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Mixed Hilfer and Caputo Fractional Riemann–Stieltjes Integro-Differential Equations with Non-Separated Boundary Conditions

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Abstract: In this paper, we investigate a sequential fractional boundary value problem which contains a combination of Hilfer and Caputo fractional derivative operators and non-separated boundary conditions. We establish the existence of a unique solution via Banach's fixed point theorem, while by applying Leray–Schauder's nonlinear alternative, we prove an existence result. Finally, examples are provided to demonstrate the results obtained.

Keywords: Hilfer fractional differential derivative; fractional integrals; existence and uniqueness; fixed-point theorems

MSC: 26A33; 34A08; 34B15

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1. Introduction

Recently, there has been great interest in fractional differential equations, since fractional orders models are more accurate than integer models. For theoretical development in fractional calculus and differential equations of fractional order, see the monographs [1–8], while for an extensive study of boundary value problems of fractional order, see the monograph [9]. In the literature, there exist a variety of fractional derivative operators such as Riemann–Liouville, Caputo, Hadamard, Erdélyi-Kober, Katugampola, Hilfer fractional derivatives, etc. As argued in [10,11], fractional derivatives of non-integer orders do not satisfy the Leibniz rule and chain rule. However, a partial answer to this fact has been discussed in [10–12]. Note that the fractional derivatives, defined in terms of integrals, are nonlocal in nature and are only valid for specific domains. The linearity of the integral operator involved in the definition of the fractional derivative may lead to its linearity. For a different fractional analysis, see [13,14]. The ψ-Riemann–Liouville integral and derivative fractional operators were introduced in [2]. The (k,ψ)-Riemann–Liouville integral and derivative fractional operators were defined in [15,16], respectively. The Hilfer fractional derivative defined in [17] extends both Riemann–Liouville and Caputo fractional derivative operators. The ψ-Hilfer fractional derivative was defined in [18]. For applications of Hilfer fractional derivatives in mathematics, physics, etc., see [19–24]. For recent results on boundary value problems for differential equations and inclusions of fractional order with Hilfer fractional derivatives, see the survey paper by Ntouyas [25].

Recently, in the papers [26–29], the authors have studied the existence and uniqueness results for Hilfer differential equations of fractional order subject to a variety of boundary conditions. In [26], the authors studied a class of fractional sequential boundary value problems involving Hilfer-type fractional derivative operators supplemented with Riemann–Stieltjes integral multi-strip boundary conditions of the form

$$\begin{cases} \left({}^{H}\mathbb{D}^{a,b} + k\,{}^{H}\mathbb{D}^{a-1,b}\right)\varphi(\omega) = \mathfrak{F}(\omega, \varphi(\omega)), \quad \omega \in [x_1, x_2], \\ \varphi(x_1) = 0, \quad \varphi(x_2) = \lambda \displaystyle\int_{x_1}^{x_2} \varphi(s)dH(s) + \sum_{i=1}^{n} \mu_i \int_{\eta_i}^{\xi_i} \varphi(s)ds. \end{cases} \tag{1}$$

In [27], the authors studied a boundary value problems involving a (k, ϑ)-Hilfer fractional derivative operator of order in $(1, 2]$, subject to nonlocal integro-multi-point boundary conditions of the form

$$\begin{cases} {}^{k,H}\mathbb{D}^{a,b;\vartheta}\varphi(\omega) = \mathfrak{F}(\omega, \varphi(\omega)), \quad \omega \in (x_1, x_2], \\ \varphi(x_1) = 0, \quad \displaystyle\int_{x_1}^{x_2} \vartheta'(s)\varphi(s)ds = \sum_{j=1}^{m} \eta_j \varphi(\xi_j), \end{cases} \tag{2}$$

while in [28], the authors studied a boundary value problem consisting of a (k, ϑ)-Hilfer generalized proportional fractional derivative operator, equipped with integro-multi-point nonlocal boundary conditions, of the form

$$\begin{cases} {}^{k,H}\mathbb{D}^{a,b,\sigma,\vartheta}_{x_1^+}\varphi(\omega) = \mathfrak{F}(\omega, \varphi(\omega)), \quad \omega \in [x_1, x_2], \\ \varphi(x_1) = 0, \quad \varphi(x_2) = \displaystyle\sum_{i=1}^{m} \lambda_i \varphi(\xi_i) + \lambda\, {}^{k}\mathfrak{I}^{b,\sigma,\vartheta}_{x_1^+}\varphi(\eta). \end{cases} \tag{3}$$

Finally, in [29], a coupled system of (k, ϑ)-Hilfer fractional derivative operators subjected to nonlocal integro-multi-point boundary conditions was investigated.

A common characteristic of all the boundary conditions above is the zero initial condition, which is necessary for the solution to be well defined. Thus, we cannot study some classes of Hilfer fractional boundary value problems, including for example boundary conditions of the form

- $\varphi(0) = -\varphi(\tau),\ \varphi'(0) = -\varphi'(\tau)$ (anti-periodic),
- $\varphi(0) + \lambda_1\varphi'(0) = 0,\ \varphi(\tau) + \lambda_1\varphi'(\tau) = 0$ (separated),
- $\varphi(0) + \lambda_1\varphi(\tau) = 0,\ \varphi'(0) + \lambda_2\varphi'(\tau) = 0$ (non-separated), etc.

To overcome this difficulty and study Hilfer fractional boundary value problems subject to boundary conditions as above, anti-periodic, separated or non-separated, we propose in the present research a combination of Hilfer and Caputo fractional derivatives, which give us the possibility to discuss boundary value problems subject to boundary conditions as above. To be more precisely, in the present paper, we investigate a sequential fractional boundary value problem which contain a combination of Hilfer and Caputo fractional derivative operators and non-separated boundary conditions of the form

$$\begin{cases} {}^{H}\mathbb{D}^{a,b,\vartheta}({}^{C}\mathbb{D}^{\delta,\vartheta}\varphi)(\omega) = \mathfrak{F}\Big(\omega, \varphi(\omega), \mathbb{I}^{c,\vartheta}\varphi(\omega), \displaystyle\int_0^{\tau} \varphi(s)dv(s)\Big),\ \omega \in [0, \tau], \\ \varphi(0) + \lambda_1\varphi(\tau) = 0, \\ {}^{C}\mathbb{D}^{\gamma+\delta-1,\vartheta}\varphi(0) + \lambda_2\,{}^{C}\mathbb{D}^{\gamma+\delta-1,\vartheta}\varphi(\tau) = 0, \end{cases} \tag{4}$$

where ${}^{H}\mathbb{D}^{a,b,\vartheta}$ and ${}^{C}\mathbb{D}^{\delta,\vartheta}, {}^{C}\mathbb{D}^{\gamma+\delta-1,\vartheta}$, $0 < a, b, \delta < 1$, $\gamma = a + b(1-a)$, $\gamma + \delta > 1$ are the ϑ-Hilfer and ϑ-Caputo fractional derivative operators, respectively. Moreover, $\lambda_1, \lambda_2 \in \mathbb{R}$, $\mathbb{I}^{c,\vartheta}$ is the Riemann–Liouville fractional integral operator of order $c > 0$ with respect to a function ϑ, $\mathfrak{F} : [0, \tau] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a nonlinear continuous function, $\int_0^{\tau} \varphi(s)dv(s)$ is the Riemann–Stieltjes integral and $v : [0, \tau] \to \mathbb{R}$ is a function of bounded variation.

We establish the existence and uniqueness results with the help of classical fixed-point theorems. First, we establish the existence of a unique solution of the fractional

Hilfer–Caputo sequential boundary value problem (4) via the Banach fixed point theorem, and next, we prove the existence of at least one solution of the fractional Hilfer–Caputo sequential boundary value problem (4) by using the Leray–Schauder nonlinear alternative. Finally, examples are provided to demonstrate the results obtained.

The novelty of the present study lies in the fact that we consider a sequential fractional boundary value problem which contains a combination of Hilfer and Caputo fractional derivative operators supplemented with non-separated boundary conditions. As far as we know, this topic is new in the literature. The method we used to establish our results is standard, but its configuration in the fractional Hilfer–Caputo sequential boundary value problem (4) is new.

The remainder of this article is organized as follows: Section 2 consists of essential concepts and definitions needed to construct our results. Also, a lemma dealing with a linear variant of the fractional Hilfer–Caputo sequential boundary value problem (4), which is the basic key to transform the nonlinear problem (4) into a fixed-point problem, is studied. In Section 3, we present our main existence and uniqueness results based on fixed-point theory. In Section 4, examples are provided to verify the reliability of the proposed results, while the paper closes with some concluding remarks in Section 5.

2. Preliminaries

Now, some essential concepts and definitions from fractional calculus are presented. Assume that $\vartheta \in C^1([0,\tau],\mathbb{R})$ with $\vartheta'(\omega) > 0$ for all $\omega \in [0,\tau]$.

Definition 1 ([2])**.** *The ϑ-Riemann–Liouville fractional integral operator of order $a > 0$ of a function $\mathfrak{g} \in C([0,\tau],\mathbb{R})$ with respect to ϑ is defined by*

$$\mathbb{I}^{a,\vartheta}\mathfrak{g}(\omega) = \frac{1}{\Gamma(a)}\int_0^{\omega} \vartheta'(s)(\vartheta(\omega)-\vartheta(s))^{a-1}\mathfrak{g}(s)ds,$$

where $\Gamma(a)$ is the Euler Gamma function given by $\Gamma(a) = \int_0^{\infty} t^{a-1}e^{-t}dt$.

Definition 2 ([18])**.** *Let $n-1 < a < n$, $n \in \mathbb{N}$ and $\mathfrak{g}, \vartheta \in C^n([0,\tau],\mathbb{R})$. The ϑ-Hilfer fractional derivative operator ${}^{H}\mathbb{D}^{a,b,\vartheta}(\cdot)$ of order a of function $\mathfrak{g}$ with a parameter $b \in [0,1]$ is defined by*

$$ {}^{H}\mathcal{D}^{a,b;\vartheta}\mathfrak{g}(\omega) = \mathbb{I}^{b(n-a);\vartheta}\left(\frac{1}{\vartheta'(\omega)}\frac{d}{d\omega}\right)^{n}\mathbb{I}^{(1-b)(n-a);\vartheta}\mathfrak{g}(\omega).$$

Definition 3 ([30])**.** *The ϑ-Caputo fractional derivative operator ${}^{C}\mathbb{D}^{a,\vartheta}(\cdot)$ of order a of a function $\mathfrak{g}$ is presented as*

$$ {}^{C}\mathcal{D}^{a;\vartheta}\mathfrak{g}(\omega) = \mathbb{I}^{n-a,\vartheta}\left(\frac{1}{\vartheta'(\omega)}\frac{d}{d\omega}\right)^{n}\mathfrak{g}(\omega), \tag{5}$$

where $n-1 < a < n$, $n \in \mathbb{N}$ and $\mathfrak{g}, \vartheta \in C^n([0,\tau],\mathbb{R})$.

Lemma 1 ([18])**.** *Let $a_1, a_2, a > 0$, $a_1 < a_2$, and $a+\delta > 1$ be constants. Then, we have*

(i) $\mathbb{I}^{a_1,\vartheta}\mathbb{I}^{a_2,\vartheta}\mathfrak{g}(\omega) = \mathbb{I}^{a_1+a_2,\vartheta}\mathfrak{g}(\omega)$;

(ii) $\mathbb{I}^{a,\vartheta}(\vartheta(\omega)-\vartheta(0))^{\delta-1} = \dfrac{\Gamma(\delta)}{\Gamma(a+\delta)}(\vartheta(\omega)-\vartheta(0))^{a+\delta-1}$,

(iii) ${}^{C}\mathcal{D}^{a,\vartheta}(\vartheta(\omega)-\vartheta(0))^{\delta-1} = \dfrac{\Gamma(\delta)}{\Gamma(\delta-a)}(\vartheta(\omega)-\vartheta(0))^{a-\delta+1}$,

(iv) ${}^{C}\mathcal{D}^{a_1,\vartheta}\mathbb{I}^{a_2,\vartheta}\mathfrak{g}(\omega) = \mathbb{I}^{a_2-a_1,\vartheta}\mathfrak{g}(\omega)$.

Lemma 2 ([31]). *Let* $\mathfrak{g} \in L(0,\tau)$, $n-1 < a \leq n$, $n \in \mathbb{N}$, $0 \leq b \leq 1$, $\gamma = a + nb - ab$, $\left(\mathbb{I}^{(n-a)(1-b),\vartheta}\mathfrak{g}\right) \in AC^k[0,\tau]$. *(*$AC^k[0,\tau]$ *is the k times absolutely continuous functions on* $[0,\tau]$*.) Then, we have*

$$\left(\mathbb{I}^{a,\vartheta}\ {}^{H}\mathcal{D}^{a,b;\vartheta}\mathfrak{g}\right)(\omega) = \mathfrak{g}(\omega) - \sum_{i=1}^{n} \frac{(\vartheta(\omega)-\vartheta(0))^{\gamma-i}}{\Gamma(\gamma-i+1)} \mathfrak{g}_{\vartheta}^{[n-i]}\left(\mathbb{I}^{(1-b)(n-a);\vartheta}\mathfrak{g}\right)(0),$$

where $\mathfrak{g}_{\vartheta}^{[n-i]} = \left(\frac{1}{\vartheta'(\omega)}\frac{d}{d\omega}\right)^{n-i}$ *and*

$$\left(\mathbb{I}^{a,\vartheta}\ {}^{C}\mathcal{D}^{a;\vartheta}\mathfrak{g}\right)(\omega) = \mathfrak{g}(\omega) - \sum_{i=0}^{n-1} \frac{\mathfrak{g}_{\vartheta}^{[i]}(0)}{i!}(\vartheta(\omega)-\varphi(0))^{i}.$$

In the following lemma, a linear variant of the sequential fractional Hilfer–Caputo boundary value problem (4) is studied. This lemma is essential to transform the nonlinear problem (4) into an integral equation and consequently into a fixed-point problem.

Lemma 3. *Let* $\mathfrak{g} \in C([0,\tau],\mathbb{R})$ *and* $\lambda_1, \lambda_2 \neq -1$. *Then, the sequential linear fractional Hilfer–Caputo boundary value problem*

$$\begin{cases} {}^{H}\mathbb{D}^{a,b,\vartheta}({}^{C}\mathbb{D}^{\delta,\vartheta}\varphi)(\omega) = \mathfrak{g}(\omega), \quad \omega \in [0,\tau], \\ \varphi(0) + \lambda_1\varphi(\tau) = 0, \\ {}^{C}\mathbb{D}^{\gamma+\delta-1,\vartheta}\varphi(0) + \lambda_2 {}^{C}\mathbb{D}^{\gamma+\delta-1,\vartheta}\varphi(\tau) = 0, \end{cases} \tag{6}$$

is equivalent to the integral equation

$$\begin{aligned} \varphi(\omega) \quad = \quad & \frac{1}{1+\lambda_1}\left[-\lambda_1\mathbb{I}^{a+\delta,\vartheta}\mathfrak{g}(\tau) + \frac{\lambda_1\lambda_2}{(1+\lambda_2)\Gamma(\gamma+\delta)}(\vartheta(\tau)-\vartheta(0))^{\gamma+\delta-1}\mathbb{I}^{a-\gamma+1,\vartheta}\mathfrak{g}(\tau)\right] \\ & -\frac{\lambda_2}{\Gamma(\gamma+\delta)(1+\lambda_2)}(\vartheta(\omega)-\vartheta(0))^{\gamma+\delta-1}\mathbb{I}^{a-\gamma+1,\vartheta}\mathfrak{g}(\tau) + \mathbb{I}^{a+\delta,\vartheta}\mathfrak{g}(\omega). \end{aligned} \tag{7}$$

Proof. Taking the fractional integral operator $\mathbb{I}^{a,\vartheta}$ on both sides of the first equation in (6) and using Lemma 2, we obtain

$${}^{C}\mathbb{D}^{\delta,\vartheta}\varphi(\omega) = c_1(\vartheta(\omega)-\vartheta(0))^{\gamma-1} + \mathbb{I}^{a,\vartheta}\mathfrak{g}(\omega), \tag{8}$$

where $\gamma = a + b(1-a)$ and $c_1 \in \mathbb{R}$. Now, by taking the fractional integral $\mathbb{I}^{\delta,\vartheta}$ on both sides of Equation (8) and applying Lemma 1, we obtain

$$\varphi(\omega) = c_2 + \frac{\Gamma(\gamma)}{\Gamma(\gamma+\delta)}c_1(\vartheta(\omega)-\vartheta(0))^{\gamma+\delta-1} + \mathbb{I}^{a+\delta,\vartheta}\mathfrak{g}(\omega). \tag{9}$$

By Lemma 1, we have

$${}^{C}\mathbb{D}^{\gamma+\delta-1,\vartheta}\varphi(\omega) = \Gamma(\gamma)c_1 + \mathbb{I}^{a-\gamma+1,\vartheta}\mathfrak{g}(\omega).$$

Now, combining the boundary conditions $\varphi(0) + \lambda_1\varphi(\tau) = 0$ and ${}^{C}\mathbb{D}^{\gamma+\delta-1,\vartheta}\varphi(0) + \lambda_2{}^{C}\mathbb{D}^{\gamma+\delta-1,\vartheta}\varphi(\tau) = 0$ with (9), we obtain

$$\begin{aligned} & c_2 + \lambda_1 c_2 + \lambda_1\frac{\Gamma(\gamma)}{\Gamma(\gamma+\delta)}(\vartheta(\tau)-\vartheta(0))^{\gamma+\delta-1}c_1 + \lambda_1\mathbb{I}^{a+\delta,\vartheta}\mathfrak{g}(\tau) = 0, \\ & \Gamma(\gamma)c_1 + \lambda_2\Gamma(\gamma)c_1 + \lambda_2\mathbb{I}^{a-\gamma+1,\vartheta}\mathfrak{g}(\tau) = 0. \end{aligned}$$

From the above equations, we obtain

$$
\begin{aligned}
c_1 &= \frac{-\lambda_2}{(1+\lambda_2)\Gamma(\gamma)}\mathbb{I}^{a-\gamma+1,\vartheta}\mathfrak{g}(\tau), \\
c_2 &= \frac{1}{1+\lambda_1}\left[-\lambda_1\mathbb{I}^{a+\delta,\vartheta}\mathfrak{g}(\tau) + \frac{\lambda_1\lambda_2}{(1+\lambda_2)\Gamma(\gamma+\delta)}(\vartheta(\tau)-\vartheta(0))^{\gamma+\delta-1}\mathbb{I}^{a-\gamma+1,\vartheta}\mathfrak{g}(\tau)\right].
\end{aligned}
$$

Replacing the values c_1 and c_2 in (9), we obtain the solution (7). On the other hand, operating the fractional differential operators ϑ-Caputo and ϑ-Hilfer of orders, δ and a, respectively, on both sides of the solution (7), we obtain the first equation in (6). It is easy to verify that (7) satisfies the existent boundary conditions in (6). Thus, the proof is completed. □

Remark 1. *In Lemma 3, we have that $\lambda_1, \lambda_2 \neq -1$, which means that our study does not cover the periodic case for the problem (4).*

Remark 2. *If the sequential fractional differential equation in (6) is interchanged as*

$$
{}^{C}\mathbb{D}^{\delta,\vartheta}\left({}^{H}\mathbb{D}^{a,b,\vartheta}\varphi\right)(\omega) = \mathfrak{g}(\omega), \tag{10}
$$

then we have

$$
\varphi(\omega) = c_2(\vartheta(\omega)-\vartheta(0))^{\gamma-1} + \frac{c_1}{\Gamma(a+1)}(\vartheta(\omega)-\vartheta(0))^{a} + \mathbb{I}^{\delta+a,\vartheta}\mathfrak{g}(\omega),
$$

where $c_1, c_2 \in \mathbb{R}$. Since $\gamma = a + b(1-a) \in (0,1)$, we have $c_2 = 0$ when $\omega \to 0$. This mean that the fractional differential Equation (10) needs an initial condition $\varphi(0) = 0$.

3. Main Results

Consider the space $\mathbb{X} = C([0,\tau],\mathbb{R})$ of all continuous functions φ from $[0,\tau]$ into $\mathbb{R}$. This space, endowed with the norm $\|\varphi\| = \sup\{|\varphi(\omega)| : \omega \in [0,\tau]\}$, is a Banach space.

Using Lemma 3, we define an operator $\mathbb{S} : \mathbb{X} \longrightarrow \mathbb{X}$ by

$$
\begin{aligned}
(\mathbb{S}\,\varphi)(\omega) &= \frac{1}{1+\lambda_1}\bigg[-\lambda_1\mathbb{I}^{a+\delta,\vartheta}\mathfrak{F}\Big(\tau,\varphi(\tau),\mathbb{I}^{c,\vartheta}\varphi(\tau),\int_0^\tau \varphi(s)dv(s)\Big) \\
&\quad + \frac{\lambda_1\lambda_2}{(1+\lambda_2)\Gamma(\gamma+\delta)}(\vartheta(\tau)-\vartheta(0))^{\gamma+\delta-1} \\
&\quad \times\mathbb{I}^{a-\gamma+1,\vartheta}\mathfrak{F}\Big(\tau,\varphi(\tau),\mathbb{I}^{c,\vartheta}\varphi(\tau),\int_0^\tau \varphi(s)dv(s)\Big)\bigg] \\
&\quad - \frac{\lambda_2}{\Gamma(\gamma+\delta)(1+\lambda_2)}(\vartheta(\omega)-\vartheta(0))^{\gamma+\delta-1} \\
&\quad \times\mathbb{I}^{a-\gamma+1,\vartheta}\mathfrak{F}\Big(\tau,\varphi(\tau),\mathbb{I}^{c,\vartheta}\varphi(\tau),\int_0^\tau \varphi(s)dv(s)\Big) \\
&\quad + \mathbb{I}^{a+\delta,\vartheta}\mathfrak{F}\Big(\omega,\varphi(\omega),\mathbb{I}^{c,\vartheta}\varphi(\omega),\int_0^\tau \varphi(s)dv(s)\Big), \quad \omega \in [0,\tau].
\end{aligned} \tag{11}
$$

The next lemma will be used in the sequent.

Lemma 4 ([32]). *If $\varphi \in V([0,\tau] \to \mathbb{R})$ and $v : [0,\tau] \to \mathbb{R}$ is a bounded variation function on $[0,\tau]$, then*

$$
\left|\int_0^\tau \varphi(s)dv(s)\right| \le \max_{\omega\in[0,\tau]}|\varphi(\omega)| \cdot V_0^\tau v,
$$

where $V_0^\tau v$ denotes the variation of function v defined by

$$
V_0^\tau v = \sup_P \sum_{i=1}^{n} |v(s_i) - v(s_{i-1})|,
$$

and $P : 0 = s_0 < s_1 < \ldots < s_n = \tau$ *is an arbitrary partition of* $[0, \tau]$.

Recall that v is called a bounded variation function on $[0, \tau]$ if $V_0^\tau v < \infty$.

In the following, to simplify the computations, we set

$$\Theta_{\mathtt{z}}(\mathtt{y}) = \frac{(\vartheta(\mathtt{y}) - \vartheta(0))^{\mathtt{z}}}{\Gamma(\mathtt{z}+1)}.$$

and

$$\begin{aligned} \mathbb{W}_0 &= 1 + \Theta_c(\tau) + V_0^\tau v, \\ \mathbb{W}_1 &= \frac{1}{|1+\lambda_1|}\left[|\lambda_1|\Theta_{a+\delta}(\tau) + \frac{|\lambda_1\lambda_2|}{|1+\lambda_2|}\Theta_{\gamma+\delta}(\tau)\Theta_{a-\gamma+1}(\tau)\right] \\ &\quad + \frac{|\lambda_2|}{|1+\lambda_2|}\Theta_{\gamma+\delta}(\tau)\Theta_{a-\gamma+1}(\tau) + \Theta_{a+\delta}(\tau). \end{aligned} \tag{12}$$

We are ready to prove our first result, the existence of a unique solution for the sequential fractional Hilfer–Caputo boundary value problem (4), via the Banach fixed-point theorem [33].

Theorem 1. *Assume that* $\mathfrak{F} : [0, \tau] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ *is such that:*
$(\mathbb{G}_1)$*There exists* $\mathbb{K} > 0$ *such that*

$$|\mathfrak{F}(\omega, \mathtt{x}_1, \mathtt{y}_1, \mathtt{z}_1) - \mathfrak{F}(\omega, \mathtt{x}_2, \mathtt{y}_2, \mathtt{z}_2)| \le \mathbb{K}(|\mathtt{x}_1 - \mathtt{x}_2| + |\mathtt{y}_1 - \mathtt{y}_2| + |\mathtt{z}_1 - \mathtt{z}_2|),$$

for all $\omega \in [0, \tau]$ *and* $\mathtt{x}_i, \mathtt{y}_i, \mathtt{z}_i \in \mathbb{R}$, $i = 1, 2$.
If

$$\mathbb{W}_0\mathbb{W}_1\mathbb{K} < 1,$$

where $\mathbb{W}_0, \mathbb{W}_1$ *are defined by (12); then, the fractional Hilfer–Caputo sequential boundary value problem (4) has a unique solution on* $[0, \tau]$.

Proof. Let $\mathbb{M} = \sup\{|\mathfrak{F}(\omega, 0, 0, 0)| : \omega \in [0, \tau]\}$ and $\mathbb{B}_{x^*} = \{\varphi \in \mathbb{X} : \|\varphi\| \le x^*\}$ with

$$x^* \ge \frac{\mathbb{W}_1\mathbb{M}}{1 - \mathbb{K}\mathbb{W}_0\mathbb{W}_1}.$$

Using $(\mathbb{G}_1)$, we have:

$$\begin{aligned} & \left|\mathfrak{F}\Big(\omega, \varphi(\omega), \mathbb{I}^{c,\vartheta}\varphi(\omega), \int_0^\tau \varphi(s)dv(s)\Big)\right| \\ \le & \left|\mathfrak{F}\Big(\omega, \varphi(\omega), \mathbb{I}^{c,\vartheta}\varphi(\omega), \int_0^\tau \varphi(s)dv(s)\Big) - \mathfrak{F}(\omega, 0, 0, 0)\right| + |\mathfrak{F}(\omega, 0, 0, 0)| \\ \le & \mathbb{K}\Big(|\varphi(\omega)| + \mathbb{I}^{c,\vartheta}|\varphi(\omega)| + \left|\int_0^\tau \varphi(s)dv(s)\right|\Big) + \mathbb{M} \\ \le & \mathbb{K}\Big(\|\varphi\| + \Theta_c(\tau)\|\varphi\| + \|\varphi\|V_0^T w\Big) + \mathbb{M} \\ \le & \mathbb{K}x^*\Big(1 + \Theta_c(\tau) + V_0^T w\Big) + \mathbb{M} \\ = & \mathbb{K}x^*\mathbb{W}_0 + \mathbb{M}. \end{aligned}$$

We will show that $\mathbb{D}\mathbb{B}_{x^*} \subseteq \mathbb{B}_{x^*}$. For all $\varphi \in \mathbb{X}$, we have

$$|(\mathbb{S}\varphi)(\omega)| \le \frac{1}{1+\lambda_1}\left[|\lambda_1|\mathbb{I}^{a+\delta,\vartheta}\left(\left|\mathfrak{F}\Big(\tau, \varphi(\tau), \mathbb{I}^{c,\vartheta}\varphi(\tau), \int_0^\tau \varphi(s)dv(s)\Big)\right|\right)\right.$$

$$
\begin{aligned}
&+\frac{|\lambda_1\lambda_2|}{|1+\lambda_2|}\Theta_{\gamma+\delta}(\tau)\mathbb{I}^{a-\gamma+1,\vartheta}\left(\left|\mathfrak{F}\left(\tau,\varphi(\tau),\mathbb{I}^{c,\vartheta}\varphi(\tau),\int_0^\tau \varphi(s)dv(s)\right)\right|\right)\Bigg]\\
&+\frac{|\lambda_2|}{|1+\lambda_2|}\Theta_{\gamma+\delta}(\tau)\mathbb{I}^{a-\gamma+1,\vartheta}\left(\left|\mathfrak{F}\left(\tau,\varphi(\tau),\mathbb{I}^{c,\vartheta}\varphi(\tau),\int_0^\tau \varphi(s)dv(s)\right)\right|\right)\\
&+\mathbb{I}^{a+\delta,\vartheta}\left(\left|\mathfrak{F}\left(\tau,\varphi(\tau),\mathbb{I}^{c,\vartheta}\varphi(\tau),\int_0^\tau \varphi(s)dv(s)\right)\right|\right)\\
\leq\;& \frac{1}{|1+\lambda_1|}\Bigg[|\lambda_1|\mathbb{I}^{a+\delta,\vartheta}(\mathbb{K}x^*\mathbb{W}_0+\mathbb{M})\\
&+\frac{|\lambda_1\lambda_2|}{|1+\lambda_2|}\Theta_{\gamma+\delta}(\tau)\mathbb{I}^{a-\gamma+1,\vartheta}(\mathbb{K}x^*\mathbb{W}_0+\mathbb{M})\Bigg]\\
&+\frac{|\lambda_2|}{|1+\lambda_2|}\Theta_{\gamma+\delta}(\tau)\mathbb{I}^{a-\gamma+1}(\mathbb{K}x^*\mathbb{W}_0+\mathbb{M})+\mathbb{I}^{a+\delta,\vartheta}(\mathbb{K}x^*\mathbb{W}_0+\mathbb{M})\\
\leq\;& (\mathbb{K}x^*\mathbb{W}_0+\mathbb{M})\Bigg\{\frac{1}{|1+\lambda_1|}\left[|\lambda_1|\Theta_{a+\delta}(\tau)+\frac{|\lambda_1\lambda_2|}{|1+\lambda_2|}\Theta_{\gamma+\delta}(\tau)\Theta_{a-\gamma+1}(\tau)\right]\\
&+\frac{|\lambda_2|}{|1+\lambda_2|}\Theta_{\gamma+\delta}(\tau)\Theta_{a-\gamma+1}(\tau)+\Theta_{a+\delta}(\tau)\Bigg\}\\
=\;& (\mathbb{K}x^*\mathbb{W}_0+\mathbb{M})\mathbb{W}_1\leq x^*.
\end{aligned}
$$

Hence, $\mathbb{S}\mathbb{B}_{x^*}\subseteq\mathbb{B}_{x^*}$. Next, we will show that the operator $\mathbb{S}$ is a contraction. For φ_1, $\varphi_2\in\mathbb{B}_{x^*}$, we have

$$
\begin{aligned}
&|(\mathbb{S}\varphi_1)(\omega)-(\mathbb{S}\varphi_2)(\omega)|\\
\leq\;& \frac{1}{1+\lambda_1}\Bigg[\Bigg|\lambda_1|\mathbb{I}^{a+\delta,\vartheta}\left(\left|\mathfrak{F}\left(\tau,\varphi_1(\tau),\mathbb{I}^{c,\vartheta}\varphi_1(\tau),\int_0^\tau \varphi_1(s)dv(s)\right)\right.\right.\\
&\left.\left.-\mathfrak{F}\left(\tau,\varphi_2(\tau),\mathbb{I}^{c,\vartheta}\varphi_2(\tau),\int_0^\tau \varphi_2(s)dv(s)\right)\right|\right)\\
&+\frac{|\lambda_1\lambda_2|}{|1+\lambda_2|}\Theta_{\gamma+\delta}(\tau)\mathbb{I}^{a-\gamma+1,\vartheta}\left(\left|\mathfrak{F}\left(\tau,\varphi_1(\tau),\mathbb{I}^{c,\vartheta}\varphi_1(\tau),\int_0^\tau \varphi_1(s)dv(s)\right)\right.\right.\\
&\left.\left.-\mathfrak{F}\left(\tau,\varphi_2(\tau),\mathbb{I}^{c,\vartheta}\varphi_2(\tau),\int_0^\tau \varphi_2(s)dv(s)\right)\right|\right]\\
&+\frac{|\lambda_2|}{|1+\lambda_2|}\Theta_{\gamma+\delta}(\tau)\mathbb{I}^{a-\gamma+1,\vartheta}\left(\left|\mathfrak{F}\left(\tau,\varphi_1(\tau),\mathbb{I}^{c,\vartheta}\varphi_1(\tau),\int_0^\tau \varphi_1((s)dv(s)\right)\right.\right.\\
&\left.\left.-\mathfrak{F}\left(\tau,\varphi_2(\tau),\mathbb{I}^{c,\vartheta}\varphi_2(\tau),\int_0^\tau \varphi_2(s)dv(s)\right)\right|\right)\\
&+\mathbb{I}^{a+\delta,\vartheta}\left(\left|\mathfrak{F}\left(\tau,\varphi_1(\tau),\mathbb{I}^{c,\vartheta}\varphi_1(\tau),\int_0^\tau \varphi_1(s)dv(s)\right)\right.\right.\\
&\left.\left.-\mathfrak{F}\left(\tau,\varphi_2(\tau),\mathbb{I}^{c,\vartheta}\varphi_2(\tau),\int_0^\tau \varphi_2(s)dv(s)\right)\right|\right)\\
\leq\;& \mathbb{K}\mathbb{W}_0\|\varphi_1-\varphi_2\|\Bigg\{\frac{1}{|1+\lambda_1|}\left[|\lambda_1|\Theta_{a+\delta}(\tau)+\frac{|\lambda_1\lambda_2|}{|1+\lambda_2|}\Theta_{\gamma+\delta}(\tau)\Theta_{a-\gamma+1}(\tau)\right]\\
&+\frac{|\lambda_2|}{|1+\lambda_2|}\Theta_{\gamma+\delta}(\tau)\Theta_{a-\gamma+1}(\tau)+\Theta_{a+\delta}(\tau)\Bigg\}\\
=\;& \mathbb{W}_0\mathbb{W}_1\mathbb{K}\|\varphi_1-\varphi_2\|.
\end{aligned}
$$

Thus, $\|(\mathbb{S}\varphi_1)-(\mathbb{S}\varphi_2)\|\leq\mathbb{K}\mathbb{W}_0\mathbb{W}_1\|\varphi_1-\varphi_2\|$, and since $\mathbb{K}\mathbb{W}_0\mathbb{W}_1<1$, $\mathbb{S}$ is a contraction. By Banach fixed-point theorem, the operator $\mathbb{S}$ has a unique solution. Thus, the fractional Hilfer–Caputo sequential boundary value problem (4) has a unique solution on $[0,\tau]$. □

Our second result, concerning the existence of at least one solution to the fractional Hilfer–Caputo sequential boundary value problem (4), is proved by using the Leray–Schauder nonlinear alternative [34].

Theorem 2. *Let $\mathfrak{F} \in C([0,\tau] \times \mathbb{R}^3, \mathbb{R})$ such that:*

($\mathbb{G}_2$) *There exist $\mathfrak{P} \in C([0,\infty),[0,\infty))$, $v_1, v_2 \in C([0,\tau]\,[0,\infty))$ such that $\mathfrak{P}$ is nondecreasing and for all $\omega \in [0,\tau]$ and $\mathrm{x}_i \in \mathbb{R},\ i = 1,2,3$, we have*

$$|\mathfrak{F}(\omega, \mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3)| \le v_1(\omega)\mathfrak{P}(|\mathrm{x}_1| + |\mathrm{x}_2| + |\mathrm{x}_3|) + v_2(\omega).$$

($\mathbb{G}_3$) *There exists $\mathbb{Z}_0 \in \mathbb{R}^+$ such that*

$$\frac{\mathbb{Z}_0}{\|v_1\|\mathfrak{P}(\mathbb{Z}_0\mathbb{W}_0) + \|v_2\|} > 1.$$

Then, the fractional sequential Hilfer–Caputo boundary value problem (4) has at least one solution on $[0,\tau]$.

Proof. The operator $\mathbb{S}$ is obviously continuous, since $\mathfrak{F}$ is continuous. Now, the compactness property of the operator $\mathbb{S}$ is proved on $\mathbb{B}_x$, where $\mathbb{B}_x = \{\varphi \in \mathbb{X} : \ \|\varphi\| \le x\}$. For all $\varphi \in \mathbb{X}$, we have

$$\begin{aligned}
|(\mathbb{S}\varphi)(\omega)| \le\ & \frac{1}{1+\lambda_1}\Bigg[|\lambda_1|\mathbb{I}^{a+\delta,\vartheta}\left(\left|\mathfrak{F}\left(\tau, \varphi(\tau), \mathbb{I}^{c,\vartheta}\varphi(\tau), \int_0^\tau \varphi(s)dv(s)\right)\right|\right) \\
&+\frac{|\lambda_1\lambda_2|}{|1+\lambda_2|}\Theta_{\gamma+\delta}(\tau)\mathbb{I}^{a-\gamma+1,\vartheta}\left(\left|\mathfrak{F}\left(\tau, \varphi(\tau), \mathbb{I}^{c,\vartheta}\varphi(\tau), \int_0^\tau \varphi(s)dv(s)\right)\right|\right)\Bigg] \\
&+\frac{|\lambda_2|}{|1+\lambda_2|}\Theta_{\gamma+\delta}(\tau)\mathbb{I}^{a-\gamma+1,\vartheta}\left(\left|\mathfrak{F}\left(\tau, \varphi(\tau), \mathbb{I}^{c,\vartheta}\varphi(\tau), \int_0^\tau \varphi(s)dv(s)\right)\right|\right) \\
&+\mathbb{I}^{a+\delta,\vartheta}\left(\left|\mathfrak{F}\left(\tau, \varphi(\tau), \mathbb{I}^{c,\vartheta}\varphi(\tau), \int_0^\tau \varphi(s)dv(s)\right)\right|\right) \\
\le\ & \frac{1}{|1+\lambda_1|}\Bigg[|\lambda_1|\Theta_{a+\delta}(\tau)\Big(\|v_1\|\mathfrak{P}(\|\varphi\|\mathbb{W}_0) + \|v_2\|\Big) \\
&+\frac{|\lambda_1\lambda_2|}{|1+\lambda_2|}\Theta_{a-\gamma+1}(\tau)\Theta_{\gamma+\delta}(\tau)\Big(\|v_1\|\mathfrak{P}(\|\varphi\|\mathbb{W}_0) + \|v_2\|\Big)\Bigg] \\
&+\frac{|\lambda_2|}{|1+\lambda_2|}\Theta_{a-\gamma+1}(\tau)\Theta_{\gamma+\delta}(\tau)\Big(\|v_1\|\mathfrak{P}(\|\varphi\|\mathbb{W}_0) + \|v_2\|\Big) \\
&+\Theta_{a+\delta}(\tau)\Big(\|v_1\|\mathfrak{P}(\|\varphi\|\mathbb{W}_0) + \|v_2\|\Big) \\
\le\ & \mathbb{W}_1\Big(\|v_1\|\mathfrak{P}(x\mathbb{W}_0) + \|v_2\|\Big) := \Phi,
\end{aligned}$$

which implies that $\|\mathbb{S}\varphi\| \le \Phi$, and thus, the operator $\mathbb{S}$ is uniformly bounded on $\mathbb{B}_x$. To show the equicontinuity property of $\mathbb{S}(\mathbb{B}_x)$, let $\omega_1, \omega_2 \in [0,\tau]$ with $\omega_1 < \omega_2$. Then, for all $\varphi \in \mathbb{B}_x$, we have

$$\begin{aligned}
& |(\mathbb{S}\varphi)(\omega_2) - (\mathbb{S}\varphi)(\omega_1)| \\
\le\ & \frac{|\lambda_2|}{|1+\lambda_2|\Gamma(\gamma+\delta)}|(\vartheta(\omega_2)-\vartheta(0))^{\gamma+\delta-1} - (\vartheta(\omega_1)-\vartheta(0))^{\gamma+\delta-1}| \\
& \times\mathbb{I}^{a-\gamma+1,\vartheta}\left(\left|\mathfrak{F}\left(\tau, \varphi(\tau), \mathbb{I}^{c,\vartheta}\varphi(\tau), \int_0^\tau \varphi(s)dv(s)\right)\right|\right) \\
& +\Bigg|\frac{1}{\Gamma(a+\delta)}\int_0^{\omega_1}\vartheta'(s)[(\vartheta(\omega_2)-\vartheta(s))^{a+\delta-1} - (\vartheta(\omega_1)-\vartheta(s))^{a+\delta-1}] \\
& \times\mathfrak{F}\left(s, \varphi(s)\ \mathbb{I}^{c,\vartheta}\varphi(s), \int_0^\tau \varphi(s)dv(s)\right)ds \\
& +\int_{\omega_1}^{\omega_2}\vartheta'(s)(\vartheta(\omega_2)-\vartheta(s))^{a+\delta-1}\mathfrak{F}\left(s, \varphi(s)\ \mathbb{I}^{c,\vartheta}\varphi(s), \int_0^\tau \varphi(s)dv(s)\right)ds\Bigg| \\
\le\ & \frac{|\lambda_2|}{|1+\lambda_2|}|\Theta_{\gamma+\delta}(\omega_2) - \Theta_{\gamma+\delta}(\omega_1)|\Theta_{a-\gamma+1}(\tau)\Big(\|v_1\|P(x\mathbb{W}_0) + \|v_2\|\Big)
\end{aligned}$$

$$+\Big(\|v_1\|P(x\mathbb{W}_0)+\|v_2\|\Big)\left[\frac{2(\vartheta(\omega_2)-\vartheta(\omega_1))^{a+\delta}}{\Gamma(a+\delta+1)}+|\Theta_{\gamma+\delta}(\omega_2)-\Theta_{\gamma+\delta}(\omega_1)|\right].$$

When $\omega_1 \longrightarrow \omega_2$, the right-hand side of the above inequality, independently of φ, tends to zero. Hence, $\mathbb{S}(\mathbb{B}_x)$ is an equicontinuous set. Consequently, the operator $\mathbb{S}$ is completely continuous, by the Arzelá–Ascoli theorem.

Finally, we indicate that the set

$$\Xi = \{\varphi \in \mathbb{X} : \ \varphi = \lambda(\mathbb{S}\varphi), \ \ 0 < \lambda < 1\}$$

is bounded. Let $\varphi \in \Xi$; then, $\varphi = \lambda(\mathbb{S}\varphi)$ for some $\lambda \in (0,1)$. Following the computations used in the first step, for all $\omega \in [0,\tau]$, we have

$$\begin{aligned} |\varphi(\omega)| &= \lambda|(\mathbb{S}\varphi)(\omega)| \\ &\leq \mathbb{W}_1\Big(\|v_1\|\mathfrak{P}(\|\varphi\|\mathbb{W}_0)+\|v_2\|\Big), \end{aligned}$$

and hence

$$\frac{\|\varphi\|}{\mathbb{W}_1\Big(\|v_1\|\mathfrak{P}(\|\varphi\|\mathbb{W}_0)+\|v_2\|\Big)} \leq 1.$$

Due to $(\mathbb{G}_3)$, $\|\varphi\| \neq Q$. Now, we define $\mathbb{E} = \{\varphi \in \mathbb{B}_x : \|\varphi\| < \mathbb{Z}_0\}$. Obviously, the operator $\mathbb{S} : \mathbb{E} \longrightarrow \mathbb{X}$ is continuous and completely continuous. Therefore, there is no $\varphi \in \partial\mathbb{E}$ such that $\varphi = \lambda(\mathbb{S}\varphi)$ with $0 < \lambda < 1$. By the Leray–Schauder nonlinear alternative, the operator $\mathbb{D}$ has a fixed point $\varphi \in \mathbb{E}$, which is a solution of the sequential fractional Hilfer–Caputo boundary value problem (4). □

The following corollaries concern some special cases of the function $\mathfrak{P}$, which is useful in checking the the existence of solutions.

Corollary 1. *If $\mathfrak{P}$ in condition $(\mathbb{G}_2)$ is given by $\mathfrak{P}(u) \equiv L, L > 0$, then the boundary value problem of sequential Hilfer and Caputo fractional operators (4) has at least one solution.*

Corollary 2. *If $\mathfrak{P}$ in condition $(\mathbb{G}_2)$ is given by $\mathfrak{P}(u) = Au + B$, where $A \geq 0$ and $B > 0$ and if $A\|v_1\|\mathbb{W}_0 < 1$, then the non-separated boundary value problem of sequential Hilfer and Caputo fractional operators (4) has at least one solution on $[0,\tau]$.*

Corollary 3. *Suppose that the function $\mathfrak{P}$ in condition $(\mathbb{G}_2)$ is given by $\mathfrak{P}(u) = Cu^2 + D$, where $C, D > 0$ are constants and $4C\|v_1\|\mathbb{W}_0^2(\|v_1\|D + \|v_2\|) < 1$. Then, the non-separated boundary value problem of sequential Hilfer and Caputo fractional operators (4) has at least one solution on $[0,\tau]$.*

4. Examples

In this section, some examples of the sequential Hilfer and Caputo fractional differential equation containing the Riemann–Stieltjes and fractional integrals with non-separated boundary conditions, by varying a nonlinear function $\mathfrak{F}$, can be considered. Consider the following sequential Hilfer and Caputo fractional boundary value problem:

$$\begin{cases} {}^{H}\mathbb{D}^{\frac{3}{4},\frac{1}{2},\omega^2+\sqrt{\omega}}({}^{C}\mathbb{D}^{\frac{1}{4},\omega^2+\sqrt{\omega}}\varphi)(\omega) = \mathfrak{F}\Big(\omega,\varphi(\omega),\mathbb{I}^{c,\vartheta}\varphi(\omega),\displaystyle\int_0^\tau \varphi(s)dv(s)\Big),\ \omega \in \left[0,\frac{3}{2}\right], \\ \varphi(0) + \dfrac{3}{7}\varphi\left(\dfrac{3}{2}\right) = 0, \\ {}^{C}\mathbb{D}^{\frac{1}{8},\omega^2+\sqrt{\omega}}\varphi(0) + \dfrac{5}{7}{}^{C}\mathbb{D}^{\frac{1}{8},\omega^2+\sqrt{\omega}}\varphi(\tau) = 0. \end{cases} \tag{13}$$

Setting $a = 3/4$, $b = 1/2$, $\delta = 1/4$, $\vartheta = \omega^2 + \sqrt{\omega}$, $\tau = 3/2$, $\lambda_1 = 3/7$, $\lambda_2 = 5/7$, then we obtain $\gamma = 7/8$, which leads to $\gamma + \delta - 1 = 1/8$. In addition, we have $\Theta_{a+\delta}(\tau) \approx 3.474744872$, $\Theta_{\gamma+\delta}(\tau) \approx 3.832247090$, $\Theta_{a-\gamma+1}(\tau) \approx 3.118972472$ and $\mathbb{W}_1 \approx 10.99153297$.

(i) Assume that the nonlinear function $\mathfrak{F}$ is given by

$$\begin{aligned} &\mathfrak{F}\left(\omega, \varphi, \mathbb{I}^{c,\vartheta}\varphi, \int_0^\tau \varphi(s)dv(s)\right) \\ = \;& 1 + \frac{1}{2}\omega^2 + \frac{1}{2(\omega+48)}\left(\frac{\varphi^2 + 2|\varphi|}{1+|\varphi|}\right) \\ &+ \frac{1}{\omega^2+49}\mathbb{I}^{\frac{2}{3},\omega^2+\sqrt{\omega}}\varphi + \frac{1}{\sqrt{\omega}+50}\int_0^{\frac{3}{2}} \varphi(s)de^{-s}. \end{aligned} \tag{14}$$

Note that the function v is $v(s) = e^{-s}$ and the order of fractional integration is $c = 2/3$. Then, we obtain $V_0^\tau v = 1 - e^{-\frac{3}{2}} \approx 0.7768698399$, $\Theta_c(\tau) \approx 2.541265551$ and $\mathbb{W}_0 \approx 4.318135391$. Further, we can check the Lipschitz condition of the function in (14) by

$$|\mathfrak{F}(\omega, x_1, y_1, z_1) - \mathfrak{F}(\omega, x_2, y_2, z_2)| \le \frac{1}{48}(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|), \quad \forall x_i, y_i, z_i \in \mathbb{R},$$

for $i = 1, 2$, with the Lipschitz constant $\mathbb{K} = 1/48$. Therefore, the relation

$$\mathbb{W}_0\mathbb{W}_1\mathbb{K} \approx 0.9888109900 < 1,$$

holds. Hence, by the conclusion of Theorem 1, we have that the mixed Hilfer–Caputo fractional Riemann–Stieltjes integro-differential equation with non-separated boundary conditions (13) with $\mathfrak{F}$ given by (14) has a unique solution on the interval $[0, 3/2]$.

(ii) Now, let the nonlinear function $\mathfrak{F}$ be given by

$$\begin{aligned} &\mathfrak{F}\left(\omega, \varphi, \mathbb{I}^{c,\vartheta}\varphi, \int_0^\tau \varphi(s)dv(s)\right) \\ = \;& (\sqrt{\omega}+1)\sin^2\varphi + (\sqrt{\omega}+2)\frac{\left|\mathbb{I}^{\frac{2}{3},\omega^2+\sqrt{\omega}}\varphi\right|}{1+\left|\mathbb{I}^{\frac{2}{3},\omega^2+\sqrt{\omega}}\varphi\right|} \\ &+ (\sqrt{\omega}+3)\cos^4\left(\int_0^{\frac{3}{2}} \varphi(s)d(s^2+1)\right) + \frac{1}{4}\omega + \frac{1}{3}. \end{aligned} \tag{15}$$

In this case, we have

$$|\mathfrak{F}(\omega, x, y, z)| \le 3(\sqrt{\omega}+3) + \frac{1}{4}\omega + \frac{1}{3}, \quad x, y, z \in \mathbb{R}.$$

Applying Corollary 1 with $\mathfrak{P} = 3$, the non-separated BVP (13), with $\mathfrak{F}$ given by (15), has at least one solution on $[0, 3/2]$.

(iii) Consider the nonlinear function $\mathfrak{F}$ expressed by

$$\begin{aligned} &\mathfrak{F}\left(\omega, \varphi, \mathbb{I}^{c,\vartheta}\varphi, \int_0^\tau \varphi(s)dv(s)\right) \\ = \;& \frac{1}{\omega+2}\left[\frac{2\varphi^2}{5(1+|\varphi|)} + \frac{1}{3}\left(\mathbb{I}^{\frac{4}{5},\omega^2+\sqrt{\omega}}\varphi\right)e^{-\varphi^4}\right. \\ &\left. + \frac{2}{7}\cos^8\varphi\int_0^{\frac{3}{2}} \varphi(s)d(\ln(1+s)) + \frac{1}{6}\right] + \frac{1}{3}\omega^2 + \frac{1}{4}. \end{aligned} \tag{16}$$

We have $v(s) = \ln(1+s)$ and $c = 4/5$, which yield $V_0^\tau v \approx 0.9162907319$, $\Theta_c(\tau) \approx 2.908102246$ and $\mathbb{W}_0 \approx 4.824392978$. Now, we obtain

$$|\mathfrak{F}(\omega, x, y, z)| \leq \frac{1}{\omega+2}\left[\frac{2}{5}(|x|+|y|+|z|)+\frac{1}{6}\right]+\frac{1}{3}\omega^2+\frac{1}{4}, \ x, y, z \in \mathbb{R},$$

and hence $\mathfrak{P}(u) = (2/5)|u| + (1/6)$, $v_1(\omega) = 1/(\omega+2)$ and $v_2(\omega) = (1/3)\omega^2 + (1/4)$. Consequently, $A = 2/5$ and $\|v_1\| = 1/2$. Then, we can compute that $A\|v_1\|\mathbb{W}_0 \approx 0.9648785956 < 1$. By using Corollary 2, the non-separated Hilfer–Caputo boundary value problem (13), with $\mathfrak{F}$ given by (16), has at least one solution on $[0, 3/2]$.

(iv) Finally, let the nonlinear function $\mathfrak{F}$ be presented by

$$\begin{aligned} &\mathfrak{F}\left(\omega, \varphi, \mathbb{I}^{c,\vartheta}\varphi, \int_0^\tau \varphi(s)dv(s)\right) \\ = \ & \frac{1}{\omega^2+15}\left[\frac{1}{3}\left(\varphi e^{-\varphi^2}+\left(\sin^4\varphi\right)\mathbb{I}^{\frac{5}{4},\omega^2+\sqrt{\omega}}\varphi+\cos^6\varphi\int_0^{\frac{3}{2}}\varphi(s)d(\sqrt[3]{s}+1)\right)^2+\frac{1}{5}\right] \\ &+\frac{1}{\sqrt{\omega}+4}. \end{aligned} \tag{17}$$

Choosing $v(s) = (\sqrt[3]{s}+1)$ and $c = 5/4$, we obtain $V_0^\tau v \approx 1.144714242$, $\Theta_c(\tau) \approx 4.187188923$ and $\mathbb{W}_0 \approx 6.331903165$. By considering

$$|\mathfrak{F}(\omega, x, y, z)| \leq \frac{1}{\omega^2+15}\left[\frac{1}{3}(|x|+|y|+|z|)^2+\frac{1}{5}\right]+\frac{1}{\sqrt{\omega}+4}, \ x, y, z \in \mathbb{R},$$

and set $\mathfrak{P}(u) = (1/3)u^2 + (1/5)$, $v_1(\omega) = 1/(\omega^2+15)$ and $v_2(\omega) = 1/(\sqrt{\omega}+4)$, we obtain $C = 1/3$, $D = 1/5$, $\|v_1\| = 1/15$ and $\|v_2\| = 1/4$. These information give $4C\|v_1\|\mathbb{W}_0^2(\|v_1\|D + \|v_2\|) \approx 0.9384731311 < 1$. The conclusion of Corollary 3 tells us that the non-separated BVP of sequential Hilfer and Caputo fractional operators (13), with $\mathfrak{F}$ given by (17), has at least one solution on $[0, 3/2]$.

5. Conclusions

In studying fractional boundary value problems involving Hilfer fractional derivative operators of order in $(1, 2]$, it is necessary to have a zero initial condition. In the present paper, we proposed a combination of Hilfer and Caputo fractional derivatives to avoid this difficulty. Thus, in this research, we investigated a sequential fractional boundary value problem subject to non-separated boundary conditions in which we combined Hilfer and Caputo fractional derivative operators. We proved the existence and uniqueness results by using fixed-point theory. The existence of a unique solution is proved via Banach's fixed point theorem, while an existence result was established via the Leray–Schauder nonlinear alternative. The obtained results are well illustrated by the constructed numerical examples.

The results are new and contribute significantly to this new research subject. For future work, we plan to apply this new method to study other kinds of boundary value problems with nonzero initial conditions as well as coupled systems of fractional differential equations containing a combination of Hilfer and Caputo fractional derivative operators.

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