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μ-Integrable Functions and Weak Convergence of Probability Measures in Complete Paranormed Spaces

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Abstract: This paper works with functions defined in metric spaces and takes values in complete paranormed vector spaces or in Banach spaces, and proves some necessary and sufficient conditions for weak convergence of probability measures. Our main result is as follows: Let X be a complete paranormed vector space and Ω an arbitrary metric space, then a sequence $\{\mu_n\}$ of probability measures is weakly convergent to a probability measure μ if and only if $\lim\limits_{n\to\infty}\int_\Omega g(s)d\mu_n = \int_\Omega g(s)d\mu$ for every bounded continuous function g: $\Omega \to X$. A special case is as the following: if X is a Banach space, Ω an arbitrary metric space, then $\{\mu_n\}$ is weakly convergent to μ if and only if $\lim\limits_{n\to\infty}\int_\Omega g(s)d\mu_n = \int_\Omega g(s)d\mu$ for every bounded continuous function g: $\Omega \to X$. Our theorems and corollaries in the article modified or generalized some recent results regarding the convergence of sequences of measures.

Keywords: finite measure; Banach space; complete paranormed space; μ-integral function; weak convergence of measures

MSC: 28A33; 46G12; 60B10

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1. Introduction and Terminology

If (Ω, Σ) is a measurable space, a sequence $\{\mu_n\}$ of probability measures is weakly convergent to probability measure μ if $\lim\limits_{n\to\infty}\int_\Omega g(s)d\mu_n = \int_\Omega g(s)d\mu$ for every bounded continuous function g: $\Omega \to (-\infty, +\infty)$. Nielsen [1] proved that if Ω is a polish metric space, and if X is a Banach space, then $\{\mu_n\}$ is weakly convergent to μ if and only if $\lim\limits_{n\to\infty}\int_\Omega g(s)d\mu_n = \int_\Omega g(s)d\mu$ for every bounded continuous function g: $\Omega \to X$. Yang [2] discussed the situation of g: $\Omega \to (-\infty, +\infty)$ and proved that the function g is not necessarily point-wise continuous. Wei [3] worked with functions taking values in a metric space X and assumed that $\{\mu_n\}$ is a tight and weak convergence of the finite dimensional distributions of $\{\mu_n\}$ to μ.

There is a rich bibliography concerning the convergence of sequences of measures, besides the above-mentioned studies [1–3]; see, for example, [4–8].

Let K be the field of real numbers or the field of complex numbers and X be a vector space over the number field K. A **paranormed space** is a pair $(X, ||\cdot||)$, where $||\cdot||$ is a function, called a **paranorm**, such that

(a) $||x|| \geq 0$, $||x|| = 0 \Leftrightarrow x = 0$;
(b) $||x+y|| \leq ||x|| + ||y||$;
(c) $||-x|| = ||x||$;
(d) $\lim\limits_{\alpha\to 0}||\alpha x|| = 0$, $\lim\limits_{x\to 0}||\alpha x|| = 0$.

Since $||x-y|| \leq ||x|| + ||y||$, for $\forall x, y \in X$, $||x-y||$ defines a **metric** in a paranormed space.

In what follows, paranormed paces will always be regarded as metric spaces with respect to the metric $||\cdot||$.

It is known that a normed vector space is a paranormed vector space, but a paranormed space is not necessarily a normed vector space. We note that, compared with the definition of "norm", "paranorm" is just without the property of positive homogeneity, replaced by the weaker conditions (c) and (d).

Any Banach space is a complete paranormed space, but the converse is not true.

A complete paranormed space is called a **Fréchet space** in Bourbaki's terminology.

Assume that Σ is the σ-algebra of all Borel measurable sets in Ω. An **additive measure** μ on Σ is called a **finite measure** if Ω ∈ Σ and μ (Ω) < ∞. (Ω, Σ, μ) is called a **finite measure space** if the measure μ on Σ is finite. If μ (Ω) = 1, then μ is said a **probability measure**. Generally speaking, we may consider that a finite measure and a probability measure are one thing.

A function g: $\Omega \to X$ is called **μ-measurable** (or **measurable** if no confusion arises) if for any scalar α

$$\{s \in \Omega; g(s) < \alpha\}$$

is a **μ-measurable** subset.

A function g: $\Omega \to X$ is called a **simple function** if there are measurable sets $B_j \in \Sigma$ with $B_i \cap B_j = \Phi$ ($i \neq j$) and $x_j \in X$ (j = 1, 2, ..., k) such that

$$g(s) = \begin{cases} x_j, & if \quad s \in B_j (j = 1, \cdots, k) \\ 0, & otherwise \end{cases}$$

The **μ-integral** (or **integral** if without confusion) of the simple function g is defined as

$$\int_\Omega g(s) d\mu = \sum_{j=1}^k x_j \mu B_j$$

A function g: $\Omega \to X$ is called **μ-integrable** (or **integrable** if no confusion arises) if there exists a sequence of $\{g_n\}$ of simple functions such that

$$\text{(a) } \lim_{n\to\infty} g_n(s) = g(s), \ \mu\text{-}\mathbf{a.e.} \text{ on } \Omega \text{ (almost everywhere on } \Omega) \tag{1}$$

i.e., there exists $A \in \Sigma$ and $\mu(A) = 0$ such that $\lim_{n\to\infty} g_n(s) = g(s)$ for $s \in \Omega \backslash A$;

(b) For every continuous seminorm p on X,

$$\lim_{n\to\infty} \int_\Omega p(g_n(s) - g(s)) d\mu = 0 \tag{2}$$

In this case, $\lim_{n\to\infty} \int_\Omega g_n(s) d\mu$ exists, and the **μ-integral** (or **integral** if no confusion arises) of g is defined as

$$\int_\Omega g(s) d\mu = \lim_{n\to\infty} \int_\Omega g_n(s) d\mu.$$

When X is a Banach space, the μ-integral is known as the **Bochner integral**.

2. Some Lemmas

In what follows, let Ω be an arbitrary metric space, Σ the σ-algebra of all Borel measurable sets in Ω, Π a family of additive finite measures on Σ, and $(X, || \cdot ||)$ a complete paranormed space.

A **seminorm** is map p: $X \to K$ satisfying

(a) $p(x) \geq 0$;
(b) $p(x + y) \leq p(x) + p(y)$;
(c) $p(\alpha x) = |\alpha| p(x)$ for any scalar α.

Compared with the definition of "norm", "seminorm" is without the property of "faithfulness": $x \neq 0$ does not imply $p(x) > 0$.

A subset $Y \subseteq X$ is said to be a bounded set, if $\forall p \in P$, there exists a constant $C_p > 0$ such that $p(y) \leq C_p, \forall y \in Y$.

The following Lemma 1 is a well-known result in functional analysis.

Lemma 1. *X is a complete paranormed space if and only if there is a family of continuous seminorms $P = \{p_n;\ n = 1, 2, \ldots\}$ on X, such that*

$$p_1(x) \leq p_2(x) \leq \cdots \leq p_n(x) \leq \cdots,\ \forall x \in X;$$

And the paranorm on X can be given by

$$\|x\| = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x)}{1 + p_n(x)},\ \forall x \in X.$$

Furthermore, for any topological net $\{x_\tau\} \subset X$, and $x \in X$, the following are equivalent

(a) $\lim\limits_{\tau} x_\tau = x$;

(b) $\lim\limits_{\tau} x_\tau - x = 0$;

(c) $\lim\limits_{\tau} p(x_\tau - x) = 0, \forall p \in P$.

A set $A \subset X$ is called separable if A has a countable dense subset, i.e., there exists a countable subset $B \subset A$ such that $\overline{B} = A$, where $\overline{B}$ is the topological closure of B. B is called a countable dense subset of A.

*For $A \subset X$, **span A** is the set of all possible linear combinations of the elements in A.*

The following Lemma 2 has an independent interest. There were different discussions of Lemma 2, which can be seen in [9,10].

Lemma 2. *Suppose $\mu \in \Pi$. A μ-measurable function g: $\Omega \to X$ is μ-integrable if and only if*

(a) *g is μ-essential separable valued, i.e., there exists $E \in \Sigma$ with $\mu E = 0$, such that $g(\Omega \backslash E)$ is a separable subset of X;*

(b) *$\int\limits_{\Omega} p(g(s))d\mu < \infty, \forall p \in P$, where P is defined as in Lemma 1.*

Proof.

The sufficiency is as follows:

Suppose $E \in \Sigma$ with $\mu E = 0$, and suppose that $g(\Omega \backslash E)$ is a separable subset of X, and $\{x_n\} \subset X$ a countable dense subset of $g(\Omega \backslash E)$. Take

$$A_{n,k} = \{s \in \Omega; p_k(g(s) - x_n) < \frac{1}{k}\},$$

since g is a μ-measurable function, $A_{n,k}$ are measurable sets, i.e., $A_{n,k} \in \Sigma$. Define

$$g_k(s) = \begin{cases} x_n, & s \in A_{n,k} \backslash \cup_{m<n} A_{m,k} \\ 0, & otherwise \end{cases},\ (n = 1,\ 2,\ \ldots)$$

Then,

$$p_i(g_k(s) - g(s)) \leq p_k(g_k(s) - g(s)) \leq \frac{1}{k}$$

μ-a.e. on Ω, for $k \geq i$. It follows that, $\forall i$

$$\lim_{k \to \infty} p_i(g_k(s) - g(s)) = 0,\ \text{μ-a.e.}$$

That is to say

$$\lim_{k\to\infty} g_k(s) = g(s),\ \text{μ-a.e} \tag{3}$$

Therefore,

$$\lim_{k\to\infty} g_k(s) - g(s) = 0,\ \text{μ-a.e.}$$

Note that g_k can be written as

$$g_k(s) = \sum_{n=1}^{\infty} x_n \chi_{B_{n,k}}$$

where $B_{n,k} = A_{n,k} \setminus \cup_{m<n} A_{m,k}$, $B_{n,i} \cap B_{n,j} = \Phi$ $(i \neq j)$, and $\chi_{B_{n,k}}$ is the **characteristic function** of the set $B_{n,k}$, i.e.,

$$\chi_{n,k}(s) = \begin{cases} 1 & s \in B_{n,k} \\ 0, & otherwise. \end{cases}$$

Since $\Omega = \cup_{n=1}^{\infty} B_{n,k}$ and $\mu\Omega < \infty$, for each k, we can choose $l(k)$ satisfying

$$\mu\left(\cup_{j=l(k)+1}^{\infty} B_{j,k}\right) < \frac{1}{k}.$$

Take

$$g_k{}'(s) = \sum_{j=1}^{l(k)} x_j \chi_{B_{j,k}},$$

Then, for each k, $g_k{}'(s)$ is a simple function and

$$\mu\{s \in \Omega; \|g_k(s) - g_k{}'(s)\| > 0\} \leq \mu\left(\cup_{j=l(k)+1}^{\infty} B_{k,j}\right) < \frac{1}{k}.$$

So, there exists a subsequence of $\{g_k - g_k{}'\}$ that converges to 0 μ-a.e. We may assume, without loss of generality, that

$$\lim_{k\to\infty} (g_k(s) - g_k{}'(s)) = 0,\ \text{μ-a.e.}$$

This and (3) imply that

$$\lim_{k\to\infty} g_k{}'(s) = g(s),\ \text{μ-a.e.}$$

On the other hand, $\forall s \in \Omega$

$$\begin{aligned}
&p_i(g_k{}'(s) - g(s)) \\
&\leq p_i(g_k{}'(s) - g_k(s)) + p_i(g_k(s) - g(s)) \\
&\leq p_i(g_k{}'(s)) + p_i(g_k(s)) + p_i(g_k(s) - g(s)) \\
&\leq 2p_i(g_k(s)) + p_i(g_k(s) - g(s)) \\
&\leq 2[p_i(g_k(s) - g(s)) + p_i(g(s))] + p_i(g_k(s) - g(s)) \\
&\leq 2p_i(g(s)) + 3p_i(g_k(s) - g(s)) \\
&\leq 2p_i(g(s)) + 3
\end{aligned}$$

Since $\int_\Omega p(g(s))d\mu < \infty$, $\forall p \in P$, an application of Lebesgue's Dominated Convergence Theorem, in our case shows that

$$\lim_{k\to\infty} \int_\Omega p_i(g_k{}'(s) - g(s))d\mu = 0,\ \forall p_i \in P.$$

This proves that g: $\Omega \to X$ is μ-integrable.

The necessity is as follows:

Suppose g: $\Omega \to X$ is μ-integrable, then the combination of

$$|\int\limits_{\Omega} p(g_n(s))d\mu - \int_{\Omega} p(g(s))d\mu| \leq \int\limits_{\Omega} p(g_n(s) - g(s))d\mu$$

and (2) imply that $\lim_{n\to\infty}\int\limits_{\Omega} p(g_n(s))d\mu$ exists, and

$$\int\limits_{\Omega} p(g(s))d\mu = \lim_{n\to\infty} \int_{\Omega} p(g_n(s))d\mu < \infty$$

Moreover, from (1), there exists $E \in \Sigma$ with $\mu E = 0$, and there exists an at most countable set $\cup_{n=1}^{\infty} g_n(\Omega)$, such that

$$\overline{\cup_{n=1}^{\infty} g_n(\Omega)} = g(\Omega \backslash E) \cup (\cup_{n=1}^{\infty} g_n(\Omega)).$$

Then, as a subset of a separable set $g(\Omega\backslash E) \cup (\cup_{n=1}^{\infty} g_n(\Omega))$, $g(\Omega\backslash E)$ is separable.

This completes the proof of Lemma 2. □

Lemma 3. *If g: $\Omega \to X$ is a bounded continuous function, then g is μ-integrable.*

Proof. Note that $p_k(g(s))$: $\Omega \to (-\infty, +\infty)$ are bounded continuous functions (k = 1, 2, ...). Therefore, each $p_k(g(s))$ is μ-integrable, and so each $p_k(g(s))$ is μ-essential separable valued, i.e., there exists $E_k \in \Sigma$ with $\mu E_k = 0$, such that $p_k(g(\Omega\backslash E_k))$ is a separable subset of X. Assume that $\left\{s_n^k\right\} \subset (-\infty, +\infty)$ is a countable dense subset of $p_k(g(\Omega\backslash E_k))$, then there exists a sequence $\left\{\omega_n^k\right\}_{n=1}^{\infty} \subset \Omega$ such that $g(\omega_n^k) = x_n^k \in X$ and $p_k(g(x_n^k)) = s_n^k$ (n = 1, 2, ...). Take

$$A_{n,k} = \{s \in \Omega\backslash E_k; p_k(g(s) - x_n^k) < \frac{1}{k}\},$$

since g is a μ-measurable function, $A_{n,k}$ are measurable sets, i.e., $A_{n,k} \in \Sigma$. Define

$$g_k(s) = \begin{cases} x_n^k, & s \in A_{n,k} \backslash \cup_{m<n} A_{m,k} \\ 0, & otherwise \end{cases}, \ (n = 1, 2, \ldots).$$

Therefore,

$$p_k(g_k(s) - g(s)) \leq \frac{1}{k}, \ (k = 1, 2, \ldots).$$

From Lemma 2, $g_k(s)$ are μ-integrable functions. The inequalities above and Lemma 2 ($p_1(x) \leq p_2(x) \leq \cdots \leq p_l(x) \leq \cdots$, $\forall x \in X$) deduce that

$$\lim_{k\to\infty} \int\limits_{\Omega} p(g_k(s) - g(s))d\mu = 0, \ \forall p \in P.$$

Using Theorem 4 in Zeng [11], the limit of a sequence of μ-integrable functions is μ-integrable in a complete paranormed space. Therefore, g: $\Omega \to X$ is a μ-integrable function. □

3. Weak Convergence of Finite Measures

A sequence $\{\mu_n\} \subset \Pi$ is called **weakly convergent** to $\mu \in \Pi$ if

$$\lim_{n\to\infty} \int_{\Omega} g(s)d\mu_n = \int_{\Omega} g(s)d\mu$$

for every bounded continuous function g: $\Omega \to (-\infty, +\infty)$.

Theorem 1. *Suppose that Ω is an arbitrary metric space, X is a complete paranormed space over the field K, and $\mu, \mu_n \in \Pi$ (n = 1, 2, ...). Then, $\{\mu_n\}$ is weakly convergent to μ if and only if*

$$\lim_{n\to\infty}\int_{\Omega} g(s)d\mu_n = \int_{\Omega} g(s)d\mu$$

for every bounded continuous function g: $\Omega \to X$.

Proof.

The necessity is as follows:

Let g: $\Omega \to X$ be a bounded continuous function. From Lemma 3, g is integrable for all $\mu, \mu_n \in \Pi$ (n = 1, 2, ...).

From the proof of Lemma 3, there exists a sequence $\{g_k(s)\}$ of functions defined as

$$g_k(s) = \begin{cases} x_j^k, & s \in A_{j,k} \setminus \cup_{i<j} A_{i,k} \\ 0, & otherwise \end{cases}, \ (j = 1, 2, \ldots).$$

with

$$p_k(g_k(s) - g(s)) \le \frac{1}{k},$$

such that

$$\lim_{k\to\infty} \int\limits_{\Omega} p(g_k(s) - g(s))d\mu = 0, \ \forall p \in P. \tag{4}$$

Given $p \in P$.

For any given $\varepsilon > 0$, take k_0 such that

$$\int\limits_{\Omega} p(g_{k_0}(s) - g(s))d\mu < \varepsilon \tag{5}$$

On the other hand, the weak convergence of $\{\mu_n\}$ to μ on Ω implies that

$$\lim_{n\to\infty} p(\int_{\Omega} g_{k_0}(s)d\mu_n - \int_{\Omega} g_{k_0}(s)d\mu) = 0.$$

Then, there exists $N > 0$, when $n \ge N$

$$p(\int_{\Omega} g_{k_0}(s)d\mu_n - \int_{\Omega} g_{k_0}(s)d\mu) < \varepsilon. \tag{6}$$

Again, from (4), $p(g_k(s) - g(s))$ converges in measure μ, which is to say

$$\lim_{k\to\infty} \mu\{s \in \Omega; p(g_k(s) - g(s)) \ge \varepsilon\} = 0.$$

Let

$$A = \{s \in \Omega; p(g_{k_0}(s) - g(s)) \ge \varepsilon\}.$$

Take k_0 to be large enough such that both (5) and the following hold true:

$$\mu A = \mu\{s \in \Omega; p(g_{k_0}(s) - g(s)) \ge \varepsilon\} < \varepsilon.$$

Since the weak convergence of $\{\mu_n\}$ to μ implies

$$\lim_{n\to\infty} \mu_n A = \mu A, \text{ and } \lim_{n\to\infty} \mu_n(\Omega) = \mu(\Omega),$$

one can take n to be large enough, say, $n \ge N$, such that

$$\mu_n A < \mu A + \varepsilon, \ \mu_n(\Omega) < \mu(\Omega) + \varepsilon.$$

Therefore, when $n \geq N$,

$$\begin{aligned} &\int_\Omega p(g_{k_0}(s) - g(s))d\mu_n \\ &= \int_A p(g_{k_0}(s) - g(s))d\mu_n + \int_{\Omega\setminus A} p(g_{k_0}(s) - g(s))d\mu_n \\ &\leq 2C_p\mu_n A + \varepsilon\mu_n(\Omega) \\ &< M\varepsilon, \end{aligned} \tag{7}$$

for a given scalar $M > 0$, where we assume that the bonded functions satisfy the conditions $p(g_k(s)) \leq C_p, p(g(s)) \leq C_p$.

Hence, from (5), (6), and (7), when $n \geq N$,

$$\begin{aligned} &p(\int_\Omega g(s)d\mu_n - \int_\Omega g(s)d\mu) \\ &\leq p(\int_\Omega g(s)d\mu_n - \int_\Omega g_{k_0}(s)d\mu_n) \\ &+p(\int_\Omega g_{k_0}(s)d\mu_n - \int_\Omega g_{k_0}(s)d\mu) \\ &+p(\int_\Omega g_{k_0}(s)d\mu - \int_\Omega g(s)d\mu) \\ &< (M+2)\varepsilon. \end{aligned}$$

That is to say, for each bounded continuous function g: $\Omega \to X$, and each $p \in P$

$$\lim_{n\to\infty} p(\int_\Omega g(s)d\mu_n - \int_\Omega g(s)d\mu) = 0.$$

Therefore, for each bounded continuous function g: $\Omega \to X$, one has

$$\lim_{n\to\infty} \int_\Omega g(s)d\mu_n = \int_\Omega g(s)d\mu.$$

The sufficiency is as follows:

Suppose that for each bounded continuous function g: $\Omega \to X$,

$$\lim_{n\to\infty} \int_\Omega g(s)d\mu_n = \int_\Omega g(s)d\mu.$$

For given function g: $\Omega \to (-\infty, +\infty)$, take $x_0 \in X$ with $x_0 \neq 0$, and define the function f: $\Omega \to X$:

$$f(s) = g(s)x_0, s \in \Omega.$$

Then, $\forall p \in P$

$$\begin{aligned} &\lim_{n\to\infty} |\int_\Omega g(s)d\mu_n - \int_\Omega g(s)d\mu| \\ &= \lim_{n\to\infty} |\int_\Omega g(s)d\mu_n - \int_\Omega g(s)d\mu| p(x_0) \\ &= \lim_{n\to\infty} p((\int_\Omega g(s)d\mu_n - \int_\Omega g(s)d\mu)x_0) \\ &= \lim_{n\to\infty} p(\int_\Omega g(s)x_0 d\mu_n - \int_\Omega g(s)x_0 d\mu) \\ &= 0, \end{aligned}$$

which completes the proof. □

From Theorem 1, we have the following Corollary 1.

Corollary 1. *Let Ω be a polish metric space, X a Banach space, and μ, μ_n are finite measures defined on Ω (n = 1, 2, ...). Then, $\{\mu_n\}$ is weakly convergent to μ if and only if*

$$\lim_{n\to\infty} \int_\Omega g(s)d\mu_n = \int_\Omega g(s)d\mu$$

for every bounded continuous function g: $\Omega \to X$.

Theorem 1 can be stated as the following Theorem 2, a result in probability distribution theory.

Theorem 2. *Suppose that Ω is an arbitrary metric space, and X is a complete paranormed space over the field K. Let $\varsigma_n \in \Omega$ be random elements (n = 1, 2, . . .), E the mathematical expectation operator, then $\{\varsigma_n\}$ converges in probability distribution to a random element $\varsigma \in \Omega$ if and only if for each bounded continuous function g: $\Omega \to X$, there holds*

$$\lim_{n\to\infty} Eg(\varsigma_n) = Eg(\varsigma).$$

From Theorem 1, we have the following Theorem 3.

Theorem 3. *Let Ω be an arbitrary metric spac, and X a Banach space. A sequence $\{\mu_n\} \subset \Pi$ is weakly convergent to $\mu \in \Pi$ if and only if*

$$\lim_{n\to\infty} \int_\Omega g(s)d\mu_n = \int_\Omega g(s)d\mu$$

for every bounded continuous function g: $\Omega \to X$.

Theorem 3 can be rewritten as the following Theorem 4.

Theorem 4. *Suppose that Ω is an arbitrary metric space and X a Banach space. Let $\varsigma_n \in \Omega$ be random elements (n = 1, 2, . . .), E the mathematical expectation operator, then $\{\varsigma_n\}$ converges in probability distribution to a random element $\varsigma \in \Omega$ if and only if for each bounded continuous function g: $\Omega \to X$, there holds*

$$\lim_{n\to\infty} Eg(\varsigma_n) = Eg(\varsigma)$$

4. Conclusions

In references [6,7], the authors discussed Riemann–Lebesgue integrals, while our discussion is about the Bochner integral (or similarly, Pettis integral) in abstract spaces.

References [4,5] also worked with convergence for sequences of measures. Although [4] supposed that their functions were defined on Hausdorff topological spaces (see the first paragraph of Section 2 in [4]), they were taking values as scalars—their proofs were carried out by using absolute values. Some results of Reference [5] are for "vector-valued functions" (see Page 14 "Section 3.1 The vector case for integrals" in [5])—the proofs were carried out by using a norm—as a Banach space has.

References [4–7] all required that measures are bounded and converge "set-wisely". Some results in [4] require the sequence to converge in value and/or uniformly and absolutely continuously, while similar results in [6] require the measures to be finite-valued and/or increasing. Our results, however, only require "a sequence of bounded measures"—weaker conditions compared with [4–7]—in which both sequences of functions and measures are considered. Our functions take values in a paranormed linear space, which is also weaker than the conditions of a normed linear space in [5] or finite dimensional space in [4,6,7].

Our results extended Proposition 3.2 in [4] and Corollary 3.8 in [5]; and modified Corollary 2.1 in [5], Theorem 3.4 and 3.5 in [6], as well as Lemma 4.1 in [7].

Theorem 1 in this article is a modification of Theorem 2.1 and 2.2 in [8]. Theorem 2.1 and 2.2 in [8] require non-negative functions *f*.

Theorem 3 is a generalization of Corollary 1. Corollary 1 is the main result in [1]. Compared with [1], we do not have the condition of "polish metric space". In this study, we obtained the same results but required weaker conditions.

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