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Global Behavior of Solutions to a Higher-Dimensional System of Difference Equations with Lucas Numbers Coefficients

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Abstract: In this paper, we derive the well-defined solutions to a θ-dimensional system of difference equations. We show that, the well-defined solutions to that system are represented in terms of Fibonacci and Lucas sequences. Moreover, we study the global stability of the solutions to that system. Finally, we give some numerical examples which confirm our theoretical results.

Keywords: Lucas numbers; Fibonacci numbers; system of difference equations; global stability

1. Introduction

Difference equations and systems of difference equations are of great importance in the field of mathematics as well as in other sciences. The applications of the difference equations appear as discrete mathematical models of many phenomena such as in biology, economics, ecology, control theory, physics, engineering, population dynamics and so forth [1–6]. This is the reason why, recently, many scientists have devoted their work to the study of the theory of difference equations, the boundedness, the periodicity and the global asymptotic stability of their solutions [7–31].

In the following, we will use the following notations: $\mathbb{N}$ for the set of natural numbers, and $\mathbb{N}_\nu$ for the set $\{n \in \mathbb{Z} : n \geq \nu\}$. Recently, there has been a growing interest in the study of finding closed-form solutions of difference equations and systems of difference equations. Some of the forms of solutions of these equations are representable via well-known integer sequences such as Fibonacci numbers (see, for example [26,32]), Horadam numbers (see, for example, [30,31]), Lucas numbers (see, for example [25,27,33]), Pell numbers and Padovan numbers (see, for example [34–36]), But in this paper, we present the solution in the form of Lucas sequences.

In [32], the authors represented the general solution of the following difference equation

$$x_{n+1} = \frac{1}{1+x_n}, \quad n \in \mathbb{N}_0, \tag{1}$$

in terms of the initial value x_0 and the Fibonacci sequence. Namely, it was proved by induction that every well-defined solution of Equation (1) can be written in the following form

$$x_n = \frac{F_n + F_{n-1}x_0}{F_{n+1} + F_n x_0}, \quad n \in \mathbb{N}_0,$$

where $\{F_n\}_{n=0}^{\infty}$ is Fibonacci sequence. They also proved that, every well-defined solution of the equation

$$x_{n+1} = \frac{1}{-1+x_n}, \quad n \in \mathbb{N}_0, \tag{2}$$

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can be written in the following form

$$x_n = \frac{F_{-n} + F_{-(n-1)}x_0}{F_{-(n+1)} + F_{-n}x_0}, \quad n \in \mathbb{N}_0,$$

where the terms of the Fibonacci sequence with negative indices are calculated by the formula

$$F_{-n} = F_{-n+2} - F_{-n+1}, \quad n \in \mathbb{N}_0,$$

with $F_0 = 0$ and $F_1 = 1$.

Stevic [30] represented the general solution of the following bilinear system of difference equations

$$x_{n+1} = \frac{\alpha + \beta y_n}{\gamma + \sigma y_n}, \quad y_{n+1} = \frac{\alpha + \beta x_n}{\gamma + \sigma x_n},$$

in terms of the initial values x_0, y_0 and the generalized Fibonacci sequence.

In [26], Khelifa and Halim analyzed the general solution to the system of difference equations

$$x^{(j)}_{n+1} = \frac{F_{m+2} + F_{m+1}x^{((j+1) \mod (p))}_{n-k}}{F_{m+3} + F_{m+2}x^{((j+1) \mod (p))}_{n-k}},$$

where $n, m \in \mathbb{N}_0$, $j = 1, \ldots, p$, and $(F_n)_{n=0}^{+\infty}$ is Fibonacci sequence. They expressed the solution to this system in terms of the Fibonacci sequence.

In this paper, we derive the well-defined solutions to the θ-dimensional system of difference equations of the form

$$x^{(q)}_{n+1} = \frac{L_{m+2} + L_{m+1}x^{((q+1) \mod (\theta))}_{n-k}}{L_{m+3} + L_{m+2}x^{((q+1) \mod (\theta))}_{n-k}}, \quad n \in \mathbb{N}_0, \tag{3}$$

where $k \in \mathbb{N}_0$, $m \in \mathbb{Z}$, $q = 1, \cdots, \theta$, $\theta \in \mathbb{N}$, $(L_n)_{n=0}^{+\infty}$ is Lucas sequence, and the initial values $x^{(q)}_{-i}$, $i = 0, \ldots, k$ are real numbers, $q = 1, \ldots, \theta$.

Clearly, our system generalizes the equations and systems studied in [25,27,32,33].

2. Preliminaries

Fibonacci sequence [6,37] is defined by

$$F_{n+1} = F_n + F_{n-1}, \quad n \in \mathbb{N}, \tag{4}$$

where $F_0 = 0$ and $F_1 = 1$. The solution to Equation (4) is given by the Binet formula of Fibonacci sequence is given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where

$$\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}.$$

Here, α is the so-called golden number.

Lucas sequence is the sequence of integer numbers defined by the recurrence relation

$$L_{n+1} = L_n + L_{n-1}, \quad n \in \mathbb{N}, \tag{5}$$

with initial conditions $L_0 = 2$ and $L_1 = 1$. The Binet formula for the Lucas sequence is given by

$$L_n = \alpha^n + \beta^n.$$

The formula of terms with negative indices in the Lucas sequence is

$$L_{-n} = (-1)^n L_n, \quad n = 1, 2, \cdots.$$

Now, we list a set of properties concerning Fibonacci and Lucas sequences that will be used through the paper [6,37].

Propertiesn 1. *Suppose that $(F_n)_{n=0}^{+\infty}$ and $(L_n)_{n=0}^{+\infty}$ are Fibonacci and Lucas sequences, and let $\eta, \kappa \in \mathbb{N}$. Then*

1. $L_{\eta+1} L_{(\kappa-1)\eta} + L_\eta L_{(\kappa-1)\eta-1} = 5F_{\kappa\eta}$,
2. $L_\eta L_{(\kappa-1)\eta} + L_{\eta-1} L_{(\kappa-1)\eta-1} = 5F_{\kappa\eta-1}$,
3. $F_{\kappa+1} L_{\eta-\kappa} + F_\kappa L_{\eta-(\kappa+1)} = L_\eta$,
4. $F_\kappa L_{\eta+3} + F_{\kappa-1} L_{\eta+2} = L_{\kappa+\eta+2}$,
5. $L_{\kappa(\eta+2)-1} + L_{\kappa(\eta+2)+1} = 5F_{\kappa(\eta+2)}$,
6. $L^2_{\kappa(\eta+2)} - L_{\kappa(\eta+2)-1} L_{\kappa(\eta+2)+1} = 5(-1)^{\kappa(\eta+2)}$,
7. $L_\kappa L_{\eta-(\kappa-1)} + L_{(\kappa-1)} L_{\eta-\kappa} = 5F_\eta$,
8. $F_\kappa F_{\eta-(\kappa-1)} + F_{\kappa-1} F_{\eta-\kappa} = F_\eta$,

Propertiesn 2. *Suppose that $(F_n)_{n=0}^{+\infty}$ and $(L_n)_{n=0}^{+\infty}$ are Fibonacci and Lucas sequences, and let $\eta, \kappa \in \mathbb{N}$. Then*

1. $5F_{2\kappa\eta} + (-1)^\kappa L_{\kappa+1} L_{\kappa(2\eta+1)} = (-1)^\kappa L_\kappa L_{\kappa(2\eta+1)+1}$,
2. $L_{\kappa(2\eta+1)} - L_{\kappa+1} F_{2\kappa\eta} = L_\kappa F_{2\kappa\eta-1}$,
3. $L_{\kappa(2\eta-1)-1} + (-1)^\kappa L_{\kappa+1} F_{2\kappa\eta-1} = (-1)^\kappa L_\kappa F_{2\kappa\eta}$,

Theorem 3 ([26])**.** *Let $(u_n)_{n \geq -\theta+1}$ be a solution to Equation*

$$u_{n+1} = \frac{F_{\theta(m+2)} + F_{\theta(m+2)-1} u_{n-(\theta-1)}}{F_{\theta(m+2)+1} + F_{\theta(m+2)} u_{n-(\theta-1)}}, \quad n \in \mathbb{N}_0. \tag{6}$$

Then for $n \in \mathbb{N}_0$,

$$u_{\theta n+j} = \frac{F_{\theta n(m+2)} + F_{\theta n(m+2)-1} u_j}{F_{\theta n(m+2)+1} + F_{\theta n(m+2)} u_j},$$

where $\theta, m \in \mathbb{N}$, $j \in \{0, 1, \cdots, \theta - 1\}$.

Proof. The proof of this theorem can be found in [26], where it is similar to Equation (17) there. □

3. Main Results

In this section, we establish the form of the solution to system (3).

Using the transformation

$$X_n^{(q)} = x^{(q)}_{(k+1)n-t}, \quad n \in \mathbb{N}_0, \tag{7}$$

where $t \in \{0, 1, \ldots, k\}$ and $q \in \{1, 2, \ldots, \theta\}$, we can write system (3) as

$$\begin{aligned} X_{n+1}^{(1)} &= \frac{L_{m+2}+L_{m+1}X_n^{(2)}}{L_{m+3}+L_{m+2}X_n^{(2)}}, \\ X_{n+1}^{(2)} &= \frac{L_{m+2}+L_{m+1}X_n^{(3)}}{L_{m+3}+L_{m+2}X_n^{(3)}}, \\ &\vdots \\ X_{n+1}^{(\theta)} &= \frac{L_{m+2}+L_{m+1}X_n^{(1)}}{L_{m+3}+L_{m+2}X_n^{(1)}}, \quad n\in\mathbb{N}_0. \end{aligned} \tag{8}$$

3.1. Solvability of System (8)

In this section, we shall derive the solution to system (8).

If we use the second recurrence relation in system (8) in the first, we obtain

$$X_{n+1}^{(1)} = \frac{F_{2m+4}+F_{2m+3}X_{n-1}^{(3)}}{F_{2m+5}+F_{2m+4}X_{n-1}^{(3)}}, \quad n\geq 1.$$

The substitution of $X_{n-1}^{(3)}$ into $X_{n+1}^{(1)}$ leads to

$$X_{n+1}^{(1)} = \frac{L_{3m+6}+L_{3m+5}X_{n-2}^{(4)}}{L_{3m+7}+L_{3m+6}X_{n-2}^{(4)}}, \quad n\geq 2.$$

Similarly, if we replace $X_{n-1}^{(4)}$ into $X_{n+1}^{(1)}$, we get

$$X_{n+1}^{(1)} = \frac{F_{4m+8}+F_{4m+7}x_{n-3}^{(5)}}{F_{4m+9}+F_{4m+8}X_{n-3}^{(5)}}, \quad n\geq 3.$$

Proceeding in the same manner, and using Property 1, system (8) can be written in the following form:

$$X_{n+1}^{(1)} = \frac{F_{2p(m+2)}+F_{2p(m+2)-1}X_{n-(2p-1)}^{(1)}}{F_{2p(m+2)+1}+F_{2p(m+2)}X_{n-(2p-1)}^{(1)}}, \quad n\geq 2p-1, \tag{9}$$

when $\theta=2p$, and

$$X_{n+1}^{(1)} = \frac{L_{(2p+1)(m+2)}+L_{(2p+1)(m+2)-1}X_{n-2p}^{(1)}}{L_{(2p+1)(m+2)+1}+L_{(2p+1)(m+2)}X_{n-2p}^{(1)}}, \quad n\geq 2p, \tag{10}$$

when $\theta=2p+1$, where $p\in\mathbb{N}$.

3.1.1. Case $\theta=2p$

Here, we consider the case $\theta=2p$. Using Theorem 3, the solution of Equation (9) can be written as

$$X_{2pn+j}^{(1)} = \frac{F_{2pn(m+2)}+F_{2pn(m+2)-1}X_j^{(1)}}{F_{2pn(m+2)+1}+F_{2pn(m+2)}X_j^{(1)}}, \quad n\in\mathbb{N}_0,$$

where $p\in\mathbb{N}$, $m\in\mathbb{Z}$ and $j\in\{0,1,\cdots,2p\}$.

Theorem 4. *Let* $\left(X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(2p)}\right)_{n\geq 0}$ *be a solution to system (8). Then*

$$X_{2pn+j}^{(q)} = \frac{L_{2p(m+2)n+j(m+2)} + L_{2p(m+2)n+j(m+2)-1}X_0^{(s)}}{L_{2p(m+2)n+j(m+2)+1} + L_{2p(m+2)n+j(m+2)}X_0^{(s)}}, \; j \in \{1,3,\dots,2p-1\},$$

and

$$X_{2pn+j}^{(q)} = \frac{F_{2p(m+2)n+j(m+2)} + F_{2p(m+2)n+j(m+2)-1}X_0^{(s)}}{F_{2p(m+2)n+j(m+2)+1} + F_{2p(m+2)n+j(m+2)}X_0^{(s)}}, \; j \in \{0,2,\dots,2p-2\},$$

where $n \in \mathbb{N}_0$, $s = (q+j) \mod (2p)$ *and* $q \in \{1,2,\dots,2p\}$.

Proof. Let $\left(X_n^{(1)}, X_n^{(2)}, \dots, X_n^{2p}\right)_{n\geq 0}$ be a solution to system (8). Then

$$X_{2pn+j}^{(1)} = \frac{F_{2pn(m+2)} + F_{2pn(m+2)-1}X_j^{(1)}}{F_{2pn(m+2)+1} + F_{2pn(m+2)}X_j^{(1)}}, \quad n \in \mathbb{N}_0\,. \tag{11}$$

We consider two different situations, depending on whether j is even or odd.

- When j is odd, we have that

$$X_j^{(1)} = \frac{L_{j(m+2)} + L_{j(m+2)-1}X_0^{(1+j)}}{L_{j(m+2)+1} + L_{j(m+2)}X_0^{(1+j)}}.$$

  This implies that for $j \in \{1,3,\cdots,2p-1\}$ we get

$$\begin{aligned} X_{2pn+j}^{(1)} = &\ \frac{(L_{j(m+2)+1} + L_{j(m+2)}X_0^{(1+j)})L_{2p(m+2)n+j(m+2)-j(m+2)}}{(L_{j(m+2)+1} + L_{j(m+2)}X_0^{(1+j)})L_{2p(m+2)n+j(m+2)-j(m+2)+1}} \\ &\ \frac{+(L_{j(m+2)} + L_{j(m+2)-1}X_0^{(1+j)})L_{2p(m+2)n+j(m+2)-j(m+2)-1}}{+(L_{j(m+2)} + L_{j(m+2)-1}X_0^{(1+j)})L_{2p(m+2)n+j(m+2)-j(m+2)}}, \\ = &\ \frac{L_{2p(m+2)n+j(m+2)} + L_{2p(m+2)n+j(m+2)-1}X_0^{(1+j)}}{L_{2p(m+2)n+j(m+2)+1} + L_{2p(m+2)n+j(m+2)}X_0^{(1+j)}}. \end{aligned}$$

- Similarly, for $j \in \{0,2,\cdots,2p-2\}$, we get

$$X_{2pn+j}^{(1)} = \frac{F_{2p(m+2)n+j(m+2)} + F_{2p(m+2)n+j(m+2)-1}X_0^{(1+j)}}{F_{2p(m+2)n+j(m+2)+1} + F_{2p(m+2)n+j(m+2)}X_0^{(1+j)}}.$$

  Then for the solution $\left(X_n^{(1)}, X_n^{(2)}, \dots, X_n^{(2p)}\right)_{n\geq 0}$, we can write

$$X_{2pn+j}^{(q)} = \frac{L_{2p(m+2)n+j(m+2)} + L_{2p(m+2)n+j(m+2)-1}X_0^{(s)}}{L_{2p(m+2)n+j(m+2)+1} + L_{2p(m+2)n+j(m+2)}X_0^{(s)}}, \; j \in \{1,3,\dots,2p-1\},$$

and

$$X_{2pn+j}^{(q)} = \frac{F_{2p(m+2)n+j(m+2)} + F_{2p(m+2)n+j(m+2)-1}X_0^{(s)}}{F_{2p(m+2)n+j(m+2)+1} + F_{2p(m+2)n+j(m+2)}X_0^{(s)}}, \; j \in \{0,2,\dots,2p-2\},$$

where $n \in \mathbb{N}_0$, $s = (q+j) \mod (2p)$ and $q \in \{1,2,\dots,2p\}$.

This completes the proof. □

Using the preceding arguments, we can provide the main result for the solvability of system (3).

Theorem 5. *Assume that $\theta = 2p$ and let $\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(2p)}\right)_{n\geq -k}$ be a solution to system (3). Then, for all $q \in \{1, 2, \ldots, 2p\}$, the solution to system (3) is*

$$x^{(q)}_{(k+1)(2pn+j)-t} = \frac{L_{2p(m+2)n+j(m+2)} + L_{2p(m+2)n+j(m+2)-1}x_{-t}^{((q+j) \mod (2p))}}{L_{2p(m+2)n+j(m+2)+1} + L_{2p(m+2)n+j(m+2)}x_{-t}^{((q+j) \mod (2p))}},$$

for $j \in \{1, 3, \cdots, 2p-1\}$, and

$$x^{(q)}_{(k+1)(2pn+j)-t} = \frac{F_{2p(m+2)n+j(m+2)} + F_{2p(m+2)n+j(m+2)-1}x_{-t}^{((q+j) \mod (2p))}}{F_{2p(m+2)n+j(m+2)+1} + F_{2p(m+2)n+j(m+2)}x_{-t}^{((q+j) \mod (2p))}},$$

for $j \in \{0, 2, \ldots, 2p-2\}$, where $t \in \{0, 1, \cdots, k\}$.

3.1.2. Case $\theta = 2p+1$

Here, we consider the case $\theta = 2p+1$.

Using the transformation

$$\Psi^j_n = X^{(1)}_{(2p+1)n+j}, \quad n \in \mathbb{N}_0, \tag{12}$$

where $j \in \{0, 1, 2, \cdots, 2p\}$, we can write Equation (10) in the form

$$\Psi^j_{n+1} = \frac{L_{(2p+1)(m+2)} + L_{(2p+1)(m+2)-1}\Psi^j_n}{L_{(2p+1)(m+2)+1} + L_{(2p+1)(m+2)}\Psi^j_n}, \quad n \in \mathbb{N}_0, \tag{13}$$

where $j \in \{0, 1, 2, \cdots, 2p\}$.

From Appendix A, the solution to Equation (13) is given by

$$\Psi^j_{2n} = \frac{F_{2(2p+1)(m+2)n} + F_{2(2p+1)(m+2)n-1}\Psi^j_0}{F_{2(2p+1)(m+2)n+1} + F_{2(2p+1)(m+2)n}\Psi^j_0}, \quad n \in \mathbb{N}_0,$$

$$\Psi^j_{2n+1} = \frac{L_{(2p+1)(m+2)(2n+1)} + L_{(2p+1)(m+2)(2n+1)-1}\Psi^j_0}{L_{(2p+1)(m+2)(2n+1)+1} + L_{(2p+1)(m+2)(2n+1)}\Psi^j_0}, \quad n \in \mathbb{N}_0,$$

where $j \in \{0, 1, \cdots, 2p\}$.

Therefore, the solution to Equation (10) is

$$X^{(1)}_{2(2p+1)n+j} = \frac{F_{2(2p+1)(m+2)n} + F_{2(2p+1)(m+2)n-1}X^{(1)}_j}{F_{2(2p+1)(m+2)n+1} + F_{2(2p+1)(m+2)n}X^{(1)}_j},$$

$$X^{(1)}_{(2p+1)(2n+1)+j} = \frac{L_{(2p+1)(m+2)(2n+1)} + L_{(2p+1)(m+2)(2n+1)-1}X^{(1)}_j}{L_{(2p+1)(m+2)(2n+1)+1} + L_{(2p+1)(m+2)(2n+1)}X^{(1)}_j}, \tag{14}$$

where $n \in \mathbb{N}_0$ and $j \in \{0, 1, \ldots, 2p\}$.

Theorem 6. *Let $\left(X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(2p+1)}\right)_{n\geq 0}$ be a solution to system (8). Then for all $j \in \{0, 2, \ldots, 2p\}$,*

$$X_{2(2p+1)n+j}^{(q)} = \frac{F_{2(2p+1)(m+2)n+j(m+2)} + F_{2(2p+1)(m+2)n+j(m+2)-1}X_0^{(s)}}{F_{2(2p+1)(m+2)n+j(m+2)+1} + F_{2(2p+1)(m+2)n+j(m+2)}X_0^{(s)}},$$

$$X_{(2p+1)(2n+1)+j}^{(q)} = \frac{L_{(2p+1)(m+2)(2n+1)+j(m+2)} + L_{(2p+1)(m+2)(2n+1)+j(m+2)-1}X_0^{(s)}}{L_{(2p+1)(m+2)(2n+1)+j(m+2)+1} + L_{(2p+1)(m+2)(2n+1)+j(m+2)}X_0^{(s)}},$$

and for $j \in \{1, 3, \dots, 2p-1\}$,

$$X_{2(2p+1)n+j}^{(q)} = \frac{L_{2(2p+1)(m+2)n+(j+1)(m+2)} + L_{2(2p+1)(m+2)n+(j+1)(m+2)-1}X_0^{(s)}}{L_{2(2p+1)(m+2)n+(j+1)(m+2)+1} + L_{2(2p+1)(m+2)n+(j+1)(m+2)}X_0^{(s)}},$$

$$X_{(2p+1)(2n+1)+j}^{(q)} = \frac{F_{(2p+1)(m+2)(2n+1)+(j+1)(m+2)} + F_{(2p+1)(m+2)(2n+1)+(j+1)(m+2)-1}X_0^{(s)}}{F_{(2p+1)(m+2)(2n+1)+(j+1)(m+2)+1} + F_{(2p+1)(m+2)(2n+1)+(j+1)(m+2)}X_0^{(s)}},$$

where $n \in \mathbb{N}_0$, $s = (q+j) \mod (2p+1)$ *and* $q \in \{1, 2, \dots, 2p+1\}$.

Proof. Let $\left(X_n^{(1)}, X_n^{(2)}, \dots, X_n^{(2p+1)}\right)_{n\geq 0}$ be a solution to system (8). We consider two different situations, depending on whether j is even or odd.

- When j is even, we have $X_j^{(1)} = \dfrac{F_{j(m+2)} + F_{j(m+2)-1}X_0^{(1+j)}}{F_{j(m+2)+1} + F_{j(m+2)}X_0^{(1+j)}}$. This implies that

$$\begin{aligned}
X_{2(2p+1)n+j}^{(1)} &= \frac{F_{2(2p+1)(m+2)n} + F_{2(2p+1)(m+2)n-1}X_j^{(1)}}{F_{2(2p+1)(m+2)n+1} + F_{2(2p+1)(m+2)n}X_j^{(1)}}, \\
&= \frac{F_{2(2p+1)(m+2)n} + F_{2(2p+1)(m+2)n-1}\left(\dfrac{F_{j(m+2)} + F_{j(m+2)-1}X_0^{(1+j)}}{F_{j(m+2)+1} + F_{j(m+2)}X_0^{(1+j)}}\right)}{F_{2(2p+1)(m+2)n+1} + F_{2(2p+1)(m+2)n}\left(\dfrac{F_{j(m+2)} + F_{j(m+2)-1}X_0^{(1+j)}}{F_{j(m+2)+1} + F_{j(m+2)}X_0^{(1+j)}}\right)}, \\
&= \frac{F_{2(2p+1)(m+2)n+j(m+2)} + F_{2(2p+1)(m+2)n+j(m+2)-1}X_0^{(1+j)}}{F_{2(2p+1)(m+2)n+j(m+2)+1} + F_{2(2p+1)(m+2)n+j(m+2)}x_0^{(1+j)}}.
\end{aligned}$$

  Similarly,

$$\begin{aligned}
X_{(2p+1)(2n+1)+j}^{(1)} &= \frac{L_{(2p+1)(m+2)(2n+1)} + L_{(2p+1)(m+2)(2n+1)-1}\left(\dfrac{F_{j(m+2)} + F_{j(m+2)-1}X_0^{(j+1)}}{F_{j(m+2)+1} + F_{j(m+2)}X_0^{(j+1)}}\right)}{L_{(2p+1)(m+2)(2n+1)+1} + L_{(2p+1)(m+2)(2n+1)}\left(\dfrac{F_{j(m+2)} + F_{j(m+2)-1}X_0^{(j+1)}}{F_{j(m+2)+1} + F_{j(m+2)}X_0^{(j+1)}}\right)}, \\
&= \frac{L_{(2p+1)(m+2)(2n+1)+j(m+2)} + L_{(2p+1)(m+2)(2n+1)+j(m+2)-1}X_0^{(1+j)}}{L_{(2p+1)(m+2)(2n+1)+j(m+2)+1} + L_{(2p+1)(m+2)(2n+1)+j(m+2)}X_0^{(1+j)}}.
\end{aligned}$$

- When j is odd, we have $X_j^{(1)} = \dfrac{F_{j(m+2)} + F_{j(m+2)-1}X_1^{(j)}}{F_{j(m+2)+1} + F_{j(m+2)}X_1^{(j)}}$. This implies that

$$
\begin{aligned}
X^{(1)}_{2(2p+1)n+j} &= \frac{F_{2(2p+1)(m+2)n} + F_{2(2p+1)(m+2)n-1}\left(\dfrac{F_{j(m+2)} + F_{j(m+2)-1}X^{(j)}_1}{F_{j(m+2)+1} + F_{j(m+2)}X^{(j)}_1}\right)}{F_{2(2p+1)(m+2)n+1} + F_{2(2p+1)(m+2)n}\left(\dfrac{F_{j(m+2)} + F_{j(m+2)-1}X^{(j)}_1}{F_{j(m+2)+1} + F_{j(m+2)}X^{(j)}_1}\right)}, \\
&= \frac{F_{2(2p+1)(m+2)n+j(m+2)} + F_{2(2p+1)(m+2)n+j(m+2)-1}X^{(j)}_1}{F_{2(2p+1)(m+2)n+j(m+2)+1} + F_{2(2p+1)(m+2)n+j(m+2)}X^{(j)}_1},
\end{aligned}
$$

and

$$
\begin{aligned}
X^{(1)}_{(2p+1)(2n+1)+j} &= \frac{L_{(2p+1)(m+2)(2n+1)} + L_{(2p+1)(m+2)(2n+1)-1}\left(\dfrac{F_{j(m+2)} + F_{j(m+2)-1}X^{(j)}_1}{F_{j(m+2)+1} + F_{j(m+2)}X^{(j)}_1}\right)}{L_{(2p+1)(m+2)(2n+1)+1} + L_{(2p+1)(m+2)(2n+1)}\left(\dfrac{F_{j(m+2)} + F_{j(m+2)-1}X^{(j)}_1}{F_{j(m+2)+1} + F_{j(m+2)}X^{(j)}_1}\right)}, \\
&= \frac{L_{\theta(m+2)(2n+1)+j(m+2)} + L_{(2p+1)(m+2)(2n+1)+j(m+2)-1}X^{(j)}_1}{L_{(2p+1)(m+2)(2n+1)+j(m+2)+1} + L_{(2p+1)(m+2)(2n+1)+j(m+2)}X^{(j)}_1}.
\end{aligned}
$$

By the same way, for $j \in \{1, 3, 5, \cdots, 2p-1\}$, we have $X^{(j)}_1 = \dfrac{L_{m+2} + L_{m+1}X^{(j+1)}_0}{L_{m+3} + L_{m+2}X^{(j+1)}_0}$.

Then

$$
\begin{aligned}
X^{(1)}_{2(2p+1)n+j} &= \frac{L_{2(2p+1)(m+2)n+(j+1)(m+2)} + L_{2(2p+1)(m+2)n+(j+1)(m+2)-1}X^{(1+j)}_0}{L_{2(2p+1)(m+2)n+(j+1)(m+2)+1} + L_{2(2p+1)(m+2)n+(j+1)(m+2)}X^{(1+j)}_0}, \\
X^{(1)}_{(2p+1)(2n+1)+j} &= \frac{F_{(2p+1)(m+2)(2n+1)+(j+1)(m+2)} + F_{(2p+1)(m+2)(2n+1)+(j+1)(m+2)-1}X^{(1+j)}_0}{F_{(2p+1)(m+2)(2n+1)+(j+1)(m+2)+1} + F_{(2p+1)(m+2)(2n+1)+(j+1)(m+2)}X^{(1+j)}_0}.
\end{aligned}
$$

Now, using the fact that

$$
X^{(q)}_{n+1} = \frac{L_{m+2} + L_{m+1}X^{((q+1) \mod (2p+1))}_n}{L_{m+3} + L_{m+2}X^{((q+1) \mod (2p+1))}_n}, \quad q \in \{1, 2, ..., 2p+1\},
$$

and after some calculations, we obtain for $j \in \{0, 2, \ldots, 2p\}$,

$$
\begin{aligned}
X^{(q)}_{2(2p+1)n+j} &= \frac{F_{2(2p+1)(m+2)n+j(m+2)} + F_{2(2p+1)(m+2)n+j(m+2)-1}X^{(s)}_0}{F_{2(2p+1)(m+2)n+j(m+2)+1} + F_{2(2p+1)(m+2)n+j(m+2)}X^{(s)}_0}, \\
X^{(q)}_{(2p+1)(2n+1)+j} &= \frac{L_{(2p+1)a(m+2)(2n+1)+j(m+2)} + L_{(2p+1)(m+2)(2n+1)+j(m+2)-1}X^{(s)}_0}{L_{(2p+1)(m+2)(2n+1)+j(m+2)+1} + L_{(2p+1)(m+2)(2n+1)+j(m+2)}x^{(s)}_0},
\end{aligned}
$$

and for $j \in \{1, 3, \ldots, 2p-1\}$,

$$
\begin{aligned}
X^{(q)}_{2(2p+1)n+j} &= \frac{L_{2(2p+1)(m+2)n+(j+1)(m+2)} + L_{2(2p+1)(m+2)n+(j+1)(m+2)-1}X^{(s)}_0}{L_{2(2p+1)(m+2)n+(j+1)(m+2)+1} + L_{2(2p+1)(m+2)n+(j+1)(m+2)}X^{(s)}_0}, \\
X^{(q)}_{(2p+1)(2n+1)+j} &= \frac{F_{(2p+1)(m+2)(2n+1)+(j+1)(m+2)} + F_{(2p+1)(m+2)(2n+1)+(j+1)(m+2)-1}X^{(s)}_0}{F_{(2p+1)(m+2)(2n+1)+(j+1)(m+2)+1} + F_{(2p+1)(m+2)(2n+1)+(j+1)(m+2)}X^{(s)}_0},
\end{aligned}
$$

where $n \in \mathbb{N}_0$, $s = (q+j) \mod (2p+1)$ and $q \in \{1, 2, \ldots, 2p+1\}$.

This completes the proof. $\square$

From all the above arguments, we can state the following theorem:

Theorem 7. *Assume that $\theta = 2p+1$ and let $\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(2p+1)}\right)_{n\geq -k}$ be a solution to system (3). Then for all $n \in \mathbb{N}_0$, the solution to system (3) is*

$$\begin{aligned}
x^{(q)}_{(k+1)(2(2p+1)n+j)-t} &= \frac{F_{2(2p+1)(m+2)n+j(m+2)} + F_{2(2p+1)(m+2)n+j(m+2)-1}x^{(s)}_{-t}}{F_{2(2p+1)(m+2)n+j(m+2)+1} + F_{2(2p+1)(m+2)n+j(m+2)}x^{(s)}_{-t}},\\
x^{(q)}_{(k+1)((2p+1)(2n+1)+j)-t} &= \frac{L_{(2p+1)(m+2)(2n+1)+j(m+2)} + L_{(2p+1)(m+2)(2n+1)+j(m+2)-1}x^{(s)}_{-t}}{L_{(2p+1)(m+2)(2n+1)+j(m+2)+1} + L_{(2p+1)(m+2)(2n+1)+j(m+2)}x^{(s)}_{-t}},
\end{aligned}$$

for all $j \in \{0, 2, \ldots, 2p\}$, and

$$\begin{aligned}
x^{(q)}_{(k+1)(2(2p+1)n+j)-t} &= \frac{L_{2(2p+1)(m+2)n+(j+1)(m+2)} + L_{2(2p+1)(m+2)n+(j+1)(m+2)-1}x^{(s)}_{-t}}{L_{2(2p+1)(m+2)n+(j+1)(m+2)+1} + L_{2(2p+1)(m+2)n+(j+1)(m+2)}x^{(s)}_{-t}},\\
x^{(q)}_{(k+1)((2p+1)(2n+1)+j)-t} &= \frac{F_{(2p+1)(m+2)(2n+1)+(j+1)(m+2)} + F_{(2p+1)(m+2)(2n+1)+(j+1)(m+2)-1}x^{(s)}_{-t}}{F_{(2p+1)(m+2)(2n+1)+(j+1)(m+2)+1} + F_{(2p+1)(m+2)(2n+1)+(j+1)(m+2)}x^{(s)}_{-t}},
\end{aligned}$$

for all $j \in \{1, 3, \ldots, 2p-1\}$, where $s = (q+j) \mod (2p+1)$, $t \in \{0, 1, \cdots, k\}$ and $q \in \{1, 2, \ldots, 2p+1\}$.

Remark 7. *The well-defined solutions to the θ-dimensional system of difference Equation (3) when $\theta = 2p$ are given by Theorem 5, and when $\theta = 2p+1$ are given by Theorem 7.*

4. Global Stability of the Well-Defined Solutions of System (3)

In this section, we study the global stability of the well-defined solutions to system (3). It is easy to show that system (3) has 2^θ fixed points. We shall study the stability of the fixed point

$$\mathcal{E} = \left(\overline{x^{(1)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(\theta)}}\right) = (-\beta, -\beta, \ldots, -\beta),$$

where $\beta = \dfrac{1-\sqrt{5}}{2}$.

Let us consider the functions

$$f_q : \mathbb{R}^{k+1} \times \mathbb{R}^{k+1} \times \ldots \times \mathbb{R}^{k+1} \longrightarrow \mathbb{R},$$

defined by

$$f_q\left(u_0^{(1)}, \ldots, u_k^{(1)}, u_0^{(2)}, \ldots, u_k^{(2)}, \ldots, u_0^{(\theta)}, \ldots, u_k^{(\theta)}\right) = \frac{L_{m+2} + L_{m+1}u_k^{((q+1) \mod (\theta))}}{L_{m+3} + L_{m+2}u_k^{((q+1) \mod (\theta))}},$$

with $q \in \{1, 2, \ldots, \theta\}$.

The following result is a direct consequence of Theorems A3 and A4 in Appendix B.

Corollary 7. *The fixed point $\mathcal{E}$ of system (3) is globally asymptotically stable.*

5. Numerical Examples

Example 7. *Consider system (3) with $m = 3, k = 4$ and $\theta = 4$. i.e., the system of difference equations*

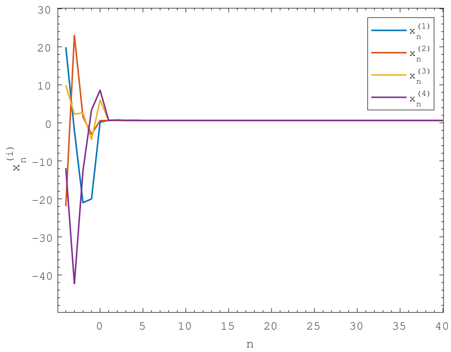
$$\begin{aligned}
x^{(1)}_{n+1} &= \frac{11 + 7x^{(2)}_{n-4}}{18 + 11x^{(2)}_{n-4}}, \quad x^{(2)}_{n+1} = \frac{11 + 7x^{(3)}_{n-4}}{18 + 11x^{(3)}_{n-4}},\\
x^{(3)}_{n+1} &= \frac{11 + 7x^{(4)}_{n-4}}{18 + 11x^{(4)}_{n-4}}, \quad x^{(4)}_{n+1} = \frac{11 + 7x^{(1)}_{n-4}}{18 + 11x^{(1)}_{n-4}},
\end{aligned} \quad n \in \mathbb{N}_0. \tag{15}$$

Figure 1 (left) represents system (15) with initial conditions $x_{-4}^{(1)} = 19.9$, $x_{-3}^{(1)} = -2$, $x_{-2}^{(1)} = -21$, $x_{-1}^{(1)} = -20$, $x_{0}^{(1)} = 0.2$, $x_{-4}^{(2)} = -21.9$, $x_{-3}^{(2)} = 23$, $x_{-2}^{(2)} = 1.6$, $x_{-1}^{(2)} = -3$, $x_{-0}^{(2)} = 0.6$, $x_{-4}^{(3)} = 9.9$, $x_{-3}^{(3)} = 2.3$, $x_{-2}^{(3)} = 2.6$, $x_{-1}^{(3)} = -4.3$, $x_{0}^{(3)} = 6$, $x_{-4}^{(4)} = -11.9$, $x_{-3}^{(4)} = -42.3$, $x_{-2}^{(4)} = -12.7$, $x_{-1}^{(4)} = 3.4$ *and* $x_{0}^{(4)} = 8.6$

Example 7. *Consider system (3) with* $m = 6, k = 7$ *and* $\theta = 5$. *i.e., the system of difference equations*

$$
\begin{aligned}
x_{n+1}^{(1)} &= \frac{47 + 29x_{n-7}^{(2)}}{76 + 47x_{n-7}^{(2)}}, \quad x_{n+1}^{(2)} = \frac{47 + 29x_{n-7}^{(3)}}{76 + 47x_{n-7}^{(3)}}, \quad x_{n+1}^{(3)} = \frac{47 + 29x_{n-7}^{(4)}}{76 + 47x_{n-7}^{(4)}}, \\
x_{n+1}^{(4)} &= \frac{47 + 29x_{n-7}^{(5)}}{76 + 47x_{n-7}^{(5)}}, \quad x_{n+1}^{(5)} = \frac{47 + 29x_{n-7}^{(1)}}{76 + 47x_{n-7}^{(1)}}, \quad n \in \mathbb{N}_0.
\end{aligned} \tag{16}
$$

Figure 1 (right) represents system (16) with initial conditions $x_{-7}^{(1)} = 9.9$, $x_{-6}^{(1)} = -12.0$, $x_{-5}^{(1)} = -11$, $x_{-4}^{(1)} = -10$, $x_{-3}^{(1)} = 0.2$, $x_{-2}^{(1)} = -6.1$, $x_{-1}^{(1)} = -10$, $x_{0}^{(1)} = 2.2$, $x_{-7}^{(2)} = -21.9$, $x_{-6}^{(2)} = 23$, $x_{-5}^{(2)} = 1.6$, $x_{-4}^{(2)} = -3$, $x_{-3}^{(2)} = 0.6$, $x_{-2}^{(2)} = -2.6$, $x_{-1}^{(2)} = -10.3$, $x_{-0}^{(2)} = -2.6$ $x_{-7}^{(3)} = -9.9$, $x_{-6}^{(3)} = 2.3$, $x_{-5}^{(3)} = 2.6$, $x_{-4}^{(3)} = -4.3$, $x_{-3}^{(3)} = 6$, $x_{-2}^{(3)} = -4.6$, $x_{-1}^{(3)} = -4.3$, $x_{0}^{(3)} = -3.12$, $x_{-7}^{(4)} = -1.9$, $x_{-6}^{(4)} = -5.3$, $x_{-5}^{(4)} = -2.6$, $x_{-4}^{(4)} = 3.3$, $x_{-3}^{(4)} = 9.6$, $x_{-2}^{(4)} = -12.6$, $x_{-1}^{(4)} = 6.3$, $x_{0}^{(4)} = 3.6$, $x_{-7}^{(5)} = -14.19$, $x_{-6}^{(5)} = -2.3$, $x_{-5}^{(5)} = 2.6$, $x_{-4}^{(5)} = -9.3$, $x_{-3}^{(5)} = -1.6$, $x_{-2}^{(5)} = 11.6$, $x_{-1}^{(5)} = 4.3$ *and* $x_{0}^{(5)} = 2.6$.



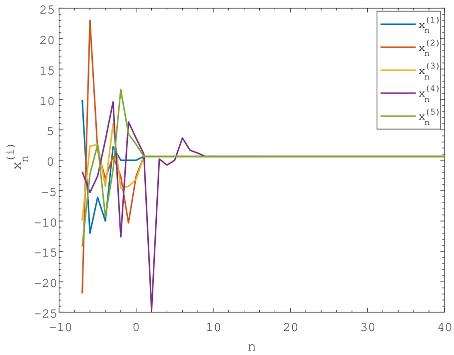


Figure 1. A graph representing the global stability of system (15) (**left**) and system (16) (**right**).

6. Conclusions

In this paper, we have derived the solutions to the system of difference equations

$$
x_{n+1}^{(q)} = \frac{L_{m+2} + L_{m+1}x_{n-k}^{((q+1) \mod (\theta))}}{L_{m+3} + L_{m+2}x_{n-k}^{((q+1) \mod (\theta))}}, \quad n \in \mathbb{N}_0,
$$

where $k \in \mathbb{N}_0$, $m \in \mathbb{Z}$, $q = 1, \cdots, \theta$, $\theta \in \mathbb{N}$, and $(L_n)_{n=0}^{+\infty}$ is Lucas sequence.

The above-mentioned system is a more general system to some systems studied recently. The well-defined solutions to that system are represented in terms of Fibonacci and Lucas sequences.

Those well-defined solutions depend on whether θ is even or odd. When θ is even, we invoked the solution of the difference equation

$$
u_{n+1} = \frac{F_{\theta(m+2)} + F_{\theta(m+2)-1}u_{n-(\theta-1)}}{F_{\theta(m+2)+1} + F_{\theta(m+2)}u_{n-(\theta-1)}}, \quad n \in \mathbb{N}_0,
$$

that was used in [26].

For odd values of θ, we invoked the solution of the difference equation

$$\Omega_{n+1} = \frac{L_{\theta(m+2)} + L_{\theta(m+2)-1}\Omega_n}{L_{\theta(m+2)+1} + L_{\theta(m+2)}\Omega_n}, \quad n \in \mathbb{N}_0.$$

The solution to that equation was derived in Appendix A.

System (3) has 2^θ fixed points; one of these fixed points is locally asymptotically stable. Moreover, we proved that it is globally asymptotically stable.

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Appendix A. (Solvability of Equation (13))

Consider the difference equation

$$\Omega_{n+1} = \frac{L_{\theta(m+2)} + L_{\theta(m+2)-1}\Omega_n}{L_{\theta(m+2)+1} + L_{\theta(m+2)}\Omega_n}, \quad n \in \mathbb{N}_0. \tag{A1}$$

Using the change of variables

$$\Omega_n = \frac{1}{L_{\theta(m+2)}}\Big(\Phi_n - L_{\theta(m+2)+1}\Big), \quad n = 0, 1, \cdots, \tag{A2}$$

we get

$$\begin{aligned}
\Phi_{n+1} &= \frac{L^2_{\theta(m+2)} + L_{\theta(m+2)-1}\Big(\Phi_n - L_{\theta(m+2)+1}\Big)}{\Phi_n} + L_{\theta(m+2)+1} \\
&= \frac{\Phi_n\Big(L_{\theta(m+2)-1} + L_{\theta(m+2)+1}\Big) + \Big(L^2_{\theta(m+2)} - L_{\theta(m+2)-1}L_{\theta(m+2)+1}\Big)}{\Phi_n}, \quad n \in \mathbb{N}_0.
\end{aligned}$$

Using the Properties 1, Equation (A1) is changed into

$$\Phi_{n+1} = \frac{5F_{\theta(m+2)}\Phi_n + 5(-1)^{\theta(m+2)}}{\Phi_n}, \quad n \in \mathbb{N}_0. \tag{A3}$$

To solve Equation (A1), we introduce the following Lemma:

Lemma A1. *Consider the linear difference equation*

$$\omega_{n+1} - 5F_r\omega_n + 5(-1)^{r+1}\omega_{n-1} = 0, \quad n \in \mathbb{N}_0, \tag{A4}$$

with initial conditions $\omega_{-1}, \omega_0 \in \mathbb{R}$. Then the solution to Equation (A4) is

$$\omega_n = \left(\frac{\sqrt{5}^n}{(-1)^r L_r}\right)\Big(\sqrt{5}\omega_{-1}\mathcal{N}_{rn} + (-1)^r\omega_0\mathcal{N}_{r(n+1)}\Big), \tag{A5}$$

where $(F_n)_{n=0}^{+\infty}$ is the Fibonacci sequence, $(L_n)_{n=0}^{+\infty}$ is the Lucas sequence,

$$\mathcal{N}_{rn} = (\alpha^{rn} - (-1)^n\beta^{rn}), \quad \alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2},$$

and

$$\mathcal{N}_{(2p+1)n} = \begin{cases} \sqrt{5}F_{rn} & \textit{if } n \textit{ is even}\,, \\ L_{rn} & \textit{if } n \textit{ is odd}\,. \end{cases}$$

Proof. The equation

$$\omega_{n+1} - 5F_r\omega_n + 5(-1)^{r+1}\omega_{n-1} = 0\,, \quad n \in \mathbb{N}_0\,,$$

is a homogeneous linear second order difference equation with constant coefficients, and initial conditions $\omega_0, \omega_{-1} \in \mathbb{R}$. We can obtain its solution by using the characteristic roots τ_1 and τ_2 of the characteristic polynomial $\tau^2 - 5F_r\tau + 5(-1)^{r+1} = 0$.

Using the identity

$$5F_\eta^2 = L_\eta^2 - 4(-1)^\eta, \;\; \eta = 0, 1, \cdots\,,$$

we get

$$\tau_1 = \sqrt{5}\left(\frac{L_r + \sqrt{5}F_r}{2}\right) = \sqrt{5}\alpha^r\,, \quad \tau_2 = -\sqrt{5}\left(\frac{L_r - \sqrt{5}F_r}{2}\right) = -\sqrt{5}\beta^r\,.$$

Then

$$\omega_n = e_1\tau_1^n + e_2\tau_2^n\,.$$

Using the initial conditions ω_0 and ω_{-1}, and after some calculations, we get

$$\begin{aligned} e_1 &= -\frac{\sqrt{5}}{(-1)^{r+1}L_r}\left(z_{-1} - \frac{\omega_0}{5}(-1)^{r+1}\tau_1\right), \\ e_2 &= -\frac{\sqrt{5}}{(-1)^{r+1}L_r}\left(\frac{\omega_0}{5}(-1)^{r+1}\tau_2 - \omega_{-1}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \omega_n &= -\frac{\sqrt{5}}{(-1)^{r+1}L_r}\left(\omega_{-1}(\tau_1^n - \tau_2^n) - \frac{\omega_0}{5}(-1)^{r+1}\left(\tau_1^{n+1} - \tau_2^{n+1}\right)\right) \\ &= -\frac{\sqrt{5}}{(-1)^{r+1}L_r}\Bigg(\omega_{-1}(\sqrt{5})^n(\alpha^{rn} - (-1)^n\beta^{rn}) + \frac{\omega_0(\sqrt{5})^{n+1}}{(\sqrt{5})^2} \\ &\quad \times (-1)^r\left(\alpha^{r(n+1)} - (-1)^{n+1}\beta^{r(n+1)}\right)\Bigg)\,. \end{aligned}$$

By defining

$$\mathcal{N}_{rn} := (\alpha^{rn} - (-1)^n\beta^{rn})\,,$$

we can write the solution to Equation (A4) as

$$\omega_n = \frac{(\sqrt{5})^n}{(-1)^r L_r}\left(\omega_{-1}\sqrt{5}\mathcal{N}_{rn} + \omega_0(-1)^r\mathcal{N}_{r(n+1)}\right).$$

This completes the proof. □

Using Lemma A1, we get

$$\Phi_n = \frac{\sqrt{5}(\omega_{-1}\sqrt{5}N_{rn} + \omega_0(-1)^r N_{r(n+1)})}{\omega_{-1}\sqrt{5}N_{r(n-1)} + \omega_0(-1)^r N_{rn}}.$$

Therefore, the general solution to Equation (A3) is

$$\Phi_{2n} = \frac{5F_{2rn} + (-1)^r\Phi_0 L_{r(2n+1)}}{L_{r(2n-1)} + (-1)^r\Phi_0 F_{2rn}}\,, \quad n \in \mathbb{N}_0, \tag{A6}$$

$$\Phi_{2n+1} = \frac{5L_{r(2n+1)} + (-1)^r 5\Phi_0 F_{2r(n+1)}}{5F_{2rn} + (-1)^r\Phi_0 L_{r(2n+1)}}\,, \quad n \in \mathbb{N}_0, \tag{A7}$$

with $r = \theta(m+2)$.

Using Lemma A1 and Equations (A6) and (A7), we can state the following theorem:

Theorem A1. *Let $(\Omega_n)_{n\geq 0}$ be a solution to Equation (A1). Then for all $n \in \mathbb{N}_0$, the solution to Equation (A1) is*

$$\begin{aligned}\Omega_{2n} &= \frac{F_{2\theta(m+2)n} + F_{2\theta(m+2)n-1}\Omega_0}{F_{2\theta(m+2)n+1} + F_{2\theta(m+2)n}\Omega_0}, \\ \Omega_{2n+1} &= \frac{L_{\theta(m+2)(2n+1)} + L_{\theta(m+2)(2n+1)-1}\Omega_0}{L_{\theta(m+2)(2n+1)+1} + L_{\theta(m+2)(2n+1)}\Omega_0},\end{aligned}$$

where $(L_n)_{n=0}^{+\infty}$ is Lucas sequence and $(F_n)_{n=0}^{+\infty}$ is Fibonacci sequence.

Proof. According to the change of variable (A2), using Property 2, we get for all $n \in \mathbb{N}_0$,

$$\begin{aligned}\Omega_{2n} &= \frac{1}{L_r}(\Phi_{2n} - L_{r+1}) \\ &= \frac{1}{L_r}\left(\frac{(5F_{2rn} - L_{r+1}L_{r(2n-1)}) + (-1)^r\Phi_0(L_{r(2n+1)} - L_{r+1}F_{2rn})}{L_{r(2n-1)} + (-1)^r\Phi_0 F_{2rn}}\right) \\ &= \frac{1}{L_r}\left(\frac{L_r L_{r(2n-1)-1} + (-1)^r\Phi_0 L_r F_{2rn-1}}{L_{r(2n-1)} + (-1)^r\Phi_0 F_{2rn}}\right) \\ &= \frac{(L_{r(2n-1)-1} + (-1)^r L_{r+1}F_{2rn-1}) + \Omega_0(-1)^r L_r F_{2rn-1}}{(L_{r(2n-1)} + (-1)^r L_{r+1}F_{2rn}) + \Omega_0(-1)^r L_r F_{2rn}} \\ &= \frac{F_{2rn} + \Omega_0 F_{2rn-1}}{F_{2rn+1} + \Omega_0 F_{2rn}}.\end{aligned}$$

Similarly,

$$\begin{aligned}\Omega_{2n+1} &= \frac{1}{L_r}(\Phi_{2n+1} - L_{r+1}) \\ &= \frac{1}{L_r}\left(\frac{5(L_{r(2n+1)} - L_{r+1}F_{2rn}) + (-1)^r\Phi_0(5F_{r(2n+2)} - L_{r+1}L_{r(2n+1)})}{5F_{2rn} + (-1)^r\Phi_0 L_{r(2n+1)}}\right) \\ &= \frac{L_r}{L_r}\left(\frac{5F_{2rn-1} + (-1)^r\Phi_0 L_{r(2n+1)-1}}{5F_{2rn} + (-1)^r\Phi_0 L_{r(2n+1)}}\right) \\ &= \frac{(5F_{2rn-1} + (-1)^r L_{r+1}L_{r(2n+1)-1}) + \Omega_0(-1)^r L_r L_{r(2n+1)-1}}{(5F_{2rn} + (-1)^r L_{r+1}L_{r(2n+1)}) + \Omega_0(-1)^r L_r L_{r(2n+1)}} \\ &= \frac{L_{r(2n+1)} + \Omega_0 L_{r(2n+1)-1}}{L_{r(2n+1)+1} + \Omega_0 L_{r(2n+1)}}.\end{aligned}$$

This completes the proof. □

Appendix B. (Locally Stability of System (3))

Lemma A2. *Suppose that $(F_n)_{n=0}^{+\infty}$ and $(L_n)_{n=0}^{+\infty}$ are Fibonacci and Lucas sequences and let $n, \kappa \in \mathbb{N}$. Then*

$$\lim_{n\to+\infty}\frac{F_{n+\kappa}}{F_n} = \lim_{n\to+\infty}\frac{L_{n+\kappa}}{L_n} = \alpha^\kappa.$$

Theorem A2. *(Linear stability)*

1. *If all the eigenvalues of the Jacobian matrix A lie in the open unit disk $|\lambda| < 1$, then the fixed point $\overline{V}$ of the linearized system is locally asymptotically stable.*

2. *If at least one eigenvalue of the Jacobian matrix A have absolute value greater than one, then the fixed point $\overline{V}$ of linearized system is unstable.*

Theorem A3. *The fixed point $\mathcal{E} = (-\beta, -\beta, \ldots, -\beta) \in \mathbb{R}^{\theta}$ is locally asymptotically stable.*

Proof. The linearized system about the fixed point

$$\overline{V} = (-\beta, -\beta, \ldots, -\beta) \in \mathbb{R}^{\theta(k+1)},$$

is given by

$$X_{n+1} = AX_n\,,$$

where

$$X_n = \left(x_n^{(1)}, x_{n-1}^{(1)}, \ldots, x_{n-k}^{(1)}, x_n^{(2)}, x_{n-1}^{(2)}, \ldots, x_{n-k}^{(2)}, \ldots, x_n^{(\theta)}, x_{n-1}^{(\theta)}, \ldots, x_{n-k}^{(\theta)}\right)^T,$$

and

$$A = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{(-1)^{m+1}}{\alpha^{2m+4}} & \cdots & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & & & 0 & & & & & & & & & 0 \\
0 & 1 & \ddots & & & & & & & & & & & \vdots \\
0 & & \ddots & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\
0 & & & 1 & 0 & & & & & & & & & \vdots \\
\vdots & & \vdots & & \vdots & & & \vdots & & & \vdots & & & \vdots \\
0 & \cdots & & & 0 & 1 & & & & & & & & 0 \\
\vdots & & & & & 0 & \ddots & 0 & \cdots & \cdots & 0 & 0 & \cdots & \frac{(-1)^{m+1}}{\alpha^{2m+4}} \\
\vdots & & & \vdots & & 0 & & 1 & & & & & & 0 \\
\vdots & & & & & & & & & & & & & \vdots \\
0 & \cdots & 0 & \frac{(-1)^{m+1}}{\alpha^{2m+4}} & 0 & 0 & & & \cdots & \cdots & 0 & & & \\
0 & & & 0 & & 0 & & & & 0 & \ddots & 0 & \vdots & \vdots \\
\vdots & & & \vdots & & \vdots & & & & & & \ddots & 0 & 0 \\
0 & 0 & \cdots & 0 & & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 & 0
\end{pmatrix}.$$

The characteristic polynomial is

$$P(\lambda) = \det(A - \lambda I_{\theta(k+1)}) = (-\lambda)^{\theta(k+1)} + (-1)^k \left(\frac{(-1)^{m+1}}{\alpha^{2m+4}}\right)^{\theta}.$$

Then

$$\left|\lambda^{\theta(k+1)}\right| = \left|\left(\frac{(-1)^{m+1}}{\alpha^{2m+4}}\right)^{\theta}\right| = \frac{1}{\alpha^{\theta(2m+4)}}.$$

Therefore, all roots of the characteristic polynomial has modulus less than one. Using Theorem A2, the fixed point $\mathcal{E}$ is locally asymptotically stable.

This completes the proof. □

Theorem A4. *The fixed point $\mathcal{E}$ of system (3) is a global attractor.*

Proof. Let $\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{\theta}\right)_{n \geq -k}$ be a well-defined solution to system (3).

We show that the well-defined solution $\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{\theta}\right)_{n \geq -k}$ converges to the fixed point $\mathcal{E}$.

It is suffices (using Theorems 5 and 7 together with Lemma A2) to see that for all $q \in \{1, 2, \ldots, \theta\}$ and $t \in \{0, 1, \ldots, k\}$,

$$\begin{aligned}
\lim_{n\to+\infty} x^{(q)}_{(k+1)(\theta n+j)-t} &= \lim_{n\to+\infty} \frac{F_{\theta n(m+2)-j(m+2)} + F_{\theta n(m+2)+j(m+2)-1} x^{s}_{-t}}{F_{\theta n(m+2)+j(m+2)+1} + F_{\theta n(m+2)+j(m+2)} x^{s}_{-t}} \\
&= \lim_{n\to+\infty} \frac{\dfrac{F_{\theta n(m+2)+j(m+2)-1}}{F_{\theta n(m+2)+j(m+2)}} x^{s}_{-t} + 1}{x^{s}_{-t} + \dfrac{F_{\theta n(m+2)+j(m+2)+1}}{F_{\theta n(m+2)+j(m+2)}}} \\
&= \frac{1 - \beta x^{s}_{-t}}{\alpha + x^{s}_{-t}} = \frac{2 - x^{s}_{-t} + \sqrt{5} x^{s}_{-t}}{1 + \sqrt{5} + 2 x^{s}_{-t}}
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to+\infty} x^{(q)}_{(k+1)(\theta n+j)-t} &= \frac{2\sqrt{5}(x^{2}_{-j} + x_{-j} - 1) - 2(x^{2}_{-j} + x_{-j} - 1)}{4(x^{2}_{-j} + x_{-j} - 1)} \\
&= \frac{-1 + \sqrt{5}}{2} = -\beta\,.
\end{aligned}$$

This completes the proof. □

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