mathematics

Review

Review of Some Control Theory Results on Uniform Stability of Impulsive Systems

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Abstract: This paper aims to review some uniform stability results for impulsive systems. For the review, we classify the models of impulsive systems into time-based impulsive systems and state-based ones, including continuous-time impulsive systems, discrete-time impulsive systems, stochastic impulsive systems, and impulsive hybrid systems. According to these models, we review, respectively, the related stability concepts and some representative results focused on uniform stability, including the results on uniform asymptotic stability, input-to-state stability (ISS), $\mathcal{KLL}$-stability (uniform stability expressed by $\mathcal{KLL}$-functions), event-stability, and event-ISS. And we formulate some questions for those not fully developed aspects on uniform stability at each subsection.

Keywords: impulsive systems; stochastic impulsive systems; impulsive hybrid systems; uniform stability; input-to-state stability; event-stability; Lyapunov function; average dwell-time; time-delay

1. Introduction

Impulsive systems have been widely studied in recent years due to the variety of applications in the fields such as mechanics, control technology, communication networks, robotics, biological population dynamics, power systems, etc. Impulsive systems can provide a natural framework for mathematical modelling of many physical phenomena. In impulsive systems, it exhibits simultaneously continuous dynamics (flow) on intervals and discrete dynamical behavior (jump) at impulsive instants. The study on impulsive systems is assuming a greater importance. The first paper where impulses were introduced into differential equations (dynamical systems) is the article written by Milman and Myshkis [1] in 1960. And the first work on stability of impulsive systems was reported in [1].

Generally, impulsive systems can be classified into time-based impulsive systems and state-based ones. The most studied models in the literature belong to the time-based impulsive systems, i.e., the impulsive instants are independent of state and the jumping behavior occurs at these state independent instants. In this review on stability, we further classify the time-based impulsive systems into three types: continuous-time impulsive systems, discrete-time impulsive systems, and stochastic impulsive systems.

For the continuous-time impulsive systems (CIS), the basic mathematical model and some fundamental results on existence and stability including uniform stability of solutions were established by, e.g., Milman, Myshkis, Lakshmikantham, Bainov, Simeonov, and Liu; see, for example, [1–4] and references therein. Recently, more attention has been put on the uniform stability of delayed CIS; see, for example, [5–10] and references therein. The uniform stability results of CIS without time-delays have non-trivially been extended to delayed cases. Most results reported for delayed impulsive systems are derived by using Razumikhin technique [11] and the method of Lyapunov-Krasovskii functional. Very recently, in [12–14], results have been reported on the stability for impulsive systems with state-dependent delays.

Note that impulses or state abrupt jumps may occur in any dynamical systems including discrete-time systems, the model of discrete-time impulsive systems (DIS) was proposed to describe the dynamics with state jumps in discrete-time dynamical systems. DIS was initially proposed in [15] and then it started the stability analysis of DIS; see, for example, [15–23] and references therein. The comparison principles for uniform stability and input-to-state stability (ISS) of DIS in [19,20] are some representative results.

Stochastic impulsive systems (SIS) may be an appropriate description of the phenomena of abrupt qualitative dynamical changes of essentially continuous time systems which are disturbed by stochastic factors. In the literature, some stability results have been reported for systems with Markovian jumps or switchings; see, for example, [24–30]. The stability for SIS has been less reported, although the results on existence and uniqueness and stability of solutions of SIS can be founded in [31–33]. Especially, in [33], the authors formulated a more general SIS, in which the flow, the jump, the switchings of models, and the impulsive instants all may be random. In [32,33], two kinds of stability concepts were defined for SIS: pth moment stability and almost sure stability. The authors of [32,33] showed that the latter is a stronger stability property than the former.

Input-to-state stability (ISS) by Sontag [34] is an important extension of uniform stability in the case of external inputs. ISS analysis has been one of most active research directions in the whole control field. ISS theories for impulsive systems with or without time-delays have been established; see, for example, [35–43] and references therein. Among the ISS results for impulsive systems, the method via average dwell time (ADT) including converse ADT by Hespanha et al. [36] is well-known. The method of ADT has become a basic tool in analyzing stability and ISS of impulsive systems in the literature; see, for example, [37–43] and references therein. However, there have been noted two aspects in using the method via ADT and converse ADT. The one is that it is often hard to test ADT or converser ADT condition in advance, as pointed out in [40]. The other is that it often requires the flow or the jump to be stable. Except the method via ADT and converse ADT, the approaches including small-gain method, Razumikhin-type technique, Lyapunov-Krasovskii functional, and Halanay-type inequalities, have been extended to derive ISS criteria for impulsive systems and impulsive hybrid networks; see, for example, [37–54]. Moreover, for discrete-time impulsive systems, the ISS concept was extended to the case of non-ISS and ISS under events, including input-to-state exponents [43], quasi-ISS [35], and event-ISS [22,43,55–57].

Different from the time-based impulsive systems, for the state-based impulsive systems, the impulsive instants are dependent on state and the jumping behavior may be driven by some state condition(s). The state-based impulsive systems was initially studied by Teel et al. [58–65] and Collins [66]. In the model by Teel et al., there are two sets: C and D. When the state belongs to the set C, the system is evolving along the flow. While the state belongs to D, the system experiences the discrete jump. This state-based impulsive model is also called as hybrid systems in the literature such as [58–65]. For the stability of hybrid systems by Teel et al., the concept of hybrid time domain was proposed. In [60–63], by using the method of (smooth) Lyapunov functions (SLF/LF), there have established some basic results including necessary and sufficient conditions of uniform stability including $\mathcal{KLL}$-stability and robustness. The results have further been extended to the case of ISS by using the method of ISS-Lyapunov function; see [64,65]. The merit of the stability results via SLF/LF approach is that it typically guarantees some form of robustness, as shown in [58–65]. However, in testing the stability for some specific hybrid system, it may not be easy to find a common SLF/LF, especially for a hybrid system with unstable subsystems.

Note that, in the literature, there is still no standard definition given for hybrid systems. Generally, a hybrid system means that the system consists of flow dynamics and discrete jumping behaviors. By such an understanding, the models for hybrid systems may be more general than those for impulsive systems. It includes state-based impulsive (hybrid) systems see, for example, [65,66], switched (hybrid) systems see, for example, [67–72], and time-based impulsive (hybrid) systems see, for example, [73–80].

One of the features for state-based impulsive systems is that it allows multiple jumps at the same impulsive instant. Thus, it is necessary to use both continuous and discrete time (CDT) variables in such systems. The hybrid time domain by Teel et al. provides a tool for the analysis of stability of hybrid systems. However, in the concept of hybrid time domain, it is hard to reflect the information from the switching or jumping events. In [22], the concept of hybrid time was generalized to the notion of hybrid-event-time, in which both CDT variables and the related number of events (switchings or jumps) are included. Based on the hybrid-event-time, the notion of stability under events (event-stability) was proposed and a basic result on necessary and sufficient condition for global uniform asymptotic stability under events (event-GUAS) was derived in [22] for discrete-time switched hybrid systems. In [55–57], more sufficient conditions for event-stability were reported by extending the average dwell time (ADT) [36] to the hybrid ADT. Note that, in the method via the hybrid ADT condition, the requirement of stable flow or stable jump as in the method via ADT [36] can be relaxed. By the hybrid ADT, it can also explain the reason why a hybrid system may be event-GUAS even if all subsystems (flow and jump) are unstable. In addition, the stability theory for hybrid systems has also been extended to hybrid networks with or without time-delays. By employing the approaches such as small-gain method [44–53], vector/multiple Lyapunov functions [67], Razumikhin-type technique [11], Lyapunov-Krasovskii functional [11], and Halanay-type inequalities [48,49], the criteria on stability and ISS for hybrid networks have been reported; see, for example, [50–54].

In the literature, many concepts related to stability issue have been developed for impulsive systems. Besides the above stated uniform stability, ISS, $\mathcal{KLL}$-stability, and event-stability, concepts such as dissipativity [81–85], contraction [86], incremental stability [87], finite-time stability [88,89], stability for impulsive systems with different jump maps [90], stability for abstract impulsive differential equations, conditional stability (dichotomy, trichotomy), and impulsive control see, for example, [91–97], have also been proposed and extensively studied. In this paper, due to the limited space and being limited to the control theory, we do not review the results related to these concepts and stability results.

The organization of this review is as follows. In Section 2, we give preliminaries and some basic models of impulsive systems. In Section 3, we review some stability results for time-based impulsive systems. In Section 4, we review some ISS results for impulsive systems. In Section 5, we review some stability results for impulsive hybrid systems. Section 6 concludes the review.

2. Preliminaries and Models of Impulsive Systems

In the sequel, let $\mathbb{R}$ denote the field of real numbers, $\mathbb{R}_+$ the subset of non-negative elements of $\mathbb{R}$, i.e., $\mathbb{R}_+ = [0, +\infty)$, and $\mathbb{R}^n$ the n-dimensional Euclidean space. $\mathbb{N} = \{0, 1, 2, \cdots\}$. For $k_1, k_2 \in \mathbb{N}$ satisfying $k_1 \le k_2$, denote $\mathcal{N}[k_1, k_2] = \{k : k \in \mathbb{N}, k_1 \le k \le k_2\}$ and $\mathcal{N}[k_1, k_2) = \{k : k \in \mathbb{N}, k_1 \le k < k_2\}$.

Given a matrix A, $\|A\|$ denotes the norm of A induced by the Euclidean vector norm, i.e., $\|A\| = [\lambda_{\max}(A^T A)]^{\frac{1}{2}}$, where $\lambda_{\max}(X)$ is the maximal eigenvalue of matrix X. Let $L_{\infty,loc}$ denote the set of Lebesgue measurable functions $w : \mathbb{R}_+ \to \mathbb{R}^n$ that are locally essentially bounded. For $\forall w \in L_{\infty,loc}$, define $\|w\|_{[s_1,s_2]} \triangleq \operatorname{ess\,sup}_{s_1 \le t \le s_2}\{\|w(t)\|\}$, $\|w\|_\infty \triangleq \operatorname{ess\,sup}_{t \in \mathbb{R}_+}\{\|w(t)\|\}$.

A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class-$\mathcal{K}$ ($\gamma \in \mathcal{K}$) if it is continuous and positive and strictly increasing and $\gamma(0) = 0$. γ is of class-$\mathcal{K}_\infty$ if it is of class-$\mathcal{K}$ and is unbounded. A function $\sigma : \mathbb{R} \to \mathbb{R}_+$ is of class-$\mathcal{L}$ ($\sigma \in \mathcal{L}$) if it is continuous, and monotonically decreasing and $\lim_{s\to\infty} \sigma(s) = 0$. It is of class-$\mathcal{L}_\infty$ if it is of class-$\mathcal{L}$ and $\lim_{s\to+\infty} \sigma(-s) = +\infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class-$\mathcal{KL}$ if $\beta(\cdot, s) \in \mathcal{K}$ for any $s \in \mathbb{R}_+$ and $\beta(a, \cdot) \in \mathcal{L}$ for $a > 0$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class-$\mathcal{KLL}$ if $\forall s_1, s_2 \in \mathbb{R}_+$ and $\forall a > 0$, $\beta(\cdot, s_1, s_2) \in \mathcal{K}$, $\beta(a, s_1, \cdot) \in \mathcal{L}$, and $\beta(a, \cdot, s_2) \in \mathcal{L}$.

A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class-$\mathcal{K}_0$ ($\gamma \in \mathcal{K}_0$) if it is continuous and strictly increasing. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class-$\mathcal{K}\mathcal{K}_0$ if $\beta(\cdot, t)$ is of class-$\mathcal{K}$ for $t \geq 0$ and $\beta(s, \cdot)$ is of class-$\mathcal{K}_0$ for $s \geq 0$.

For any random variable ξ, let $E(\xi)$ be the expectation value of ξ. For $\rho > 0$, let $S_\rho = \{x \in \mathbb{R}^n : ||x|| \leq \rho\}$, and for a random vector $\zeta \in \mathbb{R}^n$, let $\mathcal{S}_\rho = \{\zeta \in \mathbb{R}^n : E(||\zeta||) \leq \rho\}$. Let $\mathcal{K}_1 = \{\phi \in \mathcal{K} : \phi(E(s)) \geq E(\phi(s)), E(s) < +\infty, \text{for all random variable } s \geq 0\}$, $\mathcal{K}_2 = \{\phi \in \mathcal{K} : \phi(E(s)) \leq E(\phi(s)), E(s) < +\infty, \text{ for all random variable } s \geq 0\}$, and $\mathcal{K}_{2\infty} = \{\phi \in \mathcal{K} : \phi \in \mathcal{K}_\infty \wedge \mathcal{K}_2\}$. Let PC be the class of functions $p : \mathbb{R}_+ \to \mathbb{R}_+$, where p is continuous everywhere except t_k, $k \in \mathbb{N}$, at which p is left continuous and the right limit $p(t_k^+)$ exists.

Given a constant $\tau > 0$, which often represents an upper bound on the time delays, the linear space $PC([-\tau, 0], \mathbb{R}^n)$ with the norm $\|\cdot\|_\tau$ is defined by $\|\psi\|_\tau = \sup_{-\tau \leq s \leq 0} \|\psi(s)\|$. For function ψ and $m \in \mathbb{N}$ with $m \geq 1$, let $\psi^m \triangleq \underbrace{\psi \circ \cdots \circ \psi}_{m}$.

For the models of impulsive systems, the general form is "flow + jump". In an impulsive system, the "flow" is a process along which the state evolves uninterruptedly while "jump" means that state changes abruptly from the "flow" at some impulsive instants. According to whether the impulsive instants are dependent of state, impulsive systems can be roughly classified into time-based and state-based ones. In the following, we give the models of time-based and state-based impulsive systems.

For the time-based impulsive systems, the impulsive instants are independent of state. In the literature, there reported three types of time-based impulsive systems, i.e., continuous-time impulsive systems (CIS) see, for example, [1–3], discrete-time impulsive systems (DIS) see, for example, [15], and stochastic impulsive systems (SIS) see, for example, [31–33].

From [1–4], a continuous-time impulsive system (CIS) is in form of

$$\begin{cases} \dot{x} = f(t, x), \ t \notin T = \{t_k\}, \\ \Delta x = I_k(x), \ t = t_k \in T, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$; $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, satisfying $f(t, 0) = I_k(0) = 0$, $\forall t \in \mathbb{R}_+$; $\Delta x = x(t_k^+) - x(t_k^-)$ is the impulse of state $x(t)$ at t_k; $T = \{t_k\}$ is the sequence of impulsive instants satisfying

$$0 \leq t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots, \ \lim_{k \to \infty} t_k = +\infty. \tag{2}$$

From [15], a discrete-time impulsive system (DIS) can be modelled from a discrete-time dynamical system $x(k+1) = f(k, x) + g(k, x)u(k)$ under impulsive control [91,92] in the form of $u(k) = \sum_{i=1}^{\infty} \delta(k - N_i) h_i(k, x(k))$, where $\delta(t) = \begin{cases} 1, & t = 0, \\ 0, & t \neq 0. \end{cases}$ Thus, as in [15–21], DIS can be described as:

$$\begin{cases} x(k+1) = f(k, x(k)), \quad n \neq N_i, \\ \Delta x(k+1) = I_i(k, x(k)), \quad n = N_i, \\ x(N_0) = x_0, \end{cases} \tag{3}$$

where $x \in \mathbb{R}^n, \Delta x(k+1) = x(k+1) - x(k)$, $f, I_i \in C(\mathbb{N} \times \mathbb{R}^n, \mathbb{R}^n)$ satisfying $f(k, 0) \equiv 0, I_i(k, 0) \equiv 0, k, i \in \mathbb{N}$, and the discrete-time impulsive sequence $\{N_i : N_i \in \mathbb{N}\}$ satisfies:

$$0 \leq N_0 < N_1 < N_2 < \cdots < N_i < \cdots, \ \lim_{i \to \infty} N_i = \infty, \ N_{i+1} - N_i > 1. \tag{4}$$

By [31–33], stochastic impulsive system (SIS) can be used as an appropriate description of the phenomena of abrupt qualitative dynamical changes of essentially continuous time systems which are disturbed by stochastic factors. By [32], a SIS is in the form of

$$\begin{cases} \mathrm{d}x(t) = f(t, x(t))\mathrm{d}t + g(t, x(t))\mathrm{d}B(t), \ t \notin T = \{t_k\}, \\ \Delta x(t) = I_k(x(t)), \ t = t_k \in T, \ k \in \mathbb{N}, \end{cases} \tag{5}$$

where $x(t) \in \mathbb{R}^n, x(t_0^+) = x_0, \Delta x(t_k) = x(t_k{}^+) - x(t_k)$; $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ with $f(t, 0) = 0, g(t, 0) = 0$, for all $t \in \mathbb{R}_+$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies $I_k(0) = 0$; $B(t) = (B_1(t), \cdots, B_m(t))^T$ is an m-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ containing all P-null sets); and the impulsive time sequence $\{t_k\}$ satisfies (2).

For the state-based impulsive systems, the impulsive instants are dependent on the state. In the literature, e.g., [58–63], there reported such a type of impulsive systems in the form of

$$\begin{cases} \dot{x} = f(x), \ x \in C, \\ x^+ = g(x), \ x \in D, \end{cases} \tag{6}$$

where C and D are two closed sets. Note that the system in Equation (6) is also called as hybrid system in the literature, e.g., [58].

For the convenience in reading the following sections, we give a structure of impulsive systems and the related uniform stability concepts in Figures 1 and 2:

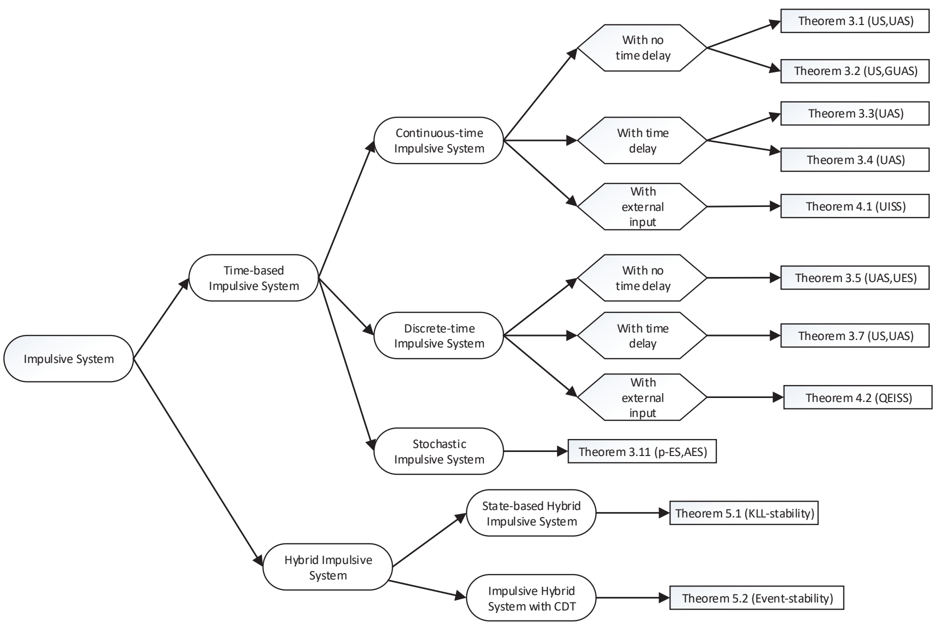

Figure 1. Structure of impulsive systems in the review.

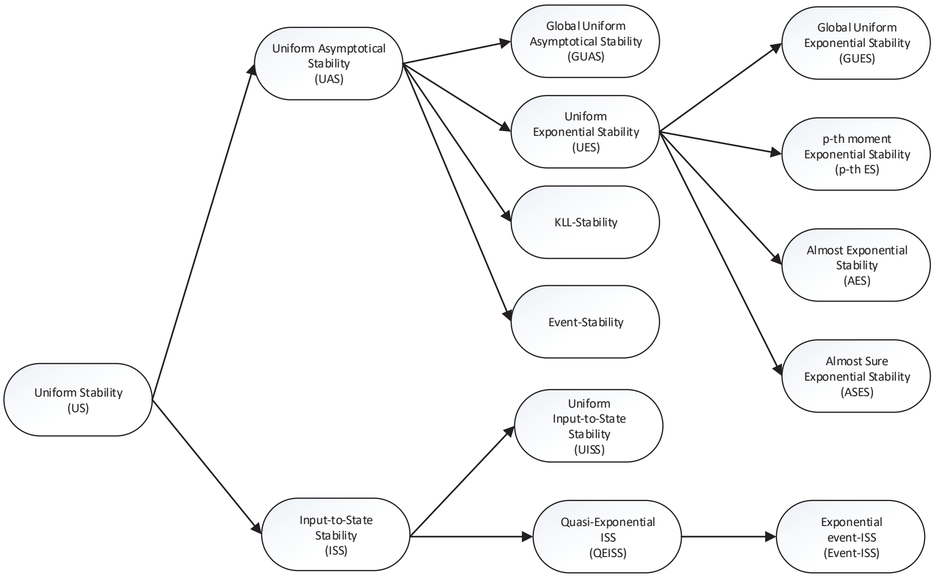

Figure 2. Structure of uniform stability in the review.

3. Review on Uniform Stability of Time-based Impulsive Systems

In the literature, the study on stability of impulsive systems is started from the continuous-time impulsive systems (CIS) (1), see, e.g., [1–4,98]. The earlier and fundamental work on stability of impulsive systems was established by Milman, Myshkis, Lakshmikantham, Bainov, Simeonov, Liu, etc., see [1–4]. After these basic work on stability, there have developed many stability resultes and analysis methods on time-based impulsive systems, including uniform stability for delayed impulsive systems, discrete-time impulsive systems (3), and stochastic impulsive systems (5). In this section, we review some representative results on uniform stability of these time-based impulsive systems.

3.1. Stability of Continuous-Time Impulsive Systems

For the stability analysis of CIS (1), it is often assumed that the solution of (1) exists uniquely and is forward complete. Let $x(t) = x(t, x_0, t_0)$ denote the solution of (1) with initial condition $x(t_0) = x_0$. The solution $x(t)$ of (1) is assume to be left-continuous, i.e., $x(t) = x(t^-)$ for all $t \geq t_0$. Note that in some literature, $x(t)$ was assume to be right-continuous, i.e., $x(t) = x(t^+)$ for all $t \geq t_0$.

The notion of stability for (1) in the literature see, e.g., [1,2], is in the sense of Lyapunov stability. The notions on uniform stability are defined as:

Definition 1. *(i) The system in Equation (1) is said to be uniformly stable (US), if* $\forall \epsilon > 0$, *there exists* $\delta = \delta(\epsilon) > 0$ *such that* $\|x(t)\| < \epsilon$ *holds for* $\forall x_0 \in \mathbb{R}^n$ *with* $\|x_0\| < \delta$, *and all* $t \geq t_0$.
(ii) The system in Equation (1) is said to be uniformly asymptotically stable (UAS), if it is US and there exists $\sigma > 0$ *such that* $\lim_{t\to\infty} \|x(t)\| = 0$ *holds uniformly for all initial conditions* (t_0, x_0) *with* $\|x_0\| \leq \sigma$.
(iii) The system in Equation (1) is said to be globally uniformly asymptotically stable (GUAS) if it is UAS and $\lim_{t\to\infty} \|x(t)\| = 0$ *holds uniformly for* $\forall (t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$.
(iv) The system in Equation (1) is said to be uniformly exponentially stable (UES), if $\forall \epsilon, \exists \delta(\epsilon) > 0, \exists \alpha > 0$, $\|x(t)\| \leq \epsilon e^{-\alpha(t-t_0)}$ *holds for* $\|x_0\| \leq \delta, \forall t \geq t_0$. *The system in Equation (1) is said to be globally uniformly exponentially stable (GUES), if* $\forall x_0 \in \mathbb{R}^n$, $\|x(t)\| \leq Ke^{-\alpha(t-t_0)}\|x_0\|$ *holds for some* $\alpha > 0$, $K > 0$ *and* $\forall t \geq t_0$.

From the literature, the method of Lyapunov function is a basic tool in analysing the stability issue of (1). Since the solution of (1) is continuous except at the sequence $T = \{t_k\}$ at which $x(t)$ is left-continuous, the Lyapunov-like function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is required to belong to a class of function ν_0 defined as:

Definition 2. *[1,2] V is said to belong to class ν_0 if*
(i) V is continuous in $(t_{k-1}, t_k] \times \mathbb{R}^n$ and satisfies for each $x \in \mathbb{R}^n$,

$$\lim_{(t,y)\to(t_k^+,x)} V(t,y) = V(t_k^+, x);$$

(ii) $V(t,x)$ is locally Lipschitzian in x.
For $V \in \nu_0$, we give the definition of Dini derivative of V as:

$$D^+V(t,x) = \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, x+h(f(t,x))) - V(t,x)].$$

The results on stability have been established for time-based CIS (1); see, e.g., [1–4,98–102]. Here, we review the following two main results:

Theorem 1. *[4] Assume that there exists a $V \in \nu_0$, satisfying:*
(i) for some $\phi_1, \phi_2 \in \mathcal{K}_\infty$, $\phi_1(\|x\|) \le V(t,x) \le \phi_2(\|x\|)$, $\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$;
(ii) there exist $c \in \mathcal{K}$ and $p \in PC$ such that $D^+V|_{(1)} \le p(t) \cdot c(V(t,x))$, $t \notin T$;
(iii) for some $\psi_k \in \mathcal{K}_0$, at $t = t_k \in T$, $V(t_k^+, x_k + I_k(x_k)) \le \psi_k(V(t_k, x_k))$;
(iv) there exist constants $\sigma > 0$ and $r_k \ge 0$, $k \in \mathbb{N}$, such that for all $z \in (0, \sigma)$

$$\int_{t_k}^{t_{k+1}} p(s)ds + \int_z^{\psi_k(z)} \frac{ds}{c(s)} \le -r_k.$$

Then the system in Equation (1) is US, and GUAS if in addition $\sum_{k=1}^{\infty} r_k = +\infty$.

Theorem 2. *[4] Assume that conditions (i) and (iii) of Theorem (1) hold while (ii) and (iv) are replaced respectively by:*
(iii) there exist $c \in \mathcal{K}, p \in PC$ such that $D^+V|_{(1)} \le -p(t) \cdot c(V(t,x))$, $t \notin T$;*
(iv) there exist constants $\sigma > 0$ and $r_k \ge 0$ such that for all $z \in (0, \sigma)$*

$$-\int_{t_k}^{t_{k+1}} p(s)ds + \int_z^{\psi_k(z)} \frac{ds}{c(s)} \le -r_k$$

Then system in Equation (1) is US, and GUAS if in addition $\sum_{k=1}^{\infty} r_k = +\infty$.

Review 1. (i) Theorems 1 and 2 are two classical and representative results for impulsive systems in the form of (1). Condition (ii) of Theorem 1 shows the flow in Equation (1) may be unstable while the jump is stable by Condition (iii). Condition (iv) is the combination of destabilizing flow and stabilizing jump. On the contrary, in Theorem 2, the flow is stable while the jump may be stable or unstable. Hence, Theorems 1 and 2 are based on three cases: stabilizing flow + stabilizing jump, stabilizing flow + destabilizing jump, destabilizing flow + stabilizing jump. Theorems 1 and 2 show that if the combination is nonpositive, i.e., the stabilizing jump/flow can overcome the destabilizing flow/jump, then the impulsive system will achieve the stability property.

(ii) From Theorems 1 and 2, one can derive the results on GUES if in Theorem 1 and Theorem 2, $\phi_i(s) = c_i s^m$, $p(s) = p$, $c(s) = s$, $\psi_k(s) = e^q s$, $r_k = r$, $\forall s \ge 0$, $\forall k \in \mathbb{N}$, $c_1 \le c_2$, $m > 0$, $p \ge 0$, $q \in \mathbb{R}$. Moreover, Condition (iv) (also (iv*)) implies a condition of dwell time in the form of nonlinearity.

Let $N(s,t]$ denote the number of impulse times in the interval $(s,t]$. In the case of GUES, by Condition (iv) of Theorem 1, we get that $q < -r$ and the following dwell time condition holds:

$$p(t-s) + (q+r)N(s,t] \le -(q+r),\ \forall t > s \ge t_0. \tag{7}$$

Similarly, in the case of GUES, Condition (iv*) means the following converse dwell time condition:

$$-p(t-s) + (q+r)N(s,t] \le 0,\ \forall t > s \ge t_0. \tag{8}$$

Note that from the conditions (7) and (8), the well-known average dwell time (ADT) including converse ADT can be derived, see (27)–(29) in Section 4. The ADT including the converse ADT for impulsive systems was proposed by Hespanha et al. [36] in the analysis of input-to-state stability (ISS). Now, it is widely used in the analysis and control design based on stability and ISS.

(iii) Except the above dwell time method, based on results on stability of impulsive systems as in Theorems 1 and 2, many applications have been reported including robust stability see, e.g., [101,102], dissipativity see, e.g., [98], impulsive synchronization see, e.g., [103–105].

Recently, more attentions have been put on the stability issue of delayed impulsive systems. Here, we review two results on uniform stability of delayed impulsive systems. For the details, refer to [5,6].

A delayed impulsive system is often in the form of

$$\begin{cases} \dot{x}(t) = f(t, x_t), & t \in [t_{k-1}, t_k), \\ \Delta x(t) = I_k(t, x_{t^-}), & t = t_k, k \in \mathbb{N}, \\ x_{t_0} = \phi, \end{cases} \tag{9}$$

where $f, I_k : \mathbb{R}_+ \times PC([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n; \phi \in PC([-\tau, 0], \mathbb{R}^n)$ is some initial function; the impulsive time instants $\{t_k\}$ satisfy (2); $\triangle x(t) = x(t^+) - x(t^-)$; and $x_t, x_{t^-} \in PC([-\tau, 0], \mathbb{R}^n)$ is defined by $x_t(s) = x(t+s), x_{t^-}(s) = x(t^- + s)$ for $-\tau \le s \le 0$.

Theorem 3. *[5,6] Assume that there exist functions $V(t,x) \in \nu_0$, $\phi_1, \phi_2, c \in \mathcal{K}_\infty$, $p \in P(\mathbb{R}_+, \mathbb{R}_+)$, $g \in \mathcal{K}$, such that the following conditions hold:*
(i) $\phi_1(\|x\|) \le V(t,x) \le \phi_2(\|x\|), \forall (t,x) \in [-\tau, \infty) \times S_\rho$;
(ii) $D^+V(t, \varphi(0)) \le p(t)c(V(t, \varphi(0))),\ \forall t \ne t_k$ in $\mathbb{R}_+$ and $\varphi \in PC([-\tau, 0], S_\rho)$ whenever $V(t, \varphi(0)) \ge g(V(t+s, \varphi(s)))$ for $s \in [-\tau, 0]$;
(iii) $V(t_k, \varphi(0) + I_k(t_k, \varphi)) \le g(V(t_k^-, \varphi(0))), \forall (t_k, \varphi) \in \mathbb{R}_+ \times PC([-\tau, 0], S_{\rho_1})$ for which $\varphi(0^-) = \varphi(0)$; and
(iv) $\tau = \sup\limits_{k \in \mathbb{N}} \{t_k - t_{k-1}\} < \infty$, $M_1 = \sup\limits_{t \ge 0} \int_t^{t+\tau} p(s)ds < M_2 = \inf\limits_{q>0} \int_{g(q)}^{q} \frac{ds}{c(s)}$.

Then the system in Equation (9) is UAS.

Theorem 4. *[5,6] Assume that there exist functions $V(t,x) \in \nu_0$, $\phi_1, \phi_2, c \in \mathcal{K}_\infty$, $p \in P(\mathbb{R}_+, \mathbb{R}_+)$, $g \in \mathcal{K}$ and $\hat{g} \in \mathcal{K}$ satisfying $s \le \hat{g}(s) < g(s), \forall s > 0$, such that the following conditions hold:*
(i) $\phi_1(\|x\|) \le V(t,x) \le \phi_2(\|x\|), \forall (t,x) \in [-\tau, \infty) \times S_\rho$;
(ii) $D^+V(t, \varphi(0)) \le -p(t)c(V(t, \varphi(0)))$, for $\forall t \ne t_k$, and $\varphi \in PC([-\tau, 0], S_\rho)$ whenever $V(t, \varphi(0)) \ge g(V(t+s, \varphi(s)))$ for $s \in [-\tau, 0]$;
(iii) $V(t_k, \varphi(0) + I_k(t_k, \varphi)) \le \hat{g}(V(t_k^-, \varphi(0))), \forall (t_k, \varphi) \in \mathbb{R}_+ \times PC([-\tau, 0], S_{\rho_1})$ for which $\varphi(0^-) = \varphi(0)$;
(iv) $\mu = \inf\limits_{k \in \mathbb{N}} \{t_k - t_{k-1}\} > 0$, $M_1 = \inf\limits_{t \ge 0} \int_t^{t+\tau} p(s)ds > M_2 = \sup\limits_{q>0} \int_{g(q)}^{q} \frac{ds}{c(s)}$.

Then the system in Equation (9) is UAS.

Review 2. (i) Compared with Theorems 1 and 2, one can see that Theorems 3 and 4 are the extensions of Theorems 1and 2 respectively to the case of time-delays. It should be noted that these extensions to the case of time-delays are non-trivially. Moreover, most reported uniform stability results, see, e.g., [5–10], for delayed impulsive systems were based on the three cases: stabilizing flow + stabilizing jump, stabilizing flow + destabilizing jump, destabilizing flow + stabilizing jump, as pointed out in Review 1.

(ii) Most results reported for delayed impulsive systems are derived by using Razumikhin technique [11] and the method of Lyapunov-Krasovskii functional. Recently, the Razumikhin-type stability results see, e.g., [5–7], including Theorems 3 and 4 have been extended to a more general delayed impulsive system [8] in the form of

$$\begin{cases} \dot{x}(t) = f(t, x_t),\ t > t_0,\ t \neq t_k, \\ \Delta x(t) = g_k(x_{t^-}, x((t-d_k)^-)),\ t = t_k, k \in \mathbb{N}, \\ x_{t_0+\theta} = \phi(\theta),\ \theta \in [-\tau, 0] \end{cases} \tag{10}$$

where functions f and g_k satisfy Assumptions (A_1)–(A_6) in [8]. The exponential stability results of (10) via Razumikhin technique were derived and the analysis on how the delayed impulses affect the stability was given, see Theorems 1 and 2 in [8]. Furthermore, in [10], the uniform stability including uniform stability, uniform asymptotic stability, and exponential stability was extended to delayed impulsive systems with *delay dependent impulses*:

$$\begin{cases} \dot{x}(t) = f(t, x_t),\ t > t_0,\ t \neq t_k, \\ x(t_k) = g_k(t_k^- - \tau, x(t_k^- - \tau)),\ \tau = \tau(t_k, x(t_k^-)), k \in \mathbb{N}, \\ x_{t_0} = \phi. \end{cases} \tag{11}$$

It should be noted that in the classical delayed impulsive systems, the time-delay function is often independent of the state. Hence, the recent results in [10] may provide some new points on stability effected by time-delays for impulsive systems.

Summary and Question 3.1. From the reported stability results, all most results on the stability and robust stability of continuous-time impulsive systems with or without time-delays are based on three basic cases as in Theorems 1–4:

Case-I: stabilizing flow + stabilizing jump;

Case-II: stabilizing flow + destabilizing jump;

Case-III: stabilizing jump + destabilizing flow.

Naturally, there are two challenging questions for stability of impulsive systems. The one is for the forth case: Case-IV: destabilizing flow + destabilizing jump. How about the stability of impulsive systems in the case of Case-IV? The second question is that the reported results on stability of impulsive systems with or without time-delays are sufficient and fewer necessary or necessary and sufficient conditions of stability are reported.

3.2. Stability of Discrete-Time Impulsive Systems

In the literature, it has been noted that impulses or state abrupt changes may occur in any dynamical systems including discrete-time systems. The model of discrete-time impulsive systems (DIS) was also proposed and there has started the stability analysis of DIS (3), see, e.g., [15–23].

For the stability of DIS (3), it is often assumed that the solution of (3) exists globally and uniquely on $\mathbb{N}$. Let $x(k) = x(k, N_0, x_0)$ denote the solution of (3) with initial condition $x(N_0) = x_0$.

In [16], the stability conditions were given for DIS (3), where $f(k, x(k)) = F(k, x(k)) + \varphi(k, x(k))$ and $I_i(k, x(k)) = B_{N_i} x(N_i)$, under the following assumptions:

(A1) $F : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying: $\forall k \in \mathbb{N}$, $F(k, 0) = 0$, and there exist matrices $A_k \in \mathbb{R}^{n\times n}$ such that $|F(k, x) - F(k, y)| \leq A_k |x - y|$.

(A2) $\varphi : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$ is an uncertain function with $\|\varphi(k, x)\| \leq \sigma_k \|x\|$, for some constant $\sigma_k > 0, k \in \mathbb{N}$.

Theorem 5. *[16] Suppose that Assumptions (A1) and (A2) hold. Furthermore, assume that there exist matrix $P > 0$ and $\varepsilon_j > 0, j \in \mathbb{N}$, such that $\sum_{j=0}^{\infty} \ln \gamma_j = -\infty$, where $\gamma_j = \begin{cases} \alpha_j, & \text{if} \quad j \neq N_i, \\ \beta_{N_i}, & \text{if} \quad j = N_i, i \in \mathbb{N}, \end{cases}$ where $\alpha_j = \frac{\lambda_{\max}\left(A_j^T|P|A_j\right)+\varepsilon_j\lambda_{\max}\left(A_j^T|P|^2A_j\right)+(\varepsilon_j^{-1}+1)\sigma_j^2\lambda_{\max}(P)}{\lambda_{\min}(P)}$, $\beta_j = \lambda_{\max}\left(P^{-1}(I+B_j)^T P(I+B_j)\right)$. Then, the system in Equation (3) is UAS. In addition, (3) is UES if if for some $\lambda > 0$, $\sum_{j=0}^{i} \ln \gamma_j \leq -\lambda i, i \in \mathbb{N}$.*

Review 3. The condition $\sum_{j=0}^{\infty} \ln \gamma_j = -\infty$ in Theorem 5 is respect to the condition $\sum_{k=1}^{\infty} r_k = +\infty$ for continuous-time impulsive systems in Theorems 1 and 2. Thus, Theorem 5 can be viewed as the version of Theorems 1–2 in the case of discrete-time. Note that the conditions in Theorem 5 for DIS (3) are still based on Case-I to Case-III mentioned in Summary and Question 3.1. Compared with the results in the literature, Theorem 5 generalized the results on stability reported in [15].

For delayed discrete-time impulsive systems, the stability issue was also investigated, see, e.g., [18–20]. Here, we review the results on uniform stability for delayed DIS in [19]:

Consider the following three delayed DIS:

$$S_1: \begin{cases} x(k+1) = f_c(k, x(k), x_k),\ k \neq N_i, \\ \Delta x(k+1) = f_d(k, x(k), x_k),\ k = N_i, \\ x(k_0) = \phi_x,\ k \in \mathbb{N}, k \geq k_0, \end{cases} \tag{12}$$

$$S_2: \begin{cases} w(k+1) = g_c(k, w(k), w_k),\ k \neq N_i, \\ \Delta w(k+1) = g_d(k, w(k), w_k),\ k = N_i, \\ w(k_0) = \phi_w,\ k \in \mathbb{N}, k \geq k_0, \end{cases} \tag{13}$$

$$S_3: \begin{cases} r(k+1) = h_c(k, r(k), r_k),\ k \neq N_i, \\ \Delta r(k+1) = h_d(k, r(k), r_k),\ k = N_i, \\ r(k_0) = \phi_r,\ k \in \mathbb{N}, k \geq k_0, \end{cases} \tag{14}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}_+^m$, $r \in \mathbb{R}_+$; f_c, $f_d \in C(\mathbb{N} \times \mathbb{R}^n \times C([-\tau, 0], \mathbb{R}^n), \mathbb{R}^n)$; g_c, $g_d \in C(\mathbb{N} \times \mathbb{R}_+^m \times C([-\tau, 0], \mathbb{R}_+^m), \mathbb{R}_+^m)$; h_c, $h_d \in C(\mathbb{N} \times \mathbb{R}_+ \times C([-\tau, 0], \mathbb{R}_+), \mathbb{R}_+)$; $\phi_x \in C([-\tau, 0], \mathbb{R}^n)$, $\phi_w \in C([-\tau, 0], \mathbb{R}^m)$, $\phi_r \in C([-\tau, 0], \mathbb{R}_+)$.

Assume $f_c(k, 0, 0) \equiv 0$, $f_d(k, 0, 0) \equiv 0$, $g_c(k, 0, 0) \equiv 0$, $g_d(k, 0, 0) \equiv 0$, $h_c(k, 0, 0) \equiv 0$, and $h_d(k, 0, 0) \equiv 0$ so that every system S_1-S_3 admits the trivial solution. Here, also assume that every system S_1-S_3 has an unique solution, respectively, denoted by $x(k) = x(k, k_0, \phi_x)$, $w(k) = w(k, k_0, \phi_w)$, $r(k) = r(k, k_0, \phi_r)$, for any given initial data: $k_0 \in \mathbb{N}$, $\phi_x \in C([-\tau, 0], \mathbb{R}^n)$, $\phi_w \in C([-\tau, 0], \mathbb{R}_+^m)$, and $\phi_r \in C([-\tau, 0], \mathbb{R}_+)$.

A vector function $l(r) = (l_1(r), \cdots, l_m(r))^T : \mathbb{R}_+ \to \mathbb{R}_+^m$ is of class-$\mathcal{K}_m\mathcal{R}$ ($l \in \mathcal{K}_m\mathcal{R}$) if $l(r) \in C(\mathbb{R}_+, \mathbb{R}_+^m)$, $l_i(0) = 0$, $l_i(r) > 0, r > 0$, and $l_i(r) \to \infty$, when $r \to +\infty$, $i = 1, 2, \cdots, m$.

Theorem 6. *[19] Assume that functions $g_c(i, v, v_k)$, $g_d(i, v, v_k)$ are nondecreasing with respect to v, v_k for any $i \in \mathbb{N}$, and furthermore suppose that there are functions $V(k, x) \in C(\mathbb{N} \times \mathbb{R}^n, \mathbb{R}_+^m)$ and $l(r) \in \mathcal{K}_m\mathcal{R}$, with $l(a+b) \geq l(a) + l(b), \forall a, b \in \mathbb{R}_+$, such that the following conditions hold:*
(i) for any $k \neq N_i$, then $V(k+1, x(k+1)) \leq g_c(k, V(k, x(k)), V_k)$, where $V_k : \mathbb{N}_{-\tau} \to \mathbb{R}^m$ and $V_k(s) = V(k+s, x_k(s))$ for any $s \in \mathbb{N}_{-\tau}$;
(ii) for $k = N_i$, then $\Delta V(k+1, x(k+1)) \leq g_d(k, V(k, x(k)), V_k)$, where $\Delta V(k+1, x(k+1)) = V(k+1, x(k+1)) - V(k, x(k))$;
(iii) for any $k \neq N_i$, then $g_c(k, l(a), (l \circ r)_k) \leq l(h_c(k, a, r_k))$, $a \in [0, r^*]$, where r^* is some positive constant or $r^* = +\infty$, and $r : \mathbb{R}_+ \to \mathbb{R}_+$, and $(l \circ r)_k(s) = l(r(k+s))$ for any $s \in \mathbb{N}_{-\tau}$;
(iv) for any $k = N_i$, then $g_d(k, l(a), (l \circ r)_k) \leq l(h_d(k, a, r_k))$, $a \in [0, r^*]$;
(v) for any $(k_0, \phi_r) \in \mathbb{N} \times C([-\tau, 0], [0, r^*])$, then $r(k, k_0, \phi_r) \in [0, r^*]$, $k \in \mathbb{N}$.

Then, $V_{\phi_x} \leq \phi_w \leq l \circ \phi_r$ *implies that*

$$V(k, x(k)) \leq w(k) \leq l(r(k)), \quad k \geq k_0, k \in \mathbb{N},$$

where $V_{\phi_x}(s) = V(k_0 + s, \phi_x(s))$ *for any* $s \in \mathbb{N}_{-\tau}$.

Theorem 7. *[19] Assume that systems (12–14) satisfy all conditions of Theorem 6 and there exist functions* $\varphi_1, \varphi_2 \in \mathcal{K}$ *such that* $\varphi_1(\|x\|) \leq \|V(k,x)\| \leq \varphi_2(\|x\|)$, $k \geq k_0 - \tau$, *then, the US (UAS) properties of the system in Equation (14) implies that the same US (UAS) properties of the system in Equation (12). Moreover, if* $D(\phi_r) = \{\phi_r : \|\phi_r\|_\tau \leq r^*\}$ *is an attractive region of the system in Equation (14),* $D(\phi_x)$ *is an attractive region of the system in Equation (12), where* $D(\phi_x) = \{\phi_x : V_{\phi_x} \leq \sup_{0 \leq \|\phi_r\|_\tau \leq r^*}\{l \circ \phi_r\}, k_0 \in \mathbb{N}\}$, *where* $V_{\phi_x}(s) = V(k_0 + s, \phi_x(s))$ *for any* $s \in \mathbb{N}_{-\tau}$.

Theorem 8. *[19] Assume that systems (12–14) satisfy all conditions of Theorem 6 and* $V(k,x)$ *and* $l(r)$ *satisfy the following conditions:*
(i) for some $\lambda_1 > 0, \lambda_2 > 0, p > 0$, $\lambda_1\|x\|^p \leq \|V(k,x)\| \leq \lambda_2\|x\|^p, k \geq k_0 - \tau$;
(ii) there exist $q \geq 1$ *and* $\Theta_1, \Theta_2 \in \mathbb{R}^m_+$, *where* $\Theta_1 = \begin{pmatrix} c_1 & c_2 & \cdots & c_m \end{pmatrix}^T$, $\Theta_2 = \begin{pmatrix} d_1 & d_2 & \cdots & d_m \end{pmatrix}^T$, *with* $c_i > 0, d_i > 0, i = 1, 2, \cdots, m$, *such that*

$$r^q\Theta_1 \leq l(r) \leq r^q\Theta_2, \quad r \in [0, r^*].$$

Then, the UES of the system in Equation (14) implies that the UES of the system in Equation (12). Moreover, if $D(\phi_r) = \{\phi_r : \|\phi_r\|_\tau < (r^*)^{\frac{1}{q}}\}$ *is an attractive region of the system in Equation (14), then* $E(\phi_x)$ *is an attractive region of the system in Equation (12), where*

$$E(\phi_x) \triangleq \{\phi_x : V_i(k_0 + s, \phi_x(s)) < c_i r^*, s \in \mathbb{N}_{-\tau}, k_0 \in \mathbb{N}, i = 1, 2, \cdots, m\},$$

where $V = (V_1, V_2, \cdots, V_m)^T$.

Review 4. (i) It is known that the comparison principle is one of basic methods for the analysis of stability of dynamical systems. Theorem 6 reported by [19] established a general comparison principle for the stability of delayed discrete-time impulsive systems. The extensions to ISS-type comparison principle for delayed DIS and networks was derived in [20]. Theorem 7 shows that the specific attractive region for UAS of (12) can be derived from the attractive region for UAS of (14). Theorem 8 is the case of GUES for Equations (13)–(14).

(ii) The comparison principle is efficient in some stability analysis, but it is often conservative. The conservative lies in that the dimension of the original dynamical system is lowered and that the conditions in the comparison principle are sufficient (not necessary). Here, in Theorem 6, the system in Equation (14) is scalar and 1-dimension while (12) can be higher n-dimension. Hence, the results in Theorems 6–8 may have some conservativeness.

Summary and Questions 3.2. Compared with continuous-time impulsive systems, there are relatively fewer stability results reported for discrete-time impulsive systems with or without time-delays, although the stability condition in Theorem 5 and the general comparison principle in Theorems 6–8 have been established. It needs further more investigations on the issues such as less conservative stability conditions, and the similar questions arisen from stability of continuous-time impulsive systems as pointed out in Summary and Question 3.1.

3.3. Stability of Stochastic Impulsive Systems

For stochastic impulsive system (SIS), the stability theory has not yet been fully developed, although in the literature, some studies have been reported on systems with Markovian jumps, see, e.g., [24–30]. The earlier work on stability of SIS can be found in [31,32]. Here, we review the stability results via comparison principle for SIS proposed in [32] and some further advances for SIS in [33].

For SIS (5), in [32], it is assumed that $0 < \Delta_{inf} = \inf_{k\in\mathbb{N}}\{t_{k+1} - t_k\} \leq \Delta_{sup} = \sup_{k\in\mathbb{N}}\{t_{k+1} - t_k\} < \infty$. Let $x(t) = x(t, t_0, x_0)$ be the solution of SIS (5) with initial condition $x(t_0^+) = x_0$. The basic notions on stability and the related Itô operator of SIS (5) were given in [30,32,106] as:

Definition 3. *[32] (i) SIS (5) is said to be stable if for given $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, t_0)$ such that for any solution $x(t)$ of (5), $E(\|x_0\|) < \delta$ implies $E(\|x(t)\|) < \epsilon$, $t \geq t_0$,*
(ii) SIS (5) is said to be asymptotically stable if it is stable and for $t_0 \in \mathbb{R}_+$, $\exists \sigma = \sigma(t_0)$ such that $E(\|x_0\|) < \sigma$ implies $\lim_{t\to\infty} E(\|x(t)\|) = 0$.
(iii) SIS (5) is said to be pth moment exponential stable if there exist constants $\alpha > 0, K \geq 1$ such that $E(\|x(t)\|^p) \leq Ke^{-\alpha(t-t_0)}\|x_0\|^p, t \geq t_0$.
(iv) SIS (5) is said to be almost exponential stable if there exist positive constants $\alpha > 0, K \geq 1$ such that $\forall t \geq t_0, t \in \mathbb{R}_+$, $\|x(t)\| \leq Ke^{-\alpha(t-t_0)}\|x_0\|$, a.s.

Definition 4. *[30,106] SIS (5) is said to be almost sure exponential stable if $\limsup\limits_{t\to\infty} \frac{1}{t}\log(\|x(t)\|) < 0$, a.s.*

Definition 5. *The class of Lyapunov-like functions ν_0 is defined by $\nu_0 = \{V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, continuous on $(t_{k-1}, t_k] \times \mathbb{R}^n, \lim_{(t,y)\to(t_k^+,x)} V(t, y) = V(t_k^+, x)$ exists, V_t, V_x, V_{xx} are continuous with finite expectations, where $(t, x) \in (t_{k-1}, t_k] \times \mathbb{R}^n$, $k \in \mathbb{N}$, and $V(t, x)$ is concave in x for each $t \geq t_0\}$.*

Definition 6. *For each $V(t, x) \in \nu_0$, $(t, x) \in (t_k, t_{k+1}] \times \mathbb{R}^n$, $(k \in \mathbb{N})$, the Itô operator $\mathcal{L}$ is defined by*

$$\mathcal{L}V(t, x) = V_t(t, x) + V_x(t, x)f(t, x) + \frac{1}{2}trace\big[g^T(t, x)V_{xx}g(t, x)\big].$$

Review 5. (i) In the literature, e.g., [30,106], the concept *almost sure exponential stable* is used to analyze the exponential stability of stochastic systems, where the almost sure exponential stability is equivalent to $\gamma = \limsup_{t\to\infty} \frac{1}{t}\log(\|x(t)\|) < 0$. Hence, the necessary condition for almost sure exponential stability is that almost all the sample paths of any solution of system starting from a nonzero state x_0 will never reach the origin.

(ii) Definitions 5 and 6 give the definition of "derivative" for stochastic impulsive systems such that the approach of Lyapunov function for continuous-time dynamical systems can be used in the analysis of SIS (5).

Theorem 9. *[32] Assume that $I_k \in C[\mathbb{R}^n, \mathbb{R}^n], k \in \mathbb{N}$, and f, g are Borel-measurable functions satisfying the local Lipschitz condition and the linear growth condition: $\forall t \in (t_{k-1}, t_k]$, $\exists c_k > 0, d_k > 0$, such that*

$$\|f(t, x) - f(t, \bar{x})\| \vee \|g(t, x) - g(t, \bar{x})\| \leq c_k\|x - \bar{x}\|,$$
$$\|f(t, x)\| \vee \|g(t, x)\| \leq d_k(1 + \|x\|).$$

Then, SIS (5) has an unique global solution for any $x_0 \in \mathbb{R}^n$.

Theorem 10. *[32] Assume that $f(t, x)$ and $g(t, x)$ are locally Lipschitz in x and there exists a function $V \in C^{1,2}[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$ such that*
(i) there exists $\phi \in \mathcal{K}_\infty$ such that $\forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, $\phi(\|x\|) \leq V(t, x)$;
(ii) there exists continuous function $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, and $h(t, s)$ is concave on s for each $t \in \mathbb{R}_+$, such that $\mathcal{L}V(t, x) \leq h(t, V(t, x))$;
(iii) there exist nondecreasing and concave functions $\psi_k : \mathbb{R}_+ \to \mathbb{R}_+$, $k \in \mathbb{N}$, such that $V(t_k, x(t_k) + I_k(x(t_k))) \leq \psi_k(V(t_k, x(t_k))), k \in \mathbb{N}$.

Then, the existence of solution $w(t)$ on $[t_0, +\infty)$ of scalar impulsive system

$$\begin{cases} \dot{w}(t) = h(t, w(t)), & t \in (t_k, t_{k+1}], \\ w(t_k^+) = \psi_k(w(t_k)), & k \in \mathbb{N}, \\ w(t_0^+) = E(V(t_0, x_0)). \end{cases} \tag{15}$$

implies that SIS (5) has an unique solution on $[t_0, +\infty)$ for any $x(t_0^+) = x_0$.

Theorem 11. *[32] Assume that there exist functions $V \in \nu_0$, and h, ψ_k with $h(t, 0), \psi_k(0) = 0$, for $\forall t \in \mathbb{R}_+, k \in \mathbb{N}$, such that (ii)-(iii) of Theorem 10 hold, while (i) of Theorem 10 is replaced by:*
(i) there exist $\mathcal{K}$-class functions ϕ_1, ϕ_2 with $\phi_1 \in \mathcal{K}_{2\infty}, \phi_2 \in \mathcal{K}_1$, such that for all $(t, x) \in R_+ \times \mathbb{R}^n$ $\phi_1(\|x(t)\|) \leq V(t, x(t)) \leq \phi_2(\|x(t)\|)$.
Then, for any $\rho > 0$, the existence of solution $w(t) \in S_{\phi_1(\rho)}$ on $[t_0, +\infty)$ of scalar impulsive system (15) implies that SIS (5) has an unique solution process $x(t) \in \mathcal{S}_\rho$ on $[t_0, +\infty)$ for $\forall x_0 \in \mathcal{S}_\rho$. Moreover, the stability properties of the system in Equation (5) imply the corresponding stability properties of SIS (5). Moreover, if $\exists p > 0, c_1 > 0, c_2 > 0$ such that $c_1\|x\|^p \leq V(t, x) \leq c_2\|x\|^p$, a.s. then, the exponential stability of system (15) implies that pth moment exponential stability of the trivial solution of SIS (5). Furthermore, if $\exists L > 0$ $\|f(t, x)\| \vee \|g(t, x)\| \leq L\|x\|, \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, then, the exponential stability of the system in Equation (15) implies that almost exponential stability of SIS (5).

Review 6. (i) Theorem 9 is an extension on the result of existence and uniqueness of solution for classical stochastic systems. Theorem 10 gives a comparison principle for existence and uniqueness of solution of SIS (5). By Theorem 10, the existence and uniqueness of solution of SIS (5) can be derived by a deterministic comparison system with known existence property of solution.

(ii) For a stochastic system without impulses, it often needs the linear growth condition as in Theorem 9 for the existence and uniqueness of solution. Here, by Theorem 10, the linear growth condition might be not necessary to guarantee the existence and uniqueness of solution of SIS (5). Theorem 11 gives a comparison principle for stability properties of SIS (5). By Theorem 11, the stability properties of SIS (5) can be derived by the corresponding stability properties of a deterministic impulsive system.

(iii) It should be noted that in SIS (5), the random factor only exists in the flow while the jump and the impulsive/jumping instants $\{t_k\}$ are not random. A more general SIS may be a system in which all the flow, the jump, and the impulsive instants $\{t_k\}$ are random. Such a stochastic system was reported in [33].

The model studied in [33] is a stochastic switched impulsive system (SSIS) subject to probabilistic state jumps:

$$\begin{cases} \mathrm{d}x(t) = f_{r(t)}(t, x(t))\mathrm{d}t + g_{r(t)}(t, x(t))\mathrm{d}B(t),\ t \notin T = \{t_k\}, \\ x^+ = \xi_k h_{r(t)}(x(t)),\ t = t_k \in T,\ k \in \mathbb{N}, \end{cases} \tag{16}$$

where $x(t) \in \mathbb{R}^n$, $T = \{t_k\}$; $r : \mathbb{R}_+ \to Q$ is a stochastic switching function and is necessarily piecewise constant with values in $Q = \{1, \cdots, N\}$ and continuous except at switching instants $\tilde{T} = \{\tilde{t}_k\}$, where $r(t)$ is left continuous, i.e., $\lim_{s \to t^-} r(s) = r(t^-) = r(t), \forall t \in \tilde{T}$; $f_i : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $g_i : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$, $h_i : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, and satisfy $\forall i \in Q, \forall t \in \mathbb{R}_+$, $f_i(t, 0) = g_i(t, 0) = h_i(t, 0) = 0$, and $\chi(t, x)$ is continuous in x for $\chi = f_i, g_i, h_i$; the impulsive/jumping time sequence $T = \{t_k\}$ satsfies (2); Brownian motion $B(t) = (B_1(t), \cdots, B_m(t))^T$ is same defined as in SIS (5); $\{\xi_k\}$ is the set of independent $\mathbb{R}$-valued random variables distributed by a common distribution function $F_\xi(s) = P\{\xi \leq s\}$.

It may be rationally assumed that lengths of intervals between consecutive jump instants are distributed independently and identically, by exponential distribution with parameter λ_J. In SSIS (16), at all impulsive instants $\{t_k\}$, the state moves to a random point in the state space according to $x^+ = \xi_k h_{r(t)}(x(t))$. Note that between any two switching instants $\tilde{t}_k$ and $\tilde{t}_{k+1}$, due to the random variables $\{\xi_k\}$, it may have multiple state jumps while the flow subsystem is keeping unchanged and the same one.

For the switching signal $\{r(t)\}_{t\geq 0}$, in the literature, it is often assumed that $r(t)$ is a left continuous Markov chain on the probability space with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\theta) = j|_{r(t)=i}\} = \begin{cases} \gamma_{ij}\theta + o(\theta), & \text{if } i \neq j, \\ 1 + \gamma_{ij}\theta + o(\theta), & \text{if } i = j, \end{cases} \tag{17}$$

where $\theta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$. Note that (17) is a standing assumption for Markov jumping/switched systems as in the literature, see, e.g., [106].

For SSIS (16), the conditions for pth moment exponential stability have been established in Theorem 1 of [33]. For more results on stability of SIS and general stochastic hybrid systems, one can also refer to [107,108].

Review 7. SSIS (16) stands for a more general model of stochastic impulsive systems. Due to the multiple random factors in impulsive instants $\{t_k\}$, jump gains $\{\xi_k\}$, and switching signals $\{r(t) \in Q\}$, some rational assumptions should be put on these random factors for the analysis of stability. Except the independence among the three types of random factors, the above other assumptions are necessary.

Summary and Question 3.3. The stability notions and the comparison principle approach for stochastic impulsive systems have been established in Theorems 9 and 11 by [32] and a more general SIS model was built in Equation (16) by [33]. There are less stability results reported on the specific affects from factors such as stochastic terms, external inputs, and signal time-delays. Moreover, the most reported stability conditions for SIS are limited to be sufficient and not necessary. And there are few results on stability reported for discrete-time stochastic impulsive systems.

4. Review on Input-to-State Stability (ISS) of Impulsive Systems

ISS analysis is to investigate how external disturbance inputs affect the system stability. ISS reflects the uniform stability property of systems when the external input is zero. Since the notion of ISS was proposed by Sontag in [34] in the late 1980s, ISS analysis of nonlinear systems with disturbance inputs has become one of the more active research topics in nonlinear feedback analysis and design. Moreover, ISS concepts have been successfully employed in the stability analysis and control synthesis of nonlinear systems with disturbance inputs and a complex structure; see, for example, [109–115].

ISS analysis has been one of active research directions on the stability issue of impulsive systems. From the literature, the earlier work on ISS for discrete-time impulsive systems was reported in [35] in 2007, where the notion of quasi-exponential ISS (also was called as event-ISS in [22]) was proposed and the related criteria were established via the method of Lyapunov-like function. ISS for continuous-time impulsive systems was reported in [36] in 2008, where the average dwell-time technique was introduced to derive criteria on exponential ISS. Here, we review respectively some ISS results for continuous-time and discrete-time impulsive systems.

For ISS analysis, CIS (1) is replaced by a CIS with the external input:

$$\begin{cases} \dot{x}(t) = f(x(t), w(t)),\ t \notin T = \{t_k\}, \\ x^+ = g(x(t^-), w(t^-)),\ t \in T. \end{cases} \tag{18}$$

where $f, g, T = \{t_k\}$ are same defined as in Equation (1), $w(t)$ is the external input satisfying $w \in L_{\infty,loc}$. It is assumed that for each $w \in L_{\infty,loc}$, the solution of (18) exists uniquely and is forward complete. Let $x(t) = x(t, x_0, w)$ denote the solution of (18) with initial condition $x(t_0) = x_0$.

Definition 7. *The system in Equation (18) is said to be ISS if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that, $\|x(t)\| \leq \beta(\|x_0\|, t - t_0) + \gamma(\|w\|_{[t_0,t]})$, $t \geq t_0$.*

Note that in the literature, ISS is equivalent to the property: there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that $\|x(t)\| \leq \max_{t \geq t_0} \{\beta(\|x_0\|, t - t_0), \gamma(\|w\|_\infty)\}$.

In [36], the candidate exponential ISS-Lyapunov function is defined as:

Definition 8. *[36] A function $V : \mathbb{R}^n \times \mathbb{R}_+$ is called as a candidate exponential ISS-Lyapunov function of (18) with rate coefficients $c, d \in \mathbb{R}$ if V is locally Lipschitz, positive definite, radially unbounded, and satisfies for some $\phi \in \mathcal{K}_\infty$,*

$$\nabla V \cdot f(x, w) \leq -cV(x) + \phi(\|w\|), \ \forall x \text{ a.e.}, \forall w, \tag{19}$$

$$V(g(x, w)) \leq e^{-d} V(x) + \phi(\|w\|), \ \forall x, w. \tag{20}$$

In (19), "$\forall x$ a.e., $\forall w$" means for every $x \in \mathbb{R}^n$ except, possibly, on a set of zero Lebesgue-measure in $\mathbb{R}^n$, i.e., (20) is satisfied with "almost everywhere".

Given a sequence $T = \{t_k\}$ and a pair of times s, t satisfying $t > s \geq t_0$, let $N(s, t]$ denote the number of impulse times t_k in the interval $(s, t]$.

The main result on ISS of CIS (18) is the following uniform ISS (UISS) theorem reported by [36].

Theorem 12. *[36] (UISS). Let V be a candidate exponential ISS-Lyapunov function for (18) with rate coefficients $c, d \in \mathbb{R}$ with $d \neq 0$. For arbitrary constants $\mu, \lambda > 0$, let $\chi[\mu, \lambda]$ denote the class of impulse time sequences $T = \{t_k\}$ satisfying*

$$-dN(s, t] - (c - \lambda)(t - s) \leq \mu, \ t \geq s \geq t_0 \tag{21}$$

Then the system in Equation (18) is uniformly ISS (UISS) over $\chi[\mu, \lambda]$.

Corollary 1. *[36] Let V be a candidate exponential ISS-Lyapunov function for(16) with rate coefficients $c, d \in \mathbb{R}$.*
(i) When $d < 0 \wedge c > 0$, (18) is UISS on $\chi[\tau^*, N_0]$ for all $\tau^* > |d|/c$ and $N_0 > 0$.
(ii) When $d > 0 \wedge c < 0$, (18) is UISS on $\chi[\tau^*, N_0]$ for all $\tau^* < d/|c|$ and $N_0 > 0$.

Review 8. (i) Theorem 12 including Corollary 1 established ISS criteria via the dwell time condition in Equation (21). Comparing Equation (21) and the dwell time conditions in Equations (7) and (8) from Theorems 1 and 2, one can see Theorem 12 is an extension to ISS of Theorems 1 and 2 in the case of GUES.

(ii) By Equation (21), one can derive two kind of average dwell time (ADT) conditions:
Case-I. ADT-condition: for some constants $\tau^* > 0, N_0 > 0$,

$$N(s, t] \leq \frac{t - s}{\tau^*} + N_0, \ \ \forall t \geq s \geq t_0. \tag{22}$$

Case-II. Converse ADT (C-ADT) condition: for some $\tau^* > 0, N_0 > 0$,

$$N(s, t] \geq \frac{t - s}{\tau^*} - N_0, \ \ \forall t \geq s \geq t_0. \tag{23}$$

The special case $N_0 = 1$ in ADT (22) reduces to a dwell-time condition in which consecutive impulses must be separated by at least τ^* units of time. By C-ADT (23), on average, there is at least one impulse per interval of length $\tau^* > 0$. Moreover, ADT (22) means that the number of impulses is restricted and upper bounded by the time variable t, which implies that the jump in CIS (18) is a destabilizing factor for ISS. On the contrary, C-ADT (23) implies that the jump in CIS (18) is a stabilizing factor for ISS.

(iii) The method of average dwell time including ADT (22) and C-ADT (23) proposed in [36] has been one of important and well-known approach in both stability analysis and control designs involving multiple dynamics.

For ISS of discrete-time impulsive systems (DIS), a DIS with form of

$$\begin{cases} x(k+1) = f(x(k), u_c(k)), \quad k \notin N = \{N_i\}, \\ \Delta x(k+1) = I_k(x(k), u_d(k)), \quad k \in N, \\ x(N_0) = x_0, \end{cases} \tag{24}$$

was investigated in [35], where $x(k) \in \mathbb{R}^n$; $f \in C[\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n]$, $I_k \in C[\mathbb{R}^n \times \mathbb{R}^l, \mathbb{R}^n]$; $u_c(k) \in \mathbb{R}^m$ and $u_d(k) \in \mathbb{R}^l$ are external disturbance inputs, and $\Delta x(k+1) = x(k+1) - x(k), k \in N = \{N_i\}$, and the sequence $\{N_i\}$ satisfies (4) and $\Delta_{i+1} \triangleq N_{i+1} - N_i > 1, i \in \mathbb{N}$. Let $x(k) \triangleq x(k, x_0, u_c, u_d)$ be the solution of the system in Equation (24) with initial condition $x(N_0) = x_0$.

Note that if there exists a positive integer i_0 such that $\Delta_{i_0+1} = +\infty$, then DIS (24) becomes a normal discrete-time system with the initial point $(N_0 = N_{i_0}, x_0)$. The ISS issue of discrete-time systems has been extensively investigated in the literature, see, for example, [110,111]. Thus, for DIS (24), in [35], it was assumed that $\Delta_{sup} \triangleq \sup_{i\in\mathbb{N}}\{\Delta_i\} < \infty$.

The ISS including QEISS for DIS (24) was defined in [35] as:

Definition 9. *[35] The system in Equation (24) is said to be ISS if there exist $\beta \in \mathcal{KL}$ and $\gamma_c, \gamma_d \in \mathcal{K}_\infty$, such that $\|x(k)\| \le \beta(\|x_0\|, k - N_0) + \gamma_c(\|u_c\|_\infty) + \gamma_d(\|u_d\|_\infty)$, $k \in \mathbb{N}$. The system in Equation (24) is said to be quasi-exponentially ISS (QEISS) if there exist constants $\alpha > 0, K > 0$, and $\gamma_c, \gamma_d \in \mathcal{K}_\infty$, such that*

$$\|x(k)\| \le Ke^{-\alpha i}\|x(N_0)\| + \gamma_c(\|u_c\|_\infty) + \gamma_d(\|u_d\|_\infty),\ \forall k \in (N_i, N_{i+1}], i \in \mathbb{N}. \tag{25}$$

Review 9. In [22], the notion of *event-ISS* was proposed for delayed DIS. In fact, this notion is originated from the notion of quasi-exponential ISS reported by [35]. The exponential event-ISS is an extension of quasi-exponential ISS. Other related stability results on event-stability including event-ISS can be found in [22,43,55–57].

Theorem 13. *[35] The system in Equation (24) is quasi-exponentially ISS if there exists a Lyapunov-like function $V(x)$ such that the following conditions are satisfied:*
(i) for some $c_1 > 0, c_2 > 0$ and $p > 0$, and $\forall x \in \mathbb{R}^n$, $c_1\|x\|^p \le V(x) \le c_2\|x\|^p$;
(ii) $\forall k \in (N_i, N_{i+1}), i \in \mathbb{N}$, $\exists \phi_{1i} \in \mathcal{K}, \exists \gamma_1 \in \mathcal{K}_\infty$, $V(x(k+1)) - V(x(k)) \le -\phi_{1i}(V(x(k))) + \gamma_1(\|u_c(k)\|)$ holds, where ϕ_{1i} satisfying $\phi_{1i}(s) \le cs$ for some constant $0 < c < 1$ and any $s \in \mathbb{R}_+$;
(iii) for $k = N_i, i \in \mathbb{N}$, $\exists \phi_{2i} \in \mathcal{K}, \exists \gamma_2 \in \mathcal{K}_\infty$, $V(x(N_i + 1)) \le \phi_{2i}(V(x(N_i))) + \gamma_2(\|u_d(N_i)\|)$;
(iv) for some $\sigma_i > 0$, $i \in \mathbb{N}$, $\phi_{2i}(V(x(N_i))) \le \sigma_i V(x(N_i)) + \sum_{j=1}^{\Delta_{i+1}-1} \phi_{1i}(V(x(N_i + j)))$ holds, where σ_i satisfies: $\prod_{j=i}^{0} \sigma_j = e^{-\alpha i + \theta_i}$, with θ_i satisfying:

$$\overline{\lim}_{i\to\infty} \frac{\theta_i}{i} < \alpha, \quad \sup_{i\in\mathbb{N}}\{\theta_i - \underline{\theta}_{i-1}\} < \infty, \tag{26}$$

where $\underline{\theta}_{i-1} = \min_{0\le j\le i-1}\{\theta_j\}$, for any $k \in \mathbb{N}$.

Theorem 14. *[35] The system in Equation (24) is quasi-exponentially ISS if there exists a Lyapunov-like function $V(x)$ such that Conditions (i) and (iii) of Theorem 13 hold while Conditions (ii) and (iv) are replaced by:*
(ii) $\forall k \in (N_i, N_{i+1}), i \in \mathbb{N}$, $\exists \phi_{1i} \in \mathcal{K}, \exists \gamma_1 \in \mathcal{K}_\infty$, $V(x(k+1)) - V(x(k)) \le \phi_{1i}(V(x(k))) + \gamma_1(\|u_c(k)\|)$;*
(iv) for some $\sigma_i > 0$, $i \in \mathbb{N}$, $\phi_{2i}(V(x(N_i))) \le \sigma_i V(x(N_i)) - \sum_{j=1}^{\Delta_{i+1}-1} \phi_{1i}(V(x(N_i + j)))$ holds, where σ_i satisfies: $\prod_{j=i}^{0} \sigma_j = e^{-\alpha i + \theta_i}$, with θ_i satisfying (26).*

Review 10. By Condition (ii) of Theorem 13, the flow in DIS (24) is a stabilizing factor for ISS. While from (iv*) of Theorem 14, the jump in DIS (24) is a stabilizing factor for ISS. Theorems 13 and 14 and Theorem 3 in [35] are the earlier ISS results for discrete-time impulsive systems.

Recently, ISS analysis for impulsive systems mainly focused on delayed impulsive systems. Some representative results were reported in [37–42] for continuous-time delayed impulsive systems and [43] for discrete-time delayed impulsive systems.

For continuous-time delayed impulsive systems, in [37], the Razumikhin technique has been extended and used to analyse the exponential ISS and iISS for delayed impulsive systems. In [38], Razumikhin-type ISS and iISS theorems were established for a more general class of delayed impulsive systems, which was called as impulsive and switching hybrid systems with time-delays. In [39,40], sufficient conditions in terms of exponential ISS-Lyapunov functions equipped with an appropriate dwell-time condition were established for ISS of impulsive systems and their networks. In [42], ISS conditions were derived for time-delayed systems with delay-dependent impulses.

For discrete-time delayed impulsive systems, the notion of input-to-state exponents (IS-e) proposed in [43]. It should be noted that IS-e includes both ISS and non-ISS and hence it is a more general concept than ISS. The exponential ISS conditions with parts suitable for infinite delays were also established in [43]. And the differences on ISS from both time-invariant and time-varying cases were given in [43], e.g., the exponential stability of a time-varying discrete-time delayed systems with zero external input cannot guarantee ISS of such a time-varying system.

Summary and Question 4.1. From the ISS concept, the ISS property of a dynamical systems is expressed by a $\mathcal{KL}$-function for the vector field and a $\mathcal{K}$-function for the external input of the system. Thus ISS is one kind of uniform stability of dynamical systems with external inputs. The ISS results reported for impulsive systems in the literature are derived from methods such as ISS-type Lyapunov function, Krasovskii-Razumikhin-Lyapunov functional, and the condition on dwell-times allocated to the flow and the jump. There are still some fundamental issues required more investigations. Most results are limited to the case of exponential ISS and the ISS conditions are sufficient, not necessary. As pointed out in [40], it may not be easy to test the dwell-time conditions. For delayed impulsive systems, the time-delay is mostly limited to be finite and bounded. And for time-varying impulsive systems, it needs more investigations on ISS issues such as the relations between of ISS and uniform stability under zero input.

5. Review on Uniform Stability of Impulsive Hybrid Systems

In this section, we review some results of uniform stability for impulsive hybrid systems. Here, the word "hybrid" is used for a large class of dynamical systems, i.e., hybrid systems.

For hybrid systems, there is still no an unified definition given in the literature. Generally, a system is called as a hybrid system if it consists of flow dynamics and discrete dynamical behaviors. The flow dynamics means the state evolves consistently according to some dynamical law, e.g., continuous-time differential equation, discrete-time difference equation. The discrete dynamical behavior includes state's jumps abruptly from the flow or switchings from one flow to another. Several types of hybrid systems have been extensively studied, e.g., switched (hybrid) systems (i.e., switched systems) see, e.g., [67–71], time-based impulsive (hybrid) systems (in which the impulsive instants are independent of state with form of (1)) see, e.g., [73–80], and state-based impulsive (hybrid) systems (in which the impulsive instants are dependent on state). The state-based impulsive hybrid systems is also called as hybrid systems in some literature, see, e.g., [65,66]. For stability of such type of hybrid systems, some basic results including necessary and sufficient conditions of uniform stability have been established in [60–63]. In [22], the notion of stability under events (event-stability) was proposed and a basic necessary and sufficient condition was derived for event-GUAS via the method of hybrid-event-time Lyapunov function. Recently, in [55–57], more sufficient conditions for event-stability were reported by extending the average dwell time (ADT) [36] to the hybrid ADT. Moreover, by employing the approaches including small-gain method [44–47], vector Lyapunov functions, Razumikhin-type technique, Lyapunov-Krasovskii functional, and Halanay-type inequalities [48,49], the stability results including ISS for hybrid networks with or without time-delays have also been established, see, e.g., [50–54].

In the following, due to the limited space, we only focus on some uniform stability results in [60–63] and the event-stability in [22,57] for impulsive hybrid systems.

5.1. $\mathcal{KLL}$-Stability of Impulsive Hybrid Systems

In [60–63], a model of state-based impulsive hybrid systems (which was called as hybrid systems in [60–63]) was built via differential-difference inclusions:

$$\begin{cases} \dot{x} \in F(x), & x \in C, \\ x^+ \in G(x), & x \in D, \end{cases} \tag{27}$$

where x is the state with $x \in \mathcal{O} \subseteq \mathbb{R}^n$, the (set-valued) maps F and G describe the flow and the jumps respectively, and $C \subset \mathcal{O}$ and $D \subset \mathcal{O}$ are the flow set and jump set respectively.

For (27), from [60–63], it requires the standing assumption called as *Hybrid Basic Conditions*: The state space $\mathcal{O}$ is open and (27) satisfies:

(A1): $C \subset \mathcal{O}$ and $D \subset \mathcal{O}$ are relatively closed in $\mathcal{O}$;

(A2): the (set-valued) map $F : \mathcal{O} \to \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $F(x)$ is nonempty and convex for all $x \in \mathcal{O}$;

(A3): the (set-valued) map $G : \mathcal{O} \to \mathbb{R}^n$ is outer semicontinuous and locally bounded, and for each $x \in D$, $G(x)$ is nonempty subset of $\mathcal{O}$.

Under the Hybrid Basic Conditions, by [60–63], the solutions to hybrid inclusion (27) are defined on hybrid time domains, as used in [58,65,69]. A subset $E \subset \mathbb{R}_+ \times \mathbb{N}$ is a compact hybrid time domain if $E = \cup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \le t_1 \le \cdots \le t_J$. It is a hybrid time domain if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \cdots, J\})$ is a compact hybrid time domain. A hybrid arc is a function ϕ defined on a hybrid time domain, and such that $\phi(\cdot, j)$ is locally absolutely continuous for each j. A hybrid arc can be viewed as a set-valued map from $\mathbb{R}_+ \times \mathbb{N}$ whose domain is a hybrid time domain. A hybrid arc $\phi : \mathrm{dom}\,\phi \to \mathcal{O}$ is a solution (trajectory) to (27) if $\phi(0,0) \in C \cup D$ and: (i) For all $j \in \mathbb{N}$ and almost all $t \in \mathbb{R}_+$ such that $(t, j) \in \mathrm{dom}\,\phi$, if $\phi(t, j) \in C$, then $\dot{\phi}(t, j) \in F(\phi(t, j))$; (ii) $\forall (t, j) \in \mathrm{dom}\,\phi$ with $(t, j+1) \in \mathrm{dom}\,\phi$, if $\phi(t, j) \in D$, then $\phi(t, j+1) \in G(\phi(t, j))$.

Definition 10. *[60–63] The continuous function $\omega : \mathcal{O} \to \mathbb{R}_+$ is a proper indicator for a closed set $\mathcal{A} \subset \mathcal{O}$ if $\omega(x) = 0 \iff x \in \mathcal{A}$. Typically: $\omega(x) = d(x, \mathcal{A})$.*

For solution $\phi(t, j)$ to (27) starting from $x \in C \cup D$, we denote $\phi(t, j, x) = \phi(t, j)$ satisfying (27) and $\phi(0, 0, x) = x$.

Definition 11. *[60–63] Let $\omega : \mathcal{O}_1 \to \mathbb{R}_+$ be continuous. The set $\mathcal{A}$ is said to be $\mathcal{KLL}$-stable w.r.t. ω on $\mathcal{O}$ if it is forward complete on $\mathcal{O}$, and there exists $\gamma \in \mathcal{KLL}$ such that for each $x \in \mathcal{O}$, all solutions ϕ starting from x satisfy*

$$\omega(\phi(t, j, x)) \le \gamma(\omega(x), t, j), \quad \forall (t, j) \in dom\phi.$$

Definition 12. *Let $\mathcal{O}_1 \subset \mathcal{O}$ be open and $\omega : \mathcal{O}_1 \to \mathbb{R}_+$ be continuous. A function V: $\mathcal{O}_1 \to \mathbb{R}_+$ is said to be a smooth Lyapunov function of (27) if it is smooth and there exist class-$\mathcal{K}_\infty$ functions α_1, α_2 such that*

$$\alpha_1(\omega(x)) \le V(x) \le \alpha_2(\omega(x)), \quad \forall x \in \mathcal{O}_1, \tag{28}$$

$$\max_{f \in F(x)} < \nabla V(x), f > \le -V(x), \quad \forall x \in \mathcal{O}_1 \cap C, \tag{29}$$

$$\sup_{g \in G(x) \cap \mathcal{O}_1} V(g) \le e^{-1} V(x) < V(x), \quad \forall x \in x \in \mathcal{O}_1 \cap D. \tag{30}$$

One of basic results in [60–63] can be summarized as:

Theorem 15. *Suppose $C \cup D = \mathcal{O}$. Let ω be any proper indicator of compact $\mathcal{A}$ w.r.t. $\mathcal{O}$ and the solution to (27) is forward complete. Then, the following are equivalent:*
(i) the set $\mathcal{A}$ is asymptotically stable for (27);
(ii) there exists a smooth Lyapunov function V of (27);
(iii) the system in Equation (27) is robustly $\mathcal{KLL}$-stable w.r.t. ω on $\mathcal{O}$.

Review 11. (i) For an impulsive hybrid system, since there may exist multiple jumps occurring at same instant of jump, it is necessary to introduce the hybrid time variable (t, j) to describe such dynamical behaviors. The concept of $\mathcal{KLL}$-stability in Definition 11 is thus proposed and based on the hybrid time variable (t, j). Clearly, $\mathcal{KLL}$-stability is one kind of UAS.

(ii) Theorem 15 gives a basic asymptotic stability result for the state-based impulsive hybrid system in Equation (27). From (i)-(ii) of Theorem 15, the set $\mathcal{A}$ is asymptotically stable if and only if there exists a smooth Lyapunov function V satisfying (29–30). Thus, the classical Lyapunov asymptotic stability result was extended to the impulsive hybrid systems. Moreover, from (iii) of Theorem 15, an asymptotic stable hybrid system also has some robustness. In [64,65], the result in Theorem 15 was extended to the case of ISS via the smooth Lyapunov function. Thus, the basic results on uniform asymptotic stability (UAS) and ISS have been established for the state-based impulsive hybrid systems (27).

Summary and Question 5.1. (i) Theorem 15 and the results in [64,65] are basic and general results on uniform asymptotic stability and ISS for impulsive hybrid system. It should be noted that it may not be easy to test the conditions of uniform stability including ISS for a specific hybrid system via the general method of smooth Lyapunov function (SLF) in Theorem 15 and the results in [64,65]. Especially for impulsive hybrid system with unstable subsystems (flows and jumps), it is generally hard to find a common SLF or LF which is strictly decreasing on unstable flows or unstable jumps as in Equations (29) and (30). Hence, it is necessary and important to find other tools and methods for the test of stability and ISS. Here, we use an example in [22] to give more illustrations:

Example 1. *[22] Consider a hybrid system with form:*

$$\begin{cases} \dot{x} = A_c x(t), \ x \in C, \\ x^+ = x(t) = A_d x(t^-), \ x \in D, \end{cases} \tag{31}$$

where $x \in \mathbb{R}^2$, $A_c = \begin{pmatrix} 0.01 & -1 \\ 1 & 0.01 \end{pmatrix}$, $A_d = \begin{pmatrix} 0 & -0.06 \\ 2 & 0 \end{pmatrix}$, *and* $C = \{(x_1, x_2)^T \in R^2 : x_1 \neq 0\}$, $D = \{(x_1, x_2)^T \in R^2 : x_1 = 0\}$, *i.e., there is a jump when $x(t)$ arrives set D.*

It is easy to know both flow and jump in (31) are unstable. However, the whole system in Equation (31) is GUES, see Figure 3:

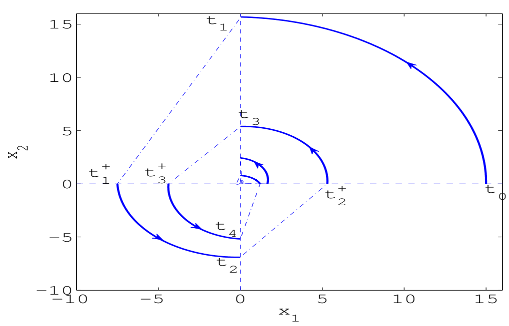

Figure 3. GUES property of the system in Equation (31).

In the case that both flow and jump are unstable, it is hard to find a smooth Lyapunov function to verify the stability property. The stability property for the system in Equation (31) belongs to the type of "destabilizing flow + destabilizing jump". This type of stability was called as event-stability in [22,55,57]. The method of hybrid-event-time Lyapunov function proposed in [22] provided a tool in analyzing event-stability including event-ISS for hybrid systems.

(ii) The system in Equation (27) is a state-based impulsive system, i.e., the impulsive/jumping instants are dependent on state. Compared with the time-based impulsive systems, see, e.g., [1–3], the stability including ISS criteria are less developed for state-based impulsive systems.

5.2. Event-Stability of Impulsive Hybrid Systems

Due to the hybrid dynamics, the stability analysis of a hybrid system is more complex than that for a pure continuous-time system (flow) or a pure discrete-time system (jump). As pointed out in [22,55–57], except the flow and the jump, the switching event between the flow and the jump may play an important role in determining stability of impulsive hybrid system. As Example 1 shows, event if all subsystems (flows and jumps) in an impulsive hybrid system are unstable, the whole system may still be UAS. In [22], the notion of event-stability was proposed to reflect this special feature of hybrid systems. The necessary and sufficient of event-GUAS was given in [22] for switched hybrid systems via hybrid-event-time Lyapunov function. In [55–57], some specific criteria on event-stability were also obtained. Here, we review some recent results on event-stability in [57].

In [57], an impulsive hybrid system with the ith stage/subsystem S_i as

$$\begin{cases} S_{ci} : \dot{x}(t,j) = f_{ci}(t, x(t,j)),\ t \in I_{ci}, j = n_{i-1}, \\ S_{di} : x(t, j+1) = f_{di}(x(t,j)),\ j \in I_{di}, t = t_i, \end{cases} \tag{32}$$

was investigated, where i is the index of the ith stage/subsystem S_i; $x(t,j) \in \mathbb{R}^n$ is the state; (t,j) is the continuous and discrete-time (CDT) variable; S_{ci} stands for the continuous time dynamics of S_i with the continuous time variable $t \in I_{ci}$ while the discrete time variable j is fixed at $j = n_{i-1}$, S_{di} is the discrete time dynamics of S_i with the discrete time variable $j \in I_{di}$ while t is fixed at $t = t_i$, where I_{ci} and I_{di} are the continuous and discrete time intervals in the ith stage S_i with form of $I_{ci} = (t_{i-1}, t_i]$, $I_{di} = \mathcal{N}[n_{i-1}, n_i)$, $i \geq 1$, in which $T^c \triangleq \{t_i : t_i \in \mathbb{R}_+\}$ and $T^d \triangleq \{n_i : n_i \in \mathbb{N}\}$ satisfy

$$0 \leq t_0 \leq t_1 \leq \cdots \leq \infty, \quad 0 \leq n_0 \leq n_1 \leq \cdots \leq \infty; \tag{33}$$

functions $f_{ci}(t,x)$ is Lipschitz continuous in x and $f_{di}(x)$ is continuous in x with $f_{ci}(t,0) \equiv 0$, $f_{di}(0) \equiv 0$; and (t_0, n_0) is the initial time. Here, assume that the solution $x(t,j)$ of (32) exists for any initial condition $x(t_0, n_0) = x_0 \in \mathbb{R}^n$ and is forward complete.

Let $T \neq \emptyset$ be the set consisting of all admissible CDT sequences (T^c, T^d) of (32) with form of $T^c = \{t_i\}$ and $T^d = \{n_i\}$. For a given (T^c, T^d), define the time set $\mathcal{T}$ of (32) as:

$$\mathcal{T} = \cup_{i\in\mathbb{N}} \mathcal{I}_i = \cup_{i\in\mathbb{N}} \mathcal{I}_{ci} \cup \mathcal{I}_{di},$$

where $\mathcal{I}_0 = \{(t_0, n_0)\}$, $\mathcal{I}_{ci} = I_{ci} \times \{n_{i-1}\}$, $\mathcal{I}_{di} = \{t_i\} \times I_{di}$, $\mathcal{I}_i = \mathcal{I}_{ci} \cup \mathcal{I}_{di}$ for all $i \geq 1$.

Definition 13. *[57] For any CDT $(t,j) \in \mathcal{T}$, the length of (t,j) is defined as: $|(t,j)| \triangleq t + j, \forall (t,j) \in \mathcal{T}$. The relation of partial-order "$\leq$" on $\mathcal{T}$ is defined as: $\forall (s_1, j_1), (s_2, j_2) \in \mathcal{T}$,*

$$(s_1, j_1) \leq (s_2, j_2) \iff |(s_1, j_1)| \leq |(s_2, j_2)|,$$
$$(s_1, j_1) < (s_2, j_2) \iff |(s_1, j_1)| < |(s_2, j_2)|.$$

Definition 14. *[22,57] For a dynamical system S with time set $\mathcal{T}$, if its dynamics (in form of differential or difference equation or inequality or inclusion) at instant $s^* \in \mathcal{T}$ is different from that at $s^* < s$ or $s < s^*$, then S is said to have an event at s^*.*

Definition 15. *[22] $(t, j, k_{(t,j)})$ is said to be hybrid-event-time, where $k_{(t,j)}$ denotes the number of events of (32) from initial time (t_0, n_0) to (t, j).*

Definition 16. *[22,57] (i) The system in Equation (32) is said to be uniformly stable (US), if $\forall \epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\forall x_0 \in \mathbb{R}^n$ with $\|x_0\| < \delta$, $\|x(t,j)\| < \epsilon$ holds, $\forall (t,j) \in \mathcal{T}$. The system in Equation (32) is said to be globally uniformly asymptotically stable (GUAS), if it is US and $\lim_{(t,j)\in\mathcal{T},|(t,j)|\to\infty} \|x(t,j)\| = 0$ holds uniformly, $\forall (t_0, n_0, x_0)$.*
(ii) The system in Equation (32) is said to be GUAS under events (event-GUAS), if it is US and $\lim_{k_{(t,j)}\to\infty} \|x(t,j)\| = 0$ holds uniformly.
(iii) The system in Equation (32) is said to be globally uniformly exponentially stable (GUES), if $\forall x_0 \in \mathbb{R}^n$, and for some constants $\alpha > 0$, $K > 0$,

$$\|x(t,j)\| \le K e^{-\alpha(|(t,j)|-|(t_0,n_0)|)} \|x_0\|, \ \forall (t,j) \in \mathcal{T}.$$

The system in Equation (32) is said to be event-GUES if for some $\alpha > 0, K > 0$,

$$\|x(t,j)\| \le K e^{-\alpha k_{(t,j)}} \|x_0\|, \ \forall (t,j) \in \mathcal{T}.$$

Definition 17. *[57] The system in Equation (32) is said to be GUAS (GUES or event-GUAS, or event-GUES) over T if $\forall (T^c, T^d) \in T$, (32) is GUAS (GUES or event-GUAS, or event-GUES) and all parameters in Definition 16 are independent of the choice of $(T^c, T^d) \in T$.*

In [57], by using the hybrid-event-time $(t, j, k_{(t,j)})$, the average dwell-time condition (ADT) (Equation (22) and converse-ADT (C-ADT) (23) in [36] were extended respectively to hybrid ADT (H-ADT) and hybrid converse ADT (H-C-ADT): for some $\tilde{M} \in R, \tilde{\tau}_a > 0$, $M \in R, \tau_a > 0$,

$$k_{(t,j)} \le \tilde{M} + \frac{|(t,j)| - |(t_0,n_0)|}{\tilde{\tau}_a}, \quad (\textit{H-ADT}), \tag{34}$$

$$k_{(t,j)} \ge M + \frac{|(t,j)| - |(t_0,n_0)|}{\tau_a}, \quad (\textit{H-C-ADT}). \tag{35}$$

Review 12. The hybrid-event-time $(t, j, k_{(t,j)})$ in Definition 15 by [22] is the extension of hybrid time (t, j) used in [58,65,69]. However, the hybrid-event-time can reflect the number of events from the initial time to current time. H-ADT (34) and H-C-ADT (35) are w.r.t. the hybrid-event-time $(t, j, k_{(t,j)})$ and reflect the relations between CDT (t, j) and the event number $k_{(t,j)}$. Clearly, for (32), if H-C-ADT (35) holds, then event-GUES (event-GUAS) implies GUES (GUAS), and if H-ADT (34) holds, then GUES (GUAS) implies event-GUES (event-GUAS). Hence, H-ADT (34) and H-C-ADT (35) build the connections between the uniform stability and the event-stability.

For $T^c = \{t_i\}$ and $T^d = \{n_i\}$, denote: $\Delta_{ci} \triangleq t_i - t_{i-1}$, $\Delta_{di} \triangleq n_i - n_{i-1}$ for all $i \ge 1$. For any constant $\delta_c > 0$, define

$$T^c_{\le \delta_c} = \{T^c : \Delta_{ci} \le \delta_c, i \ge 1\}, \ T^c_{\ge \delta_c} = \{T^c : \Delta_{ci} \ge \delta_c, i \ge 1\}.$$

Similarly, define $T^d_{\le \delta_d}$ and $T^d_{\ge \delta_d}$ for integer $\delta_d \in \mathbb{N}$.

Denote $S_c \triangleq \{S_{ci}\}$ and $S_d \triangleq \{S_{di}\}$. For every $S_{ci} \in S_c$ ($S_{di} \in S_d$), consider a Lyapunov-like function V_i^c (V_i^d): $\mathbb{R}^n \to \mathbb{R}_+$. Assume there exist $\varphi_1, \varphi_2 \in \mathcal{K}_\infty$, constants $\lambda > 0, \mu > 0$, such that

$$\varphi_1(\|x\|) \le V_i^*(x) \le \varphi_2(\|x\|),\ V_i^* = V_i^c, V_i^d, \forall x \in \mathbb{R}^n, \tag{36}$$
$$V_i^c(f_{d,i-1}(x)) \le \lambda \cdot V_{i-1}^d(f_{d,i-1}(x)),\ \forall x \in \mathbb{R}^n, \tag{37}$$
$$V_i^d(x) \le \mu \cdot V_i^c(x),\ \forall x \in \mathbb{R}^n. \tag{38}$$

For sequence $\{\theta_i\}$, define: $N_\theta^- = \{i \in \mathbb{N} : \theta_i < 0\}, N_\theta^+ = \{i \in \mathbb{N} : \theta_i \ge 0\}; N_{\theta i}^- = \{j \in N_\theta^- : j \le i\}$, $N_{\theta i}^+ = \{j \in N_\theta^+ : j \le i\}$; and for $* = -, +$,

$$\theta^* = \begin{cases} 0, \text{ if } N_\theta^* = \emptyset, \\ \sup_{i \in N_\theta^*}\{\theta_i\}, \text{ if } N_\theta^* \ne \emptyset. \end{cases}$$

Hence, for two sequences $\{a_i\}$ and $\{q_i\}$, a^-, a^+, q^- and q^+ are defined. For $\delta_c^- > 0, \delta_c^+ > 0, \delta_d^- \ge 1, \delta_d^+ \ge 1$, denote: $T^c_{(\delta_c^-, \delta_c^+)} = \{T^c : \Delta_{ci} \ge \delta_c^-, \text{if } i \in N_a^-; \Delta_{ci} \le \delta_c^+, \text{if } i \in N_a^+\}$, $T^d_{(\delta_d^-, \delta_d^+)} = \{T^d : \Delta_{di} \ge \delta_d^-, \text{if } i \in N_q^-; \Delta_{di} \le \delta_d^+, \text{if } i \in N_q^+\}$, and $\delta_{ci} = \begin{cases} \delta_c^-, \text{ if } i \in N_a^-, \\ \delta_c^+, \text{ if } i \in N_a^+, \end{cases}$ $\delta_{di} = \begin{cases} \delta_d^-, \text{ if } i \in N_q^-, \\ \delta_d^+, \text{ if } i \in N_q^+. \end{cases}$

Theorem 16. *[57] Suppose there exist V_i^c, V_i^d satisfying (36–38) with $\varphi_l(s) = w_l s^r$ for $l = 1, 2$, and some $w_1, w_2, r > 0$. For some sequences $\{a_i\}$ and $\{q_i\}$ with $a^+ < \infty$ and $-1 < q_i \le q^+ < \infty$, $i \in \mathbb{N}$, the following conditions hold:*

$$D^+ V_i^c(x)\big|_{S_{ci}} \le a_i V_i^c(x), \tag{39}$$
$$V_i^d(f_{di}(x)) \le (1 + q_i) V_i^d(x). \tag{40}$$

Then, the system in Equation (32) is event-GUES over $T = \{(T^c, T^d)\}$, where $T^c \in T^c_{(\delta_c^-, \delta_c^+)}$ and $T^d \in T^d_{(\delta_d^-, \delta_d^+)}$, if for some $\alpha > 0$ and $m_0 \in \mathbb{N}$ with $m_0 \ge 1$,

$$\frac{\sum_{l=0}^{i-1}(a_l \Delta_{cl} + \Delta_{dl} \ln(1 + q_l))}{i} + \ln(\lambda\mu) \le -\alpha,\ \forall i \ge m_0. \tag{41}$$

Moreover, the system in Equation (32) is GUES over T if H-C-ADT (35) holds.

Review 13. (i) In Theorem 16, if all the coefficients a_i and q_i are nonnegative, then all subsystems can be unstable. Hence, it was shown that the events may drive an impulsive hybrid system with all unstable subsystems to achieve stability and ISS. Here, by using Theorem 16, the event-GUES can be derived for the system in Equation (31) in Example 1:

Let $V^c(x) = V_i^c(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$, $V^d(x) = V_i^d(x) = 4x_1^2 + x_2^2$, then

$$D^+ V^c(x)\big|_{S_{ci}} \le a V_i^c(x),\ V^d(A_{di}x) \le (1 + q) V_i^d(x);$$
$$V^c(A_{di}x) \le \lambda V^d(A_{di}x), V^d(x) \le \mu V^c(x),\ x \in D,$$

where $a = 0.02$, $q = 0.44$, $\lambda = \frac{1}{8}$, and $\mu = 2$. Thus, we have $a^+ = 0.04$ and $q^+ = 0.69$. By (41), we estimate the maximal values of δ_c^+ and δ_d^+ as:

$$\delta_{c,max}^+ = 21.5391,\ \delta_d^+ = 1;\ \delta_{d,max}^+ = 2,\ \delta_c^+ = 8.4209.$$

Thus, by Theorem 16, (31) is event-GUES over $T = \{(T^c, T^d)\}$ with $T^c \in T^c_{\le 8.4209}$, $T^d \in T^d_{\le 2}$, or $T^c \in T^c_{\le 21.5391}$, $T^d \in T^d_{=1}$. In fact, for (31), we calculate the actual maximal switching interval

$\Delta_{c,\max} = \max\{\Delta_{ci} : i \in N\} \leq \frac{0.5\pi}{0.89} \approx 1.765$ and $\Delta_d = 1$. Hence, it belongs to the case of $T^c \in T^c_{\leq 8.4209}$, $T^d \in T^d_{\leq 2}$, and thus by Theorem 16, (31) is event-GUES.

(ii) The results in Theorem 16 are also shown that H-ADT (34) and H-C-ADT (35) may let different types of subsystems have flexible dwell time conditions, see [52,56].

Summary and Question 5.2. Most reported results on ISS for impulsive hybrid systems are based on ISS property of subsystems or UAS of the system under the zero external input. In Theorem 16 by [57] and Example 1, it is allowed that all subsystems are unstable and non-ISS. Moreover, the notions of input-to-state exponents in [43] and IS-$\mathcal{KLK}_0$-property including IS-$\mathcal{KL}$-stability in [56] have been proposed to address the issue of non-ISS subsystems. The factors such as time-varying dynamics and time-delays (not limited to finite time-delay) have been taken into considerations in the literature. In addition, in [54], Hanalay Lemma was extended to ISS of delayed hybrid networks with non-small gain. However, for time-varying (impulsive) hybrid systems and networks with time-delays, the analysis for uniform stability and ISS still remains relatively undeveloped, e.g., fewer necessary and sufficient conditions and converse theorems for ISS have been reported.

6. Conclusions

In this paper, we have reviewed some uniform stability results for impulsive systems, including continuous-time impulsive systems, discrete-time impulsive systems, impulsive systems with or without time-delays, stochastic impulsive systems, and impulsive hybrid systems. The reviewed concepts on uniform stability include uniform asymptotic stability, ISS, $\mathcal{KLL}$-stability, event-stability, and event-ISS.

Generally, the uniform stability results including ISS for time-based impulsive systems are sufficient and based on the existence of stabilizing flow or jump. The conditions of $\mathcal{KLL}$-stability for state-based impulsive hybrid systems are necessary and sufficient. The event-stability including event-ISS for impulsive hybrid systems allows that all subsystems are unstable and non-ISS. By reviewing the results on uniform stability, some challenging questions for stability of impulsive systems have been given, e.g., necessary or converse uniform stability for time-based impulsive systems; more stability results for the case of destabilizing flow + destabilizing jump; stability for relatively undeveloped state-based impulsive systems; and stability of impulsive hybrid systems with different impulsive instants. These questions can be some future works on the stability of impulsive systems.

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