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Unraveling the Complexity of Inverting the Sturm–Liouville Boundary Value Problem to Its Canonical Form

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Abstract: The Sturm–Liouville boundary value problem (SLBVP) stands as a fundamental cornerstone in the realm of mathematical analysis and physical modeling. Also known as the Sturm–Liouville problem (SLP), this paper explores the intricacies of this classical problem, particularly the relationship between its canonical and Liouville normal (Schrödinger) forms. While the conversion from the canonical to Schrödinger form using Liouville's transformation is well known in the literature, the inverse transformation seems to be neglected. Our study attempts to fill this gap by investigating the inverse of Liouville's transformation, that is, given any SLP in the Schrödinger form with some invariant function, we seek the SLP in its canonical form. By closely examining the second Paine–de Hoog–Anderson (PdHA) problem, we argue that retrieving the SLP in its canonical form can be notoriously difficult and can even be impossible to achieve in its exact form. Finding the inverse relationship between the two independent variables seems to be the main obstacle. We confirm this claim by considering four different scenarios, depending on the potential and density functions that appear in the corresponding invariant function. In the second PdHA problem, this invariant function takes a reciprocal quadratic binomial form. In some cases, the inverse Liouville transformation produces an exact expression for the SLP in its canonical form. In other situations, however, while an exact canonical form is not possible to obtain, we successfully derived the SLP in its canonical form asymptotically.

Keywords: Sturm–Liouville boundary value problem; Liouville's transformation; canonical form; Liouville normal (Schrödinger) form; invariant function; PdHA (Paine) problem; asymptotic expansion

MSC: 34B05; 34B24; 34E99; 34C41; 34B30; 34L40

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1. Introduction

The Sturm–Liouville boundary value problem (SLBVP) stands as a venerable and foundational topic in the realm of mathematical analysis and scientific inquiry. The SLBVP is often referred to as the Sturm–Liouville problem (SLP) in the literature, and we will follow this convention in this article. The SLP is a specific type of second-order linear ordinary differential equation (ODE) problem that arises in various areas of mathematics and physics, particularly in the study of partial differential equations (PDEs) and the eigenvalue problems associated with them. It is named after the mathematicians who pioneered the study, Jacques Charles François Sturm (1803–1855) and Joseph Liouville (1809–1882).

The origin of the SLP traces its history back to the first half of the 19th century, when both Sturm and Liouville published a sequence of papers on second-order linear ODEs that included BVPs between 1836 and 1837. The study of ODEs before this period was predominantly limited to searching for solutions in terms of analytic expressions. Sturm and Liouville were among the first mathematicians who recognized the limitations of such an approach and discerned the advantage in investigating the properties of solutions directly from the ODEs, even in the absence of closed forms for solutions [1–3].

Thanks to the advanced progress of computational tools, it was only during the second half of the 20th century that the Sturm–Liouville theory and its applications attracted significant attention. Even though the problem has become a classic, there are still dozens of research papers on the topic published annually. The mathematical framework of the SLP remains of paramount importance across various disciplines of mathematics, physics, and engineering. The SLP serves as a versatile and elegant tool for understanding and analyzing the behavior of second-order linear ODEs, with diverse applications in quantum mechanics, heat transfer, structural dynamics, vibrations of mechanical systems, and many more. Solving these problems provides the eigenvalues and eigenfunctions associated with these systems, with important physical and mathematical interpretations [4–14].

Unless otherwise mentioned, we will focus on the classical SLP in this article, which refers to the original and foundational formulation of the problem with specific properties and assumptions. The term *classical* distinguishes this foundational formulation from more generalized or extended versions of the SLP. The classical SLP serves as a fundamental model, and its properties and solutions have also been extensively studied and applied in various areas of mathematics and physics. Some key characteristics of the classical SLP include second-order linear ODEs, self-adjoint (Hermitian) differential operators, real-valued eigenvalues, and specific (e.g., Dirichlet, Neumann, Robin) boundary conditions. The generalizations of the SLP may encompass fractional differential equations [15–17], non-self-adjoint operators [18–24], complex-valued eigenvalues [25–27], or different (e.g., time-dependent, eigenparameter-dependent, nonlocal, etc.) boundary conditions [28–30].

Our study investigates the transformation from the Sturm–Liouville problem (SLP) in its canonical form to the Liouville normal (Schrödinger) form. This conversion, known as Liouville's transformation/reduction, involves transforming both the dependent and independent variables [4,31–33]. While the transformation itself is established, the inverse Liouville's transformation, which retrieves the SLP from its Schrödinger form, remains largely unexplored. This lack of attention might be explained by the assumption that reversing the steps of the transformation would be trivial. However, as we will see in the case of the generalized second Paine–de Hoog–Anderson (PdHA) problem, inverting the SLP to its canonical form can be notoriously difficult, even after solving the associated ODE for the invariant function.

This work fills a gap in the understanding of the SLP transformation between its two forms. We elucidate the process of conversion by considering the second PdHA problem, also known in the literature as the second Paine problem. We focus on a generalized version of the invariant function, focusing on the reciprocal quadratic power of the binomial term as originally presented in their 1981 paper [34]. Interestingly, depending on the combinations of potential and density functions, the inversion process can range from relatively straightforward to extremely difficult. As a consequence, the SLP can be retrieved to its canonical form in either case: with an exact expression of the so-called p-function for the former case or as an asymptotic approximation for the latter, where we propose a novel technique using asymptotic expansion. Therefore, we are able to obtain the SLP in its canonical form in either case, either with an exact expression or as an asymptotic approximation. This constitutes our primary contribution.

In her PhD thesis, Ledoux discussed a new class of numerical methods in detail for solving the SLP in both its canonical and Schrödinger forms. Using the constant and line perturbation methods, which fall under the category of piecewise perturbation methods, these techniques demonstrated efficiency, accuracy, and stability [35]. She briefly touched on Liouville's transformation, and in the appendix, she provided some functions associated with the second Paine problem in the canonical form, referring to [36]. However, no derivation was provided on how to obtain these functions. Therefore, we will delve deeper into this issue and provide the missing derivation at the end of Section 3. Interestingly, despite Everitt providing a catalog of nearly 60 examples of the SLP in both its canonical or Liouville normal forms, both the first and second Paine problems were conspicuously absent from the list [37].

Buterin and his collaborators established necessary and sufficient conditions for a self-adjoint SLP in the canonical form to be converted to another one in its Liouville normal form [38]. By concentrating on open questions in the inverse problem for transmission eigenvalues for the spherically symmetric variable-speed wave equation [39], the authors demonstrated constructively that eigenvalues, together with their multiplicities, do not uniquely determine the potential function in the Schrödinger form BVP or the density function in another SLP in its canonical form. In particular, by considering a special case of the canonical SLP with some particular values at one of the boundaries, the necessary and sufficient conditions for the convertibility between both forms of the SLP are related to the positive definiteness of the associated canonical Sturm–Liouville operator and a relationship with the solution of a particular initial value problem (IVP).

Transforming the SLP from its Liouville normal form to the canonical form has been studied in the context of perturbed potential temperature fields in atmospheric boundary layers. While close convergence was observed between asymptotic solutions using the WKB method and numerical simulations, the study only focused on a simplified case with constant density and zero potential [40]. Another study investigated the lowest-order eigenvalue under both Dirichlet and Neumann boundary conditions to further understand the second generalized Paine problem [41]. Using a method for estimating the lowest eigenvalue that incorporates the localized landscape and potential functions [42,43], the study found that the estimates tend to overshoot the actual values but exhibit excellent qualitative agreement for the second Paine problem with Dirichlet boundary conditions. However, discrepancies were observed for Neumann boundary conditions with specific combinations of small constants in the binomial term and higher values of the denominator.

This article is organized as follows. Section 2 discusses the transformation of the SLP from its canonical form to the Schrödinger form and its inverse. The so-called Liouville's transformation involves both the independent and dependent variables of the SLP. Section 3 focuses on the second PdHA problem, which generalizes the corresponding invariant function to include arbitrary positive constants while maintaining its reciprocal quadratic power. We consider four different scenarios, each of which is discussed in its own subsection. The first case examines a combination of a nonzero constant density function and a vanishing potential function. The second case explains the combination of a nonzero constant potential and quadratic density functions. The third case delves into the situation where both the potential and density functions are nonzero constants. The final case discusses when the transformation function depends reciprocally and linearly on the Schrödinger variable. Finally, Section 4 concludes our discussion and provides further remarks and future directions.

2. Sturm–Liouville Boundary Value Problem

In this section, we reconfigure the SLP in the canonical form to the one in the Schrödinger form using a transformation that involves both the independent and dependent variables, the so-called Liouville's transformation. We also verify that, using the inverse transformation, the SLP in the canonical form can be recovered from the SLP in the Liouville normal form.

Consider the general form of the classical Sturm–Liouville eigenvalue problem written in the *canonical form* or *self-adjoint form* with eigenvalue λ and the corresponding eigenfunction $u(x)$:

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = \lambda\, r(x)u, \qquad a < x < b. \tag{1}$$

The regular boundary conditions are imposed at the endpoints:

$$\delta_0\, u(a) - \delta_1 p(a)\frac{du}{dx}(a) = 0, \qquad \text{and} \qquad \gamma_0\, u(b) - \gamma_1 p(b)\frac{du}{dx}(b) = 0,$$

where δ_0 and δ_1 are real and not both zero, and similarly, $\gamma_0, \gamma_1 \in \mathbb{R}$, $\gamma_0^2 + \gamma_1^2 > 0$. The SLP (1) is regular if both a and b are finite and the functions p, q, and r are piecewise continuous on $[a, b]$, usually not zero, in particular, $p, r > 0$, but they may take constant values. The function $r = r(x)$ is called the *weight* or *density* function, whereas $q = q(x)$ is often referred to as the *potential* function by physicists and chemists.

The following lemma considers a special case when both p and r are nonzero constants. The proof of Lemma 1 can be found in Appendix A.

Lemma 1. *When both p and r are constants but nonzero, the ODE (1) can be expressed in the Liouville normal form using the change of variable $t = \eta x$, where $\eta = \sqrt{r/p} \neq 0$:*

$$-\frac{d^2 u}{dt^2} + Q(t)\, u = \lambda u, \qquad \alpha < t < \beta, \quad \textit{where} \quad Q(t) = \frac{pq}{w^2}. \tag{2}$$

Remark 1. *In the context of quantum mechanics, the ODE (2) represents the time-dependent Schrödinger equation, where $Q(t)$ denotes the potential function that depends on the time variable t and the eigenvalue λ corresponds to the energy level. However, in this article, we will deviate from common physics terminology and instead use the term "potential" to refer to the function $q(x)$ in (1) but not $Q(t)$ in (2). In what follows, it is important to note that, in this context, the independent variables x and t are not necessarily attached to particular dimension variables, that is, spatial and temporal, respectively. We can certainly interpret them as temporal variables if the involved ODE represents an evolution equation.*

The following lemma demonstrates that a transformation of the dependent variable u could yield a nonlinear ODE. The proof of Lemma 2 can be found in Appendix B.

Lemma 2. *For $p(x), r(x)$, and $u(x) > 0$, the ODE (1) can be transformed to either one of the following Riccati equations:*

$$\frac{dU}{dx} + \frac{U^2(x)}{p(x)} = q(x) - \lambda\, r(x),$$
$$\frac{dV}{dx} + \frac{1}{p}\frac{dp}{dx} V(x) + V^2(x) = \frac{1}{p}(q - \lambda r),$$

using the following change of variables in $u(x)$:

$$U(x) = \frac{p}{u}\frac{du}{dx} \qquad \textit{or} \qquad V(x) = \frac{U}{p},$$

respectively.

The following lemma shows what happens to the SLP (1) when we transform its independent variable. The proof of Lemma 3 can be found in Appendix C.

Lemma 3. *The eigenvalue problem (1) can be converted to another ODE in the following form by transforming the independent variable x to t, where $x = x(t)$:*

$$-\frac{d}{dt}\left(\frac{p}{\dot{x}}\frac{du}{dt}\right) + q\dot{x}\, u = \lambda\, r\dot{x}\, u, \tag{3}$$

where the dot represents the derivative with respect to t, that is, $\dot{x} = dx/dt$, and is assumed to take one sign on the open interval $\alpha < t < \beta$ that corresponds to $a < x < b$ in the original variable through a transformation. Furthermore, $u = u(x(t))$ and $u/\dot{x} = u(x(t))/\dot{x}(t)$. Other functions also follow a similar convention.

The following lemma demonstrates how eigenvalue problem (1) transforms into another ODE when we introduce a new dependent variable. The proof of Lemma 4 can be found in Appendix D.

Lemma 4. *The eigenvalue problem* (1) *can be converted to another ODE in the following form by transforming the dependent variable u of the form $u(x) = w(x)\,v(x)$, where $w(x)$ is a given function:*

$$-\frac{d}{dx}\left(pw^2\frac{dv}{dx}\right) + \left[qw^2 - w\frac{d}{dx}\left(p\frac{dw}{dx}\right)\right]v = \lambda\, rw^2\, v. \tag{4}$$

Lemma 5 combines both Lemma 3 and Lemma 4 and reveals what kind of eigenvalue problem will be obtained when both independent and dependent variables are transformed simultaneously.

Lemma 5. *By combining both the transformations for independent and dependent variables, that is, $x = x(t)$ and $u(x) = w(x)\,v(x)$, we obtain another transformed Sturm–Liouville ODE:*

$$-\frac{d}{dt}\left(P(t)\frac{dv}{dt}\right) + Q(t)\,v = \lambda\, R(t)v, \tag{5}$$

where

$$\begin{aligned}
P(t) &= \frac{pw^2}{\dot{x}}, \\
Q(t) &= \left[qw - \frac{1}{\dot{x}}\frac{d}{dt}\left(\frac{p}{\dot{x}}\frac{dw}{dt}\right)\right] w\dot{x}, \qquad \textit{and} \\
R(t) &= rw^2\dot{x}.
\end{aligned}$$

Proof. Using the results from Lemma 3 and Lemma 4, we can write ODE (1) as follows:

$$-\frac{1}{\dot{x}}\frac{d}{dt}\left(\frac{pw^2}{\dot{x}}\frac{dv}{dt}\right) + \left[qw - \frac{1}{\dot{x}}\frac{d}{dt}\left(\frac{p}{\dot{x}}\frac{dw}{dt}\right)\right] w\,v = \lambda\, rw^2\, v. \tag{6}$$

Multiplying (6) with $\dot{x}$, we obtain the desired result (5). This completes the proof. □

The following theorem describes how Liouville's transformation converts the SLP (1) in the canonical form to another SLP in the Liouville normal form. The proof of Theorem 1 can be found in Appendix E.

Theorem 1 (Liouville's transformation [44])**.** *The Sturm–Liouville problem in the* canonical form *with eigenvalue λ and the corresponding eigenfunction $u(x)$, with regular boundary conditions:*

$$\begin{aligned}
-\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)\,u = \lambda\, r(x)\,u, &\qquad a < x < b, \\
\delta_0\, u(a) - \delta_1\, p(a)\frac{du}{dx}(a) = 0, &\qquad \gamma_0\, u(b) - \gamma_1\, p(b)\frac{du}{dx}(b) = 0,
\end{aligned}$$

where δ_0 and δ_1 are real and not both zero, γ_0 and γ_1 are also similarly conditioned and can be converted into the Liouville normal (Schrödinger) form *by performing* Liouville's transformation

$$\begin{aligned}
-\frac{d^2v}{dt^2} + I(t)\,v = \lambda\, v, &\qquad \alpha < t < \beta, \\
\delta_2\, v(\alpha) - \delta_1\, P(\alpha)\frac{dv}{dt}(\alpha) = 0, &\qquad \gamma_2\, v(\beta) - \gamma_1\, P(\beta)\frac{dv}{dt}(\beta) = 0,
\end{aligned}$$

where both $P(\alpha)$ and $P(\beta)$ refer to the same function $P(t)$ introduced in Lemma 5,

$$\delta_2 = \left(\delta_0 w^2 - \delta_1\, pw\frac{dw}{dx}\right)\bigg|_{x=a}, \qquad \text{and} \qquad \gamma_2 = \left(\gamma_0 w^2 - \gamma_1\, pw\frac{dw}{dx}\right)\bigg|_{x=b}.$$

Here, I is the corresponding invariant function *of the SLP (1):*

$$I(t) = \frac{q}{r} + w\frac{d^2}{dt^2}\left(\frac{1}{w}\right), \tag{7}$$

and the Liouville transformation *is given by*

$$t = \int \sqrt{\frac{r}{p}}\, dx, \qquad w = (pr)^{-1/4}, \qquad \text{and} \qquad u(x) = w(x)\, v(x). \tag{8}$$

Moreover,

$$\int_a^b r(x)\, u^2(x)\, dx = \int_\alpha^\beta v^2(t)\, dt. \tag{9}$$

The following corollary discusses the opposite process, that is, given the SLP in the Liouville normal form, we can invert it back to the SLP in the canonical form. The proof of Corollary 1 can be found in Appendix F.

Corollary 1 (Inverse Liouville's transformation)**.** *The SLP expressed in its Schödinger form*

$$-\frac{d^2v}{dt^2} + I(t)\, v = \lambda\, v, \qquad \alpha < t < \beta, \tag{10}$$

$$\delta_2\, v(\alpha) - \widehat{\delta}_1\frac{dv}{dt}(\alpha) = 0, \qquad \gamma_2\, v(\beta) - \widehat{\gamma}_1\frac{dv}{dt}(\beta) = 0, \tag{11}$$

can be inverted back to the same eigenvalue problem in its canonical form

$$-\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)\, u = \lambda r(x)\, u, \qquad a < x < b, \tag{12}$$

$$\delta_0\, u(a) - \delta_1\, p(a)\frac{du}{dx}(a) = 0, \qquad \gamma_0\, u(b) - \gamma_1\, p(b)\frac{du}{dx}(b) = 0, \tag{13}$$

using the identical Liouville's transformation, where q, r, and w satisfy the following second-order differential equations:

$$\frac{q}{r} + w\frac{d^2}{dt^2}\left(\frac{1}{w}\right) = I(t),$$

$$p = \frac{1}{rw^4}, \qquad \text{and} \qquad x = \int \sqrt{\frac{p}{r}}\, dt,$$

and the coefficients δ_0, δ_1, γ_0, and γ_1 are given as follows:

$$\delta_0 = \left(\frac{\delta_2}{w^2} + \delta_1\frac{p}{w}\frac{dw}{dx}\right)\bigg|_{x=a}, \qquad \gamma_0 = \left(\frac{\gamma_2}{w^2} + \gamma_1\frac{p}{w}\frac{dw}{dx}\right)\bigg|_{x=b},$$

$$\delta_1 = \frac{\widehat{\delta}_1}{P(\alpha)}, \qquad \text{and} \qquad \gamma_1 = \frac{\widehat{\gamma}_1}{P(\beta)}.$$

3. Reciprocal Quadratic Invariant Function

In this section, we attempt to generalize the second Paine–de Hoog–Anderson (PdHA) problem, which is simply known in the literature as the second Paine problem [34]. It is an SLP expressed in Schrödinger form with Dirichlet boundary conditions. The corresponding generalized invariant function I is given by a reciprocal binomial term with positive integer power, given as follows:

$$I(t) = \frac{k}{(t+m)^n}, \qquad \text{where} \quad k,\ m > 0, \quad \text{and} \quad n \in \mathbb{N}.$$

In particular, we only focus on the case $n = 2$ but treat both the positive constants k and m as free parameters. A discussion on other values of $n > 2$ will be presented in a separate work. This choice still yields the generalized second Paine problem with a reciprocal quadratic invariant function I, and leads to the following SLP in the Liouville normal form with Dirichlet boundary conditions:

$$-\frac{d^2 v}{dt^2} + \frac{k}{(t+m)^2} v = \lambda\, v, \qquad v(0) = 0 = v(\pi). \tag{14}$$

In their original paper, Paine et al. took a special case of $k = 1$, $m = 0.1$, $n = 2$, $\alpha = 0$, and $\beta = \pi$ [34]. In what follows, we are interested in expressing the SLP (14) in the canonical form.

3.1. Vanishing Potential and Constant Density Functions

For this particular case, we have the following theorem.

Theorem 2. *Let the potential function q vanish and the density function take a constant value, that is, $q = 0$ and $r = r_0 \neq 0$, respectively. Then, using the following change of variables,*

$$\begin{aligned} t &= -m + [r_0(2\rho+1)(x+x_0)]^{1/(2\rho+1)}, \qquad \textit{and} \\ v(x) &= [r_0(2\rho+1)(x+x_0)]^{\rho/(2\rho+1)}\, u(x), \qquad x_0 \in \mathbb{R}, \end{aligned}$$

the canonical form of ODE (14) for $k \neq 3/4$ is given as follows:

$$-\frac{d}{dx}\left\{ \mathring{r}_0 [(2\rho+1)(x+x_0)]^{4\rho/(2\rho+1)} \frac{du}{dx} \right\} = \lambda\, r_0\, u, \qquad a < x < b, \tag{15}$$

where

$$\begin{aligned} \mathring{r}_0 &= r_0^{(2\rho-1)/(2\rho+1)}, \\ 2\rho+1 &= 2 \pm \sqrt{1+4k}, \\ \frac{2\rho-1}{2\rho+1} &= \frac{-(1+4k) \pm 2\sqrt{1+4k}}{3-4k}, \qquad \textit{and} \\ \frac{4\rho}{2\rho+1} &= \frac{2\left(1-4k \pm \sqrt{1+4k}\right)}{3-4k}. \end{aligned}$$

For $k = 3/4$, ODE (14) may take one of the following two distinct canonical forms:

$$-\frac{d}{dx}\left\{ 8\sqrt{r_0}(x+\widehat{x}_0)^{3/2} \frac{du}{dx} \right\} = \lambda\, r_0\, u, \qquad a < x < b, \tag{16}$$

for the following change of variables:

$$\begin{aligned} t &= -m \pm \sqrt[4]{4r_0(x+\widehat{x}_0)}, \qquad \textit{and} \\ v(x) &= [4r_0(x+\widehat{x}_0)]^{3/8}\, u(x), \qquad r_0 > 0, \quad \widehat{x}_0 \in \mathbb{R} \quad \textit{such that} \quad x+\widehat{x}_0 \geq 0, \end{aligned}$$

or

$$-\frac{d}{dx}\left\{\frac{1}{r_0}e^{-2r_0(x+\tilde{x}_0)}\frac{du}{dx}\right\} = \lambda\, r_0\, u, \qquad a < x < b, \tag{17}$$

for the following change of variables:

$$t = -m \pm e^{r_0(x+\tilde{x}_0)}, \qquad \text{and}$$
$$v(x) = e^{-r_0(x+\tilde{x}_0)/2}\, u(x), \qquad \tilde{x}_0 \in \mathbb{R}.$$

All ODEs (15)–(17) satisfy Dirichlet boundary conditions $u(a) = 0 = u(b)$.

Proof. For the case of vanishing potential function $q = 0$ and nonzero constant density function $r = r_0 \neq 0$, we seek a function w that satisfies the following ODE, which deduces from comparing the invariant functions (7) and (14)

$$w\frac{d^2}{dt^2}\left(\frac{1}{w}\right) = \frac{k}{(t+m)^2}. \tag{18}$$

Introducing a new dependent variable $\varpi = 1/w$, ODE (18) can be written as follows, which turns out to be a special case of the Cauchy–Euler equation:

$$(t+m)^2\frac{d^2\varpi}{dt^2} - k\varpi = 0. \tag{19}$$

Introducing a new independent variable $\tau = t + m$, and because $d\tau = dt$ and $d^2\tau/dt^2 = 0$, we observe that the transformed ODE takes a similar form to ODE (19), that is, $\tau^2\ddot{\varpi} - k\varpi = 0$, where double dots represent the second derivative with respect to τ. Seeking an ansatz in the form $\varpi(\tau) = \tau^{\rho}$, where $\tau = t + m$, we obtain $\rho^2 - \rho - k = 0$ as the indicial equation, which is solved as

$$\rho = \rho_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{1+4k}\right). \tag{20}$$

Observe that because $k > 0 > -1/4$, the indicial roots ρ are always real valued. The linearly independent solutions of ODE (18) are thus given by

$$w(\tau) = w_{1,2}(\tau) = \tau^{-\rho_{1,2}}. \tag{21}$$

From the Liouville transformation (8), we can express the function p and dx/dt as follows:

$$p(\tau) = \frac{1}{r_0\, w^4(\tau)} = \frac{\varpi^4(\tau)}{r_0} = \frac{\tau^{4\rho}}{r_0}, \tag{22}$$

and

$$\frac{dx}{d\tau} = \sqrt{\frac{p}{r_0}} = \frac{1}{r_0\, w^2(\tau)} = \frac{\varpi^2(\tau)}{r_0} = \frac{\tau^{2\rho}}{r_0}, \tag{23}$$

because $dx/dt = dx/d\tau$. Integrating (23) with respect to τ, we obtain a relationship between the two independent variables x and t:

$$x + x_0 = \frac{\tau^{2\rho+1}}{r_0(2\rho+1)}, \qquad x_0 \in \mathbb{R}. \tag{24}$$

The Schrödinger variable t, or τ, can be expressed explicitly in terms of the canonical variable x using (24):

$$t = -m + \left[r_0(2\rho+1)(x+x_0)\right]^{1/(2\rho+1)}.$$

Using indicial roots ρ in (20), that is, $2\rho+1=2\pm\sqrt{1+4k}$, the function p in (22) can also be expressed in terms of the canonical variable x:

$$\begin{aligned} p(x) &= \frac{1}{r_0}[r_0(2\rho+1)(x+x_0)]^{4\rho/(2\rho+1)}, \\ &= \frac{1}{r_0}[r_0(2\rho+1)(x+x_0)]^{2\left(1-4k\pm\sqrt{1+4k}\right)/(3-4k)}, \qquad k\neq\frac{3}{4}. \end{aligned}$$

Observe that the constant r_0 can be pulled out to form $\mathring{r}_0$:

$$\begin{aligned} \mathring{r}_0 &= r_0^{4\rho/(2\rho+1)-1} = r_0^{(2\rho-1)/(2\rho+1)}, \\ &= r_0^{\left\{-(1+4k)\pm2\sqrt{1+4k}\right\}/(3-4k)}, \qquad k\neq\frac{3}{4}. \end{aligned}$$

To find the left and right endpoints of the variable x, we use (24), in which they correspond to $\alpha=0$ and $\beta=\pi$, respectively. Thus,

$$\begin{aligned} a &= -x_0+\frac{m^{2\rho+1}}{r_0(2\rho+1)}, \qquad \text{and} \\ b &= -x_0+\frac{(\pi+m)^{2\rho+1}}{r_0(2\rho+1)}. \end{aligned}$$

By expressing the function w (and $\overline{\omega}$) in terms of the canonical variable x, we can investigate the boundary conditions. Using (21) and (24), we have the following expression for w:

$$w(x) = [r_0(2\rho+1)(x+x_0)]^{-\rho/(2\rho+1)},$$

which gives rise to the relationship between u and v:

$$v(x) = [r_0(2\rho+1)(x+x_0)]^{\rho/(2\rho+1)}\,u(x).$$

We also know that $\delta_1=0=\gamma_1$, whereas $\delta_2=1=\gamma_2$. Hence, we can calculate the values of δ_0 and γ_0 by evaluating $w^2(x)$ at $x=a$ and $x=b$, respectively, which gives the following values:

$$\begin{aligned} \delta_0 &= \frac{1}{w^2(a)} = \overline{\omega}^2(a) = m^{2\rho} = m^{1\pm\sqrt{1+4k}}, \qquad \text{and} \\ \gamma_0 &= \frac{1}{w^2(b)} = \overline{\omega}^2(b) = (\pi+m)^{2\rho} = (\pi+m)^{1\pm\sqrt{1+4k}}. \end{aligned}$$

However, because $\delta_1=0=\gamma_1$, these quantities are irrelevant as we can always divide with each of them, and the right-hand sides of the boundary conditions remain identical and are Dirichlet-type. Thus, $u(a)=0=u(b)$. We have completed the proof for the first part of the theorem, that is, for the case $k\neq 3/4$.

To prove the second part of the theorem, we take $k=3/4$, and the indicial roots are given by

$$\rho_1=\frac{3}{2}, \qquad \text{and} \qquad \rho_2=-\frac{1}{2}.$$

For the former, we have $2\rho_1+1=4$, $4\rho_1/(2\rho_1+1)=3/2$ and $(2\rho_1-1)/(2\rho_1+1)=1/2$. Separating the variables and integrating each side of the equation

$$\frac{dx}{d\tau}=\frac{\tau^3}{r_0},$$

we attain $\tau^4 = 4r_0(x + \widehat{x}_0)$. By assuming $r_0 > 0$ and the integration constant $\widehat{x}_0$ such that $x + \widehat{x}_0 \geq 0$, we establish the relationship between the Liouville normal form variable t and the canonical variable x:

$$t = -m \pm \sqrt[4]{4r_0(x + \widehat{x}_0)}.$$

Using (21), we acquire $w(x) = [4r_0(x + \widehat{x}_0)]^{-3/8}$, and this provides the relationship between the canonical function u and the Schrödinger function v:

$$v(x) = [4r_0(x + \widehat{x}_0)]^{3/8}\, u(x).$$

Consequently, the SLP (14) admits the following canonical form:

$$-\frac{d}{dx}\left\{8\sqrt{r_0}[(x + \widehat{x}_0)]^{3/2}\frac{du}{dx}\right\} = \lambda\, r_0\, u, \qquad a < x < b.$$

The values of the boundary points a and b are given as follows:

$$a = -\widehat{x}_0 + \frac{1}{4}\frac{m^4}{r_0}, \qquad \text{and}$$
$$b = -\widehat{x}_0 + \frac{1}{4}\frac{(\pi + m)^4}{r_0}.$$

The irrelevant constant values are $\delta_0 = m^3$ and $\gamma_0 = (\pi + m)^3$.

For the latter, that is, for $\rho_2 = -1/2$, we acquire

$$\frac{dx}{d\tau} = \frac{1}{r_0\,\tau},$$

and upon separation of variables and integration, we obtain

$$\tau = \pm e^{r_0(x + \widetilde{x}_0)}, \qquad \widetilde{x}_0 \in \mathbb{R}.$$

Hence, the relationship between the normal form variable t and the canonical variable x is given by

$$t = -m \pm e^{r_0(x + \widetilde{x}_0)}, \qquad \widetilde{x}_0 \in \mathbb{R}.$$

Using (21), we acquire $w(x) = e^{r_0(x + \widetilde{x}_0)/2}$, and this gives the relationship between $u(x)$ and $v(x)$, that is,

$$v(x) = e^{-r_0(x + \widetilde{x}_0)/2}\, u(x).$$

Moreover, because

$$p(\tau) = \frac{1}{r_0\,\tau^2},$$

we obtain

$$p(x) = \frac{1}{r_0}e^{-2r_0(x + \widetilde{x}_0)}.$$

Substituting this function p in the SLP in the canonical form, we obtain the desired result, and the boundary points a and b are given as follows:

$$a = -\widetilde{x}_0 + \frac{1}{r_0}\ln m, \qquad \text{and}$$
$$b = -\widetilde{x}_0 + \frac{1}{r_0}\ln(\pi + m).$$

The irrelevant constant values are $\delta_0 = 1/m$ and $\gamma_0 = 1/(\pi + m)$. Both ODEs (16) and (17) also admit Dirichlet boundary conditions $u(a) = 0 = u(b)$. Thus, the proof is complete. □

3.2. Constant Potential and Quadratic Density Functions

For the case of constant potential function $q = q_0 \neq 0$ and quadratic density function $r(\tau) = \tau^2$, we have three distinct subcases, depending on whether the indicial equation admits equal roots, distinct real roots, or complex conjugate roots. Each of these cases corresponds to the following relationships between k and q_0, respectively:

- Case A: $1 + 4k = 4q_0$;
- Case B: $1 + 4k > 4q_0$;
- Case C: $1 + 4k < 4q_0$.

Following a similar derivation as in Section 3.1, the function $\overline{\omega} = 1/w$ satisfies the following ODE:

$$\tau^2 \frac{d^2\overline{\omega}}{d\tau^2} + (q_0 - k)\overline{\omega} = 0, \tag{25}$$

with the indicial equation $\rho^2 - \rho - (k - q_0) = 0$ and its roots given by

$$\rho = \widehat{\rho}_{1,2} = \frac{1 \pm \sqrt{1 + 4(k - q_0)}}{2}.$$

Observe that for $q_0 = 0$, $\widehat{\rho}_{1,2}$ reduce to $\rho_{1,2}$ in (20). For Case A, $\widehat{\rho}_1 = \widehat{\rho}_2 = 1/2$. The corresponding linear independent solutions to ODE (25) are given as follows:

$$\overline{\omega}_1(\tau) = \sqrt{\tau}, \qquad \text{and} \qquad \overline{\omega}_2(\tau) = \sqrt{\tau}\,\ln\tau, \qquad \tau > 0.$$

We have the following theorem.

Theorem 3 (Equal roots A1, part 1 of Case A)**.** *By implementing the change of variables $t = -m + e^{x+x_0}$ and $v(x) = e^{(x+x_0)/2}\,u(x)$, $x_0 \in \mathbb{R}$, the SLP (14) admits the following canonical form with Dirichlet boundary conditions:*

$$-\frac{d^2u}{dx^2} + q_0\,u = \lambda\, e^{2(x+x_0)}\,u, \qquad a < x < b, \qquad u(a) = 0 = u(b),$$

where

$$a = -x_0 + \ln m, \qquad b = -x_0 + \ln(\pi + m),$$
$$\delta_0 = m, \qquad \gamma_0 = \pi + m.$$

Proof. For Case A1, using $\overline{\omega}_1(\tau) = \sqrt{\tau}$, $\tau > 0$, we observe that

$$p(\tau) = \frac{\overline{\omega}_1^4}{r} = \frac{\tau^2}{\tau^2} = 1,$$

and

$$\frac{dx}{d\tau} = \sqrt{\frac{p}{r}} = \sqrt{\frac{1}{\tau^2}} = \frac{1}{\tau}.$$

Upon integration, we obtain

$$x + x_0 = \ln\tau, \qquad \text{or} \qquad \tau = e^{x+x_0}, \qquad x_0 \in \mathbb{R}. \tag{26}$$

The functions p, r, and w expressed in terms of the canonical variable x are given as follows:

$$p(x) = 1, \qquad r(x) = e^{2(x+x_0)}, \qquad \text{and} \qquad w(x) = e^{-\frac{1}{2}(x+x_0)}.$$

To find the values of a and b, we substitute $t = \alpha = 0$ and $t = \beta = \pi$ in (26):

$$\begin{aligned} a + x_0 &= \ln(0 + m) &\implies \quad a &= -x_0 + \ln m, \\ b + x_0 &= \ln(\pi + m) &\implies \quad b &= -x_0 + \ln(\pi + m). \end{aligned}$$

Finally, Dirichlet boundary conditions are confirmed if we can show that both δ_0 and γ_0 are nonzero constants:

$$\delta_0 = \overline{\omega}_1^2(a) = m \neq 0, \qquad \text{and} \qquad \gamma_0 = \overline{\omega}_1^2(b) = \pi + m \neq 0.$$

The proof is complete. □

We have the following theorem for Case A2.

Theorem 4 (Equal roots A2, part 2 of Case A)**.** *By performing the change of variables $t = -m + e^{\sqrt[3]{[3(x+x_0)]}}$ and $v(x) = \sqrt[3]{[3(x+x_0)]}\, e^{\sqrt[3]{[3(x+x_0)]}/2}$, $x_0 \in \mathbb{R}$, the SLP (14) admits the following canonical form:*

$$-\frac{d}{dx}\left\{ [3(x+x_0)]^{4/3} \frac{du}{dx} \right\} + q_0\, u = \lambda\, e^{2\,[3(x+x_0)]^{1/3}}\, u, \qquad a < x < b,$$

with Dirichlet boundary conditions $u(a) = 0 = u(b)$, where

$$\begin{aligned} a &= -x_0 + \frac{1}{3}\ln^3 m, & b &= -x_0 + \frac{1}{3}\ln^3(\pi + m), \\ \delta_0 &= m \ln^2 m, & \gamma_0 &= (\pi + m)\ln(\pi + m). \end{aligned}$$

Proof. For Case A2, we utilize $\overline{\omega}_2(\tau) = \sqrt{\tau}\ln\tau$, $\tau > 0$. We obtain

$$p(\tau) = \frac{\overline{\omega}_2^4}{r} = \ln^4 \tau,$$

and

$$\frac{dx}{d\tau} = \sqrt{\frac{p}{r}} = \frac{\ln^2 \tau}{\tau}. \tag{27}$$

We acquire an explicit expression for x in terms of τ upon integrating (27):

$$x + x_0 = \int \frac{\ln^2 \tau}{\tau}\, d\tau = \frac{1}{3}\ln^3 \tau, \qquad x_0 \in \mathbb{R}, \tag{28}$$

or an expression for τ in terms of x, that is,

$$\tau = e^{[3(x+x_0)]^{1/3}}.$$

The functions p, r, and w in terms of the canonical variable x are given as follows:

$$\begin{aligned} p(x) &= [3(x+x_0)]^{\frac{4}{3}}, \\ r(x) &= e^{2[3(x+x_0)]^{\frac{1}{3}}}, \qquad \text{and} \\ w(x) &= [3(x+x_0)]^{-\frac{1}{3}}\, e^{-\frac{1}{2}[3(x+x_0)]^{\frac{1}{3}}}. \end{aligned}$$

The values of the endpoints $x = a$ and $x = b$ can be calculated using (28) by substituting $t = \alpha = 0$ and $t = \beta = \pi$, respectively:

$$a + x_0 = \frac{1}{3}\ln^3(0 + m) \qquad \Longrightarrow \qquad a = -x_0 + \frac{1}{3}\ln^3 m,$$
$$b + x_0 = \frac{1}{3}\ln^3(\pi + m) \qquad \Longrightarrow \qquad b = -x_0 + \frac{1}{3}\ln^3(\pi + m).$$

Finally, Dirichlet boundary conditions are guaranteed when both coefficients δ_0 and γ_0 are nonvanishing:

$$\delta_0 = \overline{\omega}_2^2(a) = m\ln^2 m \neq 0,$$
$$\gamma_0 = \overline{\omega}_2^2(b) = (\pi + m)\ln^2(\pi + m) \neq 0.$$

The proof is complete. □

For Case B, we have the following theorem.

Theorem 5 (Case B). *For $1 + 4k > 4q_0$, the SLP (14) satisfies the following canonical form:*

$$-\frac{d}{dx}\left[(2\rho - 1)^2(x + x_0)^2\frac{du}{dx}\right] + q_0\, u = \lambda[(2\rho - 1)(x + x_0)]^{2/(2\rho-1)}\, u, \qquad a < x < b, \quad (29)$$

upon administering the following change of variables:

$$t = -m + [(2\rho - 1)(x + x_0)]^{1/(2\rho-1)}, \qquad \text{and}$$
$$v(x) = [(2\rho - 1)(x + x_0)]^{\rho/(2\rho-1)}\, u(x), \qquad x_0 \in \mathbb{R},$$

where

$$a = -x_0 + \frac{1}{2\rho - 1}m^{(2\rho-1)}, \qquad b = -x_0 + \frac{1}{2\rho - 1}(\pi + m)^{(2\rho-1)},$$
$$\delta_0 = m^{2\rho}, \qquad \gamma_0 = (\pi + m)^{2\rho}.$$

ODE (29) satisfies Dirichlet boundary conditions $u(a) = 0 = u(b)$.

The proof for this theorem follows a similar argument as the proof of Theorem 2 presented in Section 3.1. We only outline the main points.

Proof. Using $\overline{\omega} = \tau^{\rho}$, we have $p = \tau^{4\rho-2}$ and $dx/d\tau = \tau^{(2\rho-2)}$. After separating the variables and integrating with respect to each variable, we obtain

$$x + x_0 = \frac{1}{2\rho - 1}\tau^{2\rho-1}, \qquad \text{or} \qquad \tau = [(2\rho - 1)(x + x_0)]^{1/(2\rho-1)}.$$

Observe that since $1 + 4k > 4q_0$, $\rho \neq 1/2$, and hence the denominator $2\rho - 1$ never vanishes. The functions p, r, and w expressed in terms of the canonical variable x are given by

$$p(x) = (2\rho - 1)^2(x + x_0)^2,$$
$$r(x) = [(2\rho - 1)(x + x_0)]^{2/(2\rho-1)},$$
$$w(x) = [(2\rho - 1)(x + x_0)]^{-\rho/(2\rho-1)}.$$

The values of a and b can be calculated by substituting $\alpha = 0$ and $\beta = \pi$ for the relationship between x and t, respectively. Similarly, the constants δ_0 and γ_0 can be found by calculating $\overline{\omega}$ at $x = a$ and $x = b$, respectively. This completes the proof. □

For Case C, the two roots of the indicial equation are complex conjugate, that is, $\rho = \frac{1}{2} \pm i\mu$, where $\mu = \frac{1}{2}\sqrt{4(q_0 - k) - 1}$. The two linearly independent solutions to ODE (25) are given by

$$\overline{\omega}_1(\tau) = \sqrt{\tau}\,\cos(\mu \ln \tau), \qquad \text{and} \qquad \overline{\omega}_2(\tau) = \sqrt{\tau}\,\sin(\mu \ln \tau), \qquad \tau > 0.$$

We have the following theorem for the case corresponding to $\overline{\omega}_1$. The terminology "in an asymptotic manner" mentioned in Theorems 6 and 7 refers to the fact that both the p- and density functions are not exact expressions. They only approach the exact values asymptotically in a specific order, in this case, up to the linear order. This limitation arises because it is simply impossible to find an exact analytical expression for the associated inverse function of the Schrödinger variable t in terms of the canonical variable x.

Theorem 6 (Complex roots C1, the first part of Case C)**.** *For $1 + 4k < 4q_0$ and $0 < \varepsilon \ll 1$, the SLP (14) could satisfy the following SLP with Dirichlet boundary conditions in the canonical form in an asymptotic manner:*

$$-\frac{d}{dx}\left[\cos^4\left(\mu \ln\left\{\tau_\varepsilon\left[1 + \frac{1}{\nu^+}(x + x_0^+)\right]\right\}\right)\frac{du}{dx}\right] + q_0\, u = \lambda \tau_\varepsilon^2\left[1 + \frac{1}{\nu^+}(x + x_0^+)\right]^2 u, \qquad a < x < b,$$

by changing the variables using

$$t = -m + \tau_\varepsilon\left[1 + \frac{1}{\nu^+}(x + x_0^+)\right], \qquad \text{and}$$
$$v(x) = \sqrt{\tau_\varepsilon\left[1 + \frac{1}{\nu^+}(x + x_0^+)\right]}\,\cos\left(\mu \ln\left\{\tau_\varepsilon\left[1 + \frac{1}{\nu^+}(x + x_0^+)\right]\right\}\right) u(x),$$

where $x_0 \in \mathbb{R}$, such that these transformation are defined with

$$\begin{aligned}
\tau_\varepsilon &= \frac{1}{2}(\pi + m + \varepsilon),\\
x_0^+ &= x_0 - \frac{1}{2}\ln \tau_\varepsilon - \frac{1}{4\mu}\sin(2\mu \ln \tau_\varepsilon),\\
\nu^+ &= \frac{1}{2}[1 + \cos(2\mu \ln \tau_\varepsilon)],\\
a &= -x_0^+ + \nu^+\left(\frac{m}{\tau_\varepsilon} - 1\right) = -x_0^+ + \nu^+\left(\frac{2}{1 + (\pi + \varepsilon)/m} - 1\right),\\
b &= -x_0^+ + \nu^+\left(\frac{\pi + m}{\tau_\varepsilon} - 1\right) = -x_0^+ + \nu^+\left(\frac{2}{1 + \varepsilon/(\pi + m)} - 1\right),\\
\delta_0 &= m\,\cos^2(\mu \ln m), \qquad \mu \ln m \neq \pi\left(n + \frac{1}{2}\right), \qquad \text{and}\\
\gamma_0 &= (\pi + m)\,\cos^2[\mu \ln(\pi + m)], \qquad \mu \ln(\pi + m) \neq \pi\left(n + \frac{1}{2}\right), \qquad n \in \mathbb{Z}.
\end{aligned}$$

Proof. For case C1, we acquire the following information:

$$\begin{aligned}
p(\tau) &= \frac{\overline{\omega}_1^4}{r} = \cos^4(\mu \ln \tau),\\
\frac{dx}{d\tau} &= \sqrt{\frac{p}{r}} = \frac{\cos^2(\mu \ln \tau)}{\tau}.
\end{aligned} \tag{30}$$

Separating the variables in (30) and integrating each side of the equation, we obtain

$$x + x_0 = \frac{1}{2}\ln \tau + \frac{1}{4\mu}\sin(2\mu \ln \tau), \qquad x_0 \in \mathbb{R}. \tag{31}$$

Because $m \le \tau \le \pi + m$, we Taylor-expand the right-hand side of (31) about $\tau = \tau_\varepsilon = (\pi + m + \varepsilon)/2$, $0 < \varepsilon \ll 1$, to obtain the following expressions:

$$\begin{aligned}
\ln \tau &= \ln \tau_\varepsilon + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{\tau}{\tau_\varepsilon} - 1 \right)^n, \qquad \left| \frac{\tau}{\tau_\varepsilon} - 1 \right| < 1, \\
&= \ln \tau_\varepsilon + \left(\frac{\tau}{\tau_\varepsilon} - 1 \right) - \frac{1}{2} \left(\frac{\tau}{\tau_\varepsilon} - 1 \right)^2 + \frac{1}{3} \left(\frac{\tau}{\tau_\varepsilon} - 1 \right)^3 + \cdots, \qquad 0 < \tau < 2\tau_\varepsilon, \\
\sin(2\mu \ln \tau) &= \sum_{n=0}^{\infty} \frac{i}{2} \left(\frac{\tau}{\tau_\varepsilon} - 1 \right)^n \left[\tau_\varepsilon^{-2i\mu} \binom{-2i\mu}{n} - \tau_\varepsilon^{2i\mu} \binom{2i\mu}{n} \right] \\
&= \sin(2\mu \ln \tau_\varepsilon) + 2\mu \cos(2\mu \ln \tau_\varepsilon) \left(\frac{\tau}{\tau_\varepsilon} - 1 \right) \\
&\quad - \mu [\cos(2\mu \ln \tau_\varepsilon) + 2\mu \sin(2\mu \ln \tau_\varepsilon)] \left(\frac{\tau}{\tau_\varepsilon} - 1 \right)^2 + \mathcal{O}\left(\frac{\tau}{\tau_\varepsilon} - 1 \right)^3.
\end{aligned}$$

Although the Taylor series expansion for $\sin(2\mu \ln \tau)$ converges for $\tau > 0$, observe that by selecting the center point as $\tau_\varepsilon = (\pi + m + \varepsilon)/2$, the value of τ falls nicely in the admissible interval, that is, $0 < m \le \tau \le \pi + m < \pi + m + \varepsilon$, for a small positive number ε. By defining the following constant quantities:

$$\begin{aligned}
x_0^+ &= x_0 - \frac{1}{2} \ln \tau_\varepsilon - \frac{1}{4\mu} \sin(2\mu \ln \tau_\varepsilon), \\
\nu^+ &= \frac{1}{2} [1 + \cos(2\mu \ln \tau_\varepsilon)], \qquad \text{and} \\
\nu_2^+ &= -\frac{1}{2} \left[\nu^+ + \mu \, \sin(2\mu \ln \tau_\varepsilon) \right],
\end{aligned}$$

we can express (31) as

$$x + x_0^+ = \nu^+ \left(\frac{\tau}{\tau_\varepsilon} - 1 \right) + \nu_2^+ \left(\frac{\tau}{\tau_\varepsilon} - 1 \right)^2 + \mathcal{O}\left(\frac{\tau}{\tau_\varepsilon} - 1 \right)^3.$$

By considering the asymptotic expansion only up to the linear term, we can express τ explicitly in terms of the canonical variable x, which is given as follows:

$$\tau = \tau_\varepsilon \left[1 + \frac{1}{\nu^+} \left(x + x_0^+ \right) \right].$$

The functions p, r, and w can now be expressed in terms of x, where they are asymptotically correct up to the linear order and are given by

$$\begin{aligned}
p(x) &= \cos^4\left(\mu \ln\left\{ \tau_\varepsilon \left[1 + \frac{1}{\nu^+} \left(x + x_0^+ \right) \right] \right\} \right), \\
r(x) &= \tau_\varepsilon^2 \left[1 + \frac{1}{\nu^+} \left(x + x_0^+ \right) \right]^2, \\
w(x) &= \frac{\sec\left(\mu \ln\left\{ \tau_\varepsilon \left[1 + \frac{1}{\nu^+} \left(x + x_0^+ \right) \right] \right\} \right)}{\sqrt{\tau_\varepsilon \left[1 + \frac{1}{\nu^+} \left(x + x_0^+ \right) \right]}},
\end{aligned}$$

where the constant $x_0 \in \mathbb{R}$ should be chosen such that $\tau > 0$, and thus the denominator of $w(x)$ is well defined. The transformation from the Schrödinger function $v(x)$ to the canonical function $u(x)$ utilizes the auxiliary function $w(x)$. This relationship is expressed as follows:

$$v(x) = \sqrt{\tau_\varepsilon \left[1 + \frac{1}{\nu^+}(x + x_0^+)\right]}\; \cos\!\left(\mu \ln\!\left\{\tau_\varepsilon \left[1 + \frac{1}{\nu^+}(x + x_0^+)\right]\right\}\right) u(x).$$

The values of $x = a$ and $x = b$ can be calculated using

$$x + x_0^+ = \nu^+ \left(\frac{t+m}{\tau_\varepsilon} - 1\right)$$

by substituting $t = 0$ and $t = \pi$, respectively:

$$\begin{aligned} a &= -x_0^+ + \nu^+ \left(\frac{m}{\tau_\varepsilon} - 1\right) = -x_0^+ + \nu^+ \left(\frac{2}{1 + (\pi+\varepsilon)/m} - 1\right), \qquad \text{and} \\ b &= -x_0^+ + \nu^+ \left(\frac{\pi+m}{\tau_\varepsilon} - 1\right) = -x_0^+ + \nu^+ \left(\frac{2}{1 + \varepsilon/(\pi+m)} - 1\right). \end{aligned}$$

Furthermore, by restricting $\mu \ln m \neq \pi(n + 1/2)$ and $\mu \ln(\pi + m) \neq \pi(n + 1/2)$, $n \in \mathbb{Z}$, it follows that both δ_0 and γ_0 are nonzero, which guarantees that the associated ODE satisfies Dirichlet boundary conditions. These constants can be easily calculated and are given by

$$\begin{aligned} \delta_0 &= \overline{\omega}_1^2(a) = m\, \cos^2(\mu \ln m), \qquad \text{and} \\ \gamma_0 &= \overline{\omega}_1^2(b) = (\pi + m)\, \cos^2[\mu \ln(\pi + m)]. \end{aligned}$$

This completes the proof. □

We have the following theorem for the case corresponding to $\overline{\omega}_2$.

Theorem 7 (Complex roots C2, the second part of Case C)**.** *For $1 + 4k < 4q_0$ and $0 < \varepsilon \ll 1$, the SLP (14) may satisfy the following SLP in the canonical form in an asymptotic manner:*

$$-\frac{d}{dx}\left[\sin^4\!\left(\mu \ln\!\left\{\tau_\varepsilon \left[1 + \frac{1}{\nu^-}(x + x_0^-)\right]\right\}\right) \frac{du}{dx}\right] + q_0\, u = \lambda\, \tau_\varepsilon^2 \left[1 + \frac{1}{\nu^-}(x + x_0^-)\right]^2 u, \qquad a < x < b,$$

using the following change of variables:

$$\begin{aligned} t &= -m + \tau_\varepsilon \left[1 + \frac{1}{\nu^-}(x + x_0^-)\right], \qquad \textit{and} \\ v(x) &= \sqrt{\tau_\varepsilon \left[1 + \frac{1}{\nu^-}(x + x_0^-)\right]}\; \sin\!\left(\mu \ln\!\left\{\tau_\varepsilon \left[1 + \frac{1}{\nu^-}(x + x_0^-)\right]\right\}\right) u(x), \end{aligned}$$

where $x_0 \in \mathbb{R}$, such that these transformations are defined and the ODE satisfies Dirichlet boundary conditions with

$$\begin{aligned} \tau_\varepsilon &= \frac{1}{2}(\pi + m + \varepsilon), \\ x_0^- &= x_0 - \frac{1}{2}\ln \tau_\varepsilon + \frac{1}{4\mu}\sin(2\mu \ln \tau_\varepsilon), \\ \nu^- &= \frac{1}{2}[1 - \cos(2\mu \ln \tau_\varepsilon)], \\ a &= -x_0^- + \nu^- \left(\frac{m}{\tau_\varepsilon} - 1\right), \\ b &= -x_0^- + \nu^- \left(\frac{\pi+m}{\tau_\varepsilon} - 1\right), \end{aligned}$$

$$\delta_0 = m\,\sin^2(\mu\ln m), \qquad \mu\ln m \neq n\pi, \qquad \textit{and}$$
$$\gamma_0 = (\pi+m)\,\sin^2[\mu\ln(\pi+m)], \qquad \mu\ln(\pi+m) \neq n\pi, \qquad n\in\mathbb{Z}.$$

Proof. The proof of Theorem 7 follows a similar argument as the proof of Theorem 6. For case C2, we obtain the following information:

$$\begin{aligned} p(\tau) &= \frac{\overline{\omega}_2^4}{r} = \sin^4(\mu\ln\tau), \\ \frac{dx}{d\tau} &= \sqrt{\frac{p}{r}} = \frac{\sin^2(\mu\ln\tau)}{\tau}. \end{aligned} \tag{32}$$

Separating the variables in (32) and integrating each side of the equation, we acquire

$$x + x_0 = \frac{1}{2}\ln\tau - \frac{1}{4\mu}\sin(2\mu\ln\tau), \qquad x_0\in\mathbb{R}. \tag{33}$$

Because $m \le \tau \le \pi+m$, we consider a Taylor-series expansion of each term on the right-hand side of (33) about $\tau = \tau_\varepsilon = \frac{1}{2}(\pi+m+\varepsilon)$ for a small positive number ε, that is, $0<\varepsilon\ll 1$:

$$\begin{aligned} \ln\tau &= \ln\tau_\varepsilon + \left(\frac{\tau}{\tau_\varepsilon}-1\right) - \frac{1}{2}\left(\frac{\tau}{\tau_\varepsilon}-1\right)^2 + \cdots, \qquad 0<\tau<2\tau_\varepsilon, \\ \sin(2\mu\ln\tau) &= \sin(2\mu\ln\tau_\varepsilon) + 2\mu\cos(2\mu\ln\tau_\varepsilon)\left(\frac{\tau}{\tau_\varepsilon}-1\right) \\ &\quad - \mu[\cos(2\mu\ln\tau_\varepsilon) + 2\mu\sin(2\mu\ln\tau_\varepsilon)]\left(\frac{\tau}{\tau_\varepsilon}-1\right)^2 + \mathcal{O}\left(\frac{\tau}{\tau_\varepsilon}-1\right)^3. \end{aligned}$$

It follows that the right-hand side of (33) can be expressed as follows:

$$x + x_0^- = \nu^-\left(\frac{\tau}{\tau_\varepsilon}-1\right) + \nu_2^-\left(\frac{\tau}{\tau_\varepsilon}-1\right)^2 + \mathcal{O}\left(\frac{\tau}{\tau_\varepsilon}-1\right)^3,$$

where

$$\begin{aligned} x_0^- &= x_0 - \frac{1}{2}\ln\tau_\varepsilon + \frac{1}{4\mu}\sin(2\mu\ln\tau_\varepsilon), \\ \nu^- &= \frac{1}{2}[1-\cos(2\mu\ln\tau_\varepsilon)], \\ \nu_2^- &= -\frac{1}{2}\big[\nu^- - \mu\sin(2\mu\ln\tau_\varepsilon)\big]. \end{aligned}$$

By considering the asymptotic expansion only up to the linear term, we can express τ explicitly in terms of x, which is given by

$$\tau = \tau_\varepsilon\left[1 + \frac{1}{\nu^-}(x+x_0^-)\right].$$

The functions p, r, and w expressed in terms of x are correct asymptotically up to the linear order and are given as follows:

$$\begin{aligned} p(x) &= \sin^4\left(\mu\ln\left\{\tau_\varepsilon\left[1+\frac{1}{\nu^-}(x+x_0^-)\right]\right\}\right), \\ r(x) &= \tau_\varepsilon^2\left[1+\frac{1}{\nu^-}(x+x_0^-)\right]^2, \end{aligned}$$

$$w(x) = \frac{\csc\left(\mu \ln\left\{\tau_\varepsilon\left[1 + \frac{1}{\nu^-}(x + x_0^-)\right]\right\}\right)}{\sqrt{\tau_\varepsilon\left[1 + \frac{1}{\nu^-}(x + x_0^-)\right]}}.$$

From the expression for w, it should be obvious by now that $x_0 \in \mathbb{R}$ should be chosen such that $\tau > 0$. The Schrödinger function v can be transformed into the canonical function u by introducing the auxiliary function w, according to the following relationship:

$$v(x) = \sqrt{\tau_\varepsilon\left[1 + \frac{1}{\nu^-}(x + x_0^-)\right]}\, \sin\left(\mu \ln\left\{\tau_\varepsilon\left[1 + \frac{1}{\nu^-}(x + x_0^-)\right]\right\}\right) u(x).$$

The values of $x = a$ and $x = b$ can be calculated using the relationship between the two independent variables by substituting $t = 0$ and $t = \pi$, respectively. It follows that

$$a = -x_0^- + \nu^-\left(\frac{m}{\tau_\varepsilon} - 1\right), \qquad \text{and}$$
$$b = -x_0^- + \nu^-\left(\frac{\pi + m}{\tau_\varepsilon} - 1\right).$$

Finally, the constants δ_0 and γ_0 can be calculated by substituting $x = a$ and $x = b$ for the square of $\overline{\omega}_2$, respectively. They are given as follows:

$$\delta_0 = \overline{\omega}_2^2(a) = m\, \sin^2(\mu \ln m) \qquad \text{and}$$
$$\gamma_0 = \overline{\omega}_2^2(b) = (\pi + m)\, \sin^2[\mu \ln(\pi + m)].$$

By imposing the restrictions $\mu \ln m \neq n\pi$ and $\mu \ln(\pi + m) \neq n\pi$, $n \in \mathbb{Z}$, these constants are nonzero, and thus the associated ODE admits Dirichlet boundary conditions. This completes the proof. □

3.3. Both Nonzero Constant Potential and Density Functions

We consider the case where both potential and density functions are nonzero constants, that is, $q = q_0 \neq 0$ and $r = r_0 \neq 0$. Consequently, the function $\overline{\omega} = 1/w$ satisfies a transformed version of the Bessel differential equation, where the first derivative of $\overline{\omega}$ with respect to the Schrödinger variable τ is absent.

$$\overline{\tau}^2 \frac{d^2\overline{\omega}}{d\overline{\tau}^2} + \left(\overline{\tau}^2 - k\right)\overline{\omega} = 0, \qquad \text{where} \quad \overline{\tau} = \sqrt{\frac{|q_0|}{|r_0|}}\tau. \tag{34}$$

Before we seek linearly independent solutions of ODE (34), we state the following theorem on a transformation of the Bessel ODE, which we will need and use later.

Theorem 8 (Bowman's transformation [45])**.** *Bowman (1958) gave a transformed version of the Bessel differential equation, which is given as follows:*

$$\overline{\tau}^2 \frac{d^2\overline{\omega}}{d\overline{\tau}^2} + (2\overline{p} + 1)\frac{d\overline{\omega}}{d\overline{\tau}} + \left(\overline{\alpha}^2\overline{\tau}^{2\overline{r}} + \overline{\beta}^2\right)\overline{\omega} = 0. \tag{35}$$

This transformed Bessel ODE (35) *possesses the following solution:*

$$\overline{\omega}(\overline{\tau}) = \frac{1}{\overline{\tau}^{\overline{p}}}\left[C_1\, J_{\overline{q}/\overline{r}}\left(\frac{\overline{\alpha}}{\overline{r}}\overline{\tau}^{\overline{r}}\right) + C_2\, Y_{\overline{q}/\overline{r}}\left(\frac{\overline{\alpha}}{\overline{r}}\overline{\tau}^{\overline{r}}\right)\right], \qquad C_1,\ C_2 \in \mathbb{R},$$

where $\overline{q} = \sqrt{\overline{p}^2 - \overline{\beta}^2}$, $J_n(\overline{\tau})$, *and* $Y_n(\overline{\tau})$, $n \in \mathbb{R}$, *are the Bessel functions of the first and second kinds, respectively [46–51].*

Comparing ODEs (34) and (35), we observe that $\overline{\alpha}^2 = 1$, $\overline{\beta}^2 = -k$, $\overline{p} = -1/2$, $\overline{q} = \frac{1}{2}\sqrt{4k+1}$, and $\overline{r} = 1$. Hence, the associated linearly independent solutions of Bessel ODE (34) are given by

$$\overline{\omega}_1(\overline{\tau}) = \sqrt{\overline{\tau}}\, J_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}), \qquad \text{and} \qquad \overline{\omega}_2(\overline{\tau}) = \sqrt{\overline{\tau}}\, Y_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}).$$

Before discussing a theorem of the SLP in connection to the Bessel function of the first kind J_n with order n, $n \geq 0$, we have the following lemma.

Lemma 6. *A combination of Bessel functions of the first kind admits the following series expansion for* $0 < m < \overline{\tau}_0 < \pi + m$*:*

$$\left[\overline{\tau}\, J_{\rho_1-\frac{1}{2}}(\overline{\tau})\right]^2 - \left[\overline{\tau}\, J_{\rho_1-\frac{3}{2}}(\overline{\tau})\right]\left[\overline{\tau}\, J_{\rho_1+\frac{1}{2}}(\overline{\tau})\right] = \overline{\tau}_0\, \mathcal{J}_0(\overline{\tau}_0) + (\overline{\tau}-\overline{\tau}_0)\mathcal{J}_1(\overline{\tau}_0) + \mathcal{O}(\overline{\tau}-\overline{\tau}_0)^2, \tag{36}$$

where

$$\begin{aligned} \rho_1 &= \frac{1}{2}\left(1+\sqrt{1+4k}\right), \qquad [\textit{see } (20)], \\ \mathcal{J}_0(\overline{\tau}_0) &= \overline{\tau}_0\left(J^2_{\rho_1-\frac{1}{2}} - J_{\rho_1-\frac{3}{2}} J_{\rho_1+\frac{1}{2}}\right)(\overline{\tau}_0), \\ \mathcal{J}_1(\overline{\tau}_0) &= (2\rho_1+1)\mathcal{J}_0(\overline{\tau}_0) + \mathcal{J}_{-1}(\overline{\tau}_0), \qquad \textit{and} \\ \mathcal{J}_{-1}(\overline{\tau}_0) &= \overline{\tau}_0^2\left(J_{\rho_1-\frac{3}{2}} J_{\rho_1+\frac{3}{2}} - J_{\rho_1-\frac{1}{2}} J_{\rho_1+\frac{1}{2}}\right)(\overline{\tau}_0). \end{aligned}$$

Proof. Let us begin with the first term on the left-hand side of (36). A series expansion of the term inside the square bracket is given by

$$\overline{\tau}\, J_{\rho_1-\frac{1}{2}}(\overline{\tau}) = \overline{\tau}_0\, J_{\rho_1-\frac{1}{2}}(\overline{\tau}_0) + (\overline{\tau}-\overline{\tau}_0)\left[\left(\rho_1+\frac{1}{2}\right)J_{\rho_1-\frac{1}{2}}(\overline{\tau}_0) - \overline{\tau}_0\, J_{\rho_1+\frac{1}{2}}(\overline{\tau}_0)\right] + \mathcal{O}(\overline{\tau}-\overline{\tau}_0)^2.$$

Squaring this expression yields

$$\left[\overline{\tau}\, J_{\rho_1-\frac{1}{2}}(\overline{\tau})\right]^2 = \overline{\tau}_0^2\, J^2_{\rho_1-\frac{1}{2}}(\overline{\tau}_0) + (\overline{\tau}-\overline{\tau}_0)\left[\overline{\tau}_0(2\rho_1+1)J^2_{\rho_1-\frac{1}{2}}(\overline{\tau}_0) - 2\overline{\tau}_0^2\, J_{\rho_1-\frac{1}{2}}(\overline{\tau}_0) J_{\rho_1+\frac{1}{2}}(\overline{\tau}_0)\right] + \mathcal{O}(\overline{\tau}-\overline{\tau}_0)^2. \tag{37}$$

Each function of the product term on the left-hand side of (36) admits the following series expansion:

$$\begin{aligned} \overline{\tau}\, J_{\rho_1-\frac{3}{2}}(\overline{\tau}) &= \overline{\tau}_0\, J_{\rho_1-\frac{3}{2}}(\overline{\tau}_0) + (\overline{\tau}-\overline{\tau}_0)\left[\left(\rho_1-\frac{1}{2}\right)J_{\rho_1-\frac{3}{2}}(\overline{\tau}_0) - \overline{\tau}_0\, J_{\rho_1-\frac{1}{2}}(\overline{\tau}_0)\right] + \mathcal{O}(\overline{\tau}-\overline{\tau}_0)^2, \\ \overline{\tau}\, J_{\rho_1+\frac{1}{2}}(\overline{\tau}) &= \overline{\tau}_0\, J_{\rho_1+\frac{1}{2}}(\overline{\tau}_0) + (\overline{\tau}-\overline{\tau}_0)\left[\left(\rho_1+\frac{3}{2}\right)J_{\rho_1+\frac{1}{2}}(\overline{\tau}_0) - \overline{\tau}_0\, J_{\rho_1+\frac{3}{2}}(\overline{\tau}_0)\right] + \mathcal{O}(\overline{\tau}-\overline{\tau}_0)^2. \end{aligned}$$

Taking the product of these two terms yields the following series expansion:

$$\begin{aligned} \left[\overline{\tau}\, J_{\rho_1-\frac{3}{2}}(\overline{\tau})\right]\left[\overline{\tau}\, J_{\rho_1+\frac{1}{2}}(\overline{\tau})\right] &= \overline{\tau}_0^2\, J_{\rho_1-\frac{3}{2}}(\overline{\tau}_0)\, J_{\rho_1+\frac{1}{2}}(\overline{\tau}_0) + (\overline{\tau}-\overline{\tau}_0)\Big\{\overline{\tau}_0(2\rho_1+1)J_{\rho_1-\frac{3}{2}}(\overline{\tau}_0)\, J_{\rho_1+\frac{1}{2}}(\overline{\tau}_0) \\ &\quad - \overline{\tau}_0^2\left[J_{\rho_1-\frac{3}{2}}(\overline{\tau}_0)\, J_{\rho_1+\frac{3}{2}}(\overline{\tau}_0) + J_{\rho_1-\frac{1}{2}}(\overline{\tau}_0)\, J_{\rho_1+\frac{1}{2}}(\overline{\tau}_0)\right]\Big\} + \mathcal{O}(\overline{\tau}-\overline{\tau}_0)^2. \end{aligned} \tag{38}$$

Subtracting (38) from (37), we obtain a series expansion for the left-hand side of (36):

$$\begin{aligned} \left[\overline{\tau}\, J_{\rho_1-\frac{1}{2}}(\overline{\tau})\right]^2 - \left[\overline{\tau}\, J_{\rho_1-\frac{3}{2}}(\overline{\tau})\right]\left[\overline{\tau}\, J_{\rho_1+\frac{1}{2}}(\overline{\tau})\right] &= \overline{\tau}_0^2\left[J^2_{\rho_1-\frac{1}{2}}(\overline{\tau}_0) - J_{\rho_1-\frac{3}{2}}(\overline{\tau}_0)\, J_{\rho_1+\frac{1}{2}}(\overline{\tau}_0)\right] \\ &+ (\overline{\tau}-\overline{\tau}_0)\Big\{\overline{\tau}_0(2\rho_1+1)\left[J^2_{\rho_1-\frac{1}{2}}(\overline{\tau}_0) - J_{\rho_1-\frac{3}{2}}(\overline{\tau}_0)\, J_{\rho_1+\frac{1}{2}}(\overline{\tau}_0)\right] \\ &\quad + \overline{\tau}_0^2\left[J_{\rho_1-\frac{3}{2}}(\overline{\tau}_0)\, J_{\rho_1+\frac{3}{2}}(\overline{\tau}_0) - J_{\rho_1-\frac{1}{2}}(\overline{\tau}_0)\, J_{\rho_1+\frac{1}{2}}(\overline{\tau}_0)\right]\Big\} + \mathcal{O}(\overline{\tau}-\overline{\tau}_0)^2. \end{aligned}$$

By defining the following quantities,

$$\mathcal{J}_0(\overline{\tau}_0) = \overline{\tau}_0\Big[J^2_{\rho_1-\frac{1}{2}}(\overline{\tau}_0) - J_{\rho_1-\frac{3}{2}}(\overline{\tau}_0)\,J_{\rho_1+\frac{1}{2}}(\overline{\tau}_0)\Big], \qquad \text{and}$$
$$\mathcal{J}_{-1}(\overline{\tau}_0) = \overline{\tau}_0^2\Big[J_{\rho_1-\frac{3}{2}}(\overline{\tau}_0)\,J_{\rho_1+\frac{3}{2}}(\overline{\tau}_0) - J_{\rho_1-\frac{1}{2}}(\overline{\tau}_0)\,J_{\rho_1+\frac{1}{2}}(\overline{\tau}_0)\Big],$$

the right-hand side of (36) can be expressed as

$$\Big[\overline{\tau}\,J_{\rho_1-\frac{1}{2}}(\overline{\tau})\Big]^2 - \Big[\overline{\tau}\,J_{\rho_1-\frac{3}{2}}(\overline{\tau})\Big]\Big[\overline{\tau}\,J_{\rho_1+\frac{1}{2}}(\overline{\tau})\Big] = \overline{\tau}_0\mathcal{J}_0(\overline{\tau}_0) + (\overline{\tau}-\overline{\tau}_0)[(2\rho_1+1)\mathcal{J}_0(\overline{\tau}_0) + \mathcal{J}_{-1}(\overline{\tau}_0)] + \mathcal{O}(\overline{\tau}-\overline{\tau}_0)^2.$$

By further defining $\mathcal{J}_1(\overline{\tau}_0) = (2\rho_1+1)\mathcal{J}_0(\overline{\tau}_0) + \mathcal{J}_{-1}(\overline{\tau}_0)$, we obtain (36). This completes our proof. □

We now have the following theorem in connection with this Bessel function of the first kind J_n. Similar to the terminology "in an asymptotic manner", which appears in Theorems 6 and 7, the wording "satisfy ... asymptotically" in Theorems 9 and 10 refers to the p-function not being an exact expression. Due to the challenge of finding exact analytical expressions when inverting the expression for the Schrödinger variable t in terms of the canonical variable x, we only use its asymptotic expression up to a specified order. Consequently, the associated p-function also becomes asymptotically accurate, which, in this case, is up to the linear order.

Theorem 9. *The SLP (14) may satisfy the following canonical form asymptotically:*

$$-\frac{d}{dx}\left(\frac{1}{r_0}(\overline{\tau}_0+\overline{x})^4 J^4_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0+\overline{x})\frac{du}{dx}\right) + q_0\,u = \lambda r_0\,u, \qquad a < x < b,$$

by implementing the following change of variables:

$$t = -m + \overline{\tau}_0 + \overline{x}, \qquad \textit{and}$$
$$v(x) = \sqrt{\overline{\tau}_0+\overline{x}}\,J_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0+\overline{x})\,u(x),$$

where $J_{\frac{1}{2}\sqrt{4k+1}}$ denotes the Bessel function of the first kind with order $\frac{1}{2}\sqrt{4k+1}$,

$$\overline{\tau} = \sqrt{\left|\frac{q_0}{r_0}\right|}\tau, \qquad 0 < m \le \tau \le \pi + m, \qquad \overline{\tau}_0 \in \sqrt{\left|\frac{q_0}{r_0}\right|}(m, \pi+m),$$
$$\overline{x} = \frac{2\sqrt{|q_0 r_0|}(x+x_0) - \overline{\tau}_0\mathcal{J}_0(\overline{\tau}_0)}{\mathcal{J}_1(\overline{\tau}_0)}, \qquad x_0 \in \mathbb{R},$$
$$a = -x_0 + \frac{1}{2r_0}[\tau_0\mathcal{J}_0(\overline{\tau}_0) + (m-\tau_0)\mathcal{J}_1(\overline{\tau}_0)],$$
$$b = -x_0 + \frac{1}{2r_0}[\tau_0\mathcal{J}_0(\overline{\tau}_0) + (\pi+m-\tau_0)\mathcal{J}_1(\overline{\tau}_0)],$$
$$\mathcal{J}_0(\overline{\tau}_0) = \overline{\tau}_0\Big[J^2_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0) - J_{-1+\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0)\,J_{1+\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0)\Big],$$
$$\mathcal{J}_1(\overline{\tau}_0) = (2\rho_1+1)\mathcal{J}_0(\overline{\tau}_0) + \mathcal{J}_{-1}(\overline{\tau}_0),$$
$$\mathcal{J}_{-1}(\overline{\tau}_0) = \overline{\tau}_0^2\Big[J_{-1+\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0)\,J_{2+\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0) - J_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0)\,J_{1+\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0)\Big],$$

$$\delta_0 = \sqrt{\left|\frac{q_0}{r_0}\right|} m \, J^2_{\frac{1}{2}\sqrt{4k+1}}\left(\sqrt{\left|\frac{q_0}{r_0}\right|} m\right),$$

$$\gamma_0 = \sqrt{\left|\frac{q_0}{r_0}\right|}(\pi + m) \, J^2_{\frac{1}{2}\sqrt{4k+1}}\left(\sqrt{\left|\frac{q_0}{r_0}\right|}(\pi + m)\right),$$

and the terms $\sqrt{|q_0/r_0|}\, m$ *and* $\sqrt{|q_0/r_0|}(\pi + m)$ *should not satisfy the zeros of* $J_{\frac{1}{2}\sqrt{4k+1}}$.

Proof. Let us consider the case of $\overline{\omega}_1$, where the function p is given by the Bessel function of the first kind $J_{\frac{1}{2}\sqrt{4k+1}}$:

$$p(\overline{\tau}) = \frac{\overline{\omega}_1^4}{r_0} = \frac{\overline{\tau}^2}{r_0} J^4_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}).$$

The relationship between the canonical variable x and Schrödinger variable t is given by

$$\sqrt{\left|\frac{q_0}{r_0}\right|} \frac{dx}{d\overline{\tau}} = \sqrt{\frac{p}{r}} = \frac{\overline{\tau}}{r_0} J^2_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}).$$

It follows that

$$\begin{aligned} x + x_0 = \int dx &= \sqrt{\left|\frac{q_0}{r_0}\right|} \int \frac{\overline{\tau}}{r_0} J^2_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}) \, d\overline{\tau}, \qquad x_0 \in \mathbb{R}, \\ &= \frac{\overline{\tau}^2}{2r_0}\sqrt{\left|\frac{q_0}{r_0}\right|}\left[J^2_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}) - J_{\frac{1}{2}\sqrt{4k+1}-1}(\overline{\tau}) \, J_{\frac{1}{2}\sqrt{4k+1}+1}(\overline{\tau})\right]. \end{aligned} \tag{39}$$

Using $\rho_1 = \frac{1}{2}\left(1 + \sqrt{1+4k}\right)$ from (20), we can express (39) as follows:

$$x + x_0 = \frac{1}{2r_0}\sqrt{\left|\frac{q_0}{r_0}\right|}\left\{\left[\overline{\tau} J_{\rho_1 - \frac{1}{2}}(\overline{\tau})\right]^2 - \left[\overline{\tau} J_{\rho_1 - \frac{3}{2}}(\overline{\tau})\right]\left[\overline{\tau} J_{\rho_1 + \frac{1}{2}}(\overline{\tau})\right]\right\}. \tag{40}$$

For $0 < \sqrt{\left|\frac{q_0}{r_0}\right|} m < \overline{\tau}_0 < \sqrt{\left|\frac{q_0}{r_0}\right|}(\pi + m)$, the right-hand side of (40) can be expressed asymptotically using Lemma 6:

$$x + x_0 = \frac{1}{2r_0}\sqrt{\left|\frac{q_0}{r_0}\right|}\left[\overline{\tau}_0 \mathcal{J}_0(\overline{\tau}_0) + (\overline{\tau} - \overline{\tau}_0)\mathcal{J}_1(\overline{\tau}_0) + \mathcal{O}(\overline{\tau} - \overline{\tau}_0)^2\right],$$

where $\mathcal{J}_0$ and $\mathcal{J}_1$ are the same quantities defined in Lemma 6, that is,

$$\begin{aligned} \mathcal{J}_0(\overline{\tau}_0) &= \overline{\tau}_0\left[J^2_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0) - J_{-1+\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0) \, J_{1+\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0)\right], \\ \mathcal{J}_1(\overline{\tau}_0) &= (2\rho_1 + 1)\mathcal{J}_0(\overline{\tau}_0) + \mathcal{J}_{-1}(\overline{\tau}_0), \qquad \text{and} \\ \mathcal{J}_{-1}(\overline{\tau}_0) &= \overline{\tau}_0^2\left[J_{-1+\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0) \, J_{2+\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0) - J_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0) \, J_{1+\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0)\right]. \end{aligned}$$

By taking the asymptotic term up to the linear term in τ, we can now express the Schrödinger variable t in terms of the canonical variable x:

$$t = -m + \overline{\tau}_0 + \overline{x},$$

where

$$\overline{x} = \frac{2\sqrt{|q_0 r_0|}(x + x_0) - \overline{\tau}_0 \mathcal{J}_0(\overline{\tau}_0)}{\mathcal{J}_1(\overline{\tau}_0)}.$$

The functions p and w can now be expressed in terms of the canonical variable x and are accurate asymptotically at the linear order. They are given as follows, respectively:

$$p(x) = \frac{1}{r_0}(\overline{\tau}_0 + \overline{x})^4 J^4_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0 + \overline{x}),$$
$$w(x) = \frac{1}{\sqrt{\overline{\tau}_0 + \overline{x}}}\, J^{-1}_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0 + \overline{x}).$$

We define a transformation between the two dependent variables, that is, the canonical u and Schrödinger v, via the auxiliary function w. The relationship is given by

$$v(x) = \sqrt{\overline{\tau}_0 + \overline{x}}\, J_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0 + \overline{x})\, u(x).$$

The left canonical boundaries $x = a$ and $x = b$ correspond to $t = 0$ and $t = \pi$, respectively. Their explicit expressions are given by:

$$a = -x_0 + \frac{1}{2r_0}[\tau_0 \mathcal{J}_0(\overline{\tau}_0) + (m - \tau_0)\mathcal{J}_1(\overline{\tau}_0)],$$
$$b = -x_0 + \frac{1}{2r_0}[\tau_0 \mathcal{J}_0(\overline{\tau}_0) + (\pi + m - \tau_0)\mathcal{J}_1(\overline{\tau}_0)].$$

Finally, to ensure Dirichlet boundary conditions, we must ascertain that the terms $\sqrt{|q_0/r_0|}\, m$ and $\sqrt{|q_0/r_0|}\,(\pi + m)$ do not satisfy as one of the zeros of $J_{\frac{1}{2}\sqrt{4k+1}}$. This verifies that both δ_0 and γ_0 are nonzero:

$$\delta_0 = \overline{\omega}_1^2(x = a) = \sqrt{\left|\frac{q_0}{r_0}\right|} m\, J^2_{\frac{1}{2}\sqrt{4k+1}}\left(\sqrt{\left|\frac{q_0}{r_0}\right|} m\right) \neq 0,$$
$$\gamma_0 = \overline{\omega}_1^2(x = b) = \sqrt{\left|\frac{q_0}{r_0}\right|}(\pi + m)\, J^2_{\frac{1}{2}\sqrt{4k+1}}\left(\sqrt{\left|\frac{q_0}{r_0}\right|}(\pi + m)\right) \neq 0.$$

The proof is complete. □

We have the following theorem in connection to the Bessel function of the second kind Y_n of order n, $n \geq 0$.

Theorem 10. *The SLP (14) could satisfy another canonical form asymptotically in connection to the Bessel function of the second kind $Y_{\frac{1}{2}\sqrt{4k+1}}$ of order $\frac{1}{2}\sqrt{4k+1}$ by enacting the change of variables $t = -m + \overline{\tau}_0 + \widehat{x}$ and $v(x) = \sqrt{\overline{\tau}_0 + \widehat{x}}\, Y_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0 + \widehat{x})\, u(x)$:*

$$-\frac{d}{dx}\left[\frac{1}{r_0}(\overline{\tau}_0 + \widehat{x})^4 Y^4_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0 + \widehat{x})\frac{du}{dx}\right] + q_0\, u = \lambda r_0\, u, \qquad a < x < b,$$

where

$$\overline{\tau} = \sqrt{\left|\frac{q_0}{r_0}\right|}\tau, \qquad 0 < m \leq \tau \leq \pi + m, \qquad \overline{\tau}_0 \in \sqrt{\left|\frac{q_0}{r_0}\right|}(m, \pi + m),$$
$$\overline{x} = \frac{2\sqrt{|q_0 r_0|}(x + x_1) - \overline{\tau}_0 \mathcal{Y}_0(\overline{\tau}_0)}{\mathcal{Y}_1(\overline{\tau}_0)}, \qquad x_1 \in \mathbb{R},$$
$$a = -x_1 + \frac{1}{2r_0}[\tau_0 \mathcal{Y}_0(\overline{\tau}_0) + (m - \tau_0)\mathcal{Y}_1(\overline{\tau}_0)],$$
$$b = -x_1 + \frac{1}{2r_0}[\tau_0 \mathcal{Y}_0(\overline{\tau}_0) + (\pi + m - \tau_0)\mathcal{Y}_1(\overline{\tau}_0)],$$
$$\mathcal{Y}_0(\overline{\tau}_0) = \overline{\tau}_0\left[Y^2_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0) - Y_{-1+\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0)\, Y_{1+\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0)\right],$$

$$\begin{aligned}
\mathcal{Y}_1(\overline{\tau}_0) &= (2\rho_1+1)\mathcal{Y}_0(\overline{\tau}_0) + \mathcal{Y}_{-1}(\overline{\tau}_0),\\
\mathcal{Y}_{-1}(\overline{\tau}_0) &= \overline{\tau}_0^2\Big[Y_{-1+\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0)\,Y_{2+\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0) - Y_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0)\,Y_{1+\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0)\Big],\\
\delta_0 &= \sqrt{\left|\frac{q_0}{r_0}\right|}m\,Y^2_{\frac{1}{2}\sqrt{4k+1}}\left(\sqrt{\left|\frac{q_0}{r_0}\right|}m\right),\\
\gamma_0 &= \sqrt{\left|\frac{q_0}{r_0}\right|}(\pi+m)\,Y^2_{\frac{1}{2}\sqrt{4k+1}}\left(\sqrt{\left|\frac{q_0}{r_0}\right|}(\pi+m)\right),
\end{aligned}$$

and the terms $\sqrt{|q_0/r_0|}\,m$ and $\sqrt{|q_0/r_0|}(\pi+m)$ must not satisfy the zeros of $Y_{\frac{1}{2}\sqrt{4k+1}}$.

Proof. The proof of Theorem 10 follows a similar argument to the proof of Theorem 9, which basically replaces $J_{\frac{1}{2}\sqrt{4k+1}}$ with $Y_{\frac{1}{2}\sqrt{4k+1}}$. We consider the case of $\overline{\omega}_2$, where the function p is now expressible in terms of $Y_{\frac{1}{2}\sqrt{4k+1}}$:

$$p(\overline{\tau}) = \frac{\overline{\omega}_2^4}{r_0} = \frac{\overline{\tau}^2}{r_0}\,Y^4_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}).$$

We have the following relationship between x and $\overline{\tau}$

$$\sqrt{\left|\frac{q_0}{r_0}\right|}\,\frac{dx}{d\overline{\tau}} = \sqrt{\frac{p}{r}} = \frac{\overline{\tau}}{r_0}\,Y^2_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}).$$

After separating the variables and integrating both sides of the equation, it follows that

$$\begin{aligned}
x + x_1 = \int dx &= \sqrt{\left|\frac{r_0}{q_0}\right|}\int \frac{\overline{\tau}}{r_0}\,Y^2_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau})\,d\overline{\tau}, \qquad x_1\in\mathbb{R},\\
&= \frac{1}{2r_0}\sqrt{\left|\frac{r_0}{q_0}\right|}\left\{\left[\overline{\tau}\,Y_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau})\right]^2 - \left[\overline{\tau}\,Y_{\frac{1}{2}\sqrt{4k+1}-1}(\overline{\tau})\right]\left[\overline{\tau}\,Y_{\frac{1}{2}\sqrt{4k+1}+1}(\overline{\tau})\right]\right\}.
\end{aligned} \tag{41}$$

A series expansion about $\overline{\tau} = \overline{\tau}_0$ where $0 < \sqrt{|q_0/r_0|}\,m < \overline{\tau}_0 < \sqrt{|q_0/r_0|}(\pi+m)$ for the right-hand side of (41) yields

$$x + x_1 = \frac{1}{2r_0}\sqrt{\left|\frac{r_0}{q_0}\right|}\left[\overline{\tau}_0\mathcal{Y}_0(\overline{\tau}_0) + (\overline{\tau}-\overline{\tau}_0)\mathcal{Y}_1(\overline{\tau}_0) + \mathcal{O}(\overline{\tau}-\overline{\tau}_0)^2\right],$$

where $\mathcal{Y}_0$ and $\mathcal{Y}_1$ are similar quantities to $\mathcal{J}_0$ and $\mathcal{J}_1$ introduced earlier in Lemma 6. They are given as follows:

$$\begin{aligned}
\mathcal{Y}_0(\overline{\tau}_0) &= \overline{\tau}_0\Big[Y^2_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0) - Y_{-1+\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0)\,Y_{1+\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0)\Big],\\
\mathcal{Y}_1(\overline{\tau}_0) &= (2\rho_1+1)\mathcal{Y}_0(\overline{\tau}_0) + \mathcal{Y}_{-1}(\overline{\tau}_0), \qquad \text{and}\\
\mathcal{Y}_{-1}(\overline{\tau}_0) &= \overline{\tau}_0^2\Big[Y_{-1+\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0)\,Y_{2+\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0) - Y_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0)\,Y_{1+\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0)\Big].
\end{aligned}$$

By considering the series expansion only up to the linear term in $\overline{\tau}$, we can express the Schrödinger variable t in terms of the canonical variable x, which is asymptotically accurate up to the linear order:

$$t = -m + \overline{\tau}_0 + \widehat{x},$$

where

$$\widehat{x} = \frac{2\sqrt{|q_0 r_0|}(x+x_1) - \overline{\tau}_0\mathcal{Y}_0(\overline{\tau}_0)}{\mathcal{Y}_1(\overline{\tau}_0)}.$$

Using this, we can now express both the p- and w-functions as functions of x, which will also be accurate up to the linear order. They are given as follows, respectively:

$$p(x) = \frac{1}{r_0}(\overline{\tau}_0 + \widehat{x})^4 Y^4_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0 + \widehat{x}),$$
$$w(x) = \frac{1}{\sqrt{\overline{\tau}_0 + \widehat{x}}} Y^{-1}_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0 + \widehat{x}).$$

Using the auxiliary function w, we can transform the Schrödinger dependent variable v into the canonical function u by the following relationship:

$$v(x) = \sqrt{\overline{\tau}_0 + \widehat{x}}\, Y_{\frac{1}{2}\sqrt{4k+1}}(\overline{\tau}_0 + \widehat{x})\, u(x).$$

The left and right canonical boundaries $x = a$ and $x = b$ can be calculated straightforwardly by substituting $t = 0$ and $t = \pi$ to $\widehat{x} + \overline{\tau}_0 = t + m$, respectively:

$$a = -x_1 + \frac{1}{2r_0}[\tau_0 \mathcal{Y}_0(\overline{\tau}_0) + (m - \tau_0)\mathcal{Y}_1(\overline{\tau}_0)],$$
$$b = -x_1 + \frac{1}{2r_0}[\tau_0 \mathcal{Y}_0(\overline{\tau}_0) + (\pi + m - \tau_0)\mathcal{Y}_1(\overline{\tau}_0)].$$

To guarantee that the ODE admits Dirichlet boundary conditions, we need to ensure that the terms $\sqrt{|q_0/r_0|}\, m$ and $\sqrt{|q_0/r_0|}(\pi + m)$ must not satisfy as one of the zeros of $Y_{\frac{1}{2}\sqrt{4k+1}}$, thus verifying that both δ_0 and γ_0 are nonzero:

$$\delta_0 = \overline{\omega}_2^2(x = a) = \sqrt{\left|\frac{q_0}{r_0}\right|} m\, Y^2_{\frac{n}{2}}\left(\sqrt{\left|\frac{q_0}{r_0}\right|} m\right) \neq 0,$$
$$\gamma_0 = \overline{\omega}_2^2(x = b) = \sqrt{\left|\frac{q_0}{r_0}\right|}(\pi + m)\, Y^2_{\frac{n}{2}}\left(\sqrt{\left|\frac{q_0}{r_0}\right|}(\pi + m)\right) \neq 0.$$

The proof is complete. □

3.4. Reciprocal Linear Function for $w(t)$

In this subsection, we consider the case where the transformation function w that appears in Lemma 4, Lemma 5, and Equations (7)–(8) is a reciprocal linear function in the independent variable t. In other words, because $\overline{\omega} = 1/w$, this also means that the function $\overline{\omega}$ is assumed to be linear in t. As a consequence of this special case, the second term of the invariant function (7) vanishes, and thus its first term, the quotient q/r, takes the form of the reciprocal binomial quadratic function in t. This special case of the second Paine problem was considered briefly by Ledoux and Ixaru et al. [35,36], but the derivation was notably absent in both works. As mentioned earlier in the introduction, we attempt to generalize this particular case and demonstrate the derivation for obtaining the p-, density, and potential functions. We also note that the title of this subsection could also be written as "Reciprocal quadratic function for $w(x)$" because the Schrödinger variable t is expressed as a quadratic function in the canonical variable x.

We have the following theorem.

Theorem 11. *For the particular case when the transformation function for the dependent variables takes the form of a reciprocal linear function, that is, $w(t) = 1/(C_0 + C_1 t)$, with C_0 and $C_1 > 0$,* *the SLP (14) in the Liouville normal form can be transformed to the canonical form by preserving the type of the boundary conditions, that is, Dirichlet. Using the following variable transformations:*

$$t = -m + \frac{C_1}{4}(x + x_0)^2, \qquad \textit{and}$$
$$v(x) = \frac{1}{4}C_1^2(x + x_0)^2\, u(x), \qquad C_1 > 0 \quad \textit{and} \quad x_0 \in \mathbb{R},$$

the associated SLP in the canonical form is given as follows:

$$-\frac{d}{dx}\left[\frac{1}{8}C_1^3(x + x_0)^3\frac{du}{dx}\right] + \frac{k}{2}C_1^3(x + x_0)\, u = \lambda\left[\frac{1}{2}C_1(x + x_0)\right]^5 u, \qquad a < x < b,$$

where

$$a = -x_0 + 2\sqrt{\frac{m}{C_1}}, \qquad b = -x_0 + 2\sqrt{\frac{\pi + m}{C_1}},$$
$$\delta_0 = (C_1\, m)^2, \qquad \gamma_0 = C_1^2(\pi + m)^2.$$

Proof. Let $w(t) = 1/(C_0 + C_1 t)$, where C_0 and C_1 are both positive constants; then, it can be calculated straightforwardly that $w\frac{d^2}{dt^2}(1/w) = 0$. Thus, the invariant function reduces to

$$\frac{q}{r} = \frac{k}{(t + m)^2}. \tag{42}$$

Furthermore, from

$$pr = \frac{1}{w^4} = (C_0 + C_1 t)^4, \tag{43}$$

we encounter an underdetermined system where we need to seek three unknown functions, that is, p, q, and r, but we only possess two equations. Although we have another relationship $dx/dt = \sqrt{p/r}$, it does not really resolve the issue unless we impose further restrictions on the relationship between the canonical variable x and the Schrödinger variable t, which would help in making an educated guess for the three functions p, q, and r. Now, let us assume that t is a function of x quadratically, and let also the potential and density functions admit similar form, that is, linear in t, but with distinct rational powers, given as follows:

$$q(t) = Q_0(C_0 + C_1 t)^{1/2}, \qquad Q_0 > 0,$$
$$r(t) = (C_0 + C_1 t)^{5/2}.$$

The quotient q/r becomes

$$\frac{q}{r} = \frac{Q_0/C_1^2}{(t + C_0/C_1)^2}. \tag{44}$$

Comparing (42) and (44), we obtain the following relationships:

$$Q_0 = C_1^2\, k, \qquad \text{and} \qquad C_0 = C_1\, m.$$

Now, because r is known, that is, $r(t) = [C_1(t + m)]^{5/2}$, from the product (43), we obtain an expression for p:

$$p(t) = [C_1(t + m)]^{3/2}.$$

We obtain the following relationship between the two independent variables:

$$\frac{dx}{dt} = \frac{1}{\sqrt{C_1(t + m)}} \qquad \Longrightarrow \qquad x + x_0 = \int \frac{dt}{\sqrt{C_1(t + m)}} = 2\sqrt{\frac{t + m}{C_1}}, \tag{45}$$

where $x_0 \in \mathbb{R}$ is an integration constant. Because $1/w(t) = C_1(t+m) = [C_1(x+x_0)/2]^2 = 1/w(x)$, it follows that

$$
\begin{aligned}
v(x) &= \frac{1}{4} C_1^2 (x+x_0)^2 \, u(x), \\
p(x) &= \left[\frac{1}{2} C_1 (x+x_0)\right]^3 = \frac{1}{8} C_1^3 (x+x_0)^3, \\
q(x) &= C_1^2 k \left[\frac{1}{2} C_1 (x+x_0)\right] = \frac{1}{2} k C_1^3 (x+x_0), \qquad \text{and} \\
r(x) &= \left[\frac{1}{2} C_1 (x+x_0)\right]^5 = \frac{1}{32} C_1^5 (x+x_0)^5.
\end{aligned}
$$

The canonical boundaries can be found by substituting $t = 0$ and $t = \pi$ in (45), respectively, and they are given as follows:

$$
a = -x_0 + 2\sqrt{\frac{m}{C_1}}, \qquad \text{and} \qquad b = -x_0 + 2\sqrt{\frac{\pi + m}{C_1}}.
$$

Finally, Dirichlet boundary conditions can be confirmed by nonzero δ_0 and γ_0:

$$
\begin{aligned}
\delta_0 &= \frac{1}{w^2(a)} = \left[\frac{1}{2} C_1 (a+x_0)\right]^4 = C_1^2 \, m^2, \\
\gamma_0 &= \frac{1}{w^2(b)} = \left[\frac{1}{2} C_1 (b+x_0)\right]^4 = C_1^2 \, (\pi+m)^2.
\end{aligned}
$$

This completes the proof. □

Remark 2. *It is often practical to set the left endpoint boundary $a = 0$, and thus, $x_0 = 2\sqrt{m/C_1}$, as it was performed in [35,36]. For this particular case, the right endpoint boundary becomes*

$$
b = \frac{2}{\sqrt{C_1}} \left(\sqrt{\pi + m} - \sqrt{m} \right) > 0.
$$

The classical Paine problem takes $k = 1$ and $m = 0.1$, and by taking a special case $C_1 = 2$, we arrive to what Ledoux and Ixaru et al. [35,36] stated, namely

$$
\begin{aligned}
&x_0 = \sqrt{0.2}, \qquad b = \sqrt{2\pi + 0.2} - \sqrt{0.2}, \\
&p(x) = \left(x + \sqrt{0.2}\right)^3, \qquad q(x) = 4\left(x + \sqrt{0.2}\right), \qquad \textit{and} \qquad r(x) = \left(x + \sqrt{0.2}\right)^5.
\end{aligned}
$$

Remark 3. *We can also generalize the powers of q and r to arbitrary positive numbers, let us say, instead of $1/2$ and $5/2$, they become n_q and n_r, respectively, where $n_r = n_q + 2$. It follows that*

$$
p(t) = [C_1(t+m)]^{4-n_r},
$$

and

$$
\frac{dx}{dt} = [C_1(t+m)]^{2-n_r},
$$

whereby, upon integration, we obtain

$$
x + x_0 = \frac{C_1^{2-n_r}}{3 - n_r} (t+m)^{3-n_r}, \qquad n_r \neq 3.
$$

Meanwhile, to avoid potential singularities for p, q, and r in the canonical variable x, we impose a further restriction: because n_q is positive, n_r must be greater than 2. A similar argument applied to the relationship between x and t yields $n_r < 3$. This narrows the range for n_r down to $2 < n_r < 3$.

However, choosing n_r values other than $5/2$ results in non-polynomial expressions for p, q, and r. While the case considered in this section is a special one, Ledoux and Ixaru et al. [35,36] made a clever choice by taking $n_q = 1/2$ and $n_r = 5/2$, resulting in polynomial functions for p, q, and r in the canonical variable.

4. Conclusions

We have considered the transformation of the Sturm–Liouville boundary value problem, often known as the SLP, from its canonical form to the Schrödinger (Liouville normal) form and vice versa. Although it is theoretically possible to retrieve the SLP in its canonical form from any given SLP in the Schrödinger form, in practice, such an attempt is not always feasible. Implementing inverse Liouville transformations can even be nearly impossible in several, or even many, cases.

For a particular case study, we investigated the second Paine–de Hoog–Anderson (PdHA) problem in a generalized manner. Also known in the literature as the Paine problem, the associated SLP with Dirichlet boundary conditions is given in its Liouville normal form instead of in its canonical form. The classical second Paine problem considered the corresponding invariant function in the form of a reciprocal binomial term with quadratic power and specific constants in the numerator and denominator, which are 1 and 0.1, respectively. We generalized these numbers to any positive constants while keeping the binomial power reciprocal quadratic.

Our study revealed that the difficulty of retrieving the SLP in the canonical form depends on the combinations between the potential and density functions. In the four special cases that we considered, inverting the SLP to the canonical form was relatively straightforward in some, while in others, such a process was impossible without adopting the technique of asymptotic expansion. One immediate consequence of the mentioned combinations is the relationship between the independent variables, that is, whether we can easily find an exact expression for the Schrödinger variable in terms of the canonical variable by simply inverting the latter from the former. Whenever this fails, we simply proceed by finding its inverse asymptotically.

The exact SLP in its canonical form occurs in three cases, that is, when the potential function vanishes but the density function is nonzero constant, when the potential function is a nonzero constant and the density function is quadratic, and when the transformation function w is reciprocal linear in t (or quadratic in x). For the second case, it occurs only in two special subcategories: when the roots of the indicial equations are either equal or real distinct. When these roots are complex conjugate, the SLP in the canonical form is only accurate asymptotically. A similar case occurs when both the potential and density functions are nonzero constants. The p-functions appear in relatively elementary forms, depending on the various cases and subcases. However, when both potential and density functions are nonzero constants, the p-function takes the form of the first and second kind Bessel functions. In all considered cases, Dirichlet boundary conditions follow accordingly.

A natural extension of this work would be to consider higher-order powers of the invariant function of the second Paine problem, such as reciprocal quartic, sextic, and even higher powers. This generalization, along with extending the reciprocal power to any positive real number, remains an open question. Further investigation could also explore other types of invariant functions found in various Liouville normal forms.

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Appendix A. Proof of Lemma 1

Proof. Because p is constant and $p \neq 0$, we can pull it out from the derivative sign and divide ODE (1) by p:

$$-\frac{d^2u}{dx^2} + \frac{q}{p}\,u = \lambda\,\frac{r}{p}\,u. \tag{A1}$$

Using the transformation $t = \eta x$, we have

$$\frac{du}{dx} = \eta\,\frac{du}{dt}, \qquad \text{and} \qquad \frac{d^2u}{dx^2} = \eta^2\frac{d^2u}{dt^2}.$$

ODE (A1) becomes

$$-\eta^2\frac{d^2u}{dt^2} + \frac{q}{p}\,u = \lambda\,\eta^2\,u.$$

By dividing with $\eta^2 \neq 0$, we obtain the desired form (2), where

$$Q(t) = \frac{q}{\eta^2\,p} = \frac{pq}{r^2}.$$

All expressions for the functions p, q, and r are to be understood as $f(x) = f(t/\eta)$, where $f = p, q$, or r. This completes the proof. □

Appendix B. Proof of Lemma 2

Proof. We show the Riccati equation in U by substituting $p(x)du/dx = u(x)\,U(x)$ in ODE (1). We observe that

$$\begin{aligned}
-\frac{d}{dx}[u(x)\,U(x)] + q(x)\,u(x) &= \lambda\,r(x)u(x),\\
-\frac{du}{dx}U(x) - u(x)\frac{dU}{dx} + qu &= \lambda ru,\\
-u\frac{dU}{dx} - \frac{u}{p}U^2(x) + qu &= \lambda ru,\\
\frac{dU}{dx} + \frac{U^2(x)}{p(x)} &= q(x) - \lambda\,r(x),
\end{aligned}$$

where we have divided by $u(x)$ to obtain the final expression. To obtain the Riccati equation in V, the following expressions are useful:

$$\begin{aligned}
V(x) &= \frac{U}{p} = \frac{1}{u}\frac{du}{dx},\\
\frac{dV}{dx} &= \frac{1}{u}\frac{d^2u}{dx^2} - \left(\frac{1}{u}\frac{du}{dx}\right)^2 = \frac{1}{u}\frac{d^2u}{dx^2} - V^2, \qquad \text{and}
\end{aligned}$$

$$\frac{1}{u}\frac{d^2u}{dx^2} = \frac{dV}{dx} + V^2.$$

Dividing ODE (1) by $u(x)$, we observe that

$$-\frac{p}{u}\frac{d^2u}{dx^2} - \frac{dp}{dx}\left(\frac{1}{u}\frac{du}{dx}\right) + q = \lambda r.$$

Dividing this expression by $-1/p$ and employing the new variable V, we obtain the Riccati equation in V:

$$\frac{dV}{dx} + \frac{1}{p}\frac{dp}{dx}V + V^2 = \frac{1}{p}(q - \lambda r).$$

The proof is complete. □

Appendix C. Proof of Lemma 3

Proof. It can be easily worked out using the fact that the differential operator $d/dx = (dt/dx)\,d/dt = (1/\dot{x})\,d/dt$, and thus $du/dx = (1/\dot{x})\,du/dt$. Substituting these expressions in ODE (1), we obtain

$$-\frac{1}{\dot{x}}\frac{d}{dt}\left(\frac{p}{\dot{x}}\frac{du}{dt}\right) + q\,u = \lambda\,r\,u. \tag{A2}$$

Multiplying both sides of (A2) with $\dot{x}$ yields the desired expression (3):

$$-\frac{d}{dt}\left(\frac{p}{\dot{x}}\frac{du}{dt}\right) + q\dot{x}\,u = \lambda\,r\dot{x}\,u.$$

The proof is complete. □

Appendix D. Proof of Lemma 4

Proof. By applying the product rule to u, we have

$$\begin{aligned}
\frac{du}{dx} &= \frac{dw}{dx}v + w\frac{dv}{dx}, \\
p\frac{du}{dx} &= pv\frac{dw}{dx} + pw\frac{dv}{dx}, \\
-\frac{d}{dx}\left(p\frac{du}{dx}\right) &= -\frac{d}{dx}\left(pv\frac{dw}{dx}\right) - \frac{d}{dx}\left(pw\frac{dv}{dx}\right).
\end{aligned}$$

Substituting these expressions in ODE (1) yields

$$-\frac{d}{dx}\left(pw\frac{dv}{dx}\right) + qw\,v - \frac{d}{dx}\left(pv\frac{dw}{dx}\right) = \lambda\,rw\,v. \tag{A3}$$

Multiplying (A3) with the function w results

$$-w\frac{d}{dx}\left(pw\frac{dv}{dx}\right) + qw^2\,v - w\frac{d}{dx}\left(pv\frac{dw}{dx}\right) = \lambda\,rw^2\,v. \tag{A4}$$

Expanding the third term on the left-hand side of (A4), we obtain (color online)

$$-w\frac{d}{dx}\left(pw\frac{dv}{dx}\right) + qw^2\,v - wv\frac{d}{dx}\left(p\frac{dw}{dx}\right) - wp\frac{dv}{dx}\frac{dw}{dx} = \lambda\,rw^2\,v.$$

Because

$$\frac{d}{dx}\left(pw^2\frac{dv}{dx}\right) = w\frac{d}{dx}\left(pw\frac{dv}{dx}\right) + pw\frac{dw}{dx}\frac{dv}{dx},$$

we observe that two terms (blue and red) cancel each other:

$$-\frac{d}{dx}\left(pw^2\frac{dv}{dx}\right)+pw\frac{dw}{dx}\frac{dv}{dx}+qw^2\,v-wv\frac{d}{dx}\left(p\frac{dw}{dx}\right)-wp\frac{dv}{dx}\frac{dw}{dx}=\lambda\,rw^2\,v. \tag{A5}$$

Rearranging the remaining terms of (A5), we obtain the desired ODE (4):

$$-\frac{d}{dx}\left(pw^2\,\frac{dv}{dx}\right)+\left[qw^2-w\frac{d}{dx}\left(p\frac{dw}{dx}\right)\right]v=\lambda\,rw^2\,v.$$

This completes the proof. □

Appendix E. Proof of Theorem 1

Proof. Using the independent variable transformation $(x\leftrightarrow t)$, we have

$$\frac{dt}{dx}=\frac{1}{dx/dt}=\frac{1}{\dot{x}}=\sqrt{\frac{r}{p}},\qquad \text{or}\qquad \dot{x}=\sqrt{\frac{p}{r}}.$$

Because $w^2=1/\sqrt{pr}$ or $1/w^2=\sqrt{pr}$, we also have

$$\frac{p}{\dot{x}}\frac{dw}{dt}=p\sqrt{\frac{r}{p}}\frac{dw}{dt}=\sqrt{rp}\frac{dw}{dt}=\frac{1}{w^2}\frac{dw}{dt}.$$

Using Lemma 5, we observe that

$$\begin{aligned}
P(t)&=\frac{pw^2}{\dot{x}}=\frac{p}{\sqrt{pr}}\sqrt{\frac{r}{p}}=1,\\
Q(t)&=qw^2\dot{x}-w\frac{d}{dt}\left(\frac{p}{\dot{x}}\frac{dw}{dt}\right)=\frac{q}{\sqrt{pr}}\sqrt{\frac{p}{r}}+w\frac{d}{dt}\left(-\frac{1}{w^2}\frac{dw}{dt}\right)\\
&=\frac{q}{r}+w\frac{d}{dt}\left[\frac{d}{dt}\left(\frac{1}{w}\right)\right]=\frac{q}{r}+w\frac{d^2}{dt^2}\left(\frac{1}{w}\right)=I(t)\\
R(t)&=rw^2\dot{x}=\frac{r}{\sqrt{pr}}\sqrt{\frac{p}{r}}=1.
\end{aligned}$$

Hence, the ODE (5) becomes

$$-\frac{d^2v}{dt^2}+I(t)\,v=\lambda\,v,\qquad \alpha<t<\beta.$$

For the boundary conditions, we multiply both of them with w and use the fact that

$$\frac{du}{dx}=\frac{dw}{dx}v+w\frac{dv}{dx}=\frac{dw}{dx}v+\frac{w}{\dot{x}}\frac{dv}{dt}.$$

By substituting the relevant boundaries, that is, $x=a$ or $t=\alpha$, we obtain

$$\begin{aligned}
\delta_0\,w^2v-\delta_1\,pw\left(\frac{dw}{dx}v+\frac{w}{\dot{x}}\frac{dv}{dt}\right)&=0,\\
\left(\delta_0\,w^2-\delta_1\,pw\frac{dw}{dx}\right)v-\delta_1\,\frac{pw^2}{\dot{x}}\frac{dv}{dt}&=0.
\end{aligned}$$

By taking

$$\delta_2=\left.\left(\delta_0w^2-\delta_1\,pw\frac{dw}{dx}\right)\right|_{x=a}\qquad\text{and}\qquad P(\alpha)=\left.\frac{pw^2}{\dot{x}}\right|_{t=\alpha}$$

we obtain the boundary condition at $t = \alpha$. A similar argument can be reached for the boundary conditions at $t = \beta$. To show the identity of the definite integrals (9), we use $dx = \sqrt{p/r}\, dt$, $u^2 = w^2 v^2$, and $w^2 = (pr)^{-1/2}$ to arrive at

$$\begin{aligned}
\int_a^b r(x)\, u^2(x)\, dx &= \int_\alpha^\beta r(t)\, w^2(t)\, v^2(t) \sqrt{\frac{p(t)}{r(t)}}\, dt \\
&= \int_\alpha^\beta \sqrt{p(t)\, r(t)}\, w^2(t)\, v^2(t)\, dt \\
&= \int_\alpha^\beta v^2(t)\, dt,
\end{aligned}$$

because $\sqrt{p(t)\, r(t)}\, w^2(t) = 1$. This completes the proof. □

Appendix F. Proof of Corollary 1

Proof. We use the following relationships for the first-order and second-order derivative operators, respectively:

$$\frac{d}{dt} = \dot{x}\frac{d}{dx} = \frac{dx}{dt}\frac{d}{dx} \qquad \text{and} \qquad \frac{d^2}{dt^2} = \ddot{x}\frac{d}{dx} + \dot{x}^2\frac{d^2}{dx^2} = \frac{d^2x}{dt^2}\frac{d}{dx} + \left(\frac{dx}{dt}\right)^2\frac{d^2}{dx^2}.$$

Substituting these expressions in ODE (10) yields

$$-\frac{d^2x}{dt^2}\frac{dv}{dx} - \left(\frac{dx}{dt}\right)^2\frac{d^2v}{dx^2} + I(x)\, v = \lambda\, v. \tag{A6}$$

Because $v(x) = u(x)/w(x)$, the first and second derivatives of v can be expressed as follows, respectively:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{u}{w}\right) &= \frac{1}{w}\frac{du}{dx} - \frac{u}{w^2}\frac{dw}{dx} \\
\frac{d^2}{dx^2}\left(\frac{u}{w}\right) &= \frac{d}{dx}\left(\frac{1}{w}\frac{du}{dx}\right) - \frac{d}{dx}\left(\frac{u}{w^2}\frac{dw}{dx}\right) \\
&= \frac{1}{w}\frac{d^2u}{dx^2} - \frac{1}{w^2}\frac{dw}{dx}\frac{du}{dx} - \frac{1}{w^2}\frac{dw}{dx}\frac{du}{dx} + \frac{2}{w^3}\left(\frac{dw}{dx}\right)^2 u - \frac{1}{w^2}\frac{d^2w}{dx^2}u \\
&= \frac{1}{w}\frac{d^2u}{dx^2} - \frac{2}{w^2}\frac{dw}{dx}\frac{du}{dx} + \frac{2}{w^3}\left(\frac{dw}{dx}\right)^2 u - \frac{1}{w^2}\frac{d^2w}{dx^2}u.
\end{aligned}$$

Substituting these expressions in ODE (A6) and multiplying it with w, we obtain

$$\begin{aligned}
&-\left(\frac{dx}{dt}\right)^2\frac{d^2u}{dx^2} + \left[\frac{2}{w}\frac{dw}{dx}\left(\frac{dx}{dt}\right)^2 - \frac{d^2x}{dt^2}\right]\frac{du}{dx} \\
&+\left\{I(x) - \left[\frac{2}{w^2}\left(\frac{dw}{dx}\right)^2 - \frac{1}{w}\frac{d^2w}{dx^2}\right]\left(\frac{dx}{dt}\right)^2 + \frac{1}{w}\frac{dw}{dx}\frac{d^2x}{dt^2}\right\}u = \lambda\, u.
\end{aligned} \tag{A7}$$

Consider again ODE (12) from the SLP in the canonical form, where we have now divided it by r:

$$-\frac{p}{r}\frac{d^2u}{dx^2} - \frac{1}{r}\frac{dp}{dx}\frac{du}{dx} + \frac{q}{r}u = \lambda u. \tag{A8}$$

We compare these two ODEs, which are Equations (A7) and (A8). Because they should be identical, we obtain the following relationship:

$$\left(\frac{dx}{dt}\right)^2 = \frac{p}{r} \qquad \text{or} \qquad \frac{dx}{dt} = \pm\sqrt{\frac{p}{r}},$$

Equating the coefficient for du/dx, we observe that

$$\begin{aligned}
\frac{2}{w}\frac{dw}{dx}r\left(\frac{dx}{dt}\right)^2 - r\frac{d^2x}{dt^2} &= -\frac{dp}{dx}, \\
\frac{2}{w}\frac{dw}{dx}p - \frac{1}{2}r\frac{d}{dx}\left(\frac{p}{r}\right) &= -\frac{dp}{dx}, \\
\frac{2p}{w}\frac{dw}{dx} + \frac{dp}{dx} - \frac{1}{2}r\left(\frac{1}{r}\frac{dp}{dx} - \frac{p}{r^2}\frac{dr}{dx}\right) &= 0, \\
\frac{2p}{w}\frac{dw}{dx} + \frac{1}{2}\frac{dp}{dx} + \frac{1}{2}\frac{p}{r}\frac{dr}{dx} &= 0, \qquad \text{(multiply with } w\text{)} \\
\frac{2pr}{w}\frac{dw}{dx} + \frac{1}{2}\left(r\frac{dp}{dx} + p\frac{dr}{dx}\right) &= 0, \qquad \text{(divide with } pr\text{)} \\
\frac{2}{w}\frac{dw}{dx} + \frac{1}{2}\frac{1}{pr}\frac{d}{dx}(pr) &= 0, \\
\frac{1}{w}\frac{dw}{dx} &= -\frac{1}{4}\frac{1}{pr}\frac{d}{dx}(pr), \\
\frac{d}{dx}\ln w &= \frac{d}{dx}\ln(pr)^{-1/4}, \\
w &= (pr)^{-1/4}.
\end{aligned}$$

Before establishing the third relationship that involves the invariant function I, we need the following first-order and second-order derivative operators. These are similar to what we used at the beginning of the proof, albeit the roles of x and t are reversed:

$$\begin{aligned}
\frac{d}{dx} &= \frac{1}{\dot{x}}\frac{d}{dt}, \qquad \text{and} \\
\frac{d^2}{dx^2} &= \frac{d}{dx}\left(\frac{1}{\dot{x}}\right)\frac{d}{dt} + \frac{1}{\dot{x}}\frac{d}{dx}\left(\frac{d}{dt}\right) = \frac{1}{\dot{x}}\frac{d}{dt}\left(\frac{1}{\dot{x}}\right)\frac{d}{dt} + \frac{1}{(\dot{x})^2}\frac{d^2}{dt^2} = \frac{1}{(\dot{x})^2}\frac{d^2}{dt^2} - \frac{\ddot{x}}{(\dot{x})^3}\frac{d}{dt}.
\end{aligned}$$

Transforming the invariant function I with an independent variable x to the one that depends on t gives the following:

$$\begin{aligned}
I(x) &= \frac{q}{r} + \left[\frac{2}{w^2}\left(\frac{dw}{dx}\right)^2 - \frac{1}{w}\frac{d^2w}{dx^2}\right]\left(\frac{dx}{dt}\right)^2 - \frac{1}{w}\frac{dw}{dx}\frac{d^2x}{dt^2} \\
I(t) &= \frac{q}{r} + \left[\frac{1}{(\dot{x})^2}\frac{2}{w^2}\left(\frac{dw}{dt}\right)^2 - \frac{1}{(\dot{x})^2}\frac{1}{w}\frac{d^2w}{dt^2} + \frac{\ddot{x}}{(\dot{x})^3}\frac{1}{w}\frac{dw}{dt}\right]\left(\frac{dx}{dt}\right)^2 - \frac{\ddot{x}}{\dot{x}}\frac{1}{w}\frac{dw}{dt} \\
&= \frac{q}{r} + \frac{2}{w^2}\left(\frac{dw}{dt}\right)^2 - \frac{1}{w}\frac{d^2w}{dt^2} = \frac{q}{r} + w\left[\frac{2}{w^3}\left(\frac{dw}{dt}\right)^2 - \frac{1}{w^2}\frac{d^2w}{dt^2}\right] \\
&= \frac{q}{r} + w\frac{d}{dt}\left(-\frac{1}{w^2}\frac{dw}{dt}\right) = \frac{q}{r} + w\frac{d}{dt}\left[\frac{d}{dt}\left(\frac{1}{w}\right)\right] = \frac{q}{r} + w\frac{d^2}{dt^2}\left(\frac{1}{w}\right).
\end{aligned}$$

To verify the boundary conditions, we utilize the following facts:

$$\frac{dv}{dt} = \dot{x}\frac{dv}{dx} \qquad \text{and} \qquad \frac{dv}{dt}(\alpha) = \left.\left(\frac{\dot{x}}{w}\frac{du}{dx} - \frac{\dot{x}}{w^2}\frac{dw}{dx}u\right)\right|_{x=a}.$$

It follows that

$$\begin{aligned}
\delta_2 v(\alpha) - \widehat{\delta}_1\frac{dv}{dt}(\alpha) &= 0, \\
\left.\left(\frac{\delta_2}{w} + \widehat{\delta}_1\frac{\dot{x}}{w^2}\frac{dw}{dx}\right)\right|_{x=a} u(a) - \widehat{\delta}_1\left.\frac{\dot{x}}{w^2}\right|_{x=a}\frac{du}{dx}(a) &= 0,
\end{aligned}$$

$$\left(\frac{\delta_2}{w}+\widehat{\delta}_1\frac{\dot{x}}{pw^2}p\frac{dw}{dx}\right)\bigg|_{x=a}u(a)-\widehat{\delta}_1\frac{\dot{x}}{pw^2}\bigg|_{x=a}p(a)\frac{du}{dx}(a)=0. \tag{A9}$$

Because $P(\alpha) = pw^2/\dot{x}\big|_{t=\alpha} = pw^2/\dot{x}\big|_{x=a}$, we can simplify an identical expression that appears on the second and third terms of the boundary condition (A9):

$$\widehat{\delta}_1\frac{\dot{x}}{pw^2}\bigg|_{x=a}=\frac{\widehat{\delta}_1}{P(a)}=\delta_1.$$

By further dividing the first two terms inside the brackets in the boundary condition (A9) by w, we obtain the desired boundary condition in the canonical form:

$$\delta_0u(a)-\delta_1p(a)\frac{du}{dx}(a)=0,$$

where

$$\delta_0=\left(\frac{\delta_2}{w^2}+\delta_1\frac{p}{w}\frac{dw}{dx}\right)\bigg|_{x=a}.$$

The second boundary condition can be derived using a similar argument, by simply replacing a, α, δ_0, $\widehat{\delta}_1$, and δ_2, with b, β, γ_0, $\widehat{\gamma}_1$, and γ_2, respectively. This completes the proof. □

References

1. Zettl, A. *Sturm-Liouville Theory*; American Mathematical Society: Providence, RI, USA, 2010.
2. Lützen, J. Sturm and Liouville's work on ordinary linear differential equations. The emergence of Sturm-Liouville theory. *Arch. Hist. Exact Sci.* **1984**, *29*, 309–376. [CrossRef]
3. Lützen, J. *Joseph Liouville 1809–1882: Master of Pure and Applied Mathematics*; Springer Science & Business Media: New York, NY, USA, 2012.
4. Pryce, J.D. *Numerical Solution of Sturm-Liouville Problems*; Oxford University Press: Oxford, UK, 1993.
5. Bailey, P.B.; Everitt, W.N.; Zettl, A. The SLEIGN2 Sturm-Liouville code. *ACM Trans. Math. Softw.* **2001**, *27*, 143–192. [CrossRef]
6. Amrein, W.O.; Hinz, A.M.; Pearson, D.B. (Eds.). *Sturm-Liouville Theory: Past and Present*; Birkhäuser Verlag: Basel, Switzerland, 2005.
7. Al-Gwaiz, M.A. *Sturm-Liouville Theory and Its Applications*; Springer: Berlin/Heidelberg, Germany, 2008.
8. Agarwal, R.P.; O'Regan, D. *Ordinary and Partial Differential Equations: With Special Functions, Fourier Series, and Boundary Value Problems*; Springer Science & Business Media: New York, NY, USA, 2009.
9. Teschl, G. *Ordinary Differential Equations and Dynamical Systems*; American Mathematical Society: Providence, RI, USA, 2012.
10. Haberman, R. *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*, 5th ed.; Pearson Higher Education: Boston, MA, USA, 2013.
11. Guenther, R.B.; Lee, J.W. *Sturm-Liouville Problems: Theory and Numerical Implementation*; CRC Press: Boca Raton, FL, USA, 2019.
12. Kravchenko, V.V. *Direct and Inverse Sturm-Liouville Problems: A Method of Solution*; Birkhäuser Verlag, Springer Nature: Cham, Switzerland, 2020.
13. Masjed-Jamei, M. *Special Functions and Generalized Sturm-Liouville Problems*; Birkhäuser: Basel, Switzerland, 2020.
14. Zettl, A. *Recent Developments in Sturm-Liouville Theory*; De Gruyter: Berlin, Germany; Boston, MA, USA, 2021.
15. Al-Mdallal, Q.M. An efficient method for solving fractional Sturm–Liouville problems. *Chaos Solitons Fractals* **2009**, *40*, 183–189. [CrossRef]
16. Klimek, M.; Agrawal, O.P. Fractional Sturm–Liouville problem. *Comput. Math. Appl.* **2013**, *66*, 795–812. [CrossRef]
17. Zayernouri, M.; Karniadakis, G.E. Fractional Sturm–Liouville eigen-problems: Theory and numerical approximation. *J. Comput. Phys.* **2013**, *252*, 495–517. [CrossRef]
18. Buterin, S.A. On the reconstruction of a non-selfadjoint Sturm-Liouville operator. *Mat. Mekhanika* **2000**, *2*, 10–13.
19. Buterin, S.A. On inverse spectral problem for non-selfadjoint Sturm–Liouville operator on a finite interval. *J. Math. Anal. Appl.* **2007**, *335*, 739–749. [CrossRef]
20. Chanane, B. Computing the spectrum of non-self-adjoint Sturm–Liouville problems with parameter-dependent boundary conditions. *J. Comput. Appl. Math.* **2007**, *206*, 229–237. [CrossRef]
21. Veliev, O.A. Non-self-adjoint Sturm-Liouville operators with matrix potentials. *Math. Notes* **2007**, *81*, 440–448. [CrossRef]
22. Albeverio, S.; Hryniv, R.; Mykytyuk, Y. On spectra of non-self-adjoint Sturm–Liouville operators. *Sel. Math.* **2008**, *13*, 571–599. [CrossRef]
23. Buterin, S.A.; Shieh, C.T.; Yurko, V.A. Inverse spectral problems for non-selfadjoint second-order differential operators with Dirichlet boundary conditions. *Bound. Value Probl.* **2013**, *2013*, 180. [CrossRef]

24. Buterin, S.; Kuznetsova, M. On Borg's method for non-selfadjoint Sturm–Liouville operators. *Anal. Math. Phys.* **2019**, *9*, 2133–2150. [CrossRef]
25. Behrndt, J.; Katatbeh, Q.; Trunk, C. Accumulation of complex eigenvalues of indefinite Sturm–Liouville operators. *J. Phys. A Math. Theor.* **2008**, *41*, 244003. [CrossRef]
26. Xie, B.; Qi, J. Non-real eigenvalues of indefinite Sturm–Liouville problems. *J. Differ. Equ.* **2013**, *255*, 2291–2301. [CrossRef]
27. Levitin, M.; Seri, M. Accumulation of complex eigenvalues of an indefinite Sturm–Liouville operator with a shifted Coulomb potential. *arXiv* **2015**, arXiv:1503.08615.
28. Sager, H.C. The Sturm-Liouville equation with time-dependent boundary conditions. *J. Math. Anal. Appl.* **1984**, *102*, 275–287. [CrossRef]
29. Binding, P.A.; Browne, P.J.; Seddighi, K. Sturm–Liouville problems with eigenparameter dependent boundary conditions. *Proc. Edinb. Math. Soc.* **1994**, *37*, 57–72. [CrossRef]
30. Štikonas, A. The Sturm-Liouville problem with a nonlocal boundary condition. *Lith. Math. J.* **2007**, *47*, 336–351. [CrossRef]
31. Adam, J. *Rays, Waves, and Scattering: Topics in Classical Mathematical Physics*; Princeton University Press: Princeton, NJ, USA, 2017.
32. Birkhoff, G.; Rota, G.-C. *Ordinary Differential Equations*, 4th ed.; John Wiley & Sons: New York, NJ, USA, 1989.
33. Everitt, W.N. On the transformation theory of ordinary second-order linear symmetric differential expressions. *Czechoslov. Math. J.* **1982**, *32*, 275–306. [CrossRef]
34. Paine, J.W.; de Hoog, F.R.; Anderssen, R.S. On the correction of finite difference eigenvalue approximations for Sturm-Liouville problems. *Computing* **1981**, *26*, 123–139. [CrossRef]
35. Ledoux, V. Study of Special Algorithms for Solving Sturm-Liouville and Schrödinger Equations. Doctoral Dissertation, Ghent University, Ghent, Belgium, 2007.
36. Ixaru, L.G.; De Meyer, H.; Berghe, G.V. SLCPM12—A program for solving regular Sturm–Liouville problems. *Comput. Phys. Commun.* **1999**, *118*, 259–277. [CrossRef]
37. Everitt, W.N. A catalogue of Sturm-Liouville differential equations. In *Sturm-Liouville Theory: Past and Present*; Amrein, W.O., Hinz, A.M., Pearson, D.B., Eds.; Birkhäuser Verlag: Basel, Switzerland, 2005; pp. 271–331.
38. Buterin, S.A.; Yang, C.F.; Yurko, V.A. On an open question in the inverse transmission eigenvalue problem. *Inverse Probl.* **2015**, *31*, 045003. [CrossRef]
39. Aktosun, T.; Gintides, D.; Papanicolaou, V.G. The uniqueness in the inverse problem for transmission eigenvalues for the spherically symmetric variable-speed wave equation. *Inverse Probl.* **2011**, *27*, 115004. [CrossRef]
40. Karjanto, N. Perturbed potential temperature field in the atmospheric boundary layer. *ZAMM-J. Appl. Math. Mech. Angew. Math. Mech.* **2022**, *102*, e202100484. [CrossRef]
41. Karjanto, N. On modified second Paine–de Hoog–Anderssen boundary value problem. *Symmetry* **2022**, *14*, 54. [CrossRef]
42. Arnold, D.N.; David, G.; Filoche, M.; Jerison, D.; Mayboroda, S. Computing spectra without solving eigenvalue problems. *SIAM J. Sci. Comput.* **2019**, *41*, B69–B92. [CrossRef]
43. Filoche, M.; Mayboroda, S. Universal mechanism for Anderson and weak localization. *Proc. Natl. Acad. Sci. USA* **2012**, *109*, 14761–14766. [CrossRef]
44. Liouville, J. Second mémoire sur le développement des fonctions ou parties de fonctions en séries dont les divers termes sont assujettis à satisfaire à une même équation différentielle du second ordre, contenant un paramètre variable. (Second dissertation on the development of functions or parts of functions in series whose various terms are subject to satisfying the same second order differential equation, containing a variable parameter). *J. Math. Pures Appl. J. Pure Appl. Math.* **1837**, *2*, 16–35. (In French)
45. Bowman, F. *Introduction to Bessel Functions*; Dover: New York, NY, USA, 1958.
46. Gray, A.; Mathews, G.B. *A Treatise on Bessel Functions and Their Applications to Physics*; Macmillan and Co.: London, UK, 1895.
47. Erdélyi, A. *Higher Transcendental Functions, Volume 2*; McGraw-Hill: New York, NY, USA, 1953.
48. Watson, G.N. *A Treatise on the Theory of Bessel Functions*, 2nd ed.; Cambridge University Press: Cambridge, UK, 1996.
49. Korenev, B.G. *Bessel Functions and Their Applications*; CRC Press: Boca Raton, FL, USA, 2002.
50. Press, W.H.; Teukolsky, S.A.; Vetterling, W.T.; Flannery, B.P. *Numerical Recipes: The Art of Scientific Computing*, 3rd ed.; Cambridge University Press: New York, NY, USA, 2007.
51. Nambudiripad, K.B.M. *Bessel Functions*; Alpha Science International Limited: Oxford, UK, 2014.