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Finite-Time Mittag–Leffler Synchronization of Neutral-Type Fractional-Order Neural Networks with Leakage Delay and Time-Varying Delays

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Abstract: This paper studies fractional-order neural networks with neutral-type delay, leakage delay, and time-varying delays. A sufficient condition which ensures the finite-time synchronization of these networks based on a state feedback control scheme is deduced using the generalized Gronwall–Bellman inequality. Then, a different state feedback control scheme is employed to realize the finite-time Mittag–Leffler synchronization of these networks by using the fractional-order extension of the Lyapunov direct method for Mittag–Leffler stability. Two numerical examples illustrate the feasibility and the effectiveness of the deduced sufficient criteria.

Keywords: fractional-order neural networks; finite-time synchronization; neutral-type neural networks; leakage delay; Mittag–Leffler function

1. Introduction

Fractional calculus studies the different possibilities of defining real or complex orders for the differentiation and integration operators. Although it has a long history, only recently it has been successfully applied to physics and engineering problems. As such, in the past few years, it became clear for engineers and scientists that some phenomena can be more accurately described by employing the fractional derivative. Fractional differential equations have been proved to better describe many systems in interdisciplinary fields, such as chemistry, biology, physics, mechanics, electromagnetism, heat transfer, acoustics, economy, and finance.

Fractional-order systems have been proven to possess infinite memory. Taking this fact into account, an extremely important improvement would be the introduction of a memory term (represented by a fractional derivative or integral) into a neural network model. Thus, fractional-order artificial neural networks were developed in [1]. Since then, many properties of this type of networks were studied: asymptotic stability and synchronization [2–7], Mittag–Leffler stability and synchronization [8–13], dissipativity [14–16], etc.

The finite-time stability and synchronization properties of fractional-order neural networks were intensely studied over the last few years. Concretely, "finite-time stability analysis of fractional-order complex-valued memristor-based neural networks with time delays" was done in [17]. Finite-time stability criteria for fractional-order delayed neural networks were also established in [18–20]. A more general model, namely fractional-order Cohen–Grossberg BAM neural networks with time delays was researched in [21], from the finite-time stability point of view. Finite-time stability analysis was undergone in [22], for fractional-order complex-valued memristor-based neural networks with both leakage and time-varying delays. Fractional-order complex-valued neural networks with time delays were discussed in [23], in terms of finite-time stability. Then, finite-time stability criteria for delayed memristor-based fractional-order neural networks were deduced in [24]. A different model, namely the fractional-order fuzzy neural networks with proportional delays, was presented in [25], by giving finite-time stability sufficient criteria for these networks. More recently, finite-time stability results were established for fractional-order complex-valued neural networks with time delay in [26]. Discrete fractional-order complex-valued neural networks with time delays were, on the other hand, the focus of [27], where the existence and finite-time stability for these networks were researched.

Finite-time synchronization was also a topic of interest in the recent past. "Finite-time synchronization of fractional-order memristor-based neural networks with time delays" was studied in [28]. Sufficient criteria that realize "finite-time projective synchronization of memristor-based fractional-order neural networks with delays" were presented in [29]. Fractional-order memristive neural networks with discontinuous activation functions were researched in [30], where criteria for their finite-time synchronization were deduced. Then, both finite-time stability and finite-time synchronization were discussed in [31], for memristor-based fractional-order fuzzy cellular neural networks. Further finite-time synchronization analysis was undergone in [32], where the models were fractional-order memristive competitive neural networks with leakage delay and time-varying delays. Finite-time projective synchronization sufficient criteria were also deduced in [33], for fractional-order complex-valued memristor-based neural networks with delay. More recently, fully complex-valued fractional-order neural networks were the focus of [34], in which finite-time synchronization conditions for these models were given. Lastly, finite-time impulsive synchronization of fractional order memristive BAM neural networks was undergone in [35].

The property of finite-time Mittag–Leffler synchronization was introduced in [36], where sufficient criteria to attain this type of synchronization were given for fractional-order memristive BAM neural networks. Then, novel methods to finite-time Mittag–Leffler synchronization were given in [37], for fractional-order quaternion-valued neural networks. As such, the subject remains largely unexplored in the existing literature.

Fractional-order neural networks with neutral-type delays were also very rarely studied in the available literature. "Delay-independent stability criteria for Riemann–Liouville fractional-order neutral-type neural networks" were established in [38]. Then, the "delay-dependent stability analysis of QUAD vector field fractional-order quaternion-valued memristive uncertain neutral-type leaky integrator echo state neural networks" was done in [39]. No other results for neutral-type fractional-order neural networks exist in the literature, to the best of our knowledge.

On the other hand, fractional-order neural networks with leakage delay were much more in the focus of researchers in the recent years. The "stability analysis of fractional-order complex-valued neural networks with both leakage and discrete delays" was undergone in [40]. New bifurcation results for fractional-order BAM neural networks with leakage delay were given in [41]. As already mentioned, "fractional-order complex-valued memristor-based neural networks with both leakage and time-varying delays" were the focus of [22], where sufficient criteria for finite-time stability were deduced for these networks. Then, also as already mentioned, in [32], fractional-order memristive competitive neural networks with leakage delay and time-varying delays were discussed, from the finite-time synchronization point of view. The "impact of leakage delay on bifurcation in high-order fractional-order BAM neural networks" was researched in [42]. "Novel results on bifurcation for fractional-order complex-valued neural networks with leakage delay" were established in [43]. Lastly, the "dynamic stability of stochastic delayed fractional-order memristor-based fuzzy BAM neural networks with leakage delay" was studied in [44].

Time delays are known to occur in practical implementations of neural networks, which can cause instability or chaotic behavior, because the amplifiers have a finite switching speed. This is the reason why we chose to consider both leakage delay and time-varying delays in our model. On the other hand, in neutral-type systems, past derivative information has also been observed to influence the present state. The properties of neural reaction processes that occur in the real world can be more accurately described by these systems. The study of these systems is more complicated than that of the usual time-delayed models because of the existence of the neutral-type delay. This type of delay is relevant in many application domains, like automatic control, population dynamics, and vibrating masses attached to an elastic bar. Neutral delay may also appear when implementing neural networks in VLSI circuits. These facts compelled us to also add neutral-type delay to our model. Taking all the above into account, we consider neutral-type fractional-order neural networks with leakage delay and time-varying delays in this paper, and study their finite-time synchronization and finite-time Mittag–Leffler synchronization based, respectively, on two general state feedback control schemes.

The main contributions of the paper are: (1) the introduction, for the first time in the literature, to the best of our knowledge, of the fractional-order neural networks with neutral-type delay, leakage delay, and time-varying delays; (2) the use of the generalized Gronwall–Bellman inequality to deduce sufficient criteria for the finite-time synchronization of the introduced networks, using a general state feedback control scheme; (3) the use of the fractional-order extension of the Lyapunov direct method for Mittag–Leffler stability in order to deduce sufficient criteria for the finite-time Mittag–Leffler synchronization of the introduced networks, using a different general state feedback control scheme; (4) the possible use of the methods developed in this paper for more general models, with impulsive effects, reaction–diffusion terms, or Markovian jump parameters.

The summary of the rest of the paper is the following. The neutral-type fractional-order neural networks with leakage delay and time-varying delays are introduced in Section 2, together with definitions regarding fractional calculus, definitions of the finite-time synchronization and the finite-time Mittag–Leffler synchronization, one assumption about the activation functions, and four useful lemmas. Then, Section 3 is dedicated to the deduction of the sufficient criteria which ensure finite-time synchronization and finite-time Mittag–Leffler synchronization, respectively, of the introduced model. Section 4 details the two numerical examples given to illustrate the feasibility and the effectiveness of the deduced sufficient criteria. The conclusions of the paper are presented in Section 5.

Notations: $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^n$ denotes the Euclidean space of dimension n. A^T represents the transpose of matrix A. I_n denotes the identity matrix of order n and 0_n the empty matrix of order n. That matrix A is positive definite (negative definite) is denoted by $A > 0$ ($A < 0$). The smallest eigenvalue of positive definite matrix P is $\lambda_{\min}(P)$. $||\cdot||$ represents the vector Euclidean norm or the matrix Frobenius norm, and $|\cdot|$ is the element-wise vector norm or the element-wise matrix norm.

2. Preliminaries

First, we will give a few definitions involving fractional calculus.

Definition 1 ([45])**.** "The fractional integral of order α for an integrable function $x : [t_0, \infty) \to \mathbb{R}$ is defined as:

$$I^{\alpha}_{t_0} x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} x(s) ds,$$

where $t \geq t_0$, $\alpha > 0$, and $\Gamma(\cdot)$ is the gamma function, defined by:

$$\Gamma(\tau) = \int_{0}^{\infty} t^{\tau-1} e^{-t} dt,$$

for $\mathrm{Re}(\tau) > 0$, where $\mathrm{Re}(\cdot)$ represents the real part."

Definition 2 ([45])**.** "The fractional Caputo derivative of order α for a function $x \in \mathcal{C}^n([t_0,\infty),\mathbb{R})$ is defined by:

$$D_{t_0}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{x^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $t > t_0$ and n is a positive integer, with $n-1 < \alpha < n$. Moreover, when $0 < \alpha < 1$, we have that:

$$D_{t_0}^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{\dot{x}(s)}{(t-s)^{\alpha}} ds."$$

Definition 3 ([45])**.** "The Mittag–Leffler function is defined by:

$$E_{\alpha}(z) = \sum_{p=0}^{\infty} \frac{z^p}{\Gamma(p\alpha+1)},$$

where $\alpha > 0$ and $z \in \mathbb{C}$. When $\alpha = 1$, we have that $E_1(z) = e^z$."

Now, the neutral-type fractional-order neural networks with leakage delay and time-varying delays will be considered as the master system:

$$\begin{aligned} D_0^{\alpha} x_i(t) &= -c_i x_i(t-\mu) + \sum_{j=1}^{N} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{N} b_{ij} f_j(x_j(t-\tau(t))) \\ &\quad + g_i D_{-\eta}^{\alpha} x_i(t-\eta) + I_i, \end{aligned} \tag{1}$$

for $\forall i = 1, \ldots, N$, where $x_i(t) \in \mathbb{R}$ represents the state of the ith neuron at time t, $c_i \in \mathbb{R}^+$ represents the self-feedback weight of the ith neuron, $a_{ij} \in \mathbb{R}$ is the weight without time delay between the ith and jth neurons, $b_{ij} \in \mathbb{R}$ is the weight with time delay between the ith and jth neurons, $g_i \in \mathbb{R}$ is the neutral-type weight of the ith neuron, $f_j : \mathbb{R} \to \mathbb{R}$ represent the nonlinear activation functions, $\forall j = 1, \ldots, N$, $I_i \in \mathbb{R}$ is the external input for the ith neuron, μ is the leakage delay, $\tau : \mathbb{R}^+ \to \mathbb{R}^+$ are the time-varying delays, and η is the neutral-type delay.

In the following, we assume that the time-varying delays $\tau : \mathbb{R}^+ \to \mathbb{R}^+$ are continuously differentiable functions and there exist $\tau > 0$ and $\tau' < 1$, such that $\tau(t) < \tau$, $\dot{\tau}(t) \le \tau'$, $\forall t > 0$. Define $\omega = \max\{\mu, \tau, \eta\}$.

The initial conditions of system (1) are given by

$$x_i(t) = \phi_i(t),\ t \in [-\omega, 0],$$

where $\phi_i \in \mathcal{C}([-\omega,0],\mathbb{R})$, for $\forall i = 1, \ldots, N$. The norm of an element $\phi \in \mathcal{C}([-\omega,0],\mathbb{R}^N)$ is defined as $||\phi|| = \sum_{i=1}^{N} \sup_{t\in[-\omega,0]} |\phi_i(t)|$.

The slave system is given by

$$\begin{aligned} D_0^{\alpha} y_i(t) &= -c_i y_i(t-\mu) + \sum_{j=1}^{N} a_{ij} f_j(y_j(t)) + \sum_{j=1}^{N} b_{ij} f_j(y_j(t-\tau(t))) \\ &\quad + g_i D_{-\eta}^{\alpha} y_i(t-\eta) + I_i - u_i(t), \end{aligned} \tag{2}$$

for $\forall i = 1, \ldots, N$, and $y_i(t) \in \mathbb{R}$ represents the state of the ith neuron at time t, and $u_i(t)$ represents a control input.

The initial conditions of system (2) are given by

$$y_i(t) = \psi_i(t),\ t \in [-\omega, 0],$$

where $\psi_i \in \mathcal{C}([-\omega,0],\mathbb{R})$, for $\forall i = 1, \ldots, N$.

If we denote by $e_i(t) = y_i(t) - x_i(t)$ for $\forall i = 1, \ldots, N$, then, based on the expressions of the master system (1) and the slave system (2), the error system has the form

$$\begin{aligned} D_0^\alpha e_i(t) &= -c_i e_i(t-\mu) + \sum_{j=1}^{N} a_{ij} \overline{f}_j(e_j(t)) + \sum_{j=1}^{N} b_{ij} \overline{f}_j(e_j(t-\tau(t))) \\ &\quad + g_i D_{-\eta}^\alpha e_i(t-\eta) - u_i(t), \end{aligned} \tag{3}$$

for $\forall i = 1, \ldots, N$, where $\overline{f}_j(e_j(t)) = f_j(e_j(t) + x_j(t)) - f_j(x_j(t))$, $\forall j = 1, \ldots, N$.

The initial conditions of error system (3) are given by

$$e_i(t) = \sigma_i(t) = \psi_i(t) - \phi_i(t),\ t \in [-\omega, 0],$$

where $\sigma_i \in \mathcal{C}([-\omega, 0], \mathbb{R})$, for $\forall i = 1, \ldots, N$.

The state feedback control scheme will be used to obtain finite-time synchronization between master system (1) and slave system (2). In this case, the controller is given by

$$u_i(t) = k_{i1} e_i(t) + k_{i2} e_i(t-\mu) + k_{i3} e_i(t-\tau(t)) + k_{i4} D_{-\eta}^\alpha e_i(t-\eta), \tag{4}$$

where $k_{i1},\ k_{i2},\ k_{i3},\ k_{i4} \in \mathbb{R}^+$ represent the control gains, for $\forall i = 1, \ldots, N$. System (3) now becomes

$$\begin{aligned} D_0^\alpha e_i(t) &= -k_{i1} e_i(t) - (c_i + k_{i2}) e_i(t-\mu) - k_{i3} e_i(t-\tau(t)) + \sum_{j=1}^{N} a_{ij} \overline{f}_j(e_j(t)) + \sum_{j=1}^{N} b_{ij} \overline{f}_j(e_j(t-\tau(t))) \\ &\quad + (g_i - k_{i4}) D_{-\eta}^\alpha e_i(t-\eta), \end{aligned} \tag{5}$$

for $\forall i = 1, \ldots, N$.

System (5) can be written in matrix form as

$$\begin{aligned} D_0^\alpha e(t) &= -K_1 e(t) - (C + K_2) e(t-\mu) - K_3 e(t-\tau(t)) + A\overline{f}(e(t)) + B\overline{f}(e(t-\tau(t))) \\ &\quad + (G - K_4) D_{-\eta}^\alpha e(t-\eta). \end{aligned} \tag{6}$$

Definition 4. The master system (1) is said to be finite-time synchronized with the slave system (2) based on the controller (4), if there exist positive constants $\{\delta, \varepsilon, T\}$, $0 < \delta < \varepsilon$, such that $||\sigma|| < \delta$ implies $||e(t)|| < \varepsilon$, $\forall t \in [0, T)$.

We will also use a different state feedback control scheme to realize finite-time Mittag–Leffler synchronization between master system (1) and slave system (2), for which the controller is given by

$$u_i(t) = k_{i1} e_i(t) + k_{i2} \mathrm{sign}(e_i(t)) |e_i(t-\mu)| + k_{i3} \mathrm{sign}(e_i(t)) |e_i(t-\tau(t))| + k_{i4} \mathrm{sign}(e_i(t)) |D_{-\eta}^\alpha e_i(t-\eta)|, \tag{7}$$

where $k_{i1},\ k_{i2},\ k_{i3},\ k_{i4} \in \mathbb{R}^+$ represent the control gains, for $\forall i = 1, \ldots, N$. System (3) becomes in this case

$$\begin{aligned} D_0^\alpha e_i(t) &= -k_{i1} e_i(t) - c_i e_i(t-\mu) - k_{i2} \mathrm{sign}(e_i(t)) |e_i(t-\mu)| - k_{i3} \mathrm{sign}(e_i(t)) |e_i(t-\tau(t))| \\ &\quad - k_{i4} \mathrm{sign}(e_i(t)) |D_{-\eta}^\alpha e_i(t-\eta)| + \sum_{j=1}^{N} a_{ij} \overline{f}_j(e_j(t)) + \sum_{j=1}^{N} b_{ij} \overline{f}_j(e_j(t-\tau(t))) \\ &\quad + g_i D_{-\eta}^\alpha e_i(t-\eta), \end{aligned} \tag{8}$$

for $\forall i = 1, \ldots, N$.

System (9) can be written in matrix form as

$$\begin{aligned} D_0^{\alpha}e(t) &= -K_1e(t) - Ce(t-\mu) - K_2\text{sign}(e(t))|e(t-\mu)| - K_3\text{sign}(e(t))|e(t-\tau(t))| \\ &\quad -K_4\text{sign}(e(t))|D_{-\eta}^{\alpha}e(t-\eta)| + A\overline{f}(e(t)) + B\overline{f}(e(t-\tau(t))) + GD_{-\eta}^{\alpha}e(t-\eta). \end{aligned} \tag{9}$$

Definition 5. The master system (1) is said to be finite-time Mittag–Leffler synchronized with the slave system (2) based on the controller (7), if there exist positive constants $\{\delta, \varepsilon, \lambda, \beta, T\}$, $0 < \delta < \varepsilon$, such that $||\sigma|| < \delta$ implies $||e(t)|| < ||\sigma||[E_{\alpha}(-\lambda t^{\alpha})]^{\beta} < \varepsilon$, $\forall t \in [0, T)$.

The following assumption has to be made in order to study the synchronization of the networks defined above.

Assumption 1. *The following Lipschitz conditions are satisfied by the activation functions f_j for any $x,\ x' \in \mathbb{R}$:*

$$||f_j(x) - f_j(x')|| \le l_j||x - x'||,$$

where, $\forall j = 1, \ldots, N$, $l_j > 0$ represent the Lipschitz constants. Also, let $L := diag(l_1, l_2, \ldots, l_N)$.

We will also need the following lemmas:

Lemma 1 ([45]). *"If $x \in \mathcal{C}^n([t_0, \infty), \mathbb{R})$, then*

$$I_{t_0}^{\alpha}D_{t_0}^{\alpha}x(t) = x(t) - \sum_{k=0}^{n-1}\frac{(t-t_0)^k}{k!}x^{(k)}(t_0),$$

where $t > t_0$ and n is a positive integer, with $n-1 < \alpha < n$. Moreover, when $0 < \alpha < 1$, we have that:

$$I_{t_0}^{\alpha}D_{t_0}^{\alpha}x(t) = x(t) - x(t_0)."$$

Lemma 2 ([46]). *"Suppose $\alpha > 0$, $a(t)$ is a nonnegative function locally integrable on $0 \le t < T$ (for some $T \le +\infty$) and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \le t < T$, $g(t) \le M$ (M is a constant), and suppose $u(t)$ is nonnegative and locally integrable on $0 \le t < T$ with*

$$u(t) \le a(t) + g(t)\int_0^t (t-s)^{\alpha-1}u(s)ds,\ 0 \le t < T.$$

Then

$$u(t) \le a(t) + \int_0^t \left[\sum_{n=1}^{\infty}\frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}a(s)\right]ds,\ 0 \le t < T.$$

Moreover, if $a(t)$ is a nondecreasing function on $0 \le t < T$, then

$$u(t) \le a(t)E_{\alpha}(g(t)\Gamma(\alpha)t^{\alpha}),\ 0 \le t < T."$$

Lemma 3 ([47]). *"If $x \in \mathcal{C}^1([t_0, \infty), \mathbb{R}^N)$, then*

$$\frac{1}{2}D_{t_0}^{\alpha}(x^T(t)x(t)) \le x^T(t)D_{t_0}^{\alpha}(x(t)),\ \forall t \ge t_0,$$

where $0 < \alpha < 1$."

Lemma 4 ([48]). *"Let $V \in \mathcal{C}([t_0, \infty), \mathbb{R})$ which satisfies*

$$D_{t_0}^{\alpha}V(t) \le -\lambda V(t),\ \forall t \ge t_0,$$

where $0 < \alpha < 1$ and $\lambda > 0$. Then

$$V(t) \le V(t_0)E_{\alpha}(-\lambda t^{\alpha}),\ \forall t \ge t_0."$$

3. Main Results

In the following, we will assume that $0 < \alpha < 1$.

First, we give a sufficient condition that ensures the finite-time synchronization of master system (1) and slave system (2), based on the controller (4).

Theorem 1. *If Assumption 1 holds, then master system (1) and slave system (2) are finite-time synchronized based on the controller (4) if $||G - K_4|| < 1$ and there exist positive constants $\{\delta, \varepsilon, T\}$ such that the following inequality holds:*

$$\frac{1+||G-K_4||}{1-||G-K_4||}E_\alpha\left(\frac{1}{1-||G-K_4||}(||K_1||+||A||||L||+||C+K_2||+||K_3||+||B||||L||)t^\alpha\right) < \frac{\varepsilon}{\delta},\ \forall t \in [0,T). \tag{10}$$

Proof. Integrating relation (6), we get that

$$\begin{aligned} I_0^\alpha D_0^\alpha e(t) &= -K_1 I_0^\alpha e(t) - (C+K_2) I_0^\alpha e(t-\mu) - K_3 I_0^\alpha e(t-\tau(t)) + A I_0^\alpha \overline{f}(e(t)) + B I_0^\alpha \overline{f}(e(t-\tau(t))) \\ &\quad +(G-K_4) I_0^\alpha D_{-\eta}^\alpha e(t-\eta), \end{aligned}$$

or, by using Lemma 1, that

$$\begin{aligned} e(t)-e(0) &= -K_1\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e(s)ds - (C+K_2)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e(s-\mu)ds \\ &\quad -K_3\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e(s-\tau(s))ds + A\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\overline{f}(e(s))ds \\ &\quad +B\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\overline{f}(e(s-\tau(s)))ds + (G-K_4) I_0^\alpha D_{-\eta}^\alpha e(t-\eta). \end{aligned} \tag{11}$$

We have that

$$\begin{aligned} I_0^\alpha D_{-\eta}^\alpha e(t-\eta) &= I_0^\alpha\left(\frac{1}{\Gamma(1-\alpha)}\int_{-\eta}^{t-\eta}\frac{\dot{e}(u)}{(t-\eta-u)^\alpha}du\right) \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\frac{1}{\Gamma(1-\alpha)}\int_{-\eta}^{s-\eta}\frac{\dot{e}(u)}{(s-\eta-u)^\alpha}du\right)ds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t (t-s)^{\alpha-1}\int_{-\eta}^{s-\eta}\frac{\dot{e}(u)}{(s-\eta-u)^\alpha}duds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{-\eta}^{t-\eta}\dot{e}(u)\int_{\eta+u}^t\frac{(t-s)^{\alpha-1}}{(s-\eta-u)^\alpha}dsdu. \end{aligned}$$

With the change of variable $s-\eta-u = v$, we get

$$I_0^\alpha D_{-\eta}^\alpha e(t-\eta) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{-\eta}^{t-\eta}\dot{e}(u)\int_0^{t-\eta-u}\frac{(t-\eta-u-v)^{\alpha-1}}{v^\alpha}dvdu.$$

Then, with the change of variable $v = (t-\eta-u)w$, we get

$$\begin{aligned} I_0^\alpha D_{-\eta}^\alpha e(t-\eta) &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{-\eta}^{t-\eta}\dot{e}(u)\int_0^{t-\eta-u}\frac{(t-\eta-u-v)^{\alpha-1}}{v^\alpha}dvdu \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{-\eta}^{t-\eta}\dot{e}(u)\int_0^1\frac{(t-\eta-u)^{\alpha-1}(1-w)^{\alpha-1}}{(t-\eta-u)^\alpha w^\alpha}(t-\eta-u)dwdu \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{-\eta}^{t-\eta}\dot{e}(u)\int_0^1 (1-w)^{\alpha-1}w^{(1-\alpha)-1}dwdu \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{-\eta}^{t-\eta}\dot{e}(u)B(1-\alpha,\alpha)du, \end{aligned}$$

where $B(x,y)$ denotes the Euler beta function. Therefore:

$$I_0^\alpha D_{-\eta}^\alpha e(t-\eta) = \int_{-\eta}^{t-\eta} \dot{e}(u)du = e(t-\eta) - e(-\eta).$$

By taking the norm of relation (12), and also taking into account the above relation, we obtain that

$$\begin{aligned}
||e(t)|| \leq\; & ||e(0)|| + ||G-K_4||||e(-\eta)|| + ||K_1||\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}||e(s)||ds \\
& + ||C+K_2||\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}||e(s-\mu)||ds \\
& + ||K_3||\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}||e(s-\tau(s))||ds \\
& + ||A||\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}||\bar{f}(e(s))||ds \\
& + ||B||\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}||\bar{f}(e(s-\tau(s)))||ds \\
& + ||G-K_4||||e(t-\eta)|| \\
\leq\; & (1+||G-K_4||)||\sigma|| + (||K_1|| + ||A||||L||)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}||e(s)||ds \\
& + ||C+K_2||\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}||e(s-\mu)||ds \\
& + (||K_3|| + ||B||||L||)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}||e(s-\tau(s))||ds + ||G-K_4||||e(t-\eta)||,
\end{aligned}$$

where, for the last inequality, we used Assumption 1.

Let $v(t) = \sup_{t-\omega\leq s\leq t}||e(s)||$, $\forall t > 0$, then $||e(t)|| \leq v(t)$, $||e(t-\mu)|| \leq v(t)$, $||e(t-\tau(t))|| \leq v(t)$, $||e(t-\eta)|| \leq v(t)$, $\forall t > 0$. Also, $v(0) = ||\sigma||$. We now have that

$$\begin{aligned}
||e(t)|| \leq\; & (1+||G-K_4||)||\sigma|| + ||G-K_4||v(t) \\
& + (||K_1|| + ||A||||L|| + ||C+K_2|| + ||K_3|| + ||B||||L||)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds,
\end{aligned}$$

which yields

$$\begin{aligned}
v(t) \leq\; & (1+||G-K_4||)||\sigma|| + ||G-K_4||v(t) \\
& + (||K_1|| + ||A||||L|| + ||C+K_2|| + ||K_3|| + ||B||||L||)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds,
\end{aligned}$$

which is equivalent with

$$\begin{aligned}
v(t) \leq\; & \frac{1+||G-K_4||}{1-||G-K_4||}||\sigma|| + \frac{1}{1-||G-K_4||} \\
& \times(||K_1|| + ||A||||L|| + ||C+K_2|| + ||K_3|| + ||B||||L||)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds. \quad (12)
\end{aligned}$$

If we denote

$$a(t) = \frac{1+||G-K_4||}{1-||G-K_4||}||\sigma||,$$

$$g(t) = \frac{1}{1-||G-K_4||}(||K_1|| + ||A||||L|| + ||C+K_2|| + ||K_3|| + ||B||||L||)\frac{1}{\Gamma(\alpha)},$$

then inequality (12) becomes:

$$v(t) \quad \leq \quad a(t) + g(t) \int_0^t (t-s)^{\alpha-1} v(s) ds.$$

Now, applying Lemma 2, we have

$$\begin{aligned} ||e(t)|| &\leq v(t) \\ &\leq a(t) E_\alpha (g(t)\Gamma(\alpha) t^\alpha) \\ &= ||\sigma|| \frac{1 + ||G - K_4||}{1 - ||G - K_4||} E_\alpha \left(\frac{1}{1 - ||G - K_4||} (||K_1|| + ||A||||L|| + ||C + K_2|| + ||K_3|| + ||B||||L||) t^\alpha \right) \\ &< \delta \cdot \frac{\varepsilon}{\delta} = \varepsilon, \ \forall t \in [0, T), \end{aligned}$$

where, for the last inequality, we used condition (10). This completes the proof of the theorem. □

Remark 1. The condition in Theorem 1 only needs the computation of the norm of 8 matrices, which is easily done using the default function in MATLAB. Also, the verification of the inequality only needs computing the Mittag–Leffler function for the biggest argument, because it is a strictly increasing function.

We now give a sufficient condition that ensures the finite-time Mittag–Leffler synchronization of master system (1) and slave system (2), based on the controller (7).

Theorem 2. *If Assumption 1 holds, then master system* (1) *and slave system* (2) *are finite-time Mittag–Leffler synchronized based on the controller* (7) *if*

$$K_1 - |A|L > 0, \ K_2 - C > 0, \ K_3 - |B|L > 0, \ K_4 - |G| > 0, \tag{13}$$

and there exist positive constants $\{\delta, \epsilon, T\}$ *such that*

$$E_\alpha(-\lambda t^\alpha) < \frac{\varepsilon^2}{\delta^2}, \ \forall t \in [0, T), \tag{14}$$

where $\lambda = 2\lambda_{\min}(K_1 - |A|L)$.

Proof. The following Lyapunov function will be considered:

$$V(t) = \frac{1}{2} e^T(t) e(t).$$

Taking into account Lemma 3, and computing the fractional-order derivative of $V(t)$ along the trajectories of system (9), it follows that

$$\begin{aligned}
D_0^\alpha V(t) &= D_0^\alpha \left(\frac{1}{2}e^T(t)e(t)\right) \\
&\leq e^T(t)D_0^\alpha e(t) \\
&= e^T(t)\left[-K_1e(t) - Ce(t-\mu) - K_2\text{sign}(e(t))|e(t-\mu)| - K_3\text{sign}(e(t))|e(t-\tau(t))|\right. \\
&\quad \left. -K_4\text{sign}(e(t))|D_{-\eta}^\alpha e(t-\eta)| + A\overline{f}(e(t)) + B\overline{f}(e(t-\tau(t))) + GD_{-\eta}^\alpha e(t-\eta)\right] \\
&= -e^T(t)K_1e(t) - e^T(t)Ce(t-\mu) - |e^T(t)|K_2|e(t-\mu)| - |e^T(t)|K_3|e(t-\tau(t))| \\
&\quad -|e^T(t)|K_4|D_{-\eta}^\alpha e(t-\eta)| + e^T(t)A\overline{f}(e(t)) + e^T(t)B\overline{f}(e(t-\tau(t))) + e^T(t)GD_{-\eta}^\alpha e(t-\eta) \qquad (15) \\
&\leq -|e^T(t)|K_1|e(t)| + |e^T(t)|(C-K_2)|e(t-\mu)| - |e^T(t)|K_3|e(t-\tau(t))| \\
&\quad -|e^T(t)|K_4|D_{-\eta}^\alpha e(t-\eta)| + |e^T(t)|(|A|L)|e(t)| + |e^T(t)|(|B|L)|e(t-\tau(t))| \\
&\quad +|e^T(t)|(|G|)|D_{-\eta}^\alpha e(t-\eta)| \\
&= -|e^T(t)|(K_1-|A|L)|e(t)| - |e^T(t)|(K_2-C)|e(t-\mu)| \\
&\quad -|e^T(t)|(K_3-|B|L)|e(t-\tau(t))| - |e^T(t)|(K_4-|G|)|D_{-\eta}^\alpha e(t-\eta)|,
\end{aligned}$$

where, for the last inequality, we used Assumption 1.

Now, taking into account conditions (13), inequality (16) becomes:

$$\begin{aligned}
D_0^\alpha V(t) &\leq -|e^T(t)|(K_1-|A|L)|e(t)| \\
&\leq -\lambda V(t),
\end{aligned}$$

where $\lambda = 2\lambda_{\min}(K_1 - |A|L) > 0$. By Lemma 4, we have that

$$V(t) \leq V(0)E_\alpha(-\lambda t^\alpha),$$

or, equivalently,

$$\begin{aligned}
\frac{1}{2}||e(t)||^2 &\leq \frac{1}{2}||e(0)||^2 E_\alpha(-\lambda t^\alpha) \\
&\leq \frac{1}{2}||\sigma||^2 E_\alpha(-\lambda t^\alpha),
\end{aligned}$$

which further leads to

$$\begin{aligned}
||e(t)|| &\leq ||\sigma||\left(E_\alpha(-\lambda t^\alpha)\right)^{\frac{1}{2}} \\
&< \delta \cdot \frac{\varepsilon}{\delta} = \varepsilon,\ \forall t \in [0,T),
\end{aligned}$$

where, for the last inequality, we used condition (14). This completes the proof of the theorem. □

Remark 2. The condition in Theorem 2 only needs to verify that 4 matrices are positive definite. Again, the verification of the inequality only needs computing the Mittag–Leffler function for the biggest argument, because it is a strictly increasing function.

4. Numerical Examples

Two numerical examples shall be given in this section to illustrate the feasibility and the effectiveness of the sufficient criteria deduced above.

Example 1. The two-neuron neutral-type fractional-order neural network having leakage delay and time-varying delays will be the master system:

$$\begin{aligned} D_0^\alpha x(t) &= -Cx(t-\mu) + \sum_{j=1}^{N} Af(x(t)) + \sum_{j=1}^{N} Bf(x(t-\tau(t))) \\ &\quad + GD_{-\eta}^\alpha x(t-\eta) + I, \end{aligned} \tag{16}$$

the fractional-order neural network with two neurons will be the slave system:

$$\begin{aligned} D_0^\alpha y(t) &= -Cy(t-\mu) + \sum_{j=1}^{N} Af(y(t)) + \sum_{j=1}^{N} Bf(y(t-\tau(t))) \\ &\quad + GD_{-\eta}^\alpha y(t-\eta) + I - u(t), \end{aligned} \tag{17}$$

and the controller will be:

$$u(t) = K_1 e(t) + K_2 e(t-\mu) + K_3 e(t-\tau(t)) + K_4 D_{-\eta}^\alpha e(t-\eta), \tag{18}$$

where $e(t) = y(t) - x(t)$, and $\alpha = 0.5$,

$$C = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix},\ A = \begin{bmatrix} 0.2 & -0.2 \\ -0.2 & 0.2 \end{bmatrix},\ B = \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix},\ G = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.5 \end{bmatrix},$$

$$f(x) = \frac{1}{1+e^{-x}},\ x \in \mathbb{R},$$

from which we deduce that $L = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix}$, and so Assumption 1 is fulfilled. The leakage delay is $\mu = 0.04$, the time-varying delays are $\tau(t) = 0.1|\cos t|$, and the neutral-type delay is $\eta = 0.05$, from where we get that $\tau = \tau' = 0.1$ and $\omega = \max\{\mu, \tau, \eta\} = 0.1$.

The values of K_1, K_2, K_3, K_4 are designed as:

$$K_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},\ K_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},\ K_3 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},\ K_4 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

With these values we get that $||G - K_4|| = 0.6 < 1$, and, if we take $\delta = 0.1$ and $\varepsilon = 100$, then condition (10) holds for $T = 2.1411$, which means that the conditions of Theorem 1 are fulfilled. Thus, we obtain that master system (16) is finite-time synchronized with slave system (17), based on controller (18).

The state trajectories and the phase trajectories of the errors e_1 and e_2 are depicted in Figures 1–3, for 8 initial values.

Example 2. Now, the master system will be the fractional-order neural network having two neurons:

$$\begin{aligned} D_0^\alpha x(t) &= -Cx(t-\mu) + \sum_{j=1}^{N} Af(x(t)) + \sum_{j=1}^{N} Bf(x(t-\tau(t))) \\ &\quad + GD_{-\eta}^\alpha x(t-\eta) + I, \end{aligned} \tag{19}$$

the slave system will be the fractional-order neural network having two neurons:

$$\begin{aligned} D_0^\alpha y(t) &= -Cy(t-\mu) + \sum_{j=1}^{N} Af(y(t)) + \sum_{j=1}^{N} Bf(y(t-\tau(t))) \\ &\quad + GD_{-\eta}^\alpha y(t-\eta) + I - u(t), \end{aligned} \tag{20}$$

and the controller will be:

$$u(t) = K_1 e(t) + K_2 \mathrm{sign}(e(t))|e(t-\mu)| + K_3 \mathrm{sign}(e(t))|e(t-\tau(t))| + K_4 \mathrm{sign}(e(t))|D^{\alpha}_{-\eta} e(t-\eta)|, \quad (21)$$

where $e(t) = y(t) - x(t)$, and $\alpha = 0.75$,

$$C = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \; A = \begin{bmatrix} 0.2 & -0.2 \\ -0.2 & 0.2 \end{bmatrix}, \; B = \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}, \; G = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$f(x) = \frac{1}{1+e^{-x}}, \; x \in \mathbb{R},$$

from which we get that $L = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix}$, which means that Assumption 1 is fulfilled. If the leakage delay is $\mu = 0.07$, the time-varying delays are $\tau(t) = 0.1|\sin t|$, and the neutral-type delay is $\eta = 0.06$, then $\tau = \tau' = 0.1$ and $\omega = \max\{\mu, \tau, \eta\} = 0.1$.

The values of K_1, K_2, K_3, K_4 are designed as:

$$K_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \; K_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, \; K_3 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \; K_4 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

With these values we get that $K_1 - |A|L > 0$, $K_2 - C > 0$, $K_3 - |B|L > 0$, $K_4 - |G| > 0$, and $\lambda = 2\lambda_{\min}(K_1 - |A|L) = 0.2 > 0$, so conditions (13) hold. Condition (14) is satisfied for any $\varepsilon > \delta$ and any $T > 0$, because $E_\alpha(-\lambda t^\alpha) \leq 1$, $\forall t \geq 0$. By applying Theorem 2, we get that master system (19) is finite-time Mittag–Leffler synchronized with slave system (20), based on controller (21).

The state trajectories and the phase trajectories of the errors e_1 and e_2 are depicted in Figures 4–6, for 8 initial values.

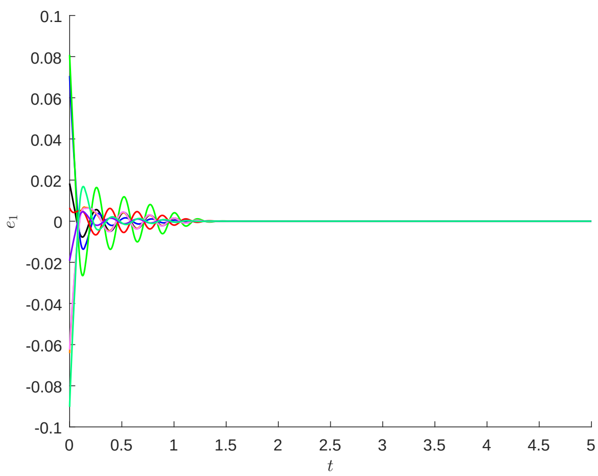

Figure 1. State trajectories of the error e_1 in Example 1, for 8 initial values (depicted with different colors). Best viewed in color.

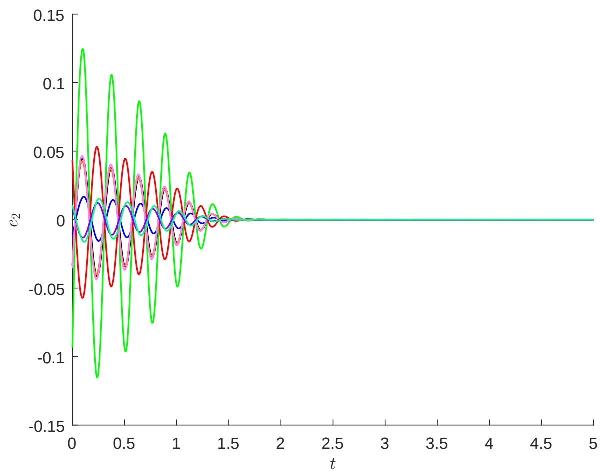

Figure 2. State trajectories of the error e_2 in Example 1, for 8 initial values (depicted with different colors). Best viewed in color.

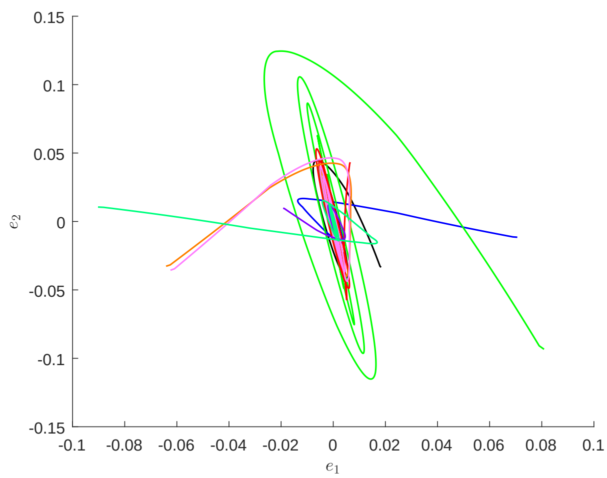

Figure 3. Phase trajectories of the errors e_1 and e_2 in Example 1, for 8 initial values (depicted with different colors). Best viewed in color.

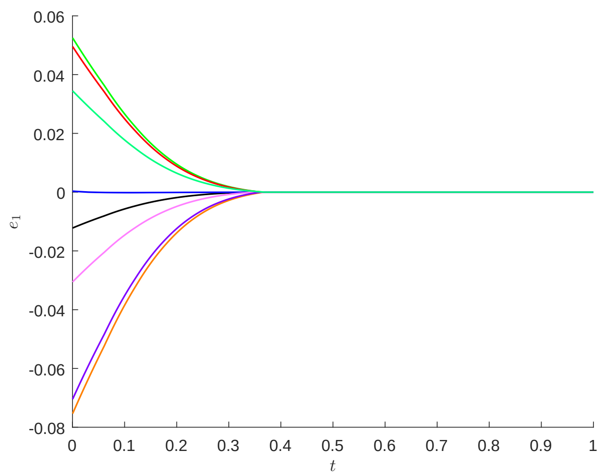

Figure 4. State trajectories of the error e_1 in Example 2, for 8 initial values (depicted with different colors). Best viewed in color.

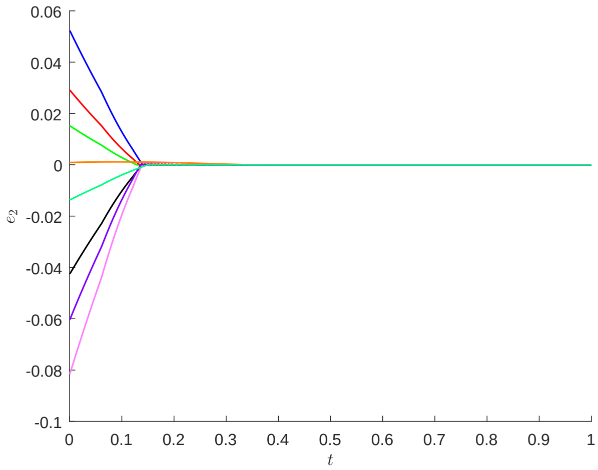

Figure 5. State trajectories of the error e_2 in Example 2, for 8 initial values (depicted with different colors). Best viewed in color.

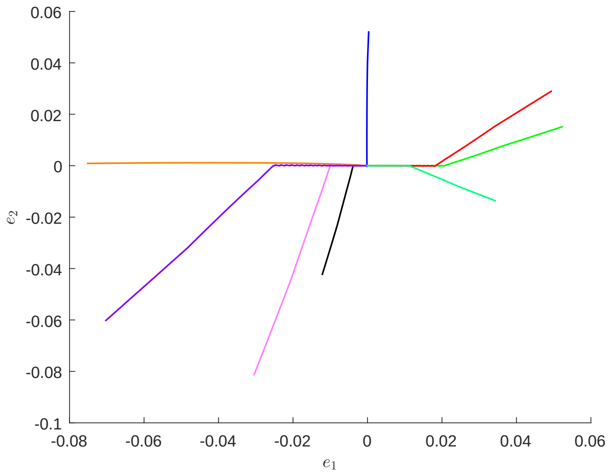

Figure 6. Phase trajectories of the errors e_1 and e_2 in Example 2, for 8 initial values (depicted with different colors). Best viewed in color.

5. Conclusions

Two sufficient criteria which ensure the finite-time synchronization and finite-time Mittag–Leffler synchronization of fractional-order neural networks with neutral-type delay, leakage delay, and time-varying delays were given, by making the assumption that the activation functions satisfy the Lipschitz conditions. The generalized Gronwall–Bellman inequality was used to realize the finite-time synchronization of the introduced networks, based on a general state feedback control scheme. Then, the fractional-order extension of the Lyapunov direct method for Mittag–Leffler stability was used to realize the finite-time Mittag–Leffler synchronization of the same networks, based on a different general state feedback control scheme. The feasibility and the effectiveness of the theoretical results was illustrated by providing two numerical examples.

The methods developed in the paper are general, and can be used to obtain sufficient criteria for the finite-time synchronization and the finite-time Mittag–Leffler synchronization of neural network models with impulsive effects, reaction–diffusion terms, or Markovian jump parameters. These developments represent promising future work directions.

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