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Evaluating Mechanism and Related Axiomatic Results under Multiple Considerations

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Abstract: Under many interactive environments in the real world, there is often a need to evaluate the minimization effects and subsequent allocation outcomes derived from these interactions under multiple considerations. For instance, in the context of product sales, it is necessary to evaluate how to minimize the manufacturing costs of various producing factors, and sometimes, from a holistic perspective, it may even be necessary to evaluate situations with minimal sales benefits. On the other hand, in order to evaluate related effects derived from interactions and subsequent allocation outcomes, many game-theoretical studies are based on interactive models to formulate evaluating mechanisms, and then they apply axiomatic processes to analyze the rationality of these mechanisms. Therefore, this study first proposes a mechanism for evaluating the minimization effects and subsequent allocation outcomes under multiple considerations. Additionally, considering that different environmental impacts result from varying participation factors, this study also presents several weighted derivatives based on participation factors and their behaviors. Concurrently, we utilize axiomatic results to demonstrate the mathematical correctness and practicality for these evaluating mechanisms.

Keywords: minimization effect; multiple consideration; evaluating mechanism; weight; axiomatic result

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1. Introduction

Under interactive environments, achieving optimal or balanced states related to the effects derived from interactions and subsequent allocations typically involves multiple minimization considerations, which may sometimes conflict with each other. For instance, a factory producing environmentally friendly products must operate under considerations of minimizing energy consumption and environmental pollution within the shortest possible timeframe. However, these considerations may incur additional costs, conflicting with cost reducing considerations. Therefore, it is necessary to evaluate optimal or balanced goals of production and sales under multiple minimization considerations. In the field of mathematics, mathematical multiattribute game-theoretical analysis is often employed to address such problems under multiple considerations. The mechanisms governing such conditions lack suitable frameworks to articulate optimal outcomes that, unlike conventional notions or perspectives, consider various objective functions. Numerous prior studies have explored multiattribute scenarios. For instance, Bednarczuk et al. [1] transformed the multiple-choice knapsack problem into a bi-objective optimization problem, whose solution set encompasses solutions of the original multiple-choice knapsack problem. Goli et al. [2] addressed the optimization of the multivariate manufacturing portfolio problem under return uncertainty. The key achievement stems from employing an enhanced artificial intelligence-robust optimization hybrid approach, introducing a new concept for assessing the risk of a manufacturing portfolio. A bi-objective mathematical formulation (maximizing return and minimizing risk) is also presented. By delving into multiattribute analysis techniques amidst diverse and complex conditions (e.g., considering multiple perspectives and incorporating multi-level participation factors), Guarini et al. [3] aimed to outline a methodology for selecting the most suitable mechanism tailored to specific evaluation requirements, often encountered in strategic decision-making contexts. A resilient combinatorial optimization modeling approach by Mustakerov et al. [4] is advanced for multi-choice yield with diverse strategy maker prerequisites. This approach is founded on formulating multiattribute linear mixed-integer optimization tasks. Tirkolaee et al. [5] highlighted the multiattribute multi-mode utility-constrained manufacturing scheduling problem with compensation planning, where tasks can be completed through various modes, aiming to minimize completion time and maximize net present value simultaneously.

Under conventional settings, each participation factor is either fully engaged or entirely excluded from engagement with other participation factors. However, under multiple considerations, various participation factors exhibit different corresponding participating levels relative to different considerations. For example, the accounting department may have different cost-evaluating principles for the manufacturing and the marketing departments. In a *multi-choice environment*, each participation factor is allowed to operate across a finite range of participating levels. Consequently, a multi-choice environment can be seen as an extension of a conventional environment. Hwang and Liao [6], Liao [7], and Nouweland et al. [8] have proposed several generalized mechanisms for traditional allocation methods, tailored to the specific requirements of multi-choice environments, to determine comprehensive outcomes for individual participatory elements. On the other hand, the same participation factors or behaviors relative to different considerations may also have varying impacts. For instance, the pollution effects caused by the same pesticide or ingredient may differ between domestic water use and industrial water use. Therefore, in performing multi-choice analysis, participation factors and its participating levels can be sensibly incorporated with the concept of weighting for analysis. Building on the preceding interpretations, there is a desire for the equitable allocation of arbitrary utility among participation factors and its levels of participation based on *weights*. Typically, weights may be assigned to either the “participation factors” or the “levels” to discern differences among the participation factors or their levels of participation, respectively.

Within the realm of game theory, there is a branch that delves into how to achieve optimal or equilibrium states using certain mechanisms within interactive environments. One of the most commonly employed methods is the so-called *axiomatic process*, which consists of the following steps: first, mathematically model the interactive environment, then define the mechanisms to achieve optimal or equilibrium states. Subsequently, formalize many principles of fairness and justice into mathematical models, giving rise to what is known as the axiomatic process. To demonstrate the mathematical correctness and practicality of these mechanisms, it is essential to prove that these mechanisms uniquely and simultaneously satisfy certain axioms, which are indispensable. In the realm of cooperative environments, the axiomatic processes for allocation mechanisms emphasize the critical notion of *steadiness* (or consistency). Steadiness, in this context, pertains to the stability and reliability of advantageous mechanisms. It can be defined as follows: within a specific environment, participation factors are expected to anticipate changes in the environment and agree to compute their rewards based on these anticipations. An allocation mechanism is said to satisfy steadiness if it assigns consistent rewards to participation factors in both the original scenario and a hypothetical *reduced environment*. Hence, steadiness is a fundamental aspect contributing to the internal “robustness” of compromises and has been thoroughly explored across various domains, including bargaining issues and resource distribution scenarios. Building upon the concept of the *equal allocation of non-separable costs* (EANSC) introduced by Ransmeier [9], Liao et al. [7] have devised two allocating mechanisms. These mechanisms involve assigning weights to participation factors and their respective levels of participation under multiattribute multi-choice situations. Taking inspiration from Moulin’s axiomatic techniques [10], Liao et al. [7] have also extended the concept of a *complement-reduced environment*. This extension aims to demonstrate that these two allocation mechanisms serve as fair and consistent mechanisms for distributing utility.

The above-mentioned existing findings raise the following question:

- Can evaluating mechanisms be devised by simultaneously incorporating weights for participation factors and their participating levels under multiattribute multi-choice considerations?

Based on the aforementioned statements, the main concepts and related achievements of this study are as follows.

1. In order to evaluate the minimization effects derived from interactions and subsequent allocating outcomes, we utilize the concept of EANSC within the framework of multiattribute multi-choice environments to propose *multiattribute equal minimization of non-separable effects* (MEMNSE).
2. Due to the fact that the same factors may have different impacts under different considerations, we integrate the concepts of participation factors, participating levels, and participating effect gaps into the MEMNSE, resulting in several different weighted forms.
3. To demonstrate the mathematical correctness and practicality of these mechanisms proposed in this study, we will use the concept of consistency to present the corresponding axiomatic results for these mechanisms.

2. Preliminaries

2.1. Definitions and Notations

Let UP denote the universe of participation factors, for instance, the set comprising humans across the Earth. Any $s \in UP$ is identified as a participation factor of UP, such as a human on Earth. For $s \in UP$ and $\zeta_s \in \mathbb{N}$, we define $\mathbb{PL}_s = \{0, 1, \cdots, \zeta_s\}$ to represent the set of participating levels for participation factor s, and $\mathbb{PL}_s^+ = \mathbb{PL}_s \setminus \{0\}$, where 0 indicates no operation.

Consider $P \subseteq UP$ as the largest set encompassing all participation factors of an interactive system within UP, like all citizens of a country on Earth. Let $\mathbb{PL}^P = \prod_{s\in P} \mathbb{PL}_s$ be the product set of participating level sets for every participation factor in P. For every $Q \subseteq P$, a participation factor alliance Q corresponds, in a standard manner, to the multi-choice alliance $e^Q \in \mathbb{PL}^P$, which is a vector indicating $e^Q_p = 1$ if $p \in Q$ and $e^Q_p = 0$ if $p \in P \setminus Q$. Denote 0_P as the zero vector in $\mathbb{R}^P$. For $m \in \mathbb{N}$, also define 0_m as the zero vector in $\mathbb{R}^m$ and $\mathbb{N}_m = \{1, 2, \cdots, m\}$.

A **multi-choice environment** is denoted as (P, ζ, θ). $P \neq \emptyset$ is a finite set of participation factors, such as a manufacturing plant. Any $s \in P$ is identified as a participation factor of P, such as a department of this plant. $\zeta = (\zeta_s)_{s\in P} \in \mathbb{PL}^P$ is a vector indicating the number of participating levels for each participation factor $s \in P$, such as the number of operating levels of each department. And $\theta : \mathbb{PL}^P \to \mathbb{R}$ is a mapping with $\theta(0_P) = 0$ that presents the effect caused by each participating level vector $\lambda = (\lambda_s)_{s\in P} \in \mathbb{PL}^P$ when each $s \in P$ operates at level λ_s, such as the manufacturing costs caused by all departments under different participating situations. A **multiattribute multi-choice environment** is denoted by (P, ζ, Θ^m), where $m \in \mathbb{N}$, $\Theta^m = (\theta^t)_{t\in\mathbb{N}_m}$ and (P, ζ, θ^t) represents a multi-choice environment for each $t \in \mathbb{N}_m$. For instance, a manufacturing plant needs to evaluate different considerations for cost minimization, which include financial aspects, manpower, equipment depreciation, and so on. The family of all multiattribute multi-choice environments is denoted as $\mathbb{ME}$.

A **mechanism** is defined as a mapping τ that assigns to each $(P, \zeta, \Theta^m) \in \mathbb{ME}$ an element

$$\tau(P, \zeta, \Theta^m) = \left(\tau^t(P, \zeta, \Theta^m)\right)_{t\in\mathbb{N}_m},$$

where $\tau^t(P, \zeta, \Theta^m) = \left(\tau^t_s(P, \zeta, \Theta^m)\right)_{s\in P} \in \mathbb{R}^P$ and $\tau^t_s(P, \zeta, \Theta^m)$ represents the outcome of participation factor s when s operates in $\left(P, \zeta, \theta^t\right)$. Let $(P, \zeta, \Theta^m) \in \mathbb{ME}$, $K \subseteq P$, and

$\lambda \in \mathbb{R}^P$. We define $S(\lambda) = \{s \in P | \lambda_s \neq 0\}$ and $\lambda_K \in \mathbb{R}^K$ as the restriction of λ to K. Given $s \in P$, we also define λ_{-s} to represent $\lambda_{P\setminus\{s\}}$. Additionally, $\sigma = (\lambda_{-s}, a) \in \mathbb{R}^P$ is defined by $\sigma_{-s} = \lambda_{-s}$ and $\sigma_s = a$.

In order to evaluate the minimization effects derived from interactions and subsequent allocation outcomes, we utilize the concept of EANSC within the framework of multiattribute multi-choice environments to propose a generalized EANSC.

Definition 1. *The* **multiattribute equal minimization of non-separable effects (MEMNSE)***,* $\overline{\beta}$*, is defined by*

$$\overline{\beta^t_s}(P,\zeta,\Theta^m) = \beta^t_s(P,\zeta,\Theta^m) + \frac{1}{|P|} \cdot \left[\theta^t(\zeta) - \sum_{k\in P} \beta^t_k(P,\zeta,\Theta^m)\right]$$

for every $(P,\zeta,\Theta^m) \in \mathbb{ME}$*, for every* $t \in \mathbb{N}_m$*, and for every* $s \in P$*. The value* $\beta^t_s(P,\zeta,\Theta^m) = min_{q\in\mathbb{PL}^+_s}\{\theta^t(\zeta_{-s},q) - \theta^t(\zeta_{-s},0)\}$ *is the* **minimal lower-aggregate marginal effect** *among all participating levels of participation factor* s *in* (P,ζ,θ^t)*. (Here, we apply bounded multi-choice environments, considered as the environments* (P,ζ,θ^t)*, such that there exists* $N_h \in \mathbb{R}$ *such that* $\theta^t(\lambda) \leq N_h$ *for every* $\lambda \in \mathbb{PL}^P$*. We apply it to guarantee that* $\beta^t_s(P,\zeta,\theta^t)$ *is well defined). Under the notion of* $\overline{\beta}$*, all participation factors firstly evaluate its minimal marginal effects and further distribute the rest of the effects equally.*

As indicated in the Introduction, the concept of weights naturally emerges in the context of evaluating effects. For example, weight allocating might be relevant in the distribution of investment plans, where weights could represent the risks of various plan options. Similarly, weights can be utilized in contracts agreed upon by townhouse owners to allocate costs for maintaining or constructing shared facilities. More applications for weights also can be found in Shapley [11]. Generally, weights can be assigned to "participation factors" or the "participating levels" to differentiate the differences among them.

Let $d : UP \to \mathbb{R}^+$ be a positive map. Then, d is termed as a **weight map for participation factors**. Similarly, let $w : \cup_{s\in UP}\mathbb{PL}^+_s \to \mathbb{R}^+$ be a positive map. Then, w is termed as a **weight map for levels**. Based on these two forms of weight maps, we consider two weighted extensions of the MEMNSE.

Definition 2.

- *The* **1-minimal weighted minimization of non-separable effects (1-MWMNSE)***,* η^d*, is defined as follows: for every* $(P,\zeta,\Theta^m) \in \mathbb{ME}$*, for every weight map* d *for participation factors, for every* $t \in \mathbb{N}_m$*, and for every participation factor* $s \in P$*,*

$$\eta^{d,t}_s(P,\zeta,\Theta^m) = \beta^t_s(P,\zeta,\Theta^m) + \frac{d(s)}{\sum\limits_{k\in P} d(k)} \cdot \left[\theta^t(\zeta) - \sum_{k\in P} \beta^t_k(P,\zeta,\Theta^m)\right].$$

  According to the definition of η^d*, all participation factors initially evaluate its minimal lower-aggregate marginal effects, and the remaining effect is evaluated proportionally based on weights for participation factors.*
- *The* **2-minimal weighted minimization of non-separable effects (2-MWMNSE)***,* η^w*, is defined as follows: for every* $(P,\zeta,\Theta^m) \in \mathbb{ME}$*, for every weight map* w *for participation factors, for every* $t \in \mathbb{N}_m$*, and for every participation factor* $s \in P$*,*

$$\eta^{w,t}_s(P,\zeta,\Theta^m) = \beta^{w,t}_s(P,\zeta,\Theta^m) + \frac{1}{|P|} \cdot \left[\theta^t(\zeta) - \sum_{k\in P} \beta^{w,t}_k(P,\zeta,\Theta^m)\right],$$

  where $\beta^{w,t}_s(P,\zeta,\Theta^m) = \min\limits_{q\in\mathbb{PL}s^+} w(q)\cdot[\theta^t(\zeta - s,q) - \theta^t(\zeta_{-s},0)]$ *is the minimal weighted lower-aggregate marginal effect among all participating levels of participation factor* s*. By* *definition of $\eta^{w,t}$, all participation factors initially evaluate its minimal weighted lower-aggregate marginal effects, and the remaining effect is evaluated equally.*

- *The* **weighted lower-aggregate multiattribute mechanism (WLAMM)**, *$\beta^{d,w}$, is defined by for every $(P, \zeta, \Theta^m) \in \mathbb{ME}$, for every weight map for participation factors d, for every weight map for levels w, for every $t \in \mathbb{N}_m$, and for every participation factor $s \in P$,*

$$\beta_s^{d,w,t}(P, \zeta, \Theta^m) = \beta_s^{w,t}(P, \zeta, \Theta^m) + \frac{d(s)}{\sum\limits_{k \in P} d(k)} \cdot \left[\theta^t(\zeta) - \sum_{k \in P} \beta_k^{w,t}(P, \zeta, \Theta^m)\right].$$

By definition of $\beta^{d,w}$, all participation factors initially evaluate its minimal weighted lower-aggregate marginal effects, and the remaining effect is evaluated proportionally based on weights for participation factors.

2.2. Motivating and Practical Examples

As mentioned in the Introduction, the main aim of multiattribute analysis is to derive optimal or balanced states when dealing with multiple considerations. Furthermore, each participation factor may have the option to participate at different participating levels under multiple considerations. Related concepts have been applied across various domains, including information engineering, environmental analysis, biomedical sciences, logistics, and strategic management sciences, all of which necessitate weighing multiple considerations to evaluate effective interactive models. For instance, companies selling central air conditioning systems must, under the considerations of minimizing manufacturing and sales costs, also reduce pollution emissions and resource consumption during the manufacturing processes while maintaining a certain grade of quality and profitability. Under such situations, different departments of the company must adopt corresponding levels of involvement relative to different considerations, exemplifying a situation involving multiple considerations and participating levels. In some cases, this may even involve three or more objectives. Hence, this study emphasizes the framework of considering multiattribute multi-choice considerations.

To illustrate related applied concepts in the framework of multiattribute multi-choice considerations, we continue applying the example mentioned above.

- Let P be the set of all departments within a company selling central air conditioning systems. Under the processes of manufacturing and selling central air conditioning systems, each department not only performs tasks aligned with its nature but also interacts in work due to different operational considerations. For example, based on sales considerations, the marketing and manufacturing departments must collaborate to devise sales strategies for products while also working with the accounting department to control sales costs. However, the operational nature of these departments during the producing processes may lead to positive or negative effects. For instance, to meet environmental standards, the manufacturing department may need to update or improve producing equipment to reduce pollution generated during the production processes, while the accounting department must effectively control costs.
- The mapping θ^t can be seen as an effect assessment function when all departments participate in the producing processes under a certain consideration. The participating levels of all departments can be represented by the vector $\alpha = (\alpha_s)_{s \in P} \in \mathbb{PL}^P$. Here, $\theta^t(\alpha)$ evaluates the effect when each department s participates at level α_s under this consideration. Modeling according to this concept, a company selling central air conditioning systems under a certain consideration can be represented as (P, ζ, θ^t). The entire company's sales producing plan under all considerations can then be presented in a multiattribute multi-choice environment (P, ζ, Θ^m).
- To evaluate the minimal effect of each department, the evaluation mechanism defined in Definition 1 can be applied. This involves assessing the minimal lower-aggregate marginal effect caused by each department respectively based on various participating

level vectors. The remaining effect allocating is then evenly evaluated among all departments, as proposed in Definition 1 (the MEMNSE).

However, it may not always be appropriate to equally evaluate the remaining effects among the concerned participation factors. Hence, it is reasonable to assign weights to participation factors or their participating levels and evaluate the remaining effect based on these weights.

- Since each department's impact varies across different participating situations, they hold different grades of related effects under different considerations. Thus, it is reasonable to generate weights through the weight map for participation factors d. The remaining effect should also be evaluated according to the weight proportions of each department, as suggested in Definition 2 (the 1-MWMNSE).
- On the other hand, since each participating level may cause varying effects under different participating environments, these participating levels naturally hold different grades of significance across different participating environments. Hence, generating weights through the weight map for levels w is also rational. The decisive effect of each department should be computed first through its minimal weighted lower-aggregate marginal effect. The remaining effect should then be evenly evaluated among all departments, as proposed in Definition 2 (the 2-MWMNSE).
- If we combine the concepts of 1-MWMNSE and 2-MWMNSE, we can first evaluate that the decisive effect of each department should be computed first through its minimal weighted lower-aggregate marginal effect. The remaining effect can then be evaluated according to the weight proportion of each department, as proposed in Definition 2 (the WLAMM).

3. Axiomatic Processes

3.1. Axiomatic Results for the MEMNSE and Its Weighted Extensions

Inspired by related axiomatic techniques of Hart and Mas-Colell [12] and Moulin [10], several axiomatic results of the MEMNSE, the 1-MWMNSE, the 2-MWMNSE, and the WLAMM are proposed to demonstrate the mathematical correctness and practicality of these mechanisms.

A mechanism τ satisfies the **multiattribute effectiveness (MEIN)** axiom if $\sum_{s\in P}\tau_s^t(P,\zeta,\Theta^m)=\theta^t(\zeta)$ for every $(P,\zeta,\Theta^m)\in\mathbb{ME}$ and for every $t\in\mathbb{N}_m$. The MEIN axiom ensures that all participation factors evaluate the entire effect completely.

Lemma 1. *The mechanisms $\overline{\beta}$, η^d, η^w, $\beta^{d,w}$ fit MEIN.*

Proof of Lemma 1. Let $(P,\zeta,\Theta^m)\in\mathbb{ME}$, $t\in\mathbb{N}_m$, d be a weight map for participation factors and w be a weight map for levels. By Definitions 1 and 2,

$$\begin{aligned}
\sum_{s\in P}\beta_s^{d,w,t}(P,\zeta,\Theta^m) &= \sum_{s\in P}\beta_s^{w,t}(P,\zeta,\Theta^m)+\sum_{s\in P}\left[\frac{d(s)}{\sum_{k\in P}d(k)}\cdot\left[\theta^t(\zeta)-\sum_{k\in P}\beta_k^{w,t}(P,\zeta,\Theta^m)\right]\right]\\
&= \sum_{s\in P}\beta_s^{w,t}(P,\zeta,\Theta^m)+\frac{\sum_{s\in P}d(s)}{\sum_{k\in P}d(k)}\cdot\left[\theta^t(\zeta)-\sum_{k\in P}\beta_k^{w,t}(P,\zeta,\Theta^m)\right]\\
&= \sum_{s\in P}\beta_s^{w,t}(P,\zeta,\Theta^m)+\theta^t(\zeta)-\sum_{k\in P}\beta_k^{w,t}(P,\zeta,\Theta^m)\\
&= \theta^t(\zeta).
\end{aligned}$$

The proof is finished. If all the weights for participation factors are set to 1 in the above proof process, the MEIN property of 2-MWMNSE can be demonstrated. Similarly, if all the weights for participating levels are set to 1 in the above proof process, the MEIN property of 1-MWMNSE can be demonstrated. Furthermore, if all the weights for both participation factors and participating levels are set to 1 in the above proof process, the MEIN property of MEMNSE can be demonstrated. □

Moulin [10] introduced the reduced environment as one in which each alliance in the subgroup could receive remunerations for its participation factors only if these remunerations are consistent with the original remunerations for all participation factors outside the subgroup. A generalized reduction is defined under multiattribute multi-choice environments as follows:

Let $(P,\zeta,\Theta^m) \in \mathbb{ME}$, $K \subseteq P$, and τ be a mechanism. The **reduced environment** $(K,\zeta_K,\Theta^m_{K,\tau})$ is defined by $\Theta^m_{K,\tau} = (\theta^t_{K,\tau})_{t\in\mathbb{N}_m}$, and for every $\lambda \in \mathbb{PL}^K$,

$$\theta^t_{K,\tau}(\lambda) = \begin{cases} 0 & \text{if } \lambda = 0_K, \\ \theta^t(\lambda,\zeta_{P\setminus K}) - \sum\limits_{s\in P\setminus K} \tau^t_s(P,\zeta,\Theta^m) & \text{otherwise,} \end{cases}$$

For any two-person group of participation factors in a environment, one defines a "reduced environment" among them by considering the amounts remaining after the rest of the participation factors are given the effect prescribed by τ. Then, τ fits *multiattribute bilateral steadiness* if, when it is applied to any reduced environment, it always yields the same effect as in the original environment. Formally, a mechanism τ satisfies the **multiattribute bilateral steadiness (MBSTN)** axiom if $\tau^t_s(K,\zeta_K,\Theta^m_{K,\tau}) = \tau^t_s(P,\zeta,\Theta^m)$ for every $(P,\zeta,\Theta^m) \in \mathbb{ME}$, for every $t \in \mathbb{N}_m$, for every $K \subseteq P$ with $|K| = 2$, and for every $s \in K$.

Lemma 2. *The mechanisms $\overline{\beta}$, η^d, η^w, $\beta^{d,w}$ fit MBSTN.*

Proof of Lemma 2. Let $(P,\zeta,\Theta^m) \in \mathbb{ME}$, $K \subseteq P$, $t \in \mathbb{N}_m$, d be a weight map for participation factors and w be a weight map for levels. Let $|P| \geq 2$ and $|K| = 2$. By Definitions 1 and 2,

$$\beta^{d,w,t}_s(K,\zeta_K,\Theta^m_{K,\beta^{d,w}}) = \beta^{w,t}_s(K,\zeta_K,\Theta^m_{K,\beta^{d,w}}) + \frac{d(s)}{\sum\limits_{k\in K} d(k)} \cdot \Big[\theta^t_{K,\beta^{d,w}}(\zeta_K) - \sum_{k\in K} \beta^{w,t}_k(K,\zeta_K,\Theta^m_{K,\beta^{d,w}})\Big] \tag{1}$$

for every $s \in K$ and for every $t \in \mathbb{N}_m$. By definitions of $\beta^{w,t}$ and $\theta^t_{K,\beta^{d,w}}$,

$$\begin{aligned} \beta^{w,t}_s(K,\zeta_K,\Theta^m_{K,\beta^{d,w}}) &= \min_{q\in\mathbb{PL}^+_s} \{w(q) \cdot [\theta^t_{K,\beta^{d,w}}(\zeta_{K\setminus\{s\}},q) - \theta^t_{K,\beta^{d,w}}(\zeta_{K\setminus\{s\}},0)]\} \\ &= \min_{q\in\mathbb{PL}^+_s} \{w(q) \cdot [\theta^t(\zeta_{-s},q) - \theta^t(\zeta_{-s},0)]\} \\ &= \beta^{w,t}_s(P,\zeta,\Theta^m). \end{aligned} \tag{2}$$

Based on Equations (1) and (2) and definitions of $\theta^t_{K,\beta^{d,w}}$ and $\beta^{d,w}$,

$$\begin{aligned} \beta^{d,w,t}_s(K,\zeta_K,\Theta^m_{K,\beta^{d,w}}) &= \beta^{w,t}_s(P,\zeta,\Theta^m) + \frac{d(s)}{\sum\limits_{k\in K} d(k)} \Big[\theta^t_{K,\beta^{d,w}}(\zeta_K) - \sum_{k\in K} \beta^{w,t}_k(P,\zeta,\Theta^m)\Big] \\ &= \beta^{w,t}_s(P,\zeta,\Theta^m) + \frac{d(s)}{\sum\limits_{k\in K} d(k)} \Big[\theta^t(\zeta) - \sum_{k\in P\setminus K} \beta^{d,w,t}_k(P,\zeta,\Theta^m) - \sum_{k\in K} \beta^{w,t}_k(P,\zeta,\Theta^m)\Big] \\ &= \beta^{w,t}_s(P,\zeta,\Theta^m) + \frac{d(s)}{\sum\limits_{k\in K} d(k)} \Big[\sum_{k\in K} \beta^{d,w,t}_k(P,\zeta,\Theta^m) - \sum_{k\in K} \beta^{w,t}_k(P,\zeta,\Theta^m)\Big] \\ &\quad \textbf{(MEIN of } \beta^{d,w}\textbf{)} \\ &= \beta^{w,t}_s(P,\zeta,\Theta^m) + \frac{d(s)}{\sum\limits_{k\in K} d(k)} \Bigg[\frac{\sum\limits_{k\in K} d(k)}{\sum\limits_{p\in P} d(p)} \cdot \Big[\theta^t(\zeta) - \sum_{p\in P} \beta^{w,t}_p(P,\zeta,\Theta^m)\Big]\Bigg] \\ &= \beta^{w,t}_s(P,\zeta,\Theta^m) + \frac{d(s)}{\sum\limits_{p\in P} d(p)} \Big[\theta^t(\zeta) - \sum_{p\in P} \beta^t_p(P,\zeta,\Theta^m)\Big] \\ &= \beta^{d,w,t}_s(P,\zeta,\Theta^m) \end{aligned}$$

for every $s \in K$ and for every $t \in \mathbb{N}_m$. If all the weights for participation factors are set to 1 in the above proof process, the MEIN property of 2-MWMNSE can be demonstrated. Similarly, if all the weights for participating levels are set to 1 in the above proof process, the MEIN property of 1-MWMNSE can be demonstrated. Furthermore, if all the weights for both participation factors and participating levels are set to 1 in the above proof process, the MBSTN property of MEMNSE can be demonstrated. □

The notion of the two-factor standardness is introduced by Hart and Mas-Colell [12] initially. It asserts that all participation factors firstly evaluate their individual effects respectively, and further evaluate the rest of effects equally under all two-factor environments. In the following, some generalizations of the two-factor standardness due to Hart and Mas-Colell [12] are introduced. A mechanism τ satisfies the **multiattribute rule for environments (MRFE)** axiom if $\tau(P,\zeta,\Theta^m) = \overline{\beta}(P,\zeta,\Theta^m)$ for every $(P,\zeta,\Theta^m) \in \mathbb{ME}$ with $|P| \leq 2$. A mechanism τ satisfies the **1-weighted rule for environments (1WRFE)** if $\tau(P,\zeta,\Theta^m) = \eta^d(P,\zeta,\Theta^m)$ for every $(P,\zeta,\Theta^m) \in \mathbb{ME}$ with $|P| \leq 2$ and for every weight map d for participation factors. A mechanism τ satisfies the **2-weighted rule for environments (2WRFE)** if $\tau(P,\zeta,\Theta^m) = \eta^w(P,\zeta,\Theta^m)$ for every $(P,\zeta,\Theta^m) \in \mathbb{ME}$ with $|P| \leq 2$ and for every weight map w for levels. A mechanism τ fits **weighted lower-aggregate rule (WLAR)** if $\tau(P,\zeta,\Theta^m) = \beta^{d,w}(P,\zeta,\Theta^m)$ for every $(P,\zeta,\Theta^m) \in \mathbb{ME}$ with $|P| \leq 2$, for every weight map for participation factors d and for every weight map for levels w.

Inspired by Hart and Mas-Colell [12] and Moulin [10], we adopt MBSTN to characterize these mechanisms.

Theorem 1.

1. *On $\mathbb{ME}$, the MEMNSE is the unique mechanism fitting MRFE and MBSTN.*
2. *On $\mathbb{ME}$, the 1-MWMNSE is the unique mechanism fitting 1WRFE and MBSTN.*
3. *On $\mathbb{ME}$, the 2-MWMNSE is the unique mechanism fitting 2WRFE and MBSTN.*
4. *On $\mathbb{ME}$, the WLAMM is the unique mechanism fitting WLAR and MBSTN.*

Proof of Theorem 1. By Lemma 2, the mechanisms $\overline{\beta}$, η^d, η^w, $\beta^{d,w}$ fit MBSTN. Clearly, the mechanisms $\overline{\beta}$, η^d, η^w, $\beta^{d,w}$ fit MRFE, 1WRFE, 2WRFE, and WLAR, respectively.

To present the uniqueness of result 4, suppose that τ fits WLAR and MBSTN. By WLAR and MBSTN of τ, it is easy to clarify that τ also fits MEIN, thus we omit it. Let $(P,\zeta,\Theta^m) \in \mathbb{ME}$, d be a weight map for participation factors and w be a weight map for levels. By WLAR of τ, $\tau(P,\zeta,\Theta^m) = \beta^{d,w}(P,\zeta,\Theta^m)$ if $|P| \leq 2$. For the situation where $|P| > 2$: let $s \in P$, $t \in \mathbb{N}_m$ and $K = \{s,p\}$ with $p \in P \setminus \{s\}$.

$$\begin{aligned} \tau_s^t(P,\zeta,\Theta^m) - \beta_s^{d,w,t}(P,\zeta,\Theta^m) &= \tau_s^t(K,\zeta_K,\Theta^m_{K,\tau}) - \beta_s^{d,w,t}(K,\zeta_K,\Theta^m_{K,\beta^{d,w}}) \ \textbf{(MBSTN of } \beta^{d,w,t} \textbf{ and } \tau\textbf{)} \\ &= \beta_s^{d,w,t}(K,\zeta_K,\Theta^m_{K,\tau}) - \beta_s^{d,w,t}(K,\zeta_K,\Theta^m_{K,\beta^{d,w}}). \ \textbf{(WLAR of } \tau\textbf{)} \end{aligned} \tag{3}$$

Similar to Equation (2)

$$\beta_s^{w,t}(K,\zeta_K,\Theta^m_{K,\tau}) = \beta_s^{w,t}(P,\zeta,\Theta^m) = \beta_s^{w,t}(K,\zeta_K,\Theta^m_{K,\beta^{d,w}}). \tag{4}$$

By Equations (3) and (4),

$$\begin{aligned} \tau_s^t(P,\zeta,\Theta^m) - \beta_s^{d,w,t}(P,\zeta,\Theta^m) &= \beta_s^{d,w,t}(K,\zeta_K,\Theta^m_{K,\tau}) - \beta_s^{d,w,t}(K,\zeta_K,\Theta^m_{K,\beta^{d,w}}) \\ &= \tfrac{d(s)}{d(s)+d(p)} \cdot \left[\theta^t_{K,\tau}(\zeta_K) - \theta^t_{K,\beta^{d,w}}(\zeta_K)\right] \\ &= \tfrac{d(s)}{d(s)+d(p)} \cdot \left[\tau_s^t(P,\zeta,\Theta^m) + \tau_p^t(P,\zeta,\Theta^m) \right. \\ &\qquad \left. - \beta_s^{d,w,t}(P,\zeta,\Theta^m) - \beta_p^{d,w,t}(P,\zeta,\Theta^m)\right]. \end{aligned}$$

Thus, $d(p) \cdot \left[\tau_s^t(P,\zeta,\Theta^m) - \beta_s^{d,w,t}(P,\zeta,\Theta^m)\right] = d(s) \cdot \left[\tau_p^t(P,\zeta,\Theta^m) - \beta_p^{d,w,t}(P,\zeta,\Theta^m)\right]$. By MEIN of $\beta^{d,w,t}$ and τ,

$$\begin{aligned} \left[\tau_s^t(P,\zeta,\Theta^m) - \beta_s^{d,w,t}(P,\zeta,\Theta^m)\right] \cdot \sum_{p \in P} d(p) &= d(s) \cdot \sum_{p \in P} \left[\tau_p^t(P,\zeta,\Theta^m) - \beta_p^{d,w,t}(P,\zeta,\Theta^m)\right] \\ &= d(s) \cdot \left[\theta^t(\zeta) - \theta^t(\zeta)\right] \\ &= 0. \end{aligned}$$

Hence, $\tau_s^t(P,\zeta,\Theta^m) = \beta_s^{d,w,t}(P,\zeta,\Theta^m)$ for every $s \in P$ and for every $t \in \mathbb{N}_m$. If all the weights for participation factors are set to 1 in the above proof process, the proof of outcome 3 could be finished. Similarly, if all the weights for participating levels are set to 1 in the above proof process, the proof of outcome 2 could be finished. Furthermore, if all the weights for both participation factors and participating levels are set to 1 in the above proof process, the proof of outcome 1 could be finished. □

In the following, some instances are exhibited to display that every one of the axioms applied in Theorem 1 is independent of the rest of the axioms.

Example 1. *We focus on the mechanism τ as follows. For every $(P,\zeta,\Theta^m) \in \mathbb{ME}$, for every weight map for participation factors d, for every weight map for levels w, for every $t \in \mathbb{N}_m$, and for every participation factor $s \in P$,*

$$\tau_s^t(P,\zeta,\Theta^m) = \begin{cases} \beta_s^{d,w,t}(P,\zeta,\Theta^m) & \text{if } |P| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, τ fits WLAR, but it does not fit MBSTN.

Example 2. *We focus on the mechanism τ as follows. For every $(P,\zeta,\Theta^m) \in \mathbb{ME}$, for every weight map for participation factors d, for every weight map for levels w, for every $t \in \mathbb{N}_m$, and for every participation factor $s \in P$,*

$$\tau_s^t(P,\zeta,\Theta^m) = \begin{cases} \eta_s^{w,t}(P,\zeta,\Theta^m) & \text{if } |P| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, τ fits 2WRFE, but it does not fit MBSTN.

Example 3. *We focus on the mechanism τ as follows. For every $(P,\zeta,\Theta^m) \in \mathbb{ME}$, for every weight map for participation factors d, for every weight map for levels w, for every $t \in \mathbb{N}_m$, and for every participation factor $s \in P$,*

$$\tau_s^t(P,\zeta,\Theta^m) = \begin{cases} \eta_s^{d,t}(P,\zeta,\Theta^m) & \text{if } |P| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, τ fits 1WRFE, but it does not fit MBSTN.

Example 4. *We focus on the mechanism τ as follows. For every $(P,\zeta,\Theta^m) \in \mathbb{ME}$, for every weight map for participation factors d, for every weight map for levels w, for every $t \in \mathbb{N}_m$, and for every participation factor $s \in P$,*

$$\tau_s^t(P,\zeta,\Theta^m) = \begin{cases} \overline{\beta_s^t}(P,\zeta,\Theta^m) & \text{if } |P| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, τ fits MRFE, but it does not fit MBSTN.

Example 5. *Define a mechanism τ to be $\tau_s^t(P,\zeta,\Theta^m) = 0$ for every $(P,\zeta,\Theta^m) \in \mathbb{ME}$, for every weight map for participation factors d, for every weight map for levels w, for every $t \in \mathbb{N}_m$, and for every participation factor $s \in P$. Clearly, τ fits MBSTN, but it does not fit MRFE, 1WRFE, 2WRFE, and WLAR.*

3.2. Different Generalization and Revised Steadiness

In Sections 2 and 3.1, various weighted generalizations are explored by introducing weights to both participation factors and their participating levels simultaneously. However, the fairness or legitimacy of these weight functions may be subject to scrutiny. The assignment of weights to participation factors and their participating levels can sometimes be arbitrary. Therefore, a concept that utilizes relative minimal marginal effects as weights under different circumstances naturally seems reasonable. "minimal marginal effects" instead of "weights", a different generalization could be considered as follows.

Definition 3. *The* **multi-choice multiattribute interior mechanism (MMIM)**, η^I, *is defined as follows: for every* $(P, \zeta, \Theta^m) \in \mathbb{ME}^*$, *for every* $t \in \mathbb{N}_m$, *and for every participation factor* $s \in P$,

$$\eta_s^{I,t}(P, \zeta, \Theta^m) = \beta_s^t(P, \zeta, \Theta^m) + \frac{\beta_s^t(P, \zeta, \Theta^m)}{\sum_{k \in P} \beta_k^t(P, \zeta, \Theta^m)} \cdot \left[\theta^t(\zeta) - \sum_{k \in P} \beta_k^t(P, \zeta, \Theta^m)\right],$$

where $\mathbb{ME}^* = \{(P, \zeta, \Theta^m) \in \mathbb{ME} | \sum_{k \in P} \beta_k^t(P, \zeta, \Theta^m) \neq 0$ *for every* $t \in \mathbb{N}_m\}$. *By definition of* η^I, *all participation factors initially evaluate its minimal lower-aggregate marginal effects, and the remaining effect is then evaluated proportionally based on these minimal lower-aggregate marginal effects.*

Next, we aim to characterize the MMIM using steadiness. A mechanism τ fits the **multiattribute interior rule (MIR)** if $\tau(P, \zeta, \Theta^m) = \eta^I(P, \zeta, \Theta^m)$ for every $(P, \zeta, \Theta^m) \in \mathbb{ME}$ with $|P| \leq 2$.

It is straightforward to verify that $\sum_{k \in K} \beta_k^t(P, \zeta, \Theta^m) = 0$ for some $(P, \zeta, \Theta^m) \in \mathbb{ME}$, for some $K \subseteq P$, and for some $t \in \mathbb{N}_m$, i.e., $\eta^{I,t}(K, \zeta_K, \Theta^m_{K,\eta})$ does not exist for some $(P, \zeta, \Theta^m) \in \mathbb{ME}$, for some $K \subseteq P$, and for some $t \in \mathbb{N}m$. Therefore, we focus on the *multiattribute revised steadiness* as follows. A mechanism τ fits the **multiattribute revised-steadiness (MRSTN)** if $(K, \zeta_K, \Theta^m_{K,\tau})$ and $\tau(K, \zeta_K, \Theta^m_{K,\tau})$ exist for some $(P, \zeta, \Theta^m) \in \mathbb{ME}$, for some $K \subseteq P$ with K=2, and for some $t \in \mathbb{N}_m$, and it holds that $\tau_s(K, \zeta_K, \Theta^m_{K,\tau}) = \tau_s(P, \zeta, \Theta^m)$ for every $s \in K$. Similar to Theorem 1, the related axiomatic process of η^I can also be presented as follows.

Theorem 2.
1. *The mechanism* η^I *fits MEIN on* $\mathbb{ME}^*$.
2. *The mechanism* η^I *fits MRSTN on* $\mathbb{ME}^*$.
3. *On* $\mathbb{ME}^*$, *the MMIM is the only mechanism fitting MIR and MRSTN.*

Proof of Theorem 2. To prove result 1, let $(P, \zeta, \Theta^m) \in \mathbb{ME}^*$ and $s \in P$. By Definition 3,

$$\begin{aligned}
\sum_{s \in P} \eta_s^{I,t}(P, \zeta, \Theta^m) &= \sum_{s \in P} \beta_s^t(P, \zeta, \Theta^m) + \sum_{s \in P} \left[\frac{\beta_s^t(P,\zeta,\Theta^m)}{\sum_{k \in P} \beta_k^t(P,\zeta,\Theta^m)} \cdot \left[\theta^t(\zeta) - \sum_{k \in P} \beta_k^t(P, \zeta, \Theta^m)\right]\right] \\
&= \sum_{s \in P} \beta_s^t(P, \zeta, \Theta^m) + \frac{\sum_{s \in P} \beta_s^t(P,\zeta,\Theta^m)}{\sum_{k \in P} \beta_k^t(P,\zeta,\Theta^m)} \cdot \left[\theta^t(\zeta) - \sum_{k \in P} \beta_k^t(P, \zeta, \Theta^m)\right] \\
&= \sum_{s \in P} \beta_s^t(P, \zeta, \Theta^m) + \theta^t(\zeta) - \sum_{k \in P} \beta_k^t(P, \zeta, \Theta^m) \\
&= \theta^t(\zeta).
\end{aligned}$$

The proof is finished. To prove result 2, let $(P, \zeta, \Theta^m) \in \mathbb{ME}^*$, $K \subseteq P$ with $|K| = 2$ and $t \in \mathbb{N}_m$. Assume that $(K, \zeta_K, \Theta^m_{K,\eta^I})$ and $\eta^I(K, \zeta_K, \Theta^m_{K,\eta^I})$ exist. By Definition 3,

$$\eta_s^{I,t}(K, \zeta_K, \Theta^m_{K,\eta^I}) = \beta_s^t(K, \zeta_K, \Theta^m_{K,\eta^I}) + \frac{\beta_s^t(K,\zeta_K,\Theta^m_{K,\eta^I})}{\sum_{k \in K} \beta_k^t(K,\zeta_K,\Theta^m_{K,\eta^I})} \cdot \left[\theta^t_{K,\eta^I}(\zeta_K) - \sum_{k \in K} \beta_k^t(K, \zeta_K, \Theta^m_{K,\eta^I})\right] \quad (5)$$

for every $s \in K$ and for every $t \in \mathbb{N}_m$. By Definitions 1, 3, and the definition of θ^t_{K,η^I},

$$\begin{aligned} \beta^t_s(K, \zeta_K, \Theta^m_{K,\eta^I}) &= \min_{q \in \mathbb{PL}^+_s} \{\theta^t_{K,\eta^I}(\zeta_{K\setminus\{s\}}, q) - \theta^t_{K,\eta^I}(\zeta_{K\setminus\{s\}}, 0)\} \\ &= \min_{q \in \mathbb{PL}^+_s} \{\theta^t(\zeta_{-s}, q) - \theta^t(\zeta_{-s}, 0)\} \\ &= \beta^t_s(P, \zeta, \Theta^m). \end{aligned} \tag{6}$$

Based on Equations (5) and (6) and definitions of θ^t_{K,η^I} and η^I,

$$\begin{aligned} \eta^{I,t}_s(K, \zeta_K, \Theta^m_{K,\eta^I}) = &\ \beta^t_s(P, \zeta, \Theta^m) + \frac{\beta^t_s(P,\zeta,\Theta^m)}{\sum\limits_{k \in K} \beta^t_k(P,\zeta,\Theta^m)} \big[\theta^t_{K,\eta^I}(\zeta_K) - \sum_{k \in K} \beta^t_k(P, \zeta, \Theta^m)\big] \\ = &\ \beta^t_s(P, \zeta, \Theta^m) + \frac{\beta^t_s(P,\zeta,\Theta^m)}{\sum\limits_{k \in K} \beta^t_k(P,\zeta,\Theta^m)} \big[\theta^t(\zeta) - \sum_{k \in P\setminus K} \eta^{I,t}_k(P, \zeta, \Theta^m) - \sum_{k \in K} \beta^t_k(P, \zeta, \Theta^m)\big] \\ = &\ \beta^t_s(P, \zeta, \Theta^m) + \frac{\beta^t_s(P,\zeta,\Theta^m)}{\sum\limits_{k \in K} \beta^t_k(P,\zeta,\Theta^m)} \big[\sum_{k \in K} \eta^{I,t}_k(P, \zeta, \Theta^m) - \sum_{k \in K} \beta^t_k(P, \zeta, \Theta^m)\big] \ \textbf{(MEIN of } \eta^I\textbf{)} \\ = &\ \beta^t_s(P, \zeta, \Theta^m) + \frac{\beta^t_s(P,\zeta,\Theta^m)}{\sum\limits_{k \in K} \beta^t_k(P,\zeta,\Theta^m)} \Big[\frac{\sum\limits_{k \in K} \beta^t_k(P,\zeta,\Theta^m)}{\sum\limits_{p \in P} \beta^t_p(P,\zeta,\Theta^m)} \cdot \big[\theta^t(\zeta) - \sum_{p \in P} \beta^t_p(P, \zeta, \Theta^m)\big]\Big] \\ = &\ \beta^t_s(P, \zeta, \Theta^m) + \frac{\beta^t_s(P,\zeta,\Theta^m)}{\sum\limits_{p \in P} \beta^t_p(P,\zeta,\Theta^m)} \big[\theta^t(\zeta) - \sum_{p \in P} \beta^t_p(P, \zeta, \Theta^m)\big] \\ = &\ \eta^{I,t}_s(P, \zeta, \Theta^m) \end{aligned}$$

for every $s \in K$ and for every $t \in \mathbb{N}_m$. The proof is finished.

To prove result 3, the mechanism η^I fits MRSTN by result 2. Clearly, the mechanism η^I fits MIR. To present the uniqueness of result 3, suppose that τ fits MIR and MRSTN. By MIR and MRSTN of τ, it is easy to clarify that τ also fits MEIN, thus we omit it. Let $(P, \zeta, \Theta^m) \in \mathbb{ME}^*$. By MIR of τ, $\tau(P, \zeta, \Theta^m) = \eta^I(P, \zeta, \Theta^m)$ if $|P| \leq 2$.

Assume that $|P| > 2$. Let $s \in P$ and $t \in \mathbb{N}_m$. Fist, we consider the case of $\beta^t_s(P, \zeta, \Theta^m) + \beta^t_p(P, \zeta, \Theta^m) \neq 0$ for some $p \in P \setminus \{s\}$ and $K = \{s, p\}$. For all $q \in K$,

$$\begin{aligned} \tau^t_q(P, \zeta, \Theta^m) - \eta^{I,t}_q(P, \zeta, \Theta^m) &= \tau^t_q(K, \zeta_K, \Theta^m_{K,\tau}) - \eta^{I,t}_q(K, \zeta_K, \Theta^m_{K,\beta^{d,w}}) \ \textbf{(MRSTN of } \eta^{I,t} \textbf{ and } \tau\textbf{)} \\ &= \eta^{I,t}_q(K, \zeta_K, \Theta^m_{K,\tau}) - \eta^{I,t}_q(K, \zeta_K, \Theta^m_{K,\eta^I}). \ \textbf{(MIR of } \tau\textbf{)} \end{aligned} \tag{7}$$

Similar to Equation (2)

$$\beta^t_q(K, \zeta_K, \Theta^m_{K,\tau}) = \beta^t_q(P, \zeta, \Theta^m) = \beta^t_q(K, \zeta_K, \Theta^m_{K,\eta^I}) \ \text{ for all } q \in K. \tag{8}$$

By Equations (7) and (8),

$$\begin{aligned} \tau^t_s(P, \zeta, \Theta^m) - \eta^{I,t}_s(P, \zeta, \Theta^m) &= \eta^{I,t}_s(K, \zeta_K, \Theta^m_{K,\tau}) - \eta^{I,t}_s(K, \zeta_K, \Theta^m_{K,\eta^I}) \\ &= \frac{\beta^t_s(P,\zeta,\Theta^m)}{\beta^t_s(P,\zeta,\Theta^m)+\beta^t_p(P,\zeta,\Theta^m)} \cdot \big[\theta^t_{K,\tau}(\zeta_K) - \theta^t_{K,\eta^I}(\zeta_K)\big] \\ &= \frac{\beta^t_s(P,\zeta,\Theta^m)}{\beta^t_s(P,\zeta,\Theta^m)+\beta^t_p(P,\zeta,\Theta^m)} \cdot \big[\tau^t_s(P, \zeta, \Theta^m) + \tau^t_p(P, \zeta, \Theta^m) \\ &\qquad - \eta^{I,t}_s(P, \zeta, \Theta^m) - \eta^{I,t}_p(P, \zeta, \Theta^m)\big]. \end{aligned}$$

So, $\beta^t_p(P, \zeta, \Theta^m) \cdot \big[\tau^t_s(P, \zeta, \Theta^m) - \eta^{I,t}_s(P, \zeta, \Theta^m)\big] = \beta^t_s(P, \zeta, \Theta^m) \cdot \big[\tau^t_p(P, \zeta, \Theta^m) - \eta^{I,t}_p(P, \zeta, \Theta^m)\big]$. By MEIN of $\eta^{I,t}$ and τ,

$$\begin{aligned} \big[\tau^t_s(P, \zeta, \Theta^m) - \eta^{I,t}_s(P, \zeta, \Theta^m)\big] \cdot \sum_{p \in P} \beta^t_p(P, \zeta, \Theta^m) &= \beta^t_s(P, \zeta, \Theta^m) \cdot \sum_{p \in P} \big[\tau^t_p(P, \zeta, \Theta^m) - \eta^{I,t}_p(P, \zeta, \Theta^m)\big] \\ &= \beta^t_s(P, \zeta, \Theta^m) \cdot \big[\theta^t(\zeta) - \theta^t(\zeta)\big] \\ &= 0. \end{aligned} \tag{9}$$

Since $(P,\zeta,\Theta^m) \in \mathbb{ME}^*$, $\sum\limits_{p\in P} \beta^t_p(P,\zeta,\Theta^m) \neq 0$. By Equation (9), $\tau^t_s(P,\zeta,\Theta^m) = \eta^{I,t}_s(P,\zeta,\Theta^m)$ for every $s \in P$ and for every $t \in \mathbb{N}_m$. Next, we consider the case of $\beta^t_s(P,\zeta,\Theta^m) + \beta^t_p(P,\zeta,\Theta^m) = 0$ for every $p \in P \setminus \{s\}$. Since $(P,\zeta,\Theta^m) \in \mathbb{ME}^*$, it is easy to check that $\beta^t_s(P,\zeta,\Theta^m) = -\beta^t_p(P,\zeta,\Theta^m)$ for every $p \in P \setminus \{s\}$ and $\beta^t_q(P,\zeta,\Theta^m) \neq 0$ for every $q \in P$. Similar to the above proof, $\tau^t_p(P,\zeta,\Theta^m) = \eta^{I,t}_p(P,\zeta,\Theta^m)$ for every $p \in P \setminus \{s\}$. By MEIN of τ and η^I,

$$\tau^t_s(P,\zeta,\Theta^m) = \theta^t(\zeta) - \sum_{p\in P\setminus\{s\}} \tau^t_p(P,\zeta,\Theta^m) = \theta^t(\zeta) - \sum_{p\in P\setminus\{s\}} \eta^{I,t}_p(P,\zeta,\Theta^m) = \eta^{I,t}_s(P,\zeta,\Theta^m).$$

The proof is finished. □

In the following, some examples are exhibited to display that every one of the properties applied in Theorem 2 is independent of the rest of properties.

Example 6. *We focus on the mechanism τ as follows. For every $(P,\zeta,\Theta^m) \in \mathbb{ME}^*$, for every $t \in \mathbb{N}_m$, and for every participation factor $s \in P$,*

$$\tau^t_s(P,\zeta,\Theta^m) = \begin{cases} \eta^{I,t}_s(P,\zeta,\Theta^m) & \text{if } |P| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, τ fits MIR, but it does not fit MRSTN.

Example 7. *Define a mechanism τ to be $\tau^t_s(P,\zeta,\Theta^m) = 0$ for every $(P,\zeta,\Theta^m) \in \mathbb{ME}^*$, for every $t \in \mathbb{N}_m$, and for every participation factor $s \in P$. Clearly, τ fits MRSTN, but it does not fit MIR.*

Subsequently, an example is provided to present **(a)** how the new mechanisms would distribute effects differently than the previous mechanisms and **(b)** differently from each other. Let $(P,\zeta,\Theta^m) \in \mathbb{ME}$ with $P = \{i,j,k\}$, $m = 2$, $\zeta = (2,1,1)$, $\mathbb{PL}_i = \{0,1_i,2_i\}$, $\mathbb{PL}_j = \{0,1_j\}$, $\mathbb{PL}_k = \{0,1_k\}$, $d(i) = 5$, $d(j) = 1$, $d(k) = 2$, $w(1_i) = 3$, $w(2_i) = 4$, $w(1_j) = 7$, $w(1_k) = 4$. Further, let $\theta^1(2,1,1) = 5$, $\theta^1(1,1,1) = 7$, $\theta^1(2,1,0) = 3$, $\theta^1(2,0,1) = 2$, $\theta^1(2,0,0) = 9$, $\theta^1(1,1,0) = 3$, $\theta^1(1,0,1) = -4$, $\theta^1(0,1,1) = 4$, $\theta^1(1,0,0) = -1$, $\theta^1(0,1,0) = 2$, $\theta^1(0,0,1) = -3$, $\theta^2(2,1,1) = 9$, $\theta^2(1,1,1) = 3$, $\theta^2(2,1,0) = 5$, $\theta^2(2,0,1) = 6$, $\theta^2(2,0,0) = 4$, $\theta^2(1,1,0) = -3$, $\theta^2(1,0,1) = 4$, $\theta^2(0,1,1) = 3$, $\theta^2(1,0,0) = 7$, $\theta^2(0,1,0) = -2$, $\theta^2(0,0,1) = 3$ and $\theta^1(0,0,0) = 0 = \theta^2(0,0,0)$. By Definitions 1–3,

$$\begin{array}{rclrclrcl}
\overline{\beta^1_i}(P,\zeta,\Theta^m) &=& \frac{2}{3}, & \overline{\beta^1_j}(P,\zeta,\Theta^m) &=& \frac{8}{3}, & \overline{\beta^1_k}(P,\zeta,\Theta^m) &=& \frac{5}{3}, \\
\overline{\beta^2_i}(P,\zeta,\Theta^m) &=& \frac{2}{3}, & \overline{\beta^2_j}(P,\zeta,\Theta^m) &=& \frac{11}{3}, & \overline{\beta^2_k}(P,\zeta,\Theta^m) &=& \frac{14}{3}, \\
\eta^{d,1}_i(P,\zeta,\Theta^m) &=& \frac{3}{8}, & \eta^{d,1}_j(P,\zeta,\Theta^m) &=& \frac{23}{8}, & \eta^{d,1}_k(P,\zeta,\Theta^m) &=& \frac{14}{8}, \\
\eta^{d,2}_i(P,\zeta,\Theta^m) &=& \frac{10}{8}, & \eta^{d,2}_j(P,\zeta,\Theta^m) &=& \frac{26}{8}, & \eta^{d,2}_k(P,\zeta,\Theta^m) &=& \frac{36}{8}, \\
\eta^{w,1}_i(P,\zeta,\Theta^m) &=& \frac{-16}{3}, & \eta^{w,1}_j(P,\zeta,\Theta^m) &=& \frac{35}{3}, & \eta^{w,1}_k(P,\zeta,\Theta^m) &=& \frac{-4}{3}, \\
\eta^{w,2}_i(P,\zeta,\Theta^m) &=& \frac{-28}{3}, & \eta^{w,2}_j(P,\zeta,\Theta^m) &=& \frac{35}{3}, & \eta^{w,2}_k(P,\zeta,\Theta^m) &=& \frac{20}{3}, \\
\beta^{d,w,1}_i(P,\zeta,\Theta^m) &=& \frac{-108}{8}, & \beta^{d,w,1}_j(P,\zeta,\Theta^m) &=& \frac{140}{8}, & \beta^{d,w,1}_k(P,\zeta,\Theta^m) &=& 1, \\
\beta^{d,w,2}_i(P,\zeta,\Theta^m) &=& \frac{-140}{8}, & \beta^{d,w,2}_j(P,\zeta,\Theta^m) &=& \frac{140}{2}, & \beta^{d,w,2}_k(P,\zeta,\Theta^m) &=& 9, \\
\eta^{I,1}_i(P,\zeta,\Theta^m) &=& \frac{5}{6}, & \eta^{I,1}_j(P,\zeta,\Theta^m) &=& \frac{15}{6}, & \eta^{I,1}_k(P,\zeta,\Theta^m) &=& \frac{10}{6}, \\
\eta^{I,2}_i(P,\zeta,\Theta^m) &=& 0, & \eta^{I,2}_j(P,\zeta,\Theta^m) &=& \frac{27}{7}, & \eta^{I,2}_k(P,\zeta,\Theta^m) &=& \frac{36}{7}.
\end{array}$$

4. Conclusions

1. Distinct from existing studies, we introduce the MEMNSE, the 1-MWMNSE, the 2-MWMNSE, the WLAMM, and associated axiomatic processes by concurrently applying weights to participation factors and their respective levels of participation in multiattribute multi-choice situations. Instead of conventional weights, we naturally

incorporate minimal marginal effects and introduce the MMIM and its related axiomatic processes in the context of multiattribute multi-choice settings. A comparative analysis is warranted against relevant findings in the existing literature.

- Traditional environmental mechanisms have primarily focused on either non-participation or universal participation among all participation factors.
- The MEMNSE, the 1-MWMNSE, the 2-MWMNSE, the WLAMM, the MMIM, and its associated axiomatic processes are initially proposed within multiattribute multi-choice environments.
    - Under the MEMNSE and the 2-MWMNSE, the remaining effect is uniformly evaluated among all participation factors.
    - Within the MEMNSE and the 1-MWMNSE, each participation factor evaluates its minimal marginal effect first.
    - Participation factors and their levels of participation are pivotal in multiattribute multi-choice environments. Hence, weights should be simultaneously applied to both participation factors and their levels of participation. Under the WLAMM, participation factors evaluate weighted minimal marginal effects initially, followed by proportional evaluation of the remaining effect based on weights of participation factors.
    - Nonetheless, weight allocating may lack naturalness. The MMIM ensures that all participation factors evaluate the minimal marginal effects initially, followed by proportional evaluation of the remaining effect based on related minimal marginal effects.

2. The mechanisms proposed in this article offer several advantages.
    - Traditional allocation mechanisms in environmental settings typically consider either non-participation or universal participation across all participation factors. This article, however, acknowledges varying levels of participation among all participation factors.
    - In numerous studies on allocating mechanisms under multi-choice environments, while it is acknowledged that participation factors have different levels of participation, most of the literature evaluates the effects of a specific participation factor at a specific level of participation. Here, we evaluate the overall effects of each participation factor across different levels of participation.
    - Reflecting real-world situations, the WLAMM is proposed to evaluate the remaining effect among participation factors and their levels of participation based on simultaneously two forms of weight functions. Furthermore, the concept of the minimal marginal effects is incorporated under the WLAMM. Considering potential questions regarding fairness or legitimacy of the weight functions, relative minimal marginal effects are utilized as weights under the MMIM.
3. However, there are some drawbacks to the proposed mechanisms. As highlighted in the advantages, each participation factor has varying levels of participation. While it is possible to determine the overall effects exerted by each participation factor, assessing the effect of a specific participation factor at a specific level of participation is challenging. Future research should explore alternative allocating mechanisms that consider both overall effects and specific levels of participation simultaneously.
4. The findings of this study also present further avenues for exploration.
    - Alternative mechanisms based on the minimal marginal effects under multiattribute and multi-choice considerations could be derived from existing mechanisms.

    Readers are encouraged to delve deeper into these aspects.

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