mathematics

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Multi Fractals of Generalized Multivalued Iterated Function Systems in *b*-Metric Spaces with Applications

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Received: 8 September 2019; Accepted: 9 October 2019; Published: 14 October 2019

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Abstract: In this paper, we obtain multifractals (attractors) in the framework of Hausdorff *b*-metric spaces. Fractals and multifractals are defined to be the fixed points of associated fractal operators, which are known as attractors in the literature of fractals. We extend the results obtained by Chifu et al. (2014) and N.A. Secelean (2015) and generalize the results of Nazir et al. (2016) by using the assumptions imposed by Dung et al. (2017) to the case of ciric type generalized multi-iterated function system (CGMIFS) composed of ciric type generalized multivalued G-contractions defined on multifractal space $\mathcal{C}(\mathcal{U})$ in the framework of a Hausdorff *b*-metric space, where $\mathcal{U} = U_1 \times U_2 \times \cdots \times U_N$, N being a finite natural number. As an application of our study, we derive collage theorem which can be used to construct general fractals and to solve inverse problem in Hausdorff *b*-metric spaces which are more general spaces than Hausdorff metric spaces.

Keywords: generalized multivalued G—Contraction; generalized multivalued iterated function systems; Hausdorff b metric space; fractal space; multifractal space; fixed point

1. Introduction

Dynamic systems characterization has been intensively investigated in diverse areas of physics [1–5], population biology [6–10], neural networks [11–14], mathematical modeling [15–17], etc. Especially fractals and multifractals play an important role in applications such as signal and image compression, creation of digital photographs, soil mechanics, fluid mechanics, computer graphics and so on. Most of these fractals and multifractals are obtained by using iterated function (or multifunction) systems (IFS). In 1981, Hutchinson [18] defined iterated function systems (IFS) and Barnsley [19] enriched the theory of IFS. This theory is known as Hutchinson–Barnsley (HB) theory. Hutchinson defined IFS as a finite collection of contractive self mappings and introduced HB operator on hyperspace of nonempty compact sets. He defined the unique fixed point of HB operator as a fractal (attractor). Thus, fixed point theory plays prominent role in the construction of fractals. For years, IFS has been an emerging technique for researchers to generate and analyze new fractal objects. In the sequel, numerous developments and extensions of IFS to construct fractals and similar sets are made (see, e.g., [20–22] and references therein).

Banach contraction principle [23] contributed a lot in fixed point theory. Several researchers enhanced the Banach contraction principle either by generalizing the domain [24–27] or by taking more general contractive conditions on mappings [28–30]. Further, several fixed point results were obtained by generalizing the concept of metric space [31]. For other new fixed point results and their applications, see [32–34].

The idea of a b-metric space was given by Czerwik [35]. This opened a new door for researchers and they published several research papers of fixed point theory (see, e.g., [35–38]). Kamran et al. [39] and Ali et al. [40] introduced F-contraction mappings in the framework of b metric spaces. They proved several fixed point results and applied their results to solve Fredholm and Volterra integral equations, respectively. In 2014, Chifu et al. [41] proved some results for multivalued fractals by using ciric type contractive conditions. Secelean [42] considered the generalized iterated function systems, defined on product of metric spaces to improve some fixed point results. Dung et al. [43] revised the results of Nazir et al. [37,44] by adding a commutativity assumption on the maps. Inspired by their work, we attempt to extend their results to find the multifractals using ciric type generalized multivalued G-contraction mappings in the framework of Hausdorff b-metric spaces.

The structure of our paper is divided into five sections. Section 2 is dedicated to some basic definitions and results concerning b-metric spaces and IFS. Section 3 deals with the notion of generalized multi-iterated function systems (GMIFS) in Hausdorff b-metric spaces. Moreover, some results regarding the existence and uniqueness of attractors (multifractals) are obtained. We derive collage theorem in Section 4. In Section 5, we conclude our findings. The results obtained by us may be further generalized and extended.

2. Preliminaries

Definition 1 ([35])**.** *Consider a nonempty set Y and let $s \in \mathbb{R}$, where $s \geq 1$. A function $d\colon Y \times Y \to \mathbb{R}^+$ is called a b-metric if following axioms are satisfied:*

(b_1) $d(p,q) = 0$ if and only if $p = q$;
(b_2) $d(p,q) = d(q,p)$; and
(b_3) $d(p,r) \leq s[d(p,q) + d(q,r)]$ (triangle inequality).

The pair (Y,d) is said to be a b-metric space.

Remark 1. *For $s = 1$, the b-metric space can be reduced in metric space. This shows that every metric space is a b-metric space, but in general the converse is not true (see [35,45,46]).*

Remark 2. *In general, every metric is a continuous functional in both variables while a b-metric need not posses this property, i.e., a b-metric space need not be continuous (see Example 2, [47]).*

Example 1 ([35])**.** *Consider a space $L_p[0,1]$ of all real functions $y(u), u \in [0,1]$ and $0 < p < 1$ such that $\int_0^1 |y(u)|^p du < \infty$, together with a metric defined by*

$$d(y,z) = \left(\int_0^1 |y(u) - z(u)|^p du \right)^{\frac{1}{p}} \forall\, y,z \in L_p[0,1],$$

then this space is not a metric space but it is a b-metric space with $s = 2^{\frac{1}{p}}$.

Definition 2 ([35,45])**.** *A sequence $\{a_n\}_{n\in\mathbb{N}}$ in a b-metric space (U,d) is:*

(i) Convergent iff for each $\epsilon > 0$ and $n(\epsilon) \in \mathbb{N}$ there exists $a \in U$ such that $d(a_n, a) < \epsilon$. i.e., $d(a_n, a) \to 0$ as $n \to +\infty$. Here, a is the limit of the sequence and can be written as $\lim_{n\to+\infty} a_n = a$.

(ii) Cauchy iff for each $\epsilon > 0$ there is some $n(\epsilon) \in \mathbb{N}$ for which $d(a_n, a_m) < \epsilon\ \forall\ n,\ m \geq n(\epsilon)$ i.e., $d(a_n, a_m) \to 0$ as $n, m \to +\infty$.

Definition 3 ([35]). *Let (U, d) be a b-metric space. Then, a subset K of U is:*

(a) Closed iff each sequence $\{a_n\}_{n\in\mathbb{N}}$ of elements of K has a limit, e.g. a, then $a \in K$. (i.e., $K = \overline{K}$)
(b) Compact iff every sequence in K has a convergent subsequence in K.

Definition 4 ([35]). *A complete b-metric space (U, d) is a b-metric space in which each Cauchy sequence is convergent in U.*

Definition 5 ([44]). *Let (U, d) be a metric space and $\mathcal{C}(U)$ be the family of all nonempty compact subsets of U. Then, for all $L, M \in \mathcal{C}(U)$, the Hausdorff metric is defined by*

$$H_d(L, M) = \max\left\{\sup_{l\in L} d(l, M), \sup_{m\in M} d(m, L)\right\}, \tag{1}$$

where $d(l, M) = \inf\{d(l, m) : m \in M\}$. The pair $(\mathcal{C}(U), H_d)$ is said to be Hausdorff metric space and also known as a Fractal space (see [19]).

Definition 6 ([19]). *The Hausdorff metric space $(\mathcal{C}(U), H_d)$ is complete iff (U, d) is complete. Analogously, $(\mathcal{C}(U), H_d)$ becomes a complete Hausdorff b-metric space iff (U, d) is a complete b-metric space.*

Definition 7 ([48]). *Consider a family $\mathcal{G}$ of all the mappings of the form $G : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following axioms:*

(G_1) G is strictly increasing mapping, i.e., $\forall\, u, v \in \mathbb{R}_+, u < v$ implies that $G(u) < G(v)$;
(G_2) $\inf G = -\infty$, i.e., if $u_n \in \mathbb{R}_+$ is a sequence, then $\lim\limits_{n\to\infty} u_n = 0$ and $\lim\limits_{n\to\infty} G(u_n) = -\infty$ both are equivalent.
(G_3) There exists $\delta \in (0, 1)$ for which $\lim\limits_{u\to 0^+} u^\delta G(u) = 0$.

G-contraction is a self map g on U, if there exists $\tau > 0$ for which following holds:

$$\tau + G(d(g(u), g(v))) \le G(d(u, v))\ \forall\, u, v \in U,\ g(u) \ne g(v). \tag{2}$$

Further, from (G_1) together with Equation (2), we have

$$d(g(u), g(v)) < d(u, v),\ \ \forall\, u, v \in U,\ g(u) \ne g(v). \tag{3}$$

This shows that every G-contraction is contractive and, therefore, continuous.

Lemma 1. *Let (U_i, d_i) be b-metric spaces for $i = 1, 2, ..., N$. Let $(\mathcal{C}(U_i), H_{d_i})$ be corresponding Hausdorff b-metric spaces. For $P_i, Q_i, R_i, S_i \subset \mathcal{C}(U_i), i = 1, 2, ..., N$, following hold:*

(a) $Q_i \subseteq R_i \Rightarrow \sup_{p_i\in P_i} d_i(p_i, R_i) \le \sup_{p_i\in P_i} d_i(p_i, Q_i)$.
(b) $\sup_{x_i\in P_i\cup Q_i} d_i(x_i, R_i) = \max\{\sup_{p_i\in P_i} d_i(p_i, R_i),\ \sup_{q_i\in Q_i} d(q_i, R_i)\}$.
(c) $H_{d_i}(P_i \cup Q_i,\ R_i \cup S_i) \le \max\{H_{d_i}(P_i, R_i),\ H_{d_i}(Q_i, S_i)\}$.

Definition 8 (see [49]). *Let (U_i, d_i) be metric spaces, where $i \in I$ (a finite indexed set). Then, the product space is the space $\mathcal{D} = \prod_{i\in I} U_i$ containing all I-tuples $\{U_i\}_{i\in I}$. Consider a metric $\rho : \mathcal{D} \to \mathbb{R}$ defined as $\rho(y_i, z_i) = \sup_{i\in I} d_i(y_i, z_i)$. Now, let $I = 1, 2, ..., N$, then*

$$\rho(y_i, z_i) = \sup_{i=1,2,...,N} d_i(y_i, z_i)\ \forall\, y_i, z_i \in \mathcal{D}, \tag{4}$$

where $y_i = (y_1, y_2, ..., y_N)$, $z_i = (z_1, z_2, ..., z_N)$ and $y_i, z_i \in \prod_{i\in I} U_i$ for $i = 1, 2, ..., N$. Then, $(\mathcal{D}, \rho)$ is a metric space with product metric ρ.

Definition 9 ([42]). *Let ρ be a product metric on $\mathcal{D}$, then a mapping $g\colon \mathcal{D} \to \mathcal{D}$, where $\mathcal{D} = \prod_{i\in I} U_i$, is considered as a generalized multivalued G-contraction if there is a mapping $G \in \mathcal{G}$ and $\tau > 0$ for which*

$$\tau + G(\rho(g(y_i), g(z_i))) \leq G(\sup_{i=1,2,\ldots,N} d_i(y_i, z_i)), \tag{5}$$

for all $y_i = (y_1, y_2, \ldots, y_N)$, $z_i = (z_1, y_2, \ldots, z_N) \in \mathcal{D}$ and $g(y_i) \neq g(z_i)$ for $i = 1, 2, 3, \ldots, N$.

Remark 3. *From Equation (5), we have*

$$G(\rho(g(y_i), g(z_i))) < G(\sup_{i=1,2,\ldots,N} d_i(y_i, z_i)).$$

Now, using (G_1), we obtain

$$\rho(g(y_i), g(z_i)) < \sup_{i=1,2,\ldots,N} d_i(y_i, z_i),$$

for all $y_i = (y_1, y_2, \ldots, y_N)$, $z_i = (z_1, z_2, \ldots, z_N) \in \mathcal{D}$ and $g(y_i) \neq g(z_i)$ for $1 \leq i \leq N$. Thus, every generalized multivalued G-contraction on a product space is contractive and hence continuous.

Definition 10 ([50]). *Let $(\mathcal{U}, \rho)$ be a product metric space on $\mathcal{U} = \prod_{i\in I} U_i$, $i = 1, 2, \ldots, N$. Consider the family $\mathcal{C}(\mathcal{U})$ of all compact subsets of $\mathcal{U}$, then the multifractal space $(\mathcal{C}(\mathcal{U}), H_\rho)$ with metric H_ρ is defined as*

$$H_\rho(\mathcal{P}, \mathcal{Q}) = \max_{i=1,2,\ldots,N} \{H_{d_i}(P_i, Q_i)\}, \tag{6}$$

where $\mathcal{P} = (P_1, P_2, \ldots, P_N)$, $\mathcal{Q} = (Q_1, Q_2, \ldots, Q_N) \in \mathcal{C}(\mathcal{U})$ and H_{d_i} is the Hausdorff distance between P_i and Q_i for $i = 1, 2, \ldots, N$.

Lemma 2 ([51]). *The metric space $(\mathcal{C}(\mathcal{U}), H_\rho)$ is a complete metric space if $(\mathcal{C}(U_i), H_{d_i})$ for each $i = 1, 2, \ldots, N$ are complete metric spaces.*

Definition 11 ([52]). *Let (U_i, d_i), $i = 1, 2, \ldots, N$ be complete metric spaces and $t_{ij}^k\colon U_j \to U_i$ with $k = 1, 2, \ldots, l_{ij}, i, j = 1, 2, \ldots, N$ be contraction mappings having contractivity factors r_{ij}^k, then multi-iterated function system (MIFS) is defined by*

$$\{U_j, j = 1, 2, \ldots, N; t_{ij}^k\colon U_j \to U_i, k = 1, 2, \ldots, l_{ij}, i, j = 1, 2, \ldots, N\},$$

having contractive factor as r, where $r = \max\{r_{ij}^k,\ k = 1, 2, \ldots, l_{ij},\ i, j = 1, 2, \ldots, N\}$.

Definition 12 ([52]). *Assume that (U_i, d_i), $i = 1, 2, \ldots, N$ are complete metric spaces and $\{U_j, j = 1, 2, \ldots, N;\ t_{ij}^k\colon U_j \to U_i, k = 1, 2, \ldots, l_{ij},\ i, j = 1, 2, \ldots, N\}$ is an MIFS. Then, the Multi-Hutchinson–Barnsley (MHB) operator of MIFS is a function $\mathcal{F}\colon \mathcal{C}(\mathcal{U}) \to \mathcal{C}(\mathcal{U})$ defined by*

$$\mathcal{F}(\mathcal{S}) = \prod_{i=1}^{N} \bigcup_{j=1}^{N} \bigcup_{k=1}^{l_{ij}} t_{ij}^k(A_i),\ \ \forall\, \mathcal{S} \in \mathcal{C}(\mathcal{U}).$$

Lemma 3. *Let $(U_i, d_i), i = 1, 2, \ldots, N$ be N complete metric spaces and $\{U_j, j = 1, 2, \ldots, N;\ t_{ij}^k\colon U_j \to U_i, k = 1, 2, \ldots, l_{ij}, i, j = 1, 2, \ldots, N\}$ be an MIFS. Then, MHB operator $\mathcal{F}$ is a contraction mapping on $(\mathcal{C}(\mathcal{U}), H_\rho)$.*

Theorem 1 ([52]). *Consider N complete metric spaces (U_i, d_i), $i = 1, 2, \ldots, N$ and $\{U_j; j = 1, 2, \ldots, N,$ $t_{ij}^k\colon U_j \to U_i, k = 1, 2, \ldots, l_{ij}, i, j = 1, 2, \ldots, N\}$ be an MIFS. Then, there exists unique compact invariant set (multi-attractor or fractal) $\mathcal{S}_\infty$ of MIFS such that $\mathcal{S}_\infty \in \mathcal{C}(\mathcal{U})$ of HB operator $\mathcal{F}$.*

Throughout this paper, we consider b-metric spaces (U_i, d_i) in such a way that b-metric is continuous functional on $U_i \times U_i$ for $i = 1, 2, ..., N$ and $(\mathcal{C}(U_i), H_{d_i}),\ i = 1, 2, ..., N$ are corresponding Hausdorff b-metric spaces such that b-metric is continuous functional on $\mathcal{C}(U_i) \times \mathcal{C}(U_i)$.

3. Main Results

Now, we obtain multifractals (attractors) for commutative self mappings defined on complete Hausdorff b-metric spaces.

Theorem 2. *The metric space $(\mathcal{C}(\mathcal{U}), H_\rho)$ is a complete b-metric space iff $(\mathcal{C}(U_i), H_{d_i})$ are complete b-metric spaces for each $i = 1, 2, ..., N$.*

Proof. Let $(\mathcal{C}(U_i), H_{d_i}), i = 1, 2, ..., N$ be complete b-metric spaces. Suppose that $\mathcal{P}_n$ is a Cauchy sequence in $(\mathcal{C}(\mathcal{U}), H_\rho)$, then by definition of a Cauchy sequence, we have for each $\epsilon > 0$, there exists $n(\epsilon) \in \mathbb{N}$ such that

$$H_\rho(\mathcal{P}_n, \mathcal{P}_m) < \epsilon,\ \forall\, n, m > n(\epsilon),$$

where $\mathcal{P}_n = (P_1^n, P_2^n, ..., P_N^n)$, $\mathcal{P}_m = (P_1^m, P_2^m, ..., P_N^m)$ and $\mathcal{P}_n, \mathcal{P}_m \in \mathcal{C}(\mathcal{U})$.

$$\begin{aligned} &\Rightarrow H_\rho((P_1^n, P_2^n, ..., P_N^n), (P_1^m, P_2^m, ..., P_N^m)) < \epsilon,\ \forall\, n, m > n(\epsilon) \\ &\qquad\qquad \Rightarrow \max\{H_{d_i}(P_i^n, P_i^m)\} < \epsilon,\ for\ \ i = 1, 2, ..., N\ and\ \ \forall\, n, m > n(\epsilon) \\ &\qquad\qquad\qquad \Rightarrow H_{d_i}(P_i^n, P_i^m) < \epsilon,\ \forall\, n, m > n(\epsilon), i = 1, 2, ..., N. \end{aligned}$$

Thus, $\{P_i^n\}_{n=0}^\infty$ is a Cauchy sequence in $\mathcal{C}(U_i)$. Now, for $i = 1, 2, ..., N,\ (\mathcal{C}(U_i), H_{d_i})$ are complete b-metric spaces, then there exists $P_i \in \mathcal{C}(U_i)$ such that $H_{d_i}(P_i^n, P_i) \to 0$ as $n \to \infty$. This gives $H_\rho(\mathcal{P}^n, \mathcal{P}) \to 0$ as $n \to \infty$, where $\mathcal{P}^n = P_i^n = (P_1^n, P_2^n, ..., P_N^n)$ and $\mathcal{P} = P_i = (P_1, P_2, ..., P_N)$. This proves that $(\mathcal{C}(\mathcal{U}), H_\rho)$ is a complete b-metric space.

By reversing the above process, we can show that $(\mathcal{C}(U_i), H_{d_i})$ are complete b-metric spaces for each $i = 1, 2, ..., N$. □

Theorem 3. *Let (U_i, d_i) be b-metric spaces for $i = 1, 2, ..., N$ and $(\mathcal{C}(U_i), H_{d_i})$ be the corresponding Hausdorff b-metric spaces. Let $t_{ij}^k \colon U_j \to U_i,\ k = 1, 2, ..., l_{ij},\ i = j = 1, 2, ..., N$ be commutative generalized multivalued G-contractions, then following hold:*

(1) t_{ij}^k maps elements of $\mathcal{C}(U_i)$ to elements in $\mathcal{C}(U_i)$.
(2) If $t_{ij}^k(P_i) = \{t_{ij}^k(p_i); p_i \in P_i, k = 1, 2, ..., l_{ij}, i, j = 1, 2, ..., N\}$ for any $P_i \in \mathcal{C}(U_i)$, then the mapping $t_{ij}^k \colon \mathcal{C}(U_j) \to \mathcal{C}(U_i), k = 1, 2, ..., l_{ij};\ i, j = 1, 2, ..., N$ is a generalized multivalued G-contraction on $(\mathcal{C}(U_i), H_{d_i})$.

Proof. (1) The mapping t_{ij}^k is continuous being generalized multivalued G-contraction and under a continuous mapping the image of a compact set is compact. Therefore, we have

$$P_i \in \mathcal{C}(U_i) \Rightarrow t_{ij}^k(P_i) \in \mathcal{C}(U_i).$$

(2) Let $P_i, Q_i \in \mathcal{C}(U_i)$. As $t_{ij}^k \colon U_j \to U_i$ is a generalized multivalued G-contraction, we have

$$0 < d_i(t_{ij}^k(p_i), t_{ij}^k(q_i)) < d_i(p_i, q_i)\ \forall p_i \in P_i, q_i \in Q_i. \tag{7}$$

Then, using Equation (7), we have

$$d_i(t_{ij}^k(p_i), t_{ij}^k(Q_i)) = \inf_{q_i \in Q_i} d_i(t_{ij}^k(p_i), t_{ij}^k(q_i)) < \inf_{q_i \in Q_i} d_i(p_i, q_i) = d_i(p_i, Q_i), \tag{8}$$

and

$$d_i(t_{ij}^k(q_i), t_{ij}^k(P_i)) = \inf_{p_i \in P_i} d_i(t_{ij}^k(q_i), t_{ij}^k(p_i)) < \inf_{p_i \in P_i} d_i(q_i, p_i) = d_i(q_i, P_i). \tag{9}$$

Now, consider

$$H_{d_i}(t_{ij}^k(P_i), t_{ij}^k(Q_i)) = \max\{\sup_{p_i \in P_i} d_i(t_{ij}^k(p_i), t_{ij}^k(Q_i)), \sup_{q_i \in Q_i} d_i(t_{ij}^k(q_i), t_{ij}^k(P_i))\}.$$

Using Equations (8) and (9), the above equation reduces to

$$\begin{aligned} H_{d_i}(t_{ij}^k(P_i), t_{ij}^k(Q_i)) &< \max\{\sup_{p_i \in P_i} d_i(p_i, Q_i), \sup_{q_i \in Q_i} d_i(q_i, P_i)\} \\ &= H_{d_i}(P_i, Q_i) \end{aligned}$$

$$\Rightarrow H_{d_i}(t_{ij}^k(P_i), t_{ij}^k(Q_i)) < H_{d_i}(P_i, Q_i). \tag{10}$$

Since G is strictly increasing, we have

$$G(H_{d_i}(t_{ij}^k(P_i), t_{ij}^k(Q_i))) < G(H_{d_i}(P_i, Q_i)). \tag{11}$$

For some $\tau > 0$, Equation (11) becomes

$$\tau + G(H_{d_i}(t_{ij}^k(P_i), t_{ij}^k(Q_i))) \leq G(H_{d_i}(P_i,\ Q_i)).$$

Hence, the mapping $t_{ij}^k\colon \mathcal{C}(U_j) \to \mathcal{C}(U_i)$ is a generalized multivalued G-contraction on $(\mathcal{C}(U_i), H_{d_i})$. □

Theorem 4. *Assume that $(\mathcal{C}(U_i), H_{d_i})$, $i = 1, 2, ..., N$ are complete Hausdorff b-metric spaces and $\{U_j, j = 1, 2, ..., N;\ t_{ij}^k\colon U_j \to U_i,\ k = 1, 2, ..., l_{ij},\ i, j = 1, 2, ..., N\}$ is a finite family of commutative generalized multivalued G-contractions. Then, the generalized multi-Hutchinson–Burnsley (GMHB) operator $\mathcal{F}\colon \mathcal{C}(\mathcal{U}) \to \mathcal{C}(\mathcal{U})$ defined by*

$$\mathcal{F}(\mathcal{S}) = \prod_{i=1}^{N} \bigcup_{j=1}^{N} \bigcup_{k=1}^{l_{ij}} t_{ij}^k(S_j),\ \ \forall\, \mathcal{S} \in \mathcal{C}(\mathcal{U}), \tag{12}$$

is a generalized multivalued G-contraction on $\mathcal{C}(\mathcal{U})$.

Proof. Let $\mathcal{P}, \mathcal{Q} \in \mathcal{C}(\mathcal{U})$, where $\mathcal{P} = (P_1, P_2, ..., P_N)$ and $\mathcal{Q} = (Q_1, Q_2, ..., Q_N)$ then using Equation (12), we observe

$$\begin{aligned} \tau + G(H_\rho(\mathcal{F}(\mathcal{P}), \mathcal{F}(\mathcal{Q}))) &= \tau + G\left\{H_\rho\left(\prod_{i=1}^{N}\bigcup_{j=1}^{N}\bigcup_{k=1}^{l_{ij}} t_{ij}^k(P_j), \prod_{i=1}^{N}\bigcup_{j=1}^{N}\bigcup_{k=1}^{l_{ij}} t_{ij}^k(Q_j)\right)\right\} \ for\ P_j, Q_j \in \mathcal{C}(U_j) \\ &= \tau + G\left[\max_{i=1,2,\ldots,N}\left\{H_{d_i}\left(\bigcup_{j=1}^{N}\bigcup_{k=1}^{l_{ij}} t_{ij}^k(P_j), \bigcup_{j=1}^{N}\bigcup_{k=1}^{l_{ij}} t_{ij}^k(Q_j)\right)\right\}\right] \ for\ P_j, Q_j \in \mathcal{C}(U_j) \\ &= \tau + G\left[\max_{i=1,2,\ldots,N}\left\{\max_{j=1,2,\ldots,N}\left(H_{d_{ij}}(\bigcup_{k=1}^{l_{ij}} t_{ij}^k(P_j), \bigcup_{k=1}^{l_{ij}} t_{ij}^k(Q_j))\right)\right\}\right] \ for\ P_j, Q_j \in \mathcal{C}(U_j). \end{aligned}$$

By using Lemma 1, we have

$$\tau + G(H_\rho(\mathcal{F}(\mathcal{P}), \mathcal{F}(\mathcal{Q}))) \leq \tau + G\left[\max_{i=1,2,\ldots,N}\left\{\max_{j=1,2,\ldots,N}\left(\max_{k=1,2,\ldots,l_{ij}}\left(H_{d_{ij}^k}\left((t_{ij}^k(P_j), t_{ij}^k(Q_j)\right)\right)\right)\right\}\right].$$

From Theorem 3, we obtain

$$\tau + G(H_\rho(\mathcal{F}(\mathcal{P}), \mathcal{F}(\mathcal{Q}))) < \tau + G\left[\max_{i=1,2,\dots,N}\left\{\max_{j=1,2,\dots,N}\left(\max_{k=1,2,\dots,l_{ij}}\left(H_{d_{ij}^k}(P_j, Q_j)\right)\right)\right\}\right] \;for\; P_j, Q_j \in \mathcal{C}(U_j)$$

$$= \tau + G\left[\max_{i=1,2,\dots,N}\left\{\max_{j=1,2,\dots,N}\left(H_{d_{ij}}(P_j, Q_j)\right)\right\}\right] \;for\; P_j, Q_j \in \mathcal{C}(U_j)$$

$$\leq \tau + G\left\{\max_{i=1,2,\dots,N}\left(H_{d_i}(P_i, Q_i)\right)\right\} \;for\; P_i, Q_i \in \mathcal{C}(U_i)$$

$$\leq G\left(H_\rho(\mathcal{P}, \mathcal{Q})\right)$$

$$\Rightarrow \tau + G(H_\rho(\mathcal{F}(\mathcal{P}), \mathcal{F}(\mathcal{Q}))) \leq G\left(H_\rho(\mathcal{P}, \mathcal{Q})\right) \;for\; \mathcal{P}, \mathcal{Q} \in \mathcal{C}(\mathcal{U}).$$

Hence, the generalized multi-Hutchinson Burnsley (GMHB) operator $\mathcal{F}\colon \mathcal{C}(\mathcal{U}) \to \mathcal{C}(\mathcal{U})$ is a generalized multivalued G-contraction on $\mathcal{C}(\mathcal{U})$. □

Definition 13. *Let $(\mathcal{C}(U_i), H_{d_i})$, $i = 1, 2, ..., N$ be Hausdorff b-metric spaces. Then, the mapping $\mathcal{F}\colon \mathcal{C}(\mathcal{U}) \to \mathcal{C}(\mathcal{U})$ is a ciric type generalized multivalued G-contraction if for $\mathcal{P}, \mathcal{Q} \in \mathcal{C}(\mathcal{U})$, $G \in \mathcal{G}$ and $\tau > 0$ the following satisfies:*

$$\tau + G(H_\rho(\mathcal{F}(\mathcal{P}), \mathcal{F}(\mathcal{Q}))) \leq F(W_{\mathcal{F}}(\mathcal{P}, \mathcal{Q})), \tag{13}$$

where

$$\begin{aligned} W_{\mathcal{F}}(\mathcal{P}, \mathcal{Q}) = \max\{&H_\rho(\mathcal{P}, \mathcal{Q}), H_\rho(\mathcal{P}, \mathcal{F}(\mathcal{P})), H_\rho(\mathcal{Q}, \mathcal{F}(\mathcal{Q})),\\ &\frac{H_\rho(\mathcal{P}, \mathcal{F}(\mathcal{Q})) + H_\rho(\mathcal{Q}, \mathcal{F}(\mathcal{P}))}{2s}, H_\rho(\mathcal{F}^2(\mathcal{P}), \mathcal{F}(\mathcal{P})),\\ &H_\rho(\mathcal{F}^2(\mathcal{P}), \mathcal{Q}), H_\rho(\mathcal{F}^2(\mathcal{P}), \mathcal{F}(\mathcal{Q}))\}. \end{aligned}$$

Definition 14. *Let $(\mathcal{C}(U_i), H_{d_i})$, $i = 1, 2, ..., N$ be Hausdorff b-metric spaces and $t_{ij}^k\colon U_j \to U_i$ with $k = 1, 2, ..., l_{ij}, i, j = 1, 2, ..., N$ be commutative ciric type generalized multivalued G-contraction mappings, then ciric type generalized multi-iterated function system (CGMIFS) is expressed as*

$$\{U_j, j = 1, 2, ..., N;\; t_{ij}^k\colon U_j \to U_i, k = 1, 2, ..., l_{ij}, i, j = 1, 2, ..., N\}. \tag{14}$$

Theorem 5. *Let $(\mathcal{C}(U_i), H_{d_i}), i = 1, 2, ..., N$ be complete Hausdorff b-metric spaces and $\{U_j; j = 1, 2, ..., N,\; t_{ij}^k\colon U_j \to U_i, k = 1, 2, ..., l_{ij},\; i, j = 1, 2, ..., N\}$ be a CGMIFS, where the mappings t_{ij}^k are commutative mappings. Then, the mapping $\mathcal{F}\colon \mathcal{C}(\mathcal{U}) \to \mathcal{C}(\mathcal{U})$ defined by*

$$\mathcal{F}(\mathcal{S}) = \prod_{i=1}^{N}\bigcup_{j=1}^{N}\bigcup_{k=1}^{l_{ij}} t_{ij}^k(S_j), \tag{15}$$

where $\mathcal{S} = S_1, S_2, ..., S_N \in \mathcal{C}(\mathcal{U})$, is a ciric type generalized multi-Hutchinson–Barnsley (CGMHB) operator.

Proof. The result holds by using Theorem 4 together with generalized multivalued G-contraction property (G_1). □

Definition 15. *A nonempty family $\mathcal{S}^* \in \mathcal{C}(\mathcal{U})$ of compact sets is said to be multi-attractor with respect to GMIFS if and only if $\mathcal{S}^* = \mathcal{F}(\mathcal{S}^*)$ where $\mathcal{S}^* = S_i = S_1, S_2, ..., S_N \in \mathcal{C}(\mathcal{U})$, i.e., $\mathcal{S}^*$ is the fixed point of associated generalized multi-Hutchinson–Barnsley (GMHB) operator $\mathcal{F}$.*

Theorem 6. *Let $(\mathcal{C}(U_i), H_{d_i})$ for $i = 1,2,...,N$ be complete Hausdorff b-metric spaces and $\{U_j, j = 1,2,...,N;\ t_{ij}^k\colon U_j \to U_i,\ k = 1,2,...,l_{ij},\ i,j = 1,2,...,N\}$ be a CGMIFS and $\mathcal{F}\colon \mathcal{C}(\mathcal{U}) \to \mathcal{C}(\mathcal{U})$ be CGMHB operator. Then, there exists unique attractor $\mathcal{S}^* \in \mathcal{C}(\mathcal{U})$, i.e.,*

$$\mathcal{S}^* = \mathcal{F}(\mathcal{S}^*) = \prod_{i=1}^{N} \bigcup_{j=1}^{N} \bigcup_{k=1}^{l_{ij}} t_{ij}^k(S_j^*),\ \ \forall\, \mathcal{S}^* \in \mathcal{C}(\mathcal{U}), \tag{16}$$

where $\mathcal{S}^* = (S_1^*, S_2^*, ..., S_N^*) \in \mathcal{C}(\mathcal{U})$.
In addition, the sequence $\{\mathcal{V}^0, \mathcal{F}(\mathcal{V}^0), \mathcal{F}^2(\mathcal{V}^0), ...\}$ of compact sets for each initial family $\mathcal{V}^0 \in \mathcal{C}(\mathcal{U})$ converges to a unique attractor $\mathcal{S}^*$ of CGMHB operator $\mathcal{F}$.

Proof. To prove the existence of an attractor $\mathcal{S}^*$, let us consider $\mathcal{V}^0 \in \mathcal{C}(\mathcal{U})$, where $\mathcal{V}^0 = (V_1^0, V_2^0, ..., V_N^0)$. If $H_\rho(\mathcal{V}^m, \mathcal{F}(\mathcal{V}^m)) = H_{d_i}(V_i^m, V_i^{m+1}) = H_{d_i}(V_i^m, \mathcal{F}(V_i^m)) = 0$ for $i = 1,2,...,N$ and $m \in \mathbb{N}$, then $V_i^m = \mathcal{F}(V_i^m)$ i.e., $\mathcal{V}^m = \mathcal{F}(\mathcal{V}^m)$. Then, $\mathcal{X}^* = \mathcal{V}^m$ is an attractor of $\mathcal{F}$, which completes the proof. Thus, let us suppose that $H_\rho(\mathcal{V}^k, \mathcal{F}(\mathcal{V}^k)) > 0\ \ \forall\, k \in \mathbb{N}$. Then from Equation (13), we have

$$\begin{aligned} \tau + G(H_\rho(\mathcal{V}^{k+1}, \mathcal{V}^{k+2})) &= \tau + G(H_\rho(\mathcal{F}(\mathcal{V}^k), \mathcal{F}(\mathcal{V}^{k+1}))) \\ &\le G(W_{\mathcal{F}}(\mathcal{V}^k, \mathcal{V}^{k+1})) = G(W_{\mathcal{F}}(V_i^k, V_i^{k+1})),\ i = 1,2,...,N. \end{aligned}$$

$$\Rightarrow \tau + G(H_\rho(\mathcal{V}^{k+1}, \mathcal{V}^{k+2})) \le G(W_{\mathcal{F}}(V_i^k, V_i^{k+1})),\ i = 1,2,...,N, \tag{17}$$

where

$$\begin{aligned} W_{\mathcal{F}}(V_i^k, V_i^{k+1}) = \max\Big\{ & H_{d_i}(V_i^k, V_i^{k+1}), H_{d_i}(V_i^k, \mathcal{F}(V_i^k)), H_{d_i}(V_i^{k+1}, \mathcal{F}(V_i^{k+1})), \\ & \frac{H_{d_i}(V_i^k, \mathcal{F}(V_i^{k+1})) + H_{d_i}(V_i^{k+1}, \mathcal{F}(V_i^k))}{2s}, H_{d_i}(\mathcal{F}^2(V_i^k), \mathcal{F}(V_i^k)), \\ & H_{d_i}(\mathcal{F}^2(V_i^k), V_i^{k+1}), H_{d_i}(\mathcal{F}^2(V_i^k), \mathcal{F}(V_i^{k+1}))\Big\} \\ = \max\Big\{ & H_{d_i}(V_i^k, V_i^{k+1}), H_{d_i}(V_i^k, V_i^{k+1}), H_{d_i}(V_i^{k+1}, V_i^{k+2}), \\ & \frac{H_{d_i}(V_i^k, V_i^{k+2}) + H_{d_i}(V_i^{k+1}, V_i^{k+1})}{2s}, H_{d_i}(V_i^{k+2}, V_i^{k+1}), \\ & H_{d_i}(V_i^{k+2}, V_i^{k+1}), H_{d_i}(V_i^{k+2}, V_i^{k+2})\Big\}. \end{aligned}$$

Thus,

$$W_{\mathcal{F}}(V_i^k, V_i^{k+1}) = \max\{H_{d_i}(V_i^k, V_i^{k+1}), H_{d_i}(V_i^{k+1}, V_i^{k+2})\}. \tag{18}$$

If $W_{\mathcal{F}}(V_i^k, V_i^{k+1}) = H_{d_i}(V_i^{k+1}, V_i^{k+2})$, then from Equation (17),

$$F(H_{d_i}(V_i^{k+1}, V_i^{k+2})) \le G(H_{d_i}(V_i^{k+1}, V_i^{k+2})) - \tau. \tag{19}$$

This gives

$$\tau \le 0,$$

which is a contradiction, since $\tau > 0$.

Therefore, we have $W_{\mathcal{T}}(V_i^k, V_i^{k+1}) = H_{d_i}(V_i^k, V_i^{k+1})$.

Now, using Equation (17), we have

$$\begin{aligned} & G(H_{d_i}(V_i^{k+1}, V_i^{k+2})) \le G(H_{d_i}(V_i^k, V_i^{k+1})) - \tau \\ \Rightarrow\ & G(H_{d_i}(V_i^{k+1}, V_i^{k+2})) < G(H_{d_i}(V_i^k, V_i^{k+1})). \end{aligned} \tag{20}$$

From Equation (20) together with (G_1),we have

$$H_{d_i}(V_i^{k+1}, V_i^{k+2}) < H_{d_i}(V_i^{k}, V_i^{k+1}),\ \forall\ k \in \mathbb{N}$$

$$\Rightarrow H_{d_i}(V_i^{k}, V_i^{k+1}) < H_{d_i}(V_i^{k-1}, V_i^{k})\ \forall\, i = 1, 2, ..., N \textit{ and } k \in \mathbb{N}. \tag{21}$$

Therefore, $\{H_{d_i}(V_i^k, V_i^{k+1})\}_{k\in\mathbb{N}}$ is a non-negative decreasing sequence and hence convergent. Now,

$$\begin{aligned} G(H_{d_i}(V_i^k, V_i^{k+1})) &\leq G(H_{d_i}(V_i^{k-1}, V_i^{k})) - \tau \\ &\leq G(H_{d_i}(V_i^{k-2}, V_i^{k-1})) - 2\tau \\ &\leq \cdots \leq G(H_{d_i}(V_i^{0}, V_i^{1})) - n\tau \end{aligned}$$

$$\Rightarrow G(H_{d_i}(V_i^k, V_i^{k+1})) \leq G(H_{d_i}(V_i^{0}, V_i^{1})) - n\tau, \tag{22}$$

which gives $\lim\limits_{k\to\infty} G(H_{d_i}(V_i^k, V_i^{k+1})) = -\infty$ and using (G_2), we have $\lim\limits_{k\to\infty} H_{d_i}(V_i^k, V_i^{k+1})) = 0$. Thus, we have

$$\lim_{k\to\infty} H_{d_i}(V_i^k, V_i^{k+1})) = \lim_{k\to\infty} H_{d_i}(V_i^k, \mathcal{F}(V_i^k)) = 0. \tag{23}$$

Now, we have to prove that $\{V_i^k\}_{k=1}^{\infty}$ is a Cauchy sequence. On the contrary, suppose that there exists $\epsilon > 0$ and two sequences $\{\alpha_i^m\}_{m=1}^{\infty}$ and $\{\beta_i^m\}_{m=1}^{\infty}$ such that

$$\alpha_i^m > \beta_i^m > m,\ H_{d_i}(V_i^{\alpha_i^m}, V_i^{\beta_i^m}) \geq \epsilon \ \textit{ and }\ H_{d_i}(V_i^{\alpha_i^m - 1}, V_i^{\beta_i^m}) < \epsilon, \tag{24}$$

for all $i = 1, 2, ..., N$ and $m \in \mathbb{N}$.

Now, using Equation (24) together with triangular inequality of a b-metric space, we have

$$\begin{aligned} \epsilon \leq H_{d_i}(V_i^{\alpha_i^m}, V_i^{\beta_i^m}) &\leq s\{H_{d_i}(V_i^{\alpha_i^m}, V_i^{\alpha_i^m - 1}) + H_{d_i}(V_i^{\alpha_i^m - 1}, V_i^{\beta_i^m})\} \\ &\leq sH_{d_i}(V_i^{\alpha_i^m}, V_i^{\alpha_i^m - 1}) + s\epsilon. \end{aligned}$$

Then, from Equation (23), we get

$$\epsilon \leq \lim_{m\to\infty} \sup H_{d_i}(V_i^{\alpha_i^m}, V_i^{\beta_i^m}) \leq s\epsilon. \tag{25}$$

Again, by using triangle inequality of a b-metric space, we have

$$\epsilon \leq H_{d_i}(V_i^{\alpha_i^m}, V_i^{\beta_i^m}) \leq s\{H_{d_i}(V_i^{\alpha_i^m}, V_i^{\beta_i^m + 1}) + H_{d_i}(V_i^{\beta_i^m + 1}, V_i^{\beta_i^m})\}. \tag{26}$$

Furthermore,

$$H_{d_i}(V_i^{\alpha_i^m}, V_i^{\beta_i^m + 1}) \leq s\{H_{d_i}(V_i^{\alpha_i^m}, V_i^{\beta_i^m}) + H_{d_i}(V_i^{\beta_i^m}, V_i^{\beta_i^m + 1})\}. \tag{27}$$

Using Equations (23) and (25) in Equation (27), we have

$$\lim_{m\to\infty} H_{d_i}(V_i^{\alpha_i^m}, V_i^{\beta_i^m + 1}) \leq s^2\epsilon. \tag{28}$$

In addition, together with Equations (23) and (26), Equation (28) becomes

$$\frac{\epsilon}{s} \leq \frac{1}{s} \lim_{m\to\infty} \sup H_{d_i}(V_i^{\alpha_i^m}, V_i^{\beta_i^m}) \leq \lim_{m\to\infty} \sup H_{d_i}(V_i^{\alpha_i^m}, V_i^{\beta_i^m + 1}) \leq s^2\epsilon$$

$$\Rightarrow \frac{\epsilon}{s} \leq \lim_{m\to\infty} \sup H_{d_i}(V_i^{\alpha_i^m}, V_i^{\beta_i^m + 1}) \leq s^2\epsilon. \tag{29}$$

Again, using the same process, we have

$$\frac{\epsilon}{s} \leq \lim_{m\to\infty} \sup H_{d_i}(V_i^{\alpha_i^m+1}, V_i^{\beta_i^m}) \leq s^2\epsilon. \tag{30}$$

Consider,

$$H_{d_i}(V_i^{\alpha_i^m}, V_i^{\beta_i^m+1}) \leq s\{H_{d_i}(V_i^{\alpha_i^m}, V_i^{\alpha_i^m+1}) + H_{d_i}(V_i^{\alpha_i^m+1}, V_i^{\beta_i^m+1})\}. \tag{31}$$

Using Equation (23) in Equation (31), we have

$$\lim_{m\to\infty} \sup H_{d_i}(V_i^{\alpha_i^m}, V_i^{\beta_i^m+1}) \leq s \lim_{m\to\infty} \sup H_{d_i}(V_i^{\alpha_i^m+1}, V_i^{\beta_i^m+1}). \tag{32}$$

Using Equations (29) and (32) becomes

$$\frac{\epsilon}{s^2} \leq \frac{1}{s} \lim_{m\to\infty} \sup H_{d_i}(V_i^{\alpha_i^m},\ V_i^{\beta_i^m+1}) \leq \lim_{m\to\infty} \sup H_{d_i}(V_i^{\alpha_i^m+1},\ V_i^{\beta_i^m+1})$$

$$\Rightarrow \frac{\epsilon}{s^2} \leq \lim_{m\to\infty} \sup H_{d_i}(V_i^{\alpha_i^m+1},\ V_i^{\beta_i^m+1}). \tag{33}$$

Consider,

$$\begin{aligned} H_{d_i}(V_i^{\alpha_i^m+1},\ V_i^{\beta_i^m+1}) &\leq s\{H_{d_i}(V_i^{\alpha_i^m+1},\ V_i^{\beta_i^m}) + H_{d_i}(V_i^{\beta_i^m},\ V_i^{\beta_i^m+1})\} \\ &\leq s^2\{H_{d_i}(V_i^{\alpha_i^m+1},\ V_i^{\alpha_i^m}) + H_{d_i}(V_i^{\alpha_i^m},\ V_i^{\beta_i^m})\} \\ &\quad + sH_{d_i}(V_i^{\beta_i^m},\ V_i^{\beta_i^m+1}). \end{aligned}$$

Using Equations (23) and (25), we obtain

$$\lim_{m\to\infty} \sup H_{d_i}(V_i^{\alpha_i^m+1},\ V_i^{\beta_i^m+1}) \leq s^3\epsilon. \tag{34}$$

Now, from Equations (33) and (34), we have

$$\frac{\epsilon}{s^2} \leq \lim_{m\to\infty} \sup H_{d_i}(V_i^{\alpha_i^m+1}, V_i^{\beta_i^m+1}) \leq s^3\epsilon. \tag{35}$$

Thus, from Equations (23) and (24), we can select $m_1 \in \mathbb{N}$ in such a way that

$$H_{d_i}(V_i^{\alpha_i^m}, \mathcal{F}(V_i^{\alpha_i^m})) < \epsilon < H_{d_i}(V_i^{\alpha_i^m}, V_i^{\beta_i^m}),\ \forall\, m \geq m_1.$$

Therefore, for all $m \geq m_1$, we have

$$G(H_{d_i}(V_i^{\alpha_i^m+1},\ V_i^{\beta_i^m+1})) \leq G(W_{\mathcal{F}}(V_i^{\alpha_i^m},\ V_i^{\beta_i^m})) - \tau(H_{d_i}(V_i^{\alpha_i^m},\ V_i^{\beta_i^m})). \tag{36}$$

Using (G_1) together with Equation (36), we have

$$
\begin{aligned}
H_{d_i}(V_i^{\alpha_i^m+1}, V_i^{\beta_i^m+1}) < W_{\mathcal{F}}(V_i^{\alpha_i^m}, V_i^{\beta_i^m})
&= \max\Big\{H_{d_i}(V_i^{\alpha_i^m}, V_i^{\beta_i^m}), H_{d_i}(V_i^{\alpha_i^m}, \mathcal{F}(V_i^{\alpha_i^m})), H_{d_i}(V_i^{\beta_i^m}, \mathcal{F}(V_i^{\beta_i^m})),\\
&\quad \frac{H_{d_i}(V_i^{\alpha_i^m}, \mathcal{F}(V_i^{\beta_i^m})) + H_{d_i}(V_i^{\beta_i^m}, \mathcal{F}(V_i^{\alpha_i^m}))}{2s}, H_{d_i}(\mathcal{F}^2(V_i^{\alpha_i^m}, \mathcal{F}(V_i^{\alpha_i^m})),\\
&\quad H_{d_i}(\mathcal{F}^2(V_i^{\alpha_i^m}), V_i^{\beta_i^m}), H_{d_i}(\mathcal{F}^2(V_i^{\alpha_i^m}), \mathcal{F}(V_i^{\beta_i^m}))\Big\}\\
&= \max\Big\{H_{d_i}(V_i^{\alpha_i^m}, V_i^{\beta_i^m}), H_{d_i}(V_i^{\alpha_i^m}, V_i^{\alpha_i^m+1}), H_{d_i}(V_i^{\beta_i^m}, V_i^{\beta_i^m+1}),\\
&\quad \frac{H_{d_i}(V_i^{\alpha_i^m}, V_i^{\beta_i^m+1})) + H_{d_i}(V_i^{\beta_i^m}, V_i^{\alpha_i^m+1})}{2s}, H_{d_i}(V_i^{\alpha_i^m+2}, V_i^{\alpha_i^m+1}),\\
&\quad H_{d_i}(V_i^{\alpha_i^m+2}, V_i^{\beta_i^m}), H_{d_i}(V_i^{\alpha_i^m+2}, V_i^{\beta_i^m+1})\Big\}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
H_{d_i}(V_i^{\alpha_i^m+1}, V_i^{\beta_i^m+1}) < W_{\mathcal{F}}(V_i^{\alpha_i^m}, V_i^{\beta_i^m})
&= \max\Big[H_{d_i}(V_i^{\alpha_i^m}, V_i^{\beta_i^m}), H_{d_i}(V_i^{\alpha_i^m}, V_i^{\alpha_i^m+1}), H_{d_i}(V_i^{\beta_i^m}, V_i^{\beta_i^m+1}),\\
&\quad \frac{H_{d_i}(V_i^{\alpha_i^m}, V_i^{\beta_i^m+1})) + H_{d_i}(V_i^{\beta_i^m}, V_i^{\alpha_i^m+1})}{2s}, H_{d_i}(V_i^{\alpha_i^m+2}, V_i^{\alpha_i^m+1}),\\
&\quad s\{H_{d_i}(V_i^{\alpha_i^m+2}, V_i^{\alpha_i^m+1}) + H_{d_i}(V_i^{\alpha_i^m+1}, V_i^{\beta_i^m})\},\\
&\quad s\{H_{d_i}(V_i^{\alpha_i^m+2}, V_i^{\alpha_i^m+1}) + H_{d_i}(V_i^{\alpha_i^m+1}, V_i^{\beta_i^m+1})\}\Big].
\end{aligned} \tag{37}
$$

Now, taking limit supremum as $m \to \infty$ on each side of Equation (37) and using Equations (23), (25) and (28), respectively, we obtain

$$
\epsilon \le \lim_{m\to\infty} \sup W_{\mathcal{F}}(V_i^{\alpha_i^m}, V_i^{\beta_i^m}) = \max\{s\epsilon,\ 0,\ 0,\ \frac{s^2\epsilon + s^2\epsilon}{2s},\ 0,\ s(0+s^2\epsilon),\ s(0+s\epsilon)\} \le s^3\epsilon.
$$

$$
\Rightarrow \epsilon \le \lim_{m\to\infty} \sup W_{\mathcal{F}}(V_i^{\alpha_i^m}, V_i^{\beta_i^m}) \le s^3\epsilon. \tag{38}
$$

Using the same argument, we can prove that

$$
\epsilon \le \lim_{m\to\infty} \inf W_{\mathcal{F}}(V_i^{\alpha_i^m}, V_i^{\beta_i^m}) \le s^3\epsilon. \tag{39}
$$

Now, taking limit supremum as $m \to \infty$ in Equation (36) and using Equations (25), (33), (35), (38) and (39), respectively, we have

$$
\begin{aligned}
G(s^3\epsilon) = G(s^5\frac{\epsilon}{s^2}) &\le G(\lim_{m\to\infty} \sup H_{d_i}(V_i^{\alpha_i^m+1}, V_i^{\beta_i^m+1}))\\
&\le G(\lim_{m\to\infty} \sup W_{\mathcal{F}}(V_i^{\alpha_i^m}, V_i^{\beta_i^m})) - \tau.
\end{aligned}
$$

This gives

$$
G(s^3\epsilon) \le G(s^3\epsilon) - \tau. \tag{40}
$$

This implies that $\tau < 0$, which is a contradiction. Thus, our supposition is wrong. Hence, $\{V_i^k\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathcal{C}(U_i)$, i.e., $\{\mathcal{V}^k\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathcal{C}(\mathcal{U})$. Since $(\mathcal{C}(\mathcal{U}), H_\rho)$ is complete, the sequence $\{\mathcal{V}^k\}_{k=1}^{\infty}$ converges to $\mathcal{S}^*$ as $k \to \infty$ for some $\mathcal{S}^* \in \mathcal{C}(\mathcal{U})$. Thus, the sequence $\{\mathcal{V}^0, \mathcal{F}(\mathcal{V}^0), \mathcal{F}^2(\mathcal{V}^0), ...\}$ of compact sets converges to $\mathcal{S}^*$, where $\mathcal{S}^* = S_i^*$.

Now, we claim that $\mathcal{S}^*$ is an attractor of $\mathcal{F}$. Arguing the contrary, we suppose that $\mathcal{S}^*$ is not the attractor of $\mathcal{F}$. Then, $H_\rho(\mathcal{S}^*, \mathcal{F}(\mathcal{S}^*)) \ne 0$, i.e., $H_{d_i}(S_i^*, \mathcal{F}(S_i^*)) \ne 0$, where $\mathcal{S}^* = S_1^*, S_2^*, ..., S_N^*$.

Then,

$$\tau + G(H_{d_i}(V_i^{k+1}, \mathcal{F}(S_i^*))) = \tau + G(H_{d_i}(\mathcal{F}(V_i^k), \mathcal{F}(V_i^*)))$$
$$\leq G(W_{\mathcal{F}}(V_i^k, S_i^*)).$$

Thus,

$$\tau + G(H_{d_i}(V_i^{k+1}, \mathcal{F}(S_i^*))) \leq G(W_{\mathcal{F}}(V_i^k, S_i^*)), \tag{41}$$

where

$$\begin{aligned} W_{\mathcal{F}}(V_i^k, X_i^*) = \max \Big\{ & H_{d_i}(V_i^k, S_i^*), H_{d_i}(V_i^k, \mathcal{F}(V_i^k)), H_{d_i}(S_i^*, \mathcal{F}(S_i^*)), \\ & \frac{H_{d_i}(V_i^k, \mathcal{F}(S_i^*)) + H_{d_i}(S_i^*, \mathcal{F}(V_i^k))}{2s}, H_{d_i}(\mathcal{F}^2(V_i^k), \mathcal{F}(V_i^k)), \\ & H_{d_i}(\mathcal{F}^2(V_i^k), S_i^*), H_{d_i}(\mathcal{F}^2(V_i^k), \mathcal{F}(S_i^*)) \Big\} \\ = \max \Big\{ & H_{d_i}(V_i^k, S_i^*), H_{d_i}(V_i^k, V_i^{k+1}), H_{d_i}(S_i^*, \mathcal{F}(S_i^*)), \\ & \frac{H_{d_i}(V_i^k, \mathcal{F}(S_i^*)) + H_{d_i}(S_i^*, V_i^{k+1})}{2s}, H_{d_i}(V_i^{k+2}, V_i^{k+1}), \\ & H_{d_i}(V_i^{k+2}, S_i^*), H_{d_i}(V_i^{k+2}, \mathcal{F}(S_i^*)) \Big\}. \end{aligned}$$

We have following possibilities:

Case I : If $W_{\mathcal{F}}(V_i^k, S_i^*) = H_{d_i}(V_i^k, S_i^*)$, then taking limit infimum as $m \to \infty$ in Equation (41), we have

$$\tau + G(H_{d_i}(S_i^*, \mathcal{F}(S_i^*))) \leq G(H_{d_i}(S_i^*, S_i^*)),$$

i.e.,

$$\tau + G(H_{d_i}(S_i^*, S_i^*)) \leq G(H_{d_i}(S_i^*, S_i^*)),$$

which is a contradiction, since $\tau > 0$.

Case II : If $W_{\mathcal{F}}(V_i^k, S_i^*) = H_{d_i}(V_i^k, V_i^{k+1})$, then by taking limit infimum as $m \to \infty$ in Equation (41), we have

$$\tau + G(H_{d_i}(S_i^*, \mathcal{F}(S_i^*))) \leq G(H_{d_i}(S_i^*, S_i^*)),$$

i.e.,

$$\tau + G(H_{d_i}(S_i^*, S_i^*)) \leq G(H_{d_i}(S_i^*, S_i^*)),$$

a contradiction.

Case III : If $W_{\mathcal{F}}(V_i^k, S_i^*) = H_{d_i}(S_i^*, \mathcal{F}(S_i^*))$, then Equation (41) reduces to

$$\tau + G(H_{d_i}(S_i^*, \mathcal{F}(S_i^*))) \leq G(H_{d_i}(\mathcal{F}(S_i^*), S_i^*)),$$

again a contradiction, since $\tau > 0$.

Case IV : If

$$W_{\mathcal{F}}(V_i^k, S_i^*) = \frac{H_{d_i}(V_i^k, \mathcal{F}(S_i^*)) + H_{d_i}(S_i^*, V_i^{k+1})}{2s},$$

then Equation (41) becomes

$$\tau + G(H_{d_i}(S_i^*, \mathcal{F}(S_i^*))) \leq G\left(\frac{H_{d_i}(S_i^*, \mathcal{F}(S_i^*)) + H_{d_i}(S_i^*, S_i^*)}{2s}\right) = G\left(\frac{H_{d_i}(S_i^*, \mathcal{F}(S_i^*))}{2s}\right),$$

which is not possible, since G is strictly increasing map.

Case V : If $W_{\mathcal{F}}(V_i^k, S_i^*) = H_{d_i}(V_i^{k+2}, V_i^{k+1})$, then from Equation (41), we have

$$\tau + G(H_{d_i}(S_i^*, \mathcal{F}(S_i^*))) \leq G(H_{d_i}(S_i^*, S_i^*)),$$

again a contradiction.

Case VI : If $W_{\mathcal{F}}(V_i^k, S_i^*) = H_{d_i}(V_i^{k+2}, S_i^*)$, then

$$\tau + G(H_{d_i}(S_i^*, \mathcal{F}(S_i^*))) \leq G(H_{d_i}(S_i^*, S_i^*)),$$

again a contradiction.

Case VII : If $W_{\mathcal{F}}(V_i^k, S_i^*) = H_{d_i}(V_i^{k+2}, \mathcal{F}(S_i^*))$, using this in Equation (41), we have

$$\tau + G(H_{d_i}(S_i^*, \mathcal{F}(S_i^*))) \leq G(H_{d_i}(S_i^*, \mathcal{F}(S_i^*)),$$

which is not possible.

Thus, our supposition is wrong. This gives, $H_{d_i}(U_i^*, \mathcal{F}(S_i^*)) = 0$, i.e., $\mathcal{F}(S_i^*) = S_i^*$ for $i = 1, 2, ..., N$. Hence, S_i^* i.e., $\mathcal{S}^*$ is an attractor of $\mathcal{F}$.

Now, we prove the uniqueness of attractor $\mathcal{S}^*$ of $\mathcal{F}$. Indeed, let $\mathcal{S}^*$ and $\mathcal{R}^*$ be two attractors of $\mathcal{F}$ with $H_\rho(\mathcal{S}^*, \mathcal{R}^*) \neq 0$. Then, using the definition of Ciric type generalized multivalued G-contraction $\mathcal{F}$, we have

$$\begin{aligned} \tau + G(H_\rho(\mathcal{S}^*, \mathcal{R}^*)) &= \tau + G(H_\rho(\mathcal{F}(\mathcal{S}^*), \mathcal{F}(\mathcal{R}^*))) \\ &\leq G(W_{\mathcal{F}}(\mathcal{S}^*, \mathcal{R}^*)) \\ \Rightarrow \tau + G(H_\rho(\mathcal{S}^*, \mathcal{R}^*)) &\leq G(W_{\mathcal{F}}(\mathcal{S}^*, \mathcal{R}^*)), \end{aligned} \tag{42}$$

where

$$\begin{aligned} W_{\mathcal{F}}(\mathcal{S}^*, \mathcal{R}^*) = \max \Big\{ & H_\rho(\mathcal{S}^*, \mathcal{R}^*), H_\rho(\mathcal{S}^*, \mathcal{F}(\mathcal{S}^*)), H_\rho(\mathcal{R}^*, \mathcal{F}(\mathcal{R}^*)), \\ & \frac{H_\rho(\mathcal{S}^*, \mathcal{F}(\mathcal{R}^*)) + H_\rho(\mathcal{R}^*, \mathcal{F}(\mathcal{S}^*))}{2s}, H_\rho(\mathcal{F}^2(\mathcal{S}^*), \mathcal{F}(\mathcal{S}^*)), \\ & H_\rho(\mathcal{F}^2(\mathcal{S}^*), \mathcal{R}^*), H_\rho(\mathcal{F}^2(\mathcal{S}^*), \mathcal{F}(\mathcal{R}^*)) \Big\} \\ = \max \{ & H_\rho(\mathcal{S}^*, \mathcal{R}^*), H_\rho(\mathcal{S}^*, \mathcal{S}^*), H_\rho(\mathcal{R}^*, \mathcal{R}^*), \\ & \frac{H_\rho(\mathcal{S}^*, \mathcal{R}^*) + H_\rho(\mathcal{R}^*, \mathcal{S}^*)}{2s}, H_\rho(\mathcal{S}^*, \mathcal{S}^*), \\ & H_\rho(\mathcal{S}^*, \mathcal{R}^*), H_\rho(\mathcal{S}^*, \mathcal{R}^*) \} \\ = H_\rho(& \mathcal{S}^*, \mathcal{R}^*) \end{aligned}$$

$$\Rightarrow \tau + G(H_\rho(\mathcal{S}^*, \mathcal{R}^*)) \leq G(H_\rho(\mathcal{S}^*, \mathcal{R}^*)),$$

which is not possible, since $\tau > 0$. This gives that $H_\rho(\mathcal{S}^*, \mathcal{R}^*) = 0$. *i.e.*, $\mathcal{S}^* = \mathcal{R}^*$. Hence, $\mathcal{F}$ has a unique attractor $\mathcal{S}^*$. □

Remark 4. *If we take $j = 1, 2, ..., N, i = 1$ in Theorem 6, then the result of Chifu et al. (Theorem 3.4, [41]) can be obtained.*

Remark 5. *If we use generalized multivalued G-contraction t_{ij}^k instead of ciric type generalized multivalued G-contraction in Theorem 6 and take j=1,2,...,N, i=1, then the result of Secelean (Theorem 3.1, [42]) can be obtained. In addition, one can generalize his result using Ciric type generalized multivalued G-contraction.*

Remark 6. *Our Theorems 3, 4 and 6 extend the results of Dung et al. [43], which are the revision of the results obtained by Nazir et al. (respectively, Theorems 9, 10 and 15, [37]) by taking $N = 1$.*

4. Applications

The image of objects found in the nature can be reconstructed by using a set of functions. This set of functions is known as iterated function system (IFS). Collage theorem (see [19,53]) enables us to approximate an image by using IFS having a specific attractor that will construct the required image, no matter what initial set is to be taken. With the help of collage theorem, one can solve inverse problems of reconstructing the fractal objects. Barnsley proved the collage theorem for Hausdorff metric space, but here we generalize this concept to a Hausdorff b-metric space, which is more general than Hausdorff metric space and obtain the collage theorem as follows:

Theorem 7. *(Collage Theorem). Suppose that* $(\mathcal{C}(U_i), H_{d_i})$ *are complete Hausdorff b-metric spaces for* $i = 1, 2, ..., N$. *Let* $(\mathcal{C}(\mathcal{U}), H_\rho)$ *be a Hausdorff b-metric space with Hausdorff metric* ρ *and* $\{U_j, j = 1, 2, ..., N; t^l_{ij} \colon U_j \to U_i, l = 1, 2, ..., k_{ij}, i, j = 1, 2, ..., N\}$ *be multi-iterated function systems (MIFS) having contractive factor* r, *where* $r = \max\{r^k_{ij}, k = 1, 2, ..., l_{ij}, i, j = 1, 2, ..., N\}$ *and* $0 \leq r < 1$. *If* $\mathcal{F} \colon \mathcal{C}(\mathcal{U}) \to \mathcal{C}(\mathcal{U})$ *is contractive operator with contractive factor* r *and* $\mathcal{V} \in \mathcal{C}(\mathcal{U})$, *then*

$$H_\rho(\mathcal{V}, \mathcal{S}^*) \leq \frac{1}{1 - sr} H_\rho(\mathcal{V}, \mathcal{F}(\mathcal{V})), \tag{43}$$

where $\mathcal{S}^* \in \mathcal{C}(\mathcal{U})$ *is an attractor of* $\mathcal{F}$.

Proof. Using triangular condition of a b-metric space, we have

$$\begin{aligned} H_{d_i}(V_i, \mathcal{F}^n(V_i)) &\leq s\{H_{d_i}(V_i, \mathcal{F}(V_i)) + H_{d_i}(\mathcal{F}(V_i), \mathcal{F}^n(V_i))\} \\ &\leq sH_{d_i}(V_i, \mathcal{F}(V_i)) + s^2\{H_{d_i}(\mathcal{F}(V_i), \mathcal{F}^2(V_i)) + H_{d_i}(\mathcal{F}^2(V_i), \mathcal{F}^n(V_i))\} \\ &\leq \cdots \leq sH_{d_i}(V_i, \mathcal{F}(V_i)) + s^2 H_{d_i}(\mathcal{F}(V_i), \mathcal{F}^2(V_i)) \\ &\quad + s^3 H_{d_i}(\mathcal{F}^2(V_i), \mathcal{F}^3(V_i)) + \cdots + s^n H_{d_i}(\mathcal{F}^{n-1}(V_i), \mathcal{F}^n(V_i)). \end{aligned}$$

This gives

$$H_{d_i}(V_i, \mathcal{F}^n(V_i)) \leq sH_{d_i}(V_i, \mathcal{F}(V_i)) + s^2 H_{d_i}(\mathcal{F}(V_i), \mathcal{F}^2(V_i)) + \cdots + s^n H_{d_i}(\mathcal{F}^{n-1}(V_i), \mathcal{F}^n(V_i)). \tag{44}$$

Since $\mathcal{F}$ is a contraction operator with contractive factor r, Equation (44) reduces to

$$\begin{aligned} H_{d_i}(V_i, \mathcal{F}^n(V_i)) &\leq (s + s^2 r + s^3 r^2 + ... + s^n r^{n-1}) H_{d_i}(V_i, \mathcal{F}(V_i)) \\ &= \frac{s(1 - sr^{n-1})}{1 - sr} H_{d_i}(V_i, \mathcal{F}(V_i)),\ for\ sr < 1 \\ \Rightarrow H_{d_i}(V_i, \mathcal{F}^n(V_i)) &\leq \frac{s(1 - sr^{n-1})}{1 - sr} H_{d_i}(V_i, \mathcal{F}(V_i)),\ for\ sr < 1. \end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$H_{d_i}(V_i, \mathcal{S}^*) \leq \frac{1}{1 - sr} H_{d_i}(V_i, \mathcal{F}(V_i)),\ for\ i = 1, 2, ..., N.$$

Now,

$$\begin{aligned} \max_{i=1,2,...,N} H_{d_i}(V_i, \mathcal{S}^*) &\leq \max_{i=1,2,...,N} \left(\frac{1}{1 - sr}\right) H_{d_i}(V_i, \mathcal{F}(V_i)) \\ \Rightarrow H_\rho(\mathcal{V}, \mathcal{S}^*) &\leq \frac{1}{1 - sr} H_\rho(\mathcal{V}, \mathcal{F}(\mathcal{V})). \end{aligned} \tag{45}$$

This theorem describes that, if the Hausdorff distance between idealized fractal and collage of the image is small, then the distance of the attractor of our IFS from the fractal will be small. It guarantees that an IFS has a unique attractor. □

5. Conclusions

In this article, a methodology for constructing multi-attractors in multifractal spaces is presented. This methodology not only states complex results, but also one can adopt this methodology to construct attractors or multi-attractors on Hausdorff b-metric spaces. In Section 4, we derive collage theorem for multi-Hutchinson Barnsley operator in Hausdorff b-metric space. Collage theorem can be applied to find a suitable IFS for obtaining desired attractor and solving inverse problem for constructing fractal objects (see Section 5, [53]). With the help of Theorem 7, by choosing suitable contractions, one can generate fractals or multifractals. In addition, for further research, our results give rise to interesting questions and generalizations to construct multifractals (attractors) either by generalizing spaces or contraction mappings. Moreover, we attempt to obtain multifractals analytically with the help of ciric type generalized multivalued G-contractions; however, construction of multifractals is still an open question.

Author Contributions: Conceptualization, S.K. and J.C.; methodology, R.C. and J.C.; validation, S.K., J.C. and C.H.; formal analysis, S.K. and J.C.; investigation, S.K., J.C. and C.H.; writing-original draft preparation, S.K., R.C. and J.C.; writing-review and editing, C.H.; supervision, J.C. and C.H.; project administration, J.C. and C.H.; funding acquisition, J.C. and C.H.

Funding: This work was supported by the National Natural Science Foundation of China (Nos. 11971076, 51839002).

Conflicts of Interest: The authors declare no conflict of interest.

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