, 2004.
22. Rassias, T.M.; Gupta, V. *Mathematical Analysis Approximation Theory and Their Applications*; Springer: New York, NY, USA, 2016.
23. Zhao, R. On a general family of function spaces. *Ann. Acad. Sci. Fennicae.* **1996**, *105*, 56.
24. Yang, W. Composition operators from $F(p,q,s)$ spaces to the nth weighted-type spaces on the unit disc. *Appl. Math. Comput.* **2011**, *218*, 1443–1448. [CrossRef]
25. Yang, W. Generalized weighted composition operators from the $F(p,q,s)$ space to the Bloch-type space. *Appl. Math. Comput.* **2012**, *218*, 4967–4972. [CrossRef]
26. Ye, S. Weighted composition operators from $F(p,q,s)$ into logarithmic Bloch space. *J. Korean Math. Soc.* **2008**, *45*, 977–991. [CrossRef]
27. Zhou, Z.; Chen, R. Weighted composition operators from $F(p,q,s)$ to Bloch type spaces on the unit ball. *Int. J. Math.* **2008**, *19*, 899–926. [CrossRef]
28. Stević, S. On some integral-type operators between a general space and Bloch-type spaces. *Appl. Math. Comput.* **2011**, *218*, 2600–2618. [CrossRef]
29. Stević, S. Boundedness and compactness of an integral-type operator from Bloch-type spaces with normal weights to $F(p,q,s)$ space. *Appl. Math. Comput.* **2012**, *218*, 5414–5421. [CrossRef]
30. Li, H.; Guo, Z. Weighted composition operators from $F(p,q,s)$ spaces to nth weighted-Orlicz spaces. *J. Comput. Anal. Appl.* **2016**, *21*, 315–323.
31. Colonna, F.; Tjani, M. Operator norms and essential norms of weighted composition operators between Banach spaces of analytic functions. *J. Math. Anal. Appl.* **2016**, *434*, 93–124. [CrossRef]