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The Approximation Characteristics of Weighted Band-Limited Function Space

Yue Li [1,†], Guanggui Chen [2,†], Yanyan Xu [1,*] and Xiangyu Pan [1]

[1] School of Science, Xihua University, Chengdu 610039, China; liyue7@stu.xhu.edu.cn (Y.L.); panxiangyu1997@gmail.com (X.P.)
[2] Yibin Campus, Xihua University, Yibin 644005, China; ggchen@mail.xhu.edu.cn
* Correspondence: xuyyts@163.com or xuyy@mail.xhu.edu.cn
† These authors contributed equally to this work.

Abstract: This article primarily investigates the width problem within weighted band-limited function space in a uniform setting. Through an analysis of the properties of s-numbers, we establish a connection between the widths of weighted band-limited function spaces and the s-numbers of infinite-dimensional diagonal operators. Furthermore, employing the discretization method, we estimate the exact asymptotic orders of Kolmogorov n-width and linear n-width in the weighted band-limited function space, which is characterized by the weight $\omega = \{\omega_k\} = \{|k|^r\}_{k\in\mathbb{Z}_0}$.

Keywords: band-limited; s-number; diagonal operators; n-width

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1. Introduction

It is well known that weighted band-limited function spaces have broad applications in communication theory, functional analysis, and data processing [1–4]. Additionally, they serve as mathematical tools for function approximation [5,6], thus attracting extensive research attention from scholars and yielding a series of elegant and profound results [7–9]. In comparison to the classical band-limited function space, weighted band-limited function space involves the application of signals or functions in the frequency domain (Fourier transform domain). In practical scenarios, the mitigation of noise influence can be achieved through judicious selection of weighted functions, particularly focusing on specific frequencies. These weighted functions are instrumental in signal reconstruction and information transmission optimization. In approximation theory, scholars have investigated the width of the weighted function space to address function approximation challenges posed by non-uniform data points and weight conditions [10,11]. The width problem of weighted function space pertains to the integration of weighted functions within function approximation and interpolation theory, aiming to enhance the performance and approximation capabilities of interpolation functions. Therefore, this paper will further these investigations by assigning numerical weights to band-limited functions, thereby creating weighted band-limited function spaces, and investigating the width problem within these weighted band-limited function spaces.

Let $\mathbb{N}$ denote the set of natural numbers, $\mathbb{N}_+$ represent the set of non-negative integers, $\mathbb{Z}$ signify the set of integers, and $\mathbb{Z}_0$ designate the set of non-zero integers. Furthermore, $\mathbb{R}$ and $\mathbb{C}$ respectively symbolize the real numbers and complex numbers.

Consider two normed linear spaces, denoted as $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, both defined over the same field. The set of all bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. The norm of a bounded linear operator A from X to Y is expressed as $\|A : X \to Y\|$ or simply $\|A\|$. The notation $X \hookrightarrow Y$ signifies the continuous embedding of X into Y.

When considering two positive functions, namely $a(x)$ and $b(x)$ defined on a common set F, the relation $a(x) \preccurlyeq b(x)$ is employed to indicate the existence of a positive constant c_1 independent of variable x, such that $a(x) \leq c_1 \cdot b(x)$. And $a(x) \succcurlyeq b(x)$ indicates that there exists a positive constant c_2 independent of variable x, such that $a(x) \geq c_2 \cdot b(x)$. Simultaneously, the notation $a(x) \asymp b(x)$ is utilized to convey the existence of two positive constants c_1 and c_2, independent of the variable x, such that $c_1 \cdot b(x) \leq a(x) \leq c_2 \cdot b(x)$.

The structure of this paper is outlined as follows. Section 2 introduces the concepts of width, s-number, and weighted band-limited function spaces. Section 3 establishes the connection between the width of weighted band-limited function spaces and the width problem of infinite-dimensional diagonal operators. Section 4 provides precise asymptotic estimates for the width of weighted band-limited function spaces under specific weight conditions.

2. The Widths, s-Numbers, and Band-Limited Function Space

We start with the basic concept of widths.

Definition 1 ([12]). *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces over the same field, $X \hookrightarrow Y$ denote the continuous embedding of X into Y and $n \in \mathbb{N}$. The Kolmogorov n-width and linear n-width of X in Y are defined as*

$$d_n(X, Y) := \inf_{L_n} \sup_{x \in B_X} \inf_{y \in L_n} \|x - y\|_Y,$$

$$a_n(X, Y) := \inf_{T_n} \sup_{x \in B_X} \|x - T_n x\|,$$

where L_n runs through all possible linear subspaces of X of dimension at most n, T_n runs over all linear operators from X to Y with rank at most n, and B_X represents the unit closed ball in X.

The notion of width, introduced by Kolmogorov in the 1940s, has garnered extensive investigation due to its close association with computational complexity. Detailed insights into width can be found in Pinkus' monograph [13].

Subsequently, we proceed to introduce s-numbers and their properties, which play a pivotal role in the proofs presented in the third section of this paper.

Definition 2 ([14]). *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces over the same field, $T \in \mathcal{L}(X, Y)$, and $n \in \mathbb{N}$. The n-th Kolmogorov number and the n-th approximation number and of the operator T are defined as*

$$d_n(T) = d_n(T : X \to Y) = \inf_{F_n} \sup_{\|x\| \in B_X} \inf_{y \in F_n} \|Tx - y\|_Y,$$

$$a_n(T) = a_n(T : X \to Y) = \inf(\|T - A\| : A \in \mathcal{L}(X, Y), rank A \leq n),$$

where F_n runs through all possible linear subspaces of Y of dimension at most n, and B_X represents the unit closed ball in X. The n-th Kolmogorov number and the n-th approximation number are collectively referred to as s-numbers.

Evidently, if $X \hookrightarrow Y$, then we have

$$d_n(X, Y) = d_n(id : X \to Y), \quad a_n(X, Y) = a_n(id : X \to Y). \tag{1}$$

Here, id represents the identity operator from X to Y. For expediency, in the ensuing discourse of this article, unless explicitly stated otherwise, the symbol $s_n(X, Y) = s_n(id) = s_n(id : X \to Y)$ denotes either $d_n(X, Y) = d_n(id) = d_n(id : X \to Y)$ or $a_n(X, Y) = a_n(id) = a_n(id : X \to Y)$.

Detailed information about s-numbers can be found in references [14,15]. Here, we introduce a particular property of s-numbers, which plays a crucial role in the proof presented in this paper.

Lemma 1 ([15]). *Let X_0, X, Y, Y_0 be Banach spaces on the same number field, $T \in \mathcal{L}(X_0, X)$, $S \in \mathcal{L}(X, Y)$, $R \in \mathcal{L}(Y, Y_0)$, and $n \in \mathbb{N}$. Then*

$$s_n(RST) \le \|R\| s_n(S) \|T\|.$$

The notation $L_p(\mathbb{R})$, where $1 < p < \infty$, denotes a classical Lebesgue space defined on the real numbers $\mathbb{R}$, characterized by integrability of p-th power, and equipped with the norm denoted by $\|\cdot\|_{L_p(\mathbb{R})}$. Similarly, we utilize $l_p(\Omega)$ to represent the conventional real sequence space defined on $\mathbb{R}$, demonstrating p-power summability, and equipped with the norm denoted by $\|\cdot\|_{l_p(\Omega)}$, where Ω belongs to the set $\{\mathbb{Z}, \mathbb{Z}_0, \mathbb{N}, \mathbb{N}_+\}$. Specifically, the notation l_p is utilized to represent $l_p(\mathbb{N}_+)$, where $\|\cdot\|_{l_p}$ functions as a concise representation for $\|\cdot\|_{l_p(\mathbb{N}_+)}$.

Subsequently, our attention will be directed towards the commencement of the discourse on function space.

Let $\sigma > 0$, $g(z)$ be an entire function on $\mathbb{C}$, for every $\varepsilon > 0$. If there is a positive constant $A{=}A(\varepsilon)$ that only related to ε , such that

$$|g(z)| \le A \exp((\sigma + \varepsilon)|z|), \quad \forall z \in \mathbb{C},$$

then $g(z)$ is said to be an entire function of exponential type σ.

Denote by B_σ the set encompassing all entire functions of exponential type σ that are bounded when restricted to $\mathbb{R}$.

Let $B_{\sigma,p}(\mathbb{R}) = B_\sigma(\mathbb{R}) \cap L_p(\mathbb{R})$, for $1 \le p \le \infty$, $\sigma > 0$. It follows that $B_{\sigma,p}(\mathbb{R})$ equipped with the norm $||\cdot||_{L_p}$ forms a Banach space. This Banach space is referred to as the band-limited function space. Specifically, the well-known Paley-Wiener space is denoted as $B_{\sigma,2}(\mathbb{R})$.

For $f \in L_1(T^d)$, the Fourier transform of f is defined as follows,

$$\hat{f}(x) = (2\pi)^{-d} \int_{T^d} f(x) e^{i(x,t)} dt,$$

and denote $\hat{f}(k), k \in \mathbb{Z}^d$, as the Fourier coefficients of f.

According to Schwartz's theorem [16], we have

$$B_{\sigma,p}(\mathbb{R}) = \Big\{ f \in L_p(\mathbb{R}), supp\hat{f} \subset [-\sigma, \sigma] \Big\},$$

where $\hat{f}$ represents the Fourier transform of f.

Lemma 2 ([5]). *Let $1 < p < \infty$, $\sigma > 0$.*

(1) Assume that $f \in B_{\sigma,p}(\mathbb{R})$, then

$$f(x) = \sum_{k \in \mathbb{Z}} f\left(\frac{k\pi}{\sigma}\right) \mathrm{sinc}\left(\sigma\left(x - \frac{k\pi}{\sigma}\right)\right), \quad \forall x \in \mathbb{R}, \tag{2}$$

and the series on the right-hand side converges absolutely and uniformly to $f(x)$ on $\mathbb{R}$, where $\mathrm{sinc}t = \begin{cases} \frac{\sin t}{t}, t \neq 0 \\ 1, \quad t = 0 \end{cases}$.

(2) For any $f \in B_{\sigma,p}(\mathbb{R})$, there exists two positive constants c_1 and c_2 depending only on p and σ, such that

$$c_1 \left(\sum_{k \in \mathbb{Z}} \left| f\left(\frac{k\pi}{\sigma}\right) \right|^p \right)^{\frac{1}{p}} \le \|f\|_{L_p} \le c_2 \left(\sum_{k \in \mathbb{Z}} \left| f\left(\frac{k\pi}{\sigma}\right) \right|^p \right)^{\frac{1}{p}}, \tag{3}$$

which implies

$$\|f\|_{L_p(\mathbb{R})} \asymp \left\|\left\{f\left(\frac{k\pi}{\sigma}\right)\right\}\right\|_{l_p(\mathbb{Z})}.$$

(3) For any $y = \{y_k\} \in l_p(\mathbb{Z})$, there exists a unique $g \in B_{\sigma,p}(\mathbb{R})$ such that

$$g\left(\frac{k\pi}{\sigma}\right) = y_k, \quad k \in \mathbb{Z}.$$

Remark 1. *(1) Lemma 2 asserts that for any sequence $y = \{y_k\} \in l_p(\mathbb{Z})$, there exists a unique function $g \in B_{\sigma,p}(\mathbb{R})$ satisfying the equations:*

$$g(x) = \sum_{k\in\mathbb{Z}} y_k \mathrm{sinc}\left(\sigma\left(x - \frac{k\pi}{\sigma}\right)\right), \quad x \in \mathbb{R},$$

and

$$g\left(\frac{k\pi}{\sigma}\right) = y_k, \quad k \in \mathbb{Z}.$$

(2) Utilizing Lemma 2, the space $\mathring{B}_{\sigma,p}(\mathbb{R})$ is defined as

$$\mathring{B}_{\sigma,p}(\mathbb{R}) := \{f \in B_{\sigma,p}(\mathbb{R}) \mid f(0) = 0\},$$

which constitutes a Banach space equipped with the norm $||\cdot||_{L_p(\mathbb{R})}$.

Let $\omega = \{\omega_k\}_{k\in\mathbb{Z}_0}$ denote a sequence of positive real numbers defined on $\mathbb{Z}_0$. Consider a function $f \in \mathring{B}_{\sigma,p}(\mathbb{R})$ with $1 \le p \le \infty$ and $\sigma > 0$. Define the function $f_\omega(x)$ as follows

$$f_\omega(x) = \sum_{k\in\mathbb{Z}_0} \omega_k f\left(\frac{k\pi}{\sigma}\right) \mathrm{sinc}\left(\sigma\left(x - \frac{k\pi}{\sigma}\right)\right), \quad x \in \mathbb{R} \tag{4}$$

where ω_k represents the k-th element of the sequence $\{\omega_k\}_{k\in\mathbb{Z}_0}$.

According to Lemma 2, if the sequence $\left\{\omega_k f\left(\frac{k\pi}{\sigma}\right)\right\} \in l_p(\mathbb{Z}_0)$, then the function f_ω belongs to the space $\mathring{B}_{\sigma,p}(\mathbb{R})$, which is a subset of $B_{\sigma,p}(\mathbb{R})$. Additionally, the expression

$$f_\omega\left(\frac{k\pi}{\sigma}\right) = \omega_k f\left(\frac{k\pi}{\sigma}\right), \quad k \in \mathbb{Z}_0.$$

holds. Moreover, it can be further stated that the norm of f_ω in the Lebesgue space $L_p(\mathbb{R})$ is asymptotically equivalent to the norm of the sequence $\left\{\omega_k f\left(\frac{k\pi}{\sigma}\right)\right\}$ in the sequence space $l_p(\mathbb{Z}_0)$, as expressed by the relation

$$\|f_\omega\|_{L_p(\mathbb{R})} \asymp \left\|\left\{\omega_k f\left(\frac{k\pi}{\sigma}\right)\right\}\right\|_{l_p(\mathbb{Z}_0)}.$$

Therefore, the wighted band-limited function space with the numerical weight ω can be defined as

$$B^{\omega}_{\sigma,p}(\mathbb{R}) := \left\{f \in \mathring{B}_{\sigma,p}(\mathbb{R}) \,\middle|\, \left\{\omega_k f\left(\frac{k\pi}{\sigma}\right)\right\} \in l_p(\mathbb{Z}_0)\right\}, \tag{5}$$

equipped with the norm

$$||f||_{p,\omega} := ||f_\omega||_{L_p(\mathbb{R})},$$

for any $f \in B^{\omega}_{\sigma,p}(\mathbb{R})$.

Obviously, $||\cdot||_{p,\omega}$ is the norm of $B^{\omega}_{\sigma,p}(\mathbb{R})$, and

$$\|f\|_{p,\omega} \asymp \left\| \left\{ \omega_k f\left(\frac{k\pi}{\sigma}\right) \right\} \right\|_{l_p(\mathbb{Z}_0)}. \tag{6}$$

Utilizing Lemma 2 in conjunction with Equation (6), the subsequent proposition can be readily demonstrated.

Proposition 1. *Let $1 < p < \infty$, $\sigma > 0$, $\omega = \{\omega_k\}$ is a sequence of positive real numbers defined on $\mathbb{Z}_0$. Then $B^{\omega}_{\sigma,p}(\mathbb{R})$ is Banach space with the norm $||\cdot||_{p,\omega}$.*

In the subsequent context, unless otherwise specified, $B^{\omega}_{\sigma,p}$ denotes the Banach space $\left(B^{\omega}_{\sigma,p}(\mathbb{R}), ||\cdot||_{p,\omega}\right)$ for brevity, where ω satisfies condition that

$$\lim_{|k|\to\infty} \omega_k = \infty, \inf_{k\in\mathbb{Z}_0} \omega_k = \rho > 0. \tag{7}$$

In the subsequent analysis, we will address the issue of continuous embedding of $B^{\omega}_{\sigma,p}(\mathbb{R})$ into $B_{\sigma,q}(\mathbb{R})$, when $1 < p, q < \infty$.

If $1 < p \le q < \infty$, according to Jackson-Nikolskii inequality and Lemma 2, for any $f \in B^{\omega}_{\sigma,p}(\mathbb{R})$, then we have

$$\begin{aligned}
\|f\|_{L_q(\mathbb{R})} &\le 2\sigma^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L_p(\mathbb{R})} \le 2\sigma^{\frac{1}{p}-\frac{1}{q}} c_2 \left(\sum_{k\in\mathbb{Z}_0} \left| f\left(\frac{k\pi}{\sigma}\right)\right|^p \right)^{\frac{1}{p}} \\
&\le 2\sigma^{\frac{1}{p}-\frac{1}{q}} c_2 \left(\sum_{k\in\mathbb{Z}_0} \left| c\rho f\left(\frac{k\pi}{\sigma}\right)\right|^p \right)^{\frac{1}{p}} = 2cc_2\sigma^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{k\in\mathbb{Z}_0} \left| \rho f\left(\frac{k\pi}{\sigma}\right)\right|^p \right)^{\frac{1}{p}} \\
&\le 2cc_2\sigma^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{k\in\mathbb{Z}_0} \left| \omega_k f\left(\frac{k\pi}{\sigma}\right)\right|^p \right)^{\frac{1}{p}} = \frac{2cc_2}{c_1}\sigma^{\frac{1}{p}-\frac{1}{q}}\|f\|_{p,\omega},
\end{aligned}$$

where c is an absolute positive constant satisfying condition $c\rho \ge 1$. Thus, we have $B^{\omega}_{\sigma,p}(\mathbb{R}) \hookrightarrow B_{\sigma,q}(\mathbb{R})$, when $1 < p \le q < \infty$.

If $1 < q < p < \infty$, let

$$\frac{1}{\omega} = \left\{ \frac{1}{\omega_k} \right\} \in l_r(\mathbb{Z}_0), \tag{8}$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, according to Hölder inequality, then for any $f \in B^{\omega}_{\sigma,p}(\mathbb{R})$, we have

$$\left\| \left\{ f\left(\frac{k\pi}{\sigma}\right) \right\} \right\|_{l_q(\mathbb{Z}_0)} \le \left\| \left\{ \omega_k f\left(\frac{k\pi}{\sigma}\right) \right\} \right\|_{l_p(\mathbb{Z}_0)} \cdot \left\| \frac{1}{\omega} \right\|_{l_r(\mathbb{Z}_0)}. \tag{9}$$

From Lemma 2 and Equation (9), we obtain

$$\begin{aligned}
\|f\|_{L_q(\mathbb{R})} &\le c_2 \left\| \left\{ f\left(\frac{k\pi}{\sigma}\right) \right\} \right\|_{l_q(\mathbb{Z}_0)} \le c_2 \left\| \left\{ \omega_k f\left(\frac{k\pi}{\sigma}\right) \right\} \right\|_{l_p(\mathbb{Z}_0)} \cdot \left\| \frac{1}{\omega} \right\|_{l_r(\mathbb{Z}_0)} \\
&\le \frac{c_2}{c_1}\|f\omega\|_{L_p} \cdot \left\| \frac{1}{\omega} \right\|_{l_r(\mathbb{Z}_0)} \\
&= \frac{c_2}{c_1}\left\| \frac{1}{\omega} \right\|_{l_r(\mathbb{Z}_0)} \|f\|_{p,\omega},
\end{aligned}$$

which shows that $B^{\omega}_{\sigma,p}(\mathbb{R}) \hookrightarrow B_{\sigma,q}(\mathbb{R})$, when $1 < q < p < \infty$.

Based on the preceding analysis, we can derive Proposition 2 concerning the continuous embedding of $B^{\omega}_{\sigma,p}(\mathbb{R})$ into $B_{\sigma,q}(\mathbb{R})$.

Proposition 2. *Let $1 < p, q < \infty$, and $\omega_k = \{\omega_k\}_{k\in\mathbb{Z}_0}$ satisfies Conditions (7) and (8). Then $B^{\omega}_{\sigma,p}(\mathbb{R}) \hookrightarrow B_{\sigma,q}(\mathbb{R})$.*

In the subsequent discussion, unless otherwise stated, we assume that $\omega = \{\omega_k\}_{k\in\mathbb{Z}_0}$ satisfies Conditions (7) and (8) in both the Kolmogorov n-width and linear n-width discussions.

3. The Relationship between the Width of Weighted Bound-Limited Function Spaces and the s-Numbers of Infinite-Dimensional Diagonal Operators

In this section, we leverage the favorable properties of s-numbers to investigate the relationship between the width of weighted band-limited function spaces and the s-numbers of infinite-dimensional diagonal operators.

Let $1 < p, q < \infty$, and $\omega = \{\omega_k\}_{k\in\mathbb{Z}_0}$ be the sequence of positive real numbers defined on $\mathbb{Z}_0$ satisfying the Conditions (7) and (8). Then the infinite-dimensional diagonal operator is defined as

$$\begin{aligned} D_{\frac{1}{\omega}} &: l_p(\mathbb{Z}_0) \to l_q(\mathbb{Z}_0), \\ x = \{\xi_k\}_{k\in\mathbb{Z}_0} &\mapsto D_{\frac{1}{\omega}} x = \left\{\tfrac{1}{\omega_k}\xi_k\right\}_{k\in\mathbb{Z}_0}. \end{aligned} \tag{10}$$

For $1 < p, q < \infty$, according to Conditions (7) and (8), the infinite-dimensional diagonal operator $D_{\frac{1}{\omega}}$ is a bounded linear operator mapping from $l_p(\mathbb{Z}_0)$ to $l_q(\mathbb{Z}_0)$. Consequently, we establish a connection between $s_n(B^{\omega}_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R}))$ and $s_n(D_{\frac{1}{\omega}} : l_p(\mathbb{Z}_0) \to l_q(\mathbb{Z}_0))$. This outcome is a pivotal result in this paper, contributing significantly to the transformation of the estimation of width orders into an estimation of s-number orders.

Theorem 1. *Let $1 < p, q < \infty$, $\omega = \{\omega_k\}_{k\in\mathbb{Z}_0}$ represent a sequence of positive real numbers defined on $\mathbb{Z}_0$ that satisfies the Conditions (7) and (8). Consider the infinite-dimensional diagonal operator $D_{\frac{1}{\omega}}$ defined by Equation (10), and $n \in \mathbb{N}$. Then, we obtain*

$$d_n\left(B^{\omega}_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) \asymp d_n\left(D_{\frac{1}{\omega}} : l_p(\mathbb{Z}_0) \to l_q(\mathbb{Z}_0)\right),$$

$$a_n\left(B^{\omega}_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) \asymp a_n\left(D_{\frac{1}{\omega}} : l_p(\mathbb{Z}_0) \to l_q(\mathbb{Z}_0)\right).$$

Proof. We consider the following operators

$$\begin{aligned} A &: B^{\omega}_{\sigma,p}(\mathbb{R}) \to l_p(\mathbb{Z}_0), \\ f &\mapsto \left\{\omega_k f\left(\frac{k\pi}{\sigma}\right)\right\}_{k\in\mathbb{Z}_0}, \end{aligned}$$

and

$$\begin{aligned} B &: l_q(\mathbb{Z}_0) \to B_{\sigma,q}(\mathbb{R}), \\ \xi = \{\xi_k\} &\mapsto B(\xi)(x), \end{aligned}$$

where $B(\xi)(x) := \sum\limits_{k\in\mathbb{Z}_0} \xi_k \operatorname{sinc}\left(\sigma\left(x - \frac{k\pi}{\sigma}\right)\right), \quad x \in \mathbb{R}.$

According to Lemma 2, it follows that the operators A and B are bounded linear operators, where $\|A\| \le \frac{1}{c_1}$ and $\|B\| \le c_2$. Furthermore, we obtain the following diagram

$$\begin{array}{ccc} B^{\omega}_{\sigma,p}(\mathbb{R}) & \xrightarrow{id} & B_{\sigma,q}(\mathbb{R}) \\ \downarrow A & & \uparrow B \\ l_p(\mathbb{Z}_0) & \xrightarrow{D_{\frac{1}{\omega}}} & l_q(\mathbb{Z}_0). \end{array}$$

From Lemma 1 and the identity $id = B \cdot D_{\frac{1}{\omega}} \cdot A$, we have

$$\begin{aligned} s_n\Big(id : B^{\omega}_{\sigma,p}(\mathbb{R}) \to B_{\sigma,q}(\mathbb{R})\Big) &\le \|A\| \cdot s_n\Big(D_{\frac{1}{\omega}} : l_p(\mathbb{Z}_0) \to l_q(\mathbb{Z}_0)\Big) \cdot \|B\| \\ &\le \frac{c_2}{c_1}\, s_n\Big(D_{\frac{1}{\omega}} : l_p(\mathbb{Z}_0) \to l_q(\mathbb{Z}_0)\Big) \\ &\preccurlyeq s_n\Big(D_{\frac{1}{\omega}} : l_p(\mathbb{Z}_0) \to l_q(\mathbb{Z}_0)\Big). \end{aligned} \tag{11}$$

To prove the reverse direction, we consider the modified diagram

$$\begin{array}{ccc} l_p(\mathbb{Z}_0) & \xrightarrow{D_{\frac{1}{\omega}}} & l_q(\mathbb{Z}_0) \\ \downarrow A^{-1} & & \uparrow B^{-1} \\ B^{\omega}_{\sigma,p}(\mathbb{R}) & \xrightarrow{id} & B_{\sigma,q}(\mathbb{R}). \end{array}$$

It is obvious that A^{-1}, B^{-1} are bounded linear operators with $\|A^{-1}\| \le c_1, \|B^{-1}\| \le \frac{1}{c_2}$. By the same argument as used above, we obtain the reverse inequality of (11) as

$$\begin{aligned} s_n\Big(D_{\frac{1}{\omega}} : l_p(\mathbb{Z}_0) \to l_q(\mathbb{Z}_0)\Big) &\le \Big\|A^{-1}\Big\| \cdot s_n\Big(id : B^{\omega}_{\sigma,p}(\mathbb{R}) \to B_{\sigma,q}(\mathbb{R})\Big) \cdot \Big\|B^{-1}\Big\| \\ &\le \frac{c_1}{c_2} s_n\Big(D_{\frac{1}{\omega}} : l_p(\mathbb{Z}_0) \to l_q(\mathbb{Z}_0)\Big) \\ &\preccurlyeq s_n\Big(id : B^{\omega}_{\sigma,p}(\mathbb{R}) \to B_{\sigma,q}(\mathbb{R})\Big). \end{aligned} \tag{12}$$

Combining Equations (11) and (12), we get

$$s_n\Big(id : B^{\omega}_{\sigma,p}(\mathbb{R}) \to B_{\sigma,q}(\mathbb{R})\Big) \asymp s_n\Big(D_{\frac{1}{\omega}} : l_p(\mathbb{Z}_0) \to l_q(\mathbb{Z}_0)\Big). \tag{13}$$

In accordance with Equation (1), the aforementioned Equation (13) can be reformulated as

$$d_n\Big(B^{\omega}_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\Big) \asymp d_n\Big(D_{\frac{1}{\omega}} : l_p(\mathbb{Z}_0) \to l_q(\mathbb{Z}_0)\Big),$$

$$a_n\Big(B^{\omega}_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\Big) \asymp a_n\Big(D_{\frac{1}{\omega}} : l_p(\mathbb{Z}_0) \to l_q(\mathbb{Z}_0)\Big).$$

□

Let $\sigma = \{\sigma_k\}_{k\in\mathbb{N}}$ be the non-increasing rearrangement of $\{1/\omega_k\}_{k\in\mathbb{Z}_0}$. Then from the result of [14], we obtain

$$s_n(D_\sigma : l_p \to l_q) = \left(\sum_{j\ge n}^{\infty} \sigma_j^{\theta}\right)^{\frac{1}{\theta}}, \tag{14}$$

where $1 < q < p < \infty$, $\frac{1}{\theta} = \frac{1}{q} - \frac{1}{p}$.

Based on the conclusion of Theorem 1, Equation (13), we can readily estimate the exact asymptotic order of the Kolmogorov n-width and linear n-width.

Corollary 1. *Let* $1 < q < p < \infty$, $\frac{1}{\theta} = \frac{1}{q} - \frac{1}{p}$, $n \in \mathbb{N}$, *and* $\sigma = \{\sigma_k\}_{k\in\mathbb{N}}$ *be the non-increasing rearrangement of* $\{1/\omega_k\}_{k\in\mathbb{Z}_0}$. *Then we have*

$$d_n\Big(B^{\omega}_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\Big) \asymp a_n\Big(B^{\omega}_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\Big) \asymp \left(\sum_{j\ge n}^{\infty} {\sigma_j}^{\theta}\right)^{\frac{1}{\theta}}.$$

Let $1 < q < p < \infty$, $n \in \mathbb{N}$, It is evident that when $\omega_k = 2^{|k|}$, $k \in \mathbb{Z}_0$, the sequence $\omega = \{\omega_k\}$ complies with Conditions (7) and (8). Similarly, if $\omega_k = |k|^r$, where $r > \max\Big\{0, \frac{1}{q} - \frac{1}{p}\Big\}$, then the sequence $\omega = \{\omega_k\}$ also satisfies Conditions (7) and (8). Consequently, we can readily deduce the following implication from Corollary 1.

Corollary 2. *Suppose that* $1 < q \le p < \infty$, $\omega = \{\omega_k\}_{k\in\mathbb{Z}_0}$.

(1) *If* $\omega_k = 2^{|k|}$, $k \in \mathbb{Z}_0$, *then*

$$d_n\left(B^{\omega}_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) \asymp a_n\left(B^{\omega}_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) \asymp 2^{-n}.$$

(2) *If* $\omega_k = |k|^r$, $k \in \mathbb{Z}_0$, *and* $r > \max\left\{0, \frac{1}{q} - \frac{1}{p}\right\}$, *then*

$$d_n\left(B^{\omega}_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) \asymp a_n\left(B^{\omega}_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) \asymp n^{-r+\frac{1}{q}-\frac{1}{p}}.$$

From Corollary 2, it is evident that, for $1 < p, q < \infty$, $r > \max\left\{0, \frac{1}{q} - \frac{1}{p}\right\}$, and $\omega = \{\omega_k\} = \{|k|^r\}_{k\in\mathbb{Z}_0}$, the space $B^{\omega}_{\sigma,p}(\mathbb{R})$ can be continuously embedded into $B_{\sigma,q}(\mathbb{R})$. Furthermore, the precise asymptotic order of the Kolmogorov n-widths and linear n-widths of $B^{\omega}_{\sigma,p}(\mathbb{R})$ in $B_{\sigma,q}(\mathbb{R})$ is determined when $1 < q \le p < \infty$. However, the exact asymptotic order of the Kolmogorov n-widths and linear n-widths of $B^{\omega}_{\sigma,p}(\mathbb{R})$ in $B_{\sigma,q}(\mathbb{R})$ remains unresolved when $1 < p < q < \infty$. This unresolved issue will be addressed in the subsequent section of the paper. For the sake of convenience, the weighted band-limited function space $B^{\omega}_{\sigma,p}(\mathbb{R})$ is temporarily denoted as $B^{r}_{\sigma,p}(\mathbb{R})$.

4. Exact Asymptotic Order of the Width of $B^r_{\sigma,p}(\mathbb{R})$ in $B_{\sigma,q}(\mathbb{R})$ $(1 < p < q < \infty)$

In Section 3, we derived the precise asymptotic order of the width of $B^r_{\sigma,p}(\mathbb{R})$ within $B_{\sigma,q}(\mathbb{R})$ when $1 < q \le p < \infty$. Extending this investigation, we proceed in this section to estimate the asymptotic order of the width of $B^r_{\sigma,p}(\mathbb{R})$ within $B_{\sigma,q}(\mathbb{R})$ when $1 < p < q < \infty$, employing the discretization method.

To ensure comprehensive coverage, we examine the precise asymptotic order of the width of $B^r_{\sigma,p}(\mathbb{R})$ within $B_{\sigma,q}(\mathbb{R})$ for the general case $1 < p, q < \infty$, which represents another primary outcome of this study.

Theorem 2. *Let* $1 < p < q < \infty$, $r > \max\left\{0, \frac{1}{q} - \frac{1}{p}\right\}$, $n \in \mathbb{N}$. *Then we have*

$$d_n\left(B^{r}_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) \asymp \begin{cases} n^{-r+\frac{1}{q}-\frac{1}{p}}, & 1 < q \le p < \infty, \\ n^{-r}, & 1 < p < q \le 2, \\ n^{-r+\frac{1}{q}-\frac{1}{2}}, & 1 < p < 2 \le q < \infty, r > \frac{1}{q} + \frac{1}{2}, \\ n^{-r+\frac{1}{q}-\frac{1}{p}}, & 2 \le p < q < \infty, r > \left(\frac{1}{q} + \frac{1}{2}\right)\lambda_{p,q}, \end{cases}$$

and

$$a_n\left(B^{r}_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) \asymp \begin{cases} n^{-r+\frac{1}{q}-\frac{1}{p}}, & 1 < q \le p < \infty, \\ n^{-r}, & 1 < p < q \le 2, \\ n^{-r+\frac{1}{q}-\frac{1}{2}}, & 1 < p < 2 \le q < \infty, r > \frac{1}{q} + \frac{1}{2}, \\ n^{-r}, & 2 \le p < q < \infty, \end{cases}$$

where $\lambda_{p,q} = \frac{1/p-1/q}{1/2-1/q}$.

Remark 2. *When* $1 < q \le p < \infty$, *the results of Theorem 2 are entirely consistent with those of Corollary 2.*

This section will employ the discretization method to establish Theorem 2, thereby converting the problem of estimating the width in infinite-dimensional spaces into one of estimating width in finite-dimensional spaces. To this end, we first recall results pertaining to the width of finite-dimensional spaces.

Let l_p^m, $1 \le p \le \infty$, be the classical finite-dimensional space equipped with the norm $\|\cdot\|_{l_p^m}$ on $\mathbb{R}^m$ as

$$\|x\|_{l_p^m} = \begin{cases} \left(\sum\limits_{k=1}^{m} |x_k|^p\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \max\limits_{1 \le k \le m} |x_k|, & p = \infty. \end{cases}$$

Denoting by B_p^m the unit ball in l_p^m, the n-width of l_p^m in l_q^m has been thoroughly investigated by numerous scholars, and the findings can be summarized as follows.

Lemma 3 ([13,17–20])**.** *Let $1 < p, q < \infty, n \in \mathbb{N}$, and $0 \le 2n \le m$. Then we have*

$$d_n\left(B_p^m, l_q^m\right) \asymp \begin{cases} (m-n)^{\frac{1}{q}-\frac{1}{p}}, & 1 < q < p < \infty, \\ 1, & 1 < p < q \le 2, \\ \min\left\{1, m^{\frac{1}{q}} n^{-\frac{1}{2}}\right\}, & 1 < p < 2 \le q < \infty, \\ \left\{\min\left\{1, m^{\frac{1}{q}} n^{-\frac{1}{2}}\right\}\right\}^{\lambda_{p,q}}, & 2 \le p < q < \infty. \end{cases}$$

and

$$a_n\left(B_p^m, l_q^m\right) \asymp \begin{cases} (m-n)^{\frac{1}{q}-\frac{1}{p}}, & 1 < q < p < \infty, \\ 1, & 1 < p < q \le 2, \\ \min\left\{1, m^{\frac{1}{q}} n^{-\frac{1}{2}}\right\}, & 1 < p < 2 \le q < \infty, \\ 1, & 2 \le p < q < \infty. \end{cases}$$

where $\lambda_{p,q} = \frac{1/p-1/q}{1/2-1/q}$.

To establish the discretization lemma for estimating the upper bound of Theorem 2, we begin by partitioning the entirety of non-zero points into blocks.

For each integer $k \in \mathbb{N}_+$, define the set S_k as

$$S_k = \{n \in \mathbb{Z}_0 : 2^{k-1} \le |n| < 2^k\}.$$

It is evident that the cardinality of S_k is 2^k, denoted as $|S_k| = 2^k$. Moreover, for $k, k' \in \mathbb{N}_+, k \ne k'$, we obtain

$$S_k \cap S_{k'} = \varnothing, \bigcup_{k=1}^{\infty} S_k = \mathbb{Z}_0.$$

For convenience, we denote

$$e_n(x) = \operatorname{sinc}\left(\sigma\left(x - \frac{n\pi}{\sigma}\right)\right), x \in \mathbb{R}, n \in \mathbb{Z}_0.$$

For any $f \in \mathring{B}_{\sigma,p}(\mathbb{R})(1 < p < \infty)$, and $k \in \mathbb{N}_+$, let $\delta_k f(x)$ denote the "block" for $f(x)$, namely

$$\delta_k f(x) = \sum_{n \in S_k} f\left(\frac{n\pi}{\sigma}\right) e_n(x), \quad x \in \mathbb{R}.$$

Then we have

$$
\begin{aligned}
\|f\|_{\sigma,p} &\asymp \left(\sum_{n\in\mathbb{Z}_0} \left| f\left(\frac{n\pi}{\sigma}\right) \right|^p \right)^{\frac{1}{p}} = \left(\sum_{k\in\mathbb{N}_+} \sum_{n\in S_k} \left| f\left(\frac{n\pi}{\sigma}\right) \right|^p \right)^{\frac{1}{p}} \\
&= \left(\sum_{k\in\mathbb{N}_+} \| \left\{ f\left(\frac{n\pi}{\sigma}\right) \right\}_{n\in S_k} \|^p_{l_p^{|S_k|}} \right)^{\frac{1}{p}} \asymp \left(\sum_{k\in S_k} \| \delta_k f \|^p_{\sigma,p} \right)^{\frac{1}{p}}.
\end{aligned} \tag{15}
$$

Let $k \in S_k$. Define F_k as $F_k := span\{e_n(x) | n \in S_k\}$. It is obvious that the dimension of F_k is $|S_k| = 2^k$.

Consider the mapping

$$
\begin{aligned}
I_k : \qquad & F_k \to \mathbb{R}^{|S_k|} \\
f(x) = \sum_{n\in S_k} c_n e_n(x) &\mapsto \{c_n\}_{n\in S_k}.
\end{aligned}
$$

It is evident that I_k is a linear isomorphism between F_k and $\mathbb{R}^{|S_k|}$. Moreover, for any $f(x) = \sum_{n\in S_k} c_n e_n(x)$ and $g(x) = \sum_{n\in S_k} c_n^{'} e_n(x), x \in \mathbb{R}$, we have

$$
\|f\|_{p,\omega} \asymp \left(\sum_{n\in S_k} |n|^{rp} |c_n|^p \right)^{\frac{1}{p}} \asymp 2^{rk} \left(\sum_{n\in S_k} |c_n|^p \right)^{\frac{1}{p}} \asymp 2^{rk} \|I_k f\|_{l_p^{|S_k|}}, 1 < p < \infty, \tag{16}
$$

$$
\|g\|_{L_p} \asymp \left(\sum_{n\in S_k} |c_n'|^p \right)^{\frac{1}{p}} \asymp \|I_k g\|_{l_q^{|S_k|}}, 1 < q < \infty. \tag{17}
$$

According to the definitions of Kolmogorov n-width, linear n-width and Equations (16) and (17), for $n_k \in \mathbb{N}$, we have

$$
s_{n_k}\left(B^r_{\sigma,p}(\mathbb{R}) \bigcap F_k, B_{\sigma,q}(\mathbb{R}) \bigcap F_k \right) \preccurlyeq 2^{-rk} s_{n_k}\left(B_p^{|S_k|}, l_q^{|S_k|} \right), 1 < p, q < \infty, \tag{18}
$$

where s_{n_k} represents d_{n_k} or a_{n_k}.

Therefore, from Equations (15) and (18), we can obtain the discretization lemma for the upper bound of estimating Theorem 2.

Lemma 4. *Let $1 < p, q < \infty$, $n \in \mathbb{N}$, $r > \max\{0, 1/q - 1/p\}$, and let $\{n_k\}$ be a sequence of non-negative integers defined on $\mathbb{N}$ such that $\sum\limits_{k=1}^{\infty} n_k \leq n$ with $n_k \leq |S_k|$, $k \in \mathbb{N}_+$. Then*

$$
s_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R}) \right) \preccurlyeq \sum_{k=1}^{\infty} 2^{-rk} s_{n_k}\left(B_p^{|S_k|}, l_q^{|S_k|} \right),
$$

where s_{n_k} represents d_{n_k} or a_{n_k}.

In the subsequent discourse, we establish a discretization lemma aimed at deriving an estimative lower bound for Theorem 2.

Lemma 5. *Let $1 < p, q < \infty$, $n \in \mathbb{N}$, $k \in \mathbb{N}_+$, and $n \asymp 2^k$, $|S_k| \geq 2n$, $r > \max\left(0, \frac{1}{q} - \frac{1}{p}\right)$. Then*

$$
s_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R}) \right) \succcurlyeq 2^{-rk} s_n\left(B_p^{|S_k|}, l_q^{|S_k|} \right).
$$

Proof. By Equation (15), we obtain

$$
s_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R}) \right) \geq s_n\left(B^r_{\sigma,p}(\mathbb{R}) \cap F_k, B_{\sigma,q}(\mathbb{R}) \cap F_k \right). \tag{19}
$$

From the Equations (16), (17) and (19), we obtain

$$\begin{aligned} s_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) &\geq s_n\left(B^r_{\sigma,p}(\mathbb{R}) \cap F_k, B_{\sigma,q}(\mathbb{R}) \cap F_k\right) \\ &\succcurlyeq 2^{-rk}\, s_n\left(B_p^{|S_k|} l_q^{|S_k|}\right), \end{aligned}$$

which completes the proof of Lemma 5. □

We are currently in a position to formally demonstrate Theorem 2, which stands as the primary culmination of findings of this paper.

Proof of Theorem 2. For $n \in \mathbb{N}$, let $2^{k'} \asymp n$, $k' \in \mathbb{N}_+$, and

$$n_k = \begin{cases} |S_k|, 1 \leq k \leq k', \\ \left[n \cdot 2^{k'-k}\right], k > k'. \end{cases}$$

It is easy to see that $\{n_k\}$ satisfies the conditions of Lemma 4 and when $k > k'$, we have

$$\frac{|S_k|}{n_k} = \frac{2^k}{n \cdot 2^{k'-k}} \geq \frac{2^k}{2^{k'} \cdot 2^{k'-k}} = \frac{2^{2k}}{2^{2k'}} \geq 2,$$

which means that $|S_k| \geq 2n_k$.

Step-I: Firstly, we primarily focus on estimating the exact asymptotic order of the Kolmogorov n-widths $d_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right)$.

According to Lemma 4 and the consideration that $d_m\left(l_p^m, l_q^m\right) = 0$, we can obtain the upper bound of $d_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right)$ as

$$d_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) \preccurlyeq \sum_{k \in \mathbb{N}_+} 2^{-rk} d_{n_k}\left(B_p^{|S_k|}, l_q^{|S_k|}\right) = \sum_{k > k'} 2^{-rk} d_{n_k}\left(B_p^{|S_k|}, l_q^{|S_k|}\right). \tag{20}$$

In the analysis of $d_{n_k}\left(B_p^{|S_k|}, l_q^{|S_k|}\right)$, four distinct scenarios are deliberated upon for estimation.

Case I: For $1 < q \leq p < \infty$, by Equation (20) and Lemma 3, we have

$$\begin{aligned} d_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) &\preccurlyeq \sum_{k > k'} 2^{-rk} d_{n_k}\left(B_p^{|S_k|}, l_q^{|S_k|}\right) \\ &\preccurlyeq \sum_{k > k'} 2^{-rk} (|S_k| - n_k)^{\frac{1}{q} - \frac{1}{p}} \\ &\preccurlyeq \sum_{k > k'} 2^{-rk} |S_k|^{\frac{1}{q} - \frac{1}{p}} \\ &= \sum_{k > k'} 2^{-rk} 2^{\left(\frac{1}{q} - \frac{1}{p}\right)k} \\ &= \sum_{k > k'} 2^{-\left(r - \frac{1}{q} + \frac{1}{p}\right)k} \\ &\preccurlyeq 2^{-\left(r - \frac{1}{q} + \frac{1}{p}\right)k'} \\ &\asymp n^{-\left(r - \frac{1}{q} + \frac{1}{p}\right)}. \end{aligned}$$

Case II: For $1 < p < q \leq 2$, according to Equation (20) and Lemma 3, we have

$$\begin{aligned} d_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) &\preccurlyeq \sum_{k>k'} 2^{-rk} d_{n_k}\left(B_p^{|S_k|}, l_q^{|S_k|}\right) \\ &\preccurlyeq \sum_{k>k'} 2^{-rk} \preccurlyeq 2^{-rk'} \asymp n^{-r}. \end{aligned}$$

Case III: For $1 < p < 2 \leq q < \infty$, according to Equation (20) and Lemma 3, we have

$$\begin{aligned} d_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) &\preccurlyeq \sum_{k>k'} 2^{-rk} \min\left\{1, |S_k|^{\frac{1}{q}} n_k^{-\frac{1}{2}}\right\} \\ &\preccurlyeq \sum_{k>k'} 2^{-rk} |S_k|^{\frac{1}{q}} n_k^{-\frac{1}{2}} \\ &= \sum_{k>k'} 2^{-rk} 2^{\frac{k}{q}} n^{-\frac{1}{2}} 2^{\frac{k}{2}-\frac{k'}{2}} \\ &= \sum_{k>k'} n^{-\frac{1}{2}} 2^{-\frac{k'}{2}-\left(r-\frac{1}{q}-\frac{1}{2}\right)k} \\ &\preccurlyeq n^{-\frac{1}{2}} 2^{-\frac{k'}{2}} 2^{-\left(r-\frac{1}{q}-\frac{1}{2}\right)k'} \\ &= n^{-\frac{1}{2}} 2^{-\frac{k'}{2}} 2^{-\left(r-\frac{1}{q}\right)k'+\frac{1}{2}k'} \\ &= n^{-\frac{1}{2}} 2^{-\left(r-\frac{1}{q}\right)k'} \\ &\asymp n^{-r^{+}+\frac{1}{q}-\frac{1}{2}}. \end{aligned}$$

Case IV: For $2 \leq p < q < \infty$, according to Equation (20) and Lemma 3, we have

$$\begin{aligned} d_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) &\preccurlyeq \sum_{k>k'} 2^{-rk}\left\{\min\left\{1, m^{\frac{1}{q}} n^{-\frac{1}{2}}\right\}\right\}^{\lambda_{p,q}} \preccurlyeq \sum_{k>k'} 2^{-rk}\left(|S_k|^{\frac{1}{q}} n_k^{-\frac{1}{2}}\right)^{\lambda_{p,q}} \\ &= \sum_{k>k'} 2^{-rk}\left(2^{\frac{1}{q}k} n^{-\frac{1}{2}} 2^{\frac{1}{2}k-\frac{1}{2}k'}\right)^{\lambda_{p,q}} \preccurlyeq \sum_{k>k'} 2^{-rk}\left(2^{\frac{1}{q}k} n^{-1} 2^{\frac{1}{2}k}\right)^{\lambda_{p,q}} \\ &= n^{-\lambda_{p,q}} \sum_{k>k'} 2^{-rk+\frac{\lambda_{p,q}}{q}k+\frac{\lambda_{p,q}}{2}k} \preccurlyeq n^{-\lambda_{p,q}} 2^{-\left(r-\frac{\lambda_{p,q}}{q}-\frac{\lambda_{p,q}}{2}\right)k'} \\ &\asymp n^{\lambda_{p,q}} n^{-r+\left(\frac{1}{q}+\frac{1}{2}\right)\lambda_{p,q}} = n^{-r+\left(\frac{1}{q}+\frac{1}{2}\right)\lambda_{p,q}} \\ &= n^{-r+\frac{1}{q}-\frac{1}{p}}. \end{aligned}$$

Similarly, we estimate the lower bound of $d_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right)$ in four cases. For $n \in \mathbb{N}$, let $k \in \mathbb{N}_+$, such that $2^k \asymp n$, and $2^k \geq 2n$. By Lemma 5, we have

$$d_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) \succcurlyeq 2^{-rk} d_n\left(B_p^{|S_k|}, l_q^{|S_k|}\right). \tag{21}$$

Case I: For $1 < q \leq p < \infty$, by Equation (21) and Lemma 3, we have

$$\begin{aligned} d_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) &\succcurlyeq 2^{-rk}(|S_k|-n)^{\frac{1}{q}-\frac{1}{p}} \\ &= 2^{-rk}\left(2^k-n\right)^{\frac{1}{q}-\frac{1}{p}} \\ &\geq 2^{-rk}(2n-n)^{\frac{1}{q}-\frac{1}{p}} \\ &\geq n^{-r+\frac{1}{q}-\frac{1}{p}}. \end{aligned}$$

Case II: For $1 < p < q \le 2$, by equation by Equation (21) and Lemma 3, we have

$$d_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) \succcurlyeq 2^{-rk} \asymp n^{-r}.$$

Case III: For $1 < p < 2 \le q < \infty$, given the asymptotic relation $2^k \asymp n$, there exists an absolute positive constant b_1 such that

$$2^k \le b_1 n.$$

Consequently, the expression

$$|S_k|^{\frac{1}{q}} n^{-\frac{1}{2}} = 2^{\frac{k}{q}} n^{-\frac{1}{2}} \le b_1^{\frac{1}{q}} n^{\frac{1}{q}-\frac{1}{2}} < b_1^{\frac{1}{q}}$$

holds true.

This inequality implies that

$$\frac{1}{b_1^{1/q}} |S_k|^{\frac{1}{q}} n^{-\frac{1}{2}} < 1. \tag{22}$$

Therefore, by Equation (21) and Lemma 3, we have

$$\begin{aligned} d_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) &\succcurlyeq 2^{-rk} |S_k|^{\frac{1}{q}} n^{-\frac{1}{2}} \\ &= 2^{-rk+\frac{k}{q}} n^{-\frac{1}{2}} \\ &\succcurlyeq n^{-r+\frac{1}{q}-\frac{1}{2}}. \end{aligned}$$

Case IV: For $2 \le p < q < \infty$, by Equation (21) and Lemma 3, we have

$$\begin{aligned} d_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) &\succcurlyeq 2^{-rk}\left(|S_k|^{\frac{1}{q}} n^{-\frac{1}{2}}\right)^{\lambda_{p,q}} \succcurlyeq 2^{-rk} 2^{\left(\frac{k}{q}-\frac{k}{2}\right)\lambda_{p,q}} \\ &= 2^{-\left(r+\frac{1}{q}-\frac{1}{p}\right)k} \succcurlyeq n^{-r+\frac{1}{q}-\frac{1}{p}} \end{aligned}$$

From the derivation above, we obtain the exact asymptotic order of the Kolmogorov n-widths $d_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right)$ as

$$d_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) \asymp \begin{cases} n^{-r+\frac{1}{q}-\frac{1}{p}}, & 1 < q \le p < \infty, \\ n^{-r}, & 1 < p < q \le 2, \\ n^{-r+\frac{1}{q}-\frac{1}{2}}, & 1 < p < 2 \le q < \infty, r > \frac{1}{q} + \frac{1}{2}, \\ n^{-r+\frac{1}{q}-\frac{1}{p}}, & 2 \le p < q < \infty, r > \left(\frac{1}{q}+\frac{1}{2}\right)\lambda_{p,q}. \end{cases}$$

where $\lambda_{p,q} = \frac{1/p-1/q}{1/2-1/q}$.

Step-II: Next, we turn to focus on estimating the exact asymptotic order of the linear n-widths $a_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right)$.

According to Lemma 4 and the consideration that $a_m\left(l_p^m, l_q^m\right) = 0$, we can obtain the upper bound of $a_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right)$ as

$$a_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) \preccurlyeq \sum_{k=1}^{\infty} 2^{-rk} a_{n_k}\left(B_p^{|s_k|}, l_q^{|s_k|}\right). \tag{23}$$

Case I: For $1 < q \le p < \infty$, by Equation (23) and Lemma 3, we have

$$
\begin{aligned}
a_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) &\preccurlyeq \sum_{k>k'} 2^{-rk}\left(|S_k| - n^k\right)^{\frac{1}{q}-\frac{1}{p}} \preccurlyeq \sum_{k>k'} 2^{-rk}|S_k|^{\frac{1}{q}-\frac{1}{p}} \\
&\preccurlyeq \sum_{k>k'} 2^{-\left(r-\frac{1}{q}+\frac{1}{p}\right)k} \preccurlyeq 2^{-\left(r-\frac{1}{q}+\frac{1}{p}\right)k'} \preccurlyeq n^{-r+\frac{1}{q}-\frac{1}{p}}.
\end{aligned}
$$

Case II: For $1 < p < q \leq 2$, by Equation (23) and Lemma 3, we have

$$
\begin{aligned}
a_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) &\preccurlyeq \sum_{k>k'} 2^{-rk} \\
&\preccurlyeq 2^{-rk'} \asymp n^{-r}.
\end{aligned}
$$

Case III: For $1 < p < 2 \leq q < \infty$, by Equation (23) and Lemma 3, we have

$$
\begin{aligned}
a_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) &\preccurlyeq \sum_{k>k'} 2^{-rk}\min\left\{1, |S_k|^{\frac{1}{q}} n_k^{-\frac{1}{2}}\right\} \\
&\leq \sum_{k>k'} 2^{-rk}|S_k|^{\frac{1}{q}} n_k^{-\frac{1}{2}} = \sum_{k>k'} 2^{-rk} 2^{\frac{k}{q}} n^{-\frac{1}{2}} 2^{\frac{k}{2}-\frac{k'}{2}} \\
&= n^{-\frac{1}{2}} 2^{-\frac{k'}{2}} \sum_{k>k'} 2^{-\left(r-\frac{1}{q}-\frac{1}{2}\right)k} \preccurlyeq n^{-\frac{1}{2}} 2^{-\frac{k'}{2}} 2^{-\left(r-\frac{1}{q}-\frac{1}{2}\right)k'} \\
&= n^{-\frac{1}{2}} 2^{-\frac{k'}{2}} 2^{-\left(r-\frac{1}{q}\right)k'+\frac{1}{2}k'} = n^{-\frac{1}{2}} 2^{-\left(r-\frac{1}{q}\right)k'} \\
&\asymp n^{-r+\frac{1}{q}-\frac{1}{2}}.
\end{aligned}
$$

Case IV: For $2 \leq p < q < \infty$, by Equation (23) and Lemma 3, we have

$$
a_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) \preccurlyeq 2^{-rk} \preccurlyeq 2^{-rk'} \preccurlyeq n^{-r}.
$$

We estimate the lower bound of $a_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right)$ in four cases. For $n \in \mathbb{N}$, let $k \in \mathbb{N}_+$, such that $2^k \asymp n$, and $r > \max\left\{0, \frac{1}{q} - \frac{1}{p}\right\}$. By Lemma 5, we have

$$
a_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) \succcurlyeq 2^{-rk} a_n\left(B_p^{|S_k|}, l_q^{|S_k|}\right). \tag{24}
$$

Case I: For $1 < q \leq p < \infty$, by Equation (24) and Lemma 3, we have

$$
\begin{aligned}
a_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) &\succcurlyeq 2^{-rk}(|S_k| - n)^{\frac{1}{q}-\frac{1}{p}} = 2^{-rk}\left(2^k - n\right)^{\frac{1}{q}-\frac{1}{p}} \\
&\geq 2^{-rk}(2n - n)^{\frac{1}{q}-\frac{1}{p}} \geq n^{-r+\frac{1}{q}-\frac{1}{p}}.
\end{aligned}
$$

Case II: For $1 < p < q \leq 2$, by Equation (24) and Lemma 3, we have

$$
a_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) \succcurlyeq 2^{-rk} \asymp n^{-r}.
$$

Case III: For $1 < p < 2 \leq q < \infty$, by Equations (22), (24) and Lemma 3, we have

$$
\begin{aligned}
a_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) &\succcurlyeq 2^{-rk}|S_k|^{\frac{1}{q}} n^{-\frac{1}{2}} = 2^{-rk+\frac{k}{q}} n^{-\frac{1}{2}} \\
&\succcurlyeq n^{-r+\frac{1}{q}-\frac{1}{2}}.
\end{aligned}
$$

Case IV: For $2 \leq p < q < \infty$, by Equation (24) and Lemma 3, we have

$$a_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) \succcurlyeq 2^{-rk} \succcurlyeq 2^{-rk'} \succcurlyeq n^{-r}.$$

From the derivation above, we obtain the exact asymptotic order of the linear n-widths $a_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right)$ as

$$a_n\left(B^r_{\sigma,p}(\mathbb{R}), B_{\sigma,q}(\mathbb{R})\right) \asymp \begin{cases} n^{-r+\frac{1}{q}-\frac{1}{p}}, & 1 < q \le p < \infty, \\ n^{-r}, & 1 < p < q \le 2, \\ n^{-r+\frac{1}{q}-\frac{1}{2}}, & 1 < p < 2 \le q < \infty, r > \frac{1}{q} + \frac{1}{2}, \\ n^{-r}, & 2 \le p < q < \infty. \end{cases}$$

which completes the proof of Theorem 2. □

5. Conclusions

The concept of band-limited function spaces constitutes a crucial paradigm in the fields of approximation theory and signal processing, forming the theoretical underpinnings for functional analysis, computational complexity, and optimal algorithms. Simultaneously, it maintains significant connections with branches such as communication, data processing, and information theory. In this paper, we endow classical band-limited function spaces with weights to obtain weighted band-limited function spaces. Leveraging the favorable properties of s-numbers, we establish the relationship between the width of weighted band-limited function spaces and the s-numbers of infinite-dimensional diagonal operators. Furthermore, within a uniform setting, we provide exact asymptotic orders for the Kolmogorov n-width and linear n-width in the weighted band-limited function spaces endowed with the weight function $\omega = \{|k|^r\}$, where $k \in \mathbb{Z}_0$. Considering the width characteristics of function classes in a uniform setting elucidates the optimal errors for the "worst" elements, while errors and costs in algorithms exhibit distinct features in different frameworks. Therefore, future investigations may delve into the discussion of n-width in band-limited function spaces under various settings to advance the development of width theory.

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