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A Novel Method for Solutions of Fourth-Order Fractional Boundary Value Problems

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Abstract: In this paper, we find the solutions of fourth order fractional boundary value problems by using the reproducing kernel Hilbert space method. Firstly, the reproducing kernel Hilbert space method is introduced and then the method is applied to this kind problems. The experiments are discussed and the approximate solutions are obtained to be more correct compared to the other obtained results in the literature.

Keywords: fourth-order fractional boundary value problems; reproducing kernel method; operators

MSC: 47B32; 46E22; 30E25

1. Introduction

Boundary value problems come into view in many areas of science and engineering [1]. Many numerical methods have been presented for solving boundary value problems recently [2]. Wazwaz [3] has presented a modified decomposition method for investigating a special class of fourth order boundary value problems. The series solution of fourth order boundary value problems has been investigated by Adomian decomposition method in [4]. The differential transform method was implemented in [5] for series solution of fourth order boundary value problems. The approximate solution of two-point fourth order boundary value problems utilizing non-polynomial quintic spline functions have been given in [6,7]. Lodhi and Mishra [8] have enhanced a numerical method for solving fourth order boundary value problems utilizing quintic B-spline functions. A numerical technique depending on non-polynomial spline functions has been applied in [9] for the numerical solution of self-adjoint singularly perturbed fourth order boundary value problems. Akram and Amin [10] have applied quintic spline collocation method for approximate solution of singularly perturbed fourth order boundary value problems.

Fractional order boundary value problems have taken much interest by many investigators due to their implementations in many areas of science and engineering. Recently, much attentions have been given to investigate fractional order differential equations [11].

We take into consideration the following fourth order fractional boundary value problems arising in plate deflection theory by the reproducing kernel Hilbert space method.

$$u^{(4)}(z) + D^\gamma p(z)u(z) = g(z), \quad z \in [a, b], \quad (1)$$

subject to the end conditions

$$u(a) - a_1 = u(b) - a_2 = u''(a) - b_1 = u''(b) - b_2 = 0. \quad (2)$$

where a'_i, b'_i , ($i = 1, 2$) are real constants and $0 \leq \gamma \leq 1$. $p(z)$ and $g(z)$ are continuous on the $[a, b]$.

Let H be a Hilbert space of functions on some set X , which every point consideration at points in X are continuous in the norm of H . Such space H , can be given by a positive definite kernel. This fact is first presented by Aronszajn [12]. These Hilbert spaces are called the reproducing kernel Hilbert spaces [13].

Arqub [14] has found the approximate solutions of DASs with nonclassical boundary conditions using a novel reproducing kernel algorithm. Arqub [15] has investigated the reproducing kernel algorithm for handling differential algebraic systems of ordinary differential equations. Arqub [16] has applied a fitted reproducing kernel Hilbert space method for the solutions of some certain classes of time-fractional partial differential equations subject to initial and Neumann boundary conditions. Azarnavid et al. [17] have investigated the Picard-reproducing kernel Hilbert space method for solving generalized singular nonlinear Lane–Emden type equations. Azarnavid et al. [18] have found multiplicity results by shooting reproducing kernel Hilbert space method for the catalytic reaction in a flat particle. Azarnavid et al. [19] have used an iterative kernel based method for fourth order nonlinear equation with nonlinear boundary condition.

We arrange the paper as: The main notions of fractional calculus are given in Section 2. The reproducing kernel Hilbert space method for fourth-order fractional order differential equations is presented in Section 3. The numerical results and discussions are presented in Section 4. The conclusion is presented in the last section.

2. Preliminaries and Notations

We give some definitions of fractional calculus in this section. There are many definitions of fractional derivatives. The definitions of Riemann–Liouville fractional integral and Caputo’s fractional derivative are presented as [17]:

Definition 1. The Riemann–Liouville left and right fractional integral of order $\gamma > 0$ is given as [11]:

$$I_{a+}^{\gamma} u(z) = \frac{1}{\Gamma(\gamma)} \int_a^z (z - \tau)^{\gamma-1} u(\tau) d\tau, \quad m-1 < \gamma \leq m, \quad m \in N, \quad (3)$$

and

$$I_{b-}^{\gamma} u(z) = -\frac{1}{\Gamma(\gamma)} \int_b^z (z - \tau)^{\gamma-1} u(\tau) d\tau, \quad m-1 < \gamma \leq m, \quad m \in N. \quad (4)$$

where Γ defines the Gamma function.

Definition 2. The Riemann–Liouville fractional derivative of order $\gamma > 0$ is presented as [11]:

$$D_a^{\gamma} u(z) = \frac{1}{\Gamma(m-\gamma)} \frac{d^m}{dz^m} \int_a^z (z - \tau)^{m-\gamma-1} u(\tau) d\tau, \quad m-1 < \gamma \leq m, \quad m \in N. \quad (5)$$

Definition 3. The Caputo’s fractional derivative of order $\gamma > 0$ is presented as [11]:

$$D_a^{\gamma} u(z) = \frac{1}{\Gamma(m-\gamma)} \int_a^z (z - \tau)^{m-\alpha-1} \frac{d^m u(\tau)}{d\tau^m} d\tau, \quad m-1 < \alpha \leq m, \quad m \in N. \quad (6)$$

Lemma 1. Assume that f and g are analytic functions on $(a-h, a+h)$ and let $0 < \gamma \leq 1$. Then we have [11]:

$$\begin{aligned} D_a^{\gamma} [f(z)g(z)] &= \frac{(z-a)^{-\gamma}}{\Gamma(1-\gamma)} g(a)(f(z) - f(a)) + (D_a^{\alpha} g(z))f(z) \\ &+ \sum_{l=1}^{\infty} \binom{\alpha}{l} (I_a^{1-\alpha} g(z)) D_a^l f(z). \end{aligned} \quad (7)$$

3. Reproducing Kernel Hilbert Space Method

We will construct reproducing kernel Hilbert spaces to solve the problem.

Definition 4. $V_2^1[0, 1]$ is the first reproducing kernel Hilbert space that we need.

$$V_2^1[0, 1] = \{r \in AC[0, 1] : r' \in L^2[0, 1]\}.$$

The inner product and norm for this reproducing kernel Hilbert space are defined by:

$$\langle r, q \rangle_{V_2^1} = r(0)q(0) + \int_0^1 r'(z)q'(z)dz, \quad r, q \in V_2^1[0, 1]$$

and

$$\|r\|_{V_2^1} = \sqrt{\langle r, r \rangle_{V_2^1}}, \quad r \in V_2^1[0, 1].$$

Lemma 2. The reproducing kernel function U_z of $V_2^1[0, 1]$ is obtained as:

$$U_t(z) = \begin{cases} 1 + z, & z \leq t, \\ 1 + t, & z > t. \end{cases}$$

Definition 5. We construct the reproducing kernel Hilbert space $V_2^5[0, 1]$ as:

$$V_2^5[0, 1] = \{r \in AC[0, 1] : r', r'', r''', r^{(4)} \in AC[0, 1], r^{(5)} \in L^2[0, 1], \\ r(0) = r(1) = r''(0) = r''(1) = 0\}.$$

The inner product and the norm for this special Hilbert space is defined as:

$$\langle r, q \rangle_{V_2^5} = \sum_{i=0}^4 r^{(i)}(0)q^{(i)}(0) + \int_0^1 r^{(5)}(z)q^{(5)}(z)dz, \quad r, q \in V_2^5[0, 1]$$

and

$$\|r\|_{V_2^5} = \sqrt{\langle r, r \rangle_{V_2^5}}, \quad r \in V_2^5[0, 1].$$

Theorem 1. We obtain the reproducing kernel function for the reproducing kernel Hilbert space $V_2^5[0, 1]$ by:

$$A_y(x) = \begin{cases} a_y(x), & x \leq y, \\ b_y(x), & x > y. \end{cases}$$

where,

$$\begin{aligned}
 a_y(x) = & \frac{x^9}{362880} + \frac{635131x^6y^4}{28582381440} + \frac{11x^6y^9}{142911907200} - \frac{11x^6y^8}{15879100800} + \frac{x^7y^2}{10080} \\
 & + \frac{231x^6y}{8270365} - \frac{105619x^7y^7}{3001150051200} + \frac{91723x^7y^6}{428735721600} + \frac{23x^5y^9}{31758201600} \\
 & - \frac{88249x^7y^5}{1442911907200} - \frac{88249x^7y^4}{28582381440} - \frac{91723x^7y^3}{3572797680} - \frac{193x^7y^9}{1000383350400} \\
 & + \frac{193x^7y^8}{111153705600} - \frac{579x^7y}{8270365} - \frac{2969x^8y}{66692223360} + \frac{13x^8y^9}{190549209600} \\
 & - \frac{13x^8y^8}{21172134400} + \frac{193x^8y^7}{111153705600} - \frac{11x^8y^6}{15879100800} - \frac{69x^8y^5}{10586067200} \\
 & - \frac{69x^8y^4}{2117213440} + \frac{11x^8y^3}{132325840} - \frac{13x^9y^9}{1714942886400} + \frac{13x^9y^8}{190549209600} \\
 & - \frac{193x^9y^7}{1000383350400} + \frac{11x^9y^6}{142911907200} + \frac{23x^9y^5}{31758201600} - \frac{635131x^3y^4}{238186512} \\
 & + \frac{23x^9y^4}{6351640320} - \frac{11x^9y^3}{1190932560} - \frac{91x^9y}{33081460} + \frac{2969xy}{1654073} - \frac{127013x^6y^3}{2381865120} \\
 & - \frac{91xy^9}{33081460} + \frac{819xy^8}{33081460} - \frac{579xy^7}{8270365} + \frac{231xy^6}{8270365} + \frac{635131x^6y^5}{142911907200} \\
 & + \frac{4347xy^5}{16540730} + \frac{4347xy^4}{3308146} - \frac{5544xy^3}{1654073} + \frac{127013x^3y^3}{19848876} - \frac{636517x^6y^6}{428735721600} \\
 & - \frac{91723x^3y^7}{3572797680} + \frac{636517x^3y^6}{3572797680} - \frac{635131x^3y^5}{1190932560} - \frac{635131x^5y^3}{1190932560} \\
 & - \frac{11x^3y^9}{1190932560} + \frac{11x^3y^8}{132325840} - \frac{5544x^3y}{1654073} - \frac{222389x^4y^4}{158791008} + \frac{91723x^6y^7}{428735721600} \\
 & - \frac{88249x^4y^7}{28582381440} + \frac{635131x^4y^6}{28582381440} - \frac{319739x^4y^5}{4763730240} - \frac{635131x^4y^3}{238186512} \\
 & + \frac{23x^4y^9}{6351640320} - \frac{69x^4y^8}{2117213440} + \frac{4347x^4y}{3308146} + \frac{222389x^5y^4}{793955040} + \frac{4347x^5y}{16540730} \\
 & - \frac{88249x^5y^7}{142911907200} + \frac{635131x^5y^6}{142911907200} - \frac{319739x^5y^5}{23818651200} - \frac{69x^5y^8}{10586067200}.
 \end{aligned}$$

The reproducing kernel function $A_y(x)$ is symmetric. Therefore, when exchange x and y in $a_y(x)$, we will obtain $b_y(x)$.

Proof. We have

$$\langle r, A_y \rangle_{V_2^5} = \sum_{i=0}^4 A_y^{(i)}(0) r^{(i)}(0) + \int_0^1 A_y^{(5)}(z) r^{(5)}(z) dz,$$

by Definition 5. We obtain

$$\begin{aligned}\langle r, A_y \rangle_{V_2^5} &= A_y(0)r(0) + A_y'(0)r'(0) + A_y''(0)r''(0) \\ &\quad + A_y^{(3)}(0)r'''(0) + A_y^{(4)}(0)r^{(4)}(0) \\ &\quad + A_y^{(5)}(1)r^{(4)}(1) - A_y^{(5)}(0)r^{(4)}(0) \\ &\quad - A_y^{(6)}(1)r^{(3)}(1) + A_y^{(6)}(0)r^{(3)}(0) \\ &\quad + A_y^{(7)}(1)r''(1) - A_y^{(7)}(0)r''(0) \\ &\quad - A_y^{(8)}(1)r'(1) + A_y^{(8)}(0)r'(0) + \int_0^1 A_y^{(9)}(z)r'(z)dz,\end{aligned}$$

by integration by parts. Since $r(0) = r(1) = r''(0) = r''(1) = 0$, we get

$$\begin{aligned}\langle r, A_y \rangle_{V_2^5} &= A_y'(0)r'(0) + A_y^{(3)}(0)r^{(3)}(0) + A_y^{(4)}(0)r^{(4)}(0) \\ &\quad + A_y^{(5)}(1)r^{(4)}(1) - A_y^{(5)}(0)r^{(4)}(0) \\ &\quad - A_y^{(6)}(1)r^{(3)}(1) + A_y^{(6)}(0)r^{(3)}(0) \\ &\quad - A_y^{(8)}(1)r'(1) + A_y^{(8)}(0)r'(0) + \int_0^1 A_y^{(9)}(z)r'(z)dz.\end{aligned}$$

Since

$$A_y^{(5)}(1) = A_y^{(6)}(1) = A_y^{(8)}(1) = 0,$$

we get

$$\begin{aligned}\langle r, A_y \rangle_{V_2^5} &= A_y'(0)r'(0) + A_y^{(3)}(0)r^{(3)}(0) + A_y^{(4)}(0)r^{(4)}(0) \\ &\quad - A_y^{(5)}(0)r^{(4)}(0) + A_y^{(6)}(0)r^{(3)}(0) \\ &\quad + A_y^{(8)}(0)r'(0) + \int_0^1 A_y^{(9)}(z)r'(z)dz.\end{aligned}$$

We have

$$\begin{aligned}A_y(0) &= \frac{2969y}{1654073} + \frac{4347y^4}{3308146} - \frac{91y^9}{33081460} + \frac{819y^8}{33081460} \\ &\quad + \frac{4347y^5}{16540730} - \frac{5544y^3}{1654073} - \frac{579y^7}{8270365} + \frac{231y^6}{8270365}, \\ A_y^{(3)}(0) &= -\frac{33264y}{1654073} - \frac{635131y^4}{39697752} - \frac{11y^9}{198488760} + \frac{33y^8}{66162920} \\ &\quad - \frac{635131y^5}{198488760} + \frac{127013y^3}{3308146} - \frac{91723y^7}{595466280} + \frac{636517y^6}{595466280}, \\ A_y^{(4)}(0) &= \frac{52164y}{1654073} + \frac{222389y^4}{6616292} + \frac{23y^9}{264651680} - \frac{207y^8}{264651680} \\ &\quad - \frac{319739y^5}{198488760} - \frac{635131y^3}{9924438} - \frac{88249y^7}{1190932560} + \frac{635131y^6}{1190932560},\end{aligned}$$

$$A_y^{(5)}(0) = \frac{52164y}{1654073} + \frac{222389y^4}{6616292} + \frac{23y^9}{264651680} - \frac{207y^8}{264651680} \\ - \frac{319739y^5}{198488760} - \frac{635131y^3}{9924438} - \frac{88249y^7}{1190932560} + \frac{635131y^6}{1190932560},$$

$$A_y^{(6)}(0) = \frac{33264y}{1654073} + \frac{635131y^4}{39697752} + \frac{11y^9}{198488760} - \frac{33y^8}{66162920} \\ + \frac{635131y^5}{198488760} - \frac{127013y^3}{3308146} + \frac{91723y^7}{595466280} - \frac{636517y^6}{595466280},$$

$$A_y^{(8)}(0) = -\frac{2969y}{1654073} - \frac{4347y^4}{3308146} + \frac{91y^9}{33081460} - \frac{819y^8}{33081460} \\ - \frac{4347y^5}{16540730} + \frac{5544y^3}{1654073} + \frac{579y^7}{8270365} - \frac{231y^6}{8270365}.$$

Therefore, we reach

$$\langle r, A_y \rangle_{V_2^5} = \int_0^y A_y^{(9)}(z) r'(z) dz + \int_y^1 A_y^{(9)}(z) r'(z) dz.$$

We have

$$A_y^{(9)}(z) = \begin{cases} 1 + k(y), & z < y, \\ k(y), & z > y. \end{cases}$$

where

$$k(y) = \frac{819y^8}{33081460} - \frac{1651104y}{1654073} + \frac{4347y^5}{165540730} + \frac{4347y^4}{3308146} \\ - \frac{91y^9}{33081460} - \frac{5544y^3}{1654073} - \frac{579y^7}{8270365} + \frac{231y^6}{8270365}.$$

Therefore, we find

$$\langle r, A_y \rangle_{V_2^5} = \int_0^y (1 + k(y)) r'(z) dz + \int_y^1 (k(y)) r'(z) dz,$$

$$\langle r, A_y \rangle_{V_2^5} = (1 + k(y))(r(y) - r(0)) + k(y)(r(1) - r(y)),$$

$$\langle r, A_y \rangle_{V_2^5} = r(y).$$

Thus, the proof is completed. $\square$

We investigate the solutions of the problem (1) and (2) in the reproducing kernel Hilbert space $V_2^5[0, 1]$. We need to homogenize the conditions to apply the reproducing kernel Hilbert space method. We use the following transformation to homogenize these conditions.

$$v(z) = u(z) - f(z)$$

where

$$\begin{aligned} f(z) &= \frac{a^2bb_1 + 2a^2bb_2 - a^2zb_1 - 2a^2zb_2 - 2ab^2b_1}{6(a-b)} \\ &- \frac{ab^2b_2 + 2abzb_1 - 2azb_2 + 3az^2b_2 + 2b^2zb_1}{6(a-b)} \\ &+ \frac{zb^2b_2 - 3bz^2b_1 + z^3b_1 - z^3b_2 + 6aa_2}{6(a-b)} - \frac{6ba_1 + 6za_1 - 6za_2}{6(a-b)}. \end{aligned}$$

After using the above transformation we obtain the following problem:

$$Hv = v^{(4)}(z) + D^\gamma p(z)v(z) = M(z), \quad z \in [a, b], \quad (8)$$

subject to the end conditions

$$v(a) = v(b) = v''(a) = v''(b) = 0 \quad (9)$$

where

$$M(z) = g(z) - D^\gamma p(z)f(z).$$

Lemma 3. *H is a bounded linear operator.*

Proof. We will show

$$\|Hv\|_{V_2^1[0,1]}^2 \leq P \|v\|_{V_2^5[0,1]}^2,$$

where P is a positive constant. We know

$$\|Hv\|_{V_2^1[0,1]}^2 = \langle Hv, Hv \rangle_{V_2^1[0,1]} = [Hv(0)]^2 + \int_0^1 [Hv'(z)]^2 dz.$$

We obtain

$$v(y) = \langle v(\cdot), A_y(\cdot) \rangle_{V_2^5[0,1]}$$

and

$$Hv(y) = \langle v(\cdot), HA_y(\cdot) \rangle_{V_2^5[0,1]}.$$

Therefore, we reach

$$|Hv| \leq \|v\|_{V_2^5[0,1]} \|HA_y\|_{V_2^5[0,1]} = P_1 \|v\|_{V_2^5[0,1]}.$$

Thus, we acquire

$$[Hv(0)]^2 \leq P_1^2 \|v\|_{V_2^5[0,1]}^2.$$

Since

$$(Hv)'(z) = \langle v(\cdot), (HA_y)'(\cdot) \rangle_{V_2^5[0,1]},$$

we obtain

$$|(Hv)'| \leq \|v\|_{E_2^5[0,1]} \|(HA_y)'\|_{E_2^5[0,1]} = P_2 \|v\|_{V_2^5[0,1]}.$$

Then, we find

$$[Hv]^2 \leq P_2^2 \|v\|_{V_2^5[0,1]}^2.$$

At last we obtain

$$\|Hv\|_{V_2^1[0,1]}^2 = [Hv(0)]^2 + \int_0^1 [(Hv)'(z)]^2 dz \leq (P_1^2 + P_2^2) \|v\|_{V_2^5[0,1]}^2,$$

where $P = P_1^2 + P_2^2$ is a positive constant. This completes the proof. $\square$

We construct $a_i(z) = U_{z_i}(z)$ and $\psi_i(z) = H^* a_i(z)$, where H^* is conjugate operator of H . The orthonormal system $\{\hat{\psi}_i(z)\}_{i=1}^\infty$ of $E_2^5[0, 1]$ can be obtained by Gram–Schmidt orthogonalization operation of $\{\psi_i(z)\}_{i=1}^\infty$,

$$\hat{\psi}_i(z) = \sum_{k=1}^i \beta_{ik} \psi_k(z), \quad (\beta_{ii} > 0, \quad i = 1, 2, \dots). \quad (10)$$

Theorem 2. If $v(z)$ is the exact solution of (8) and (9), then we obtain

$$v(z) = \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} M(z_k) \hat{\psi}_i(z). \quad (11)$$

where $\{z_i\}_{i=1}^\infty$ is dense in $[0, 1]$.

Proof. We prove this theorem by using the reproducing property, the features of adjoint operator and the complete system as:

$$\begin{aligned} v(z) &= \sum_{i=1}^\infty \langle v(z), \hat{\psi}_i(z) \rangle_{V_2^5[0,1]} \hat{\psi}_i(z) \\ &= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} \langle v(z), \psi_k(z) \rangle_{V_2^5[0,1]} \hat{\psi}_i(z) \\ &= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} \langle v(z), H^* a_k(z) \rangle_{V_2^5[0,1]} \hat{\psi}_i(z) \\ &= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} \langle H v(z), a_k(z) \rangle_{V_2^1[0,1]} \hat{\psi}_i(z) \\ &= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} \langle M(z), U_{z_k} \rangle_{V_2^1[0,1]} \hat{\psi}_i(z) \\ &= \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} U(z_k) \hat{\psi}_i(z). \end{aligned}$$

This completes the proof. $\square$

The approximate solution $v_n(z)$ can be obtained as:

$$v_n(z) = \sum_{i=1}^n \sum_{k=1}^i \beta_{ik} M(z_k) \hat{\psi}_i(z). \quad (12)$$

4. Numerical Experiments

Two test examples have been taken into consideration to demonstrate the accuracy of the reproducing kernel Hilbert space method in this section. The numerical simulation has been executed in MAPLE 18.

Example 1. We consider the fourth order fractional boundary value problem as [17]:

$$u^{(4)}(z) + 0.05 D^\gamma u(z) = g(z), \quad z \in [0, 1], \quad (13)$$

with the boundary conditions

$$u(0) = 0, \quad u(1) = 0, \quad u''(0) = 0, \quad u''(1) = 8. \quad (14)$$

We have the exact solution of the problem as $z^4(z-1)$.

The absolute errors for different values of γ applying the reproducing kernel Hilbert space method are given in the Table 1. Figures 1–3 demonstrate the approximate and exact solution for $\gamma = 1.0$ and $\gamma = 0.99$.

Example 2. We take into consideration the following fourth order fractional boundary value problem [17]:

$$u^{(4)}(z) + D^\gamma zu(z) = g(z), \quad z \in [0, 1], \quad (15)$$

with the boundary conditions

$$u(0) = 0, \quad u(1) = 0, \quad u''(0) = 0, \quad u''(1) = 26(\gamma - 1). \quad (16)$$

We have the exact solution of the problem as $z^6(z^\gamma - z^{2-\gamma})$. The absolute errors applying the reproducing kernel Hilbert space method are presented in the Table 2. Figures 4 and 5 show numerical solutions for $\gamma = 0.3$.

Table 1. Absolute errors for Example 1.

x	$\gamma = 0.99$	$\gamma = 1.0$
0.0	0.0000000000000000	0.0000000000000000
0.1	$1.172251330 \times 10^{-4}$	$1.079370100 \times 10^{-4}$
0.2	$2.050389600 \times 10^{-4}$	$1.892279000 \times 10^{-4}$
0.3	$2.501585000 \times 10^{-4}$	$2.301822740 \times 10^{-4}$
0.4	$2.502148300 \times 10^{-4}$	$2.286283000 \times 10^{-4}$
0.5	$2.134363900 \times 10^{-4}$	$1.922819800 \times 10^{-4}$
0.6	$1.537822000 \times 10^{-4}$	$1.353536000 \times 10^{-4}$
0.7	$8.948229000 \times 10^{-5}$	$7.512914000 \times 10^{-5}$
0.8	$3.720321000 \times 10^{-5}$	$2.799920000 \times 10^{-5}$
0.9	$7.854290000 \times 10^{-6}$	$3.736680000 \times 10^{-6}$
1.0	$1.591755364 \times 10^{-8}$	$4.209367700 \times 10^{-7}$

Table 2. Absolute errors for Example 2.

x	$\gamma = 0.3$
0.0	$1.542329002 \times 10^{-11}$
0.1	$1.509810086 \times 10^{-4}$
0.2	$2.928833663 \times 10^{-4}$
0.3	$4.165814337 \times 10^{-4}$
0.4	$5.106740380 \times 10^{-4}$
0.5	$5.655431300 \times 10^{-4}$
0.6	$5.714676400 \times 10^{-4}$
0.7	$5.202103100 \times 10^{-4}$
0.8	$4.058811200 \times 10^{-4}$
0.9	$2.300090700 \times 10^{-4}$
1.0	$2.085180231 \times 10^{-7}$

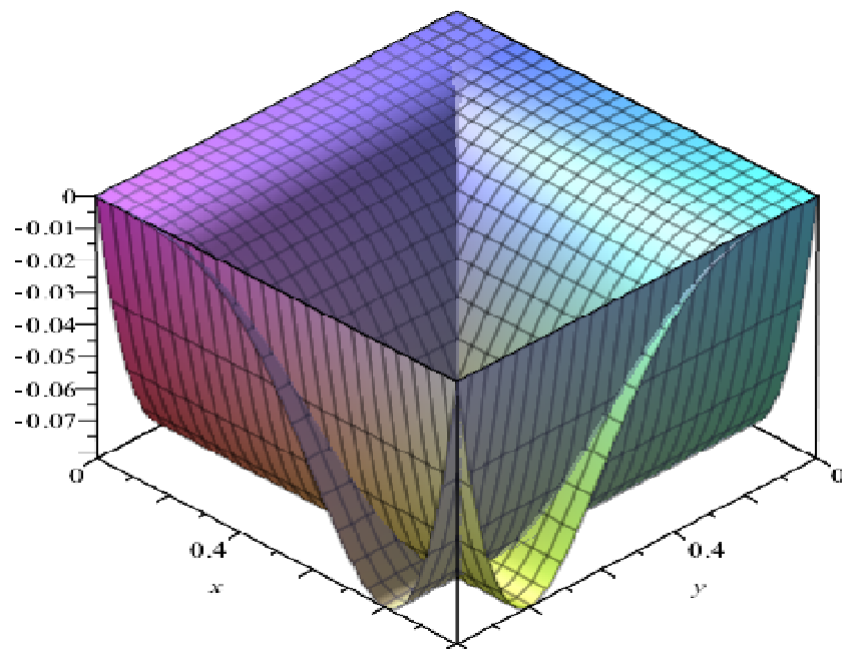


Figure 1. Exact Solutions and Approximate Solutions of Example 1 for $\gamma = 1$.

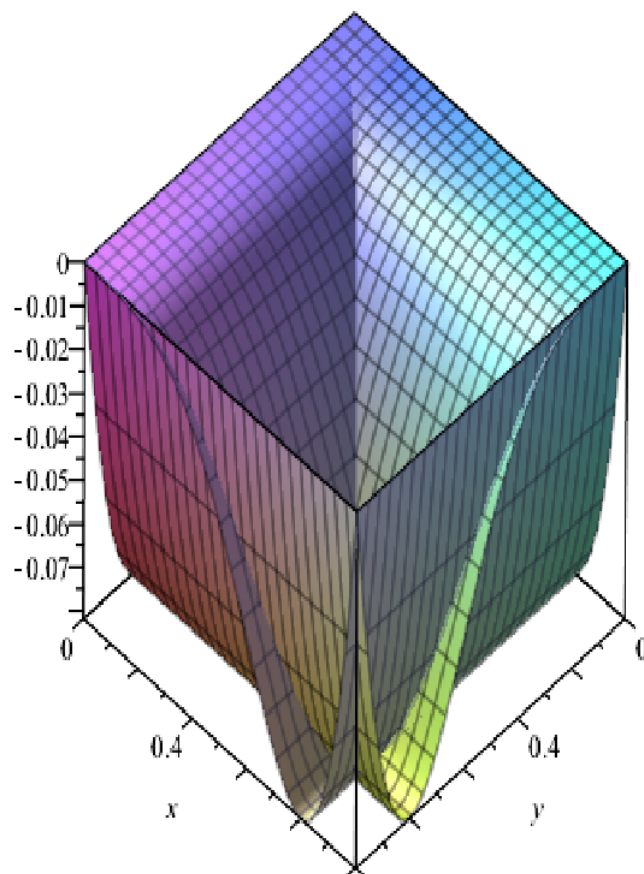


Figure 2. Exact Solutions and Approximate Solutions of Example 1 for $\gamma = 0.99$.

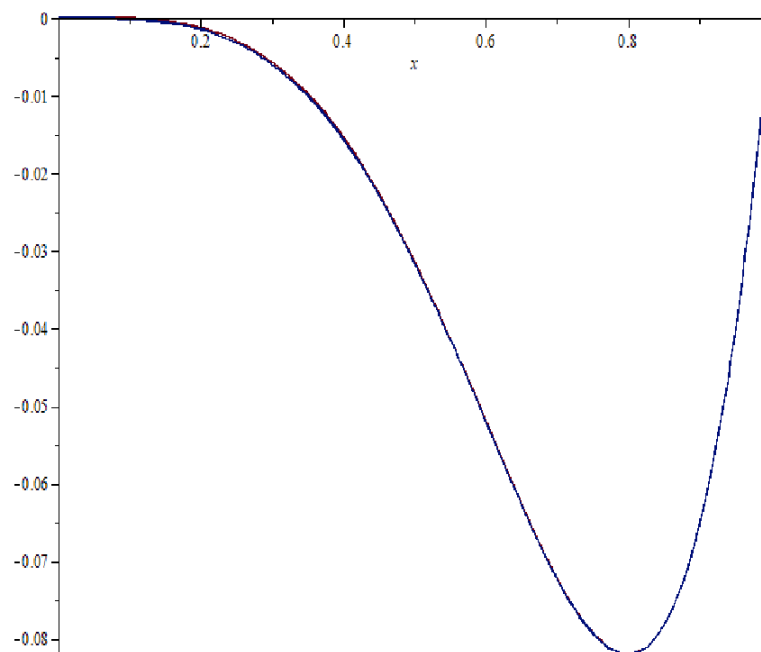


Figure 3. Exact Solutions and Approximate Solutions of Example 1 for $\gamma = 0.99$.

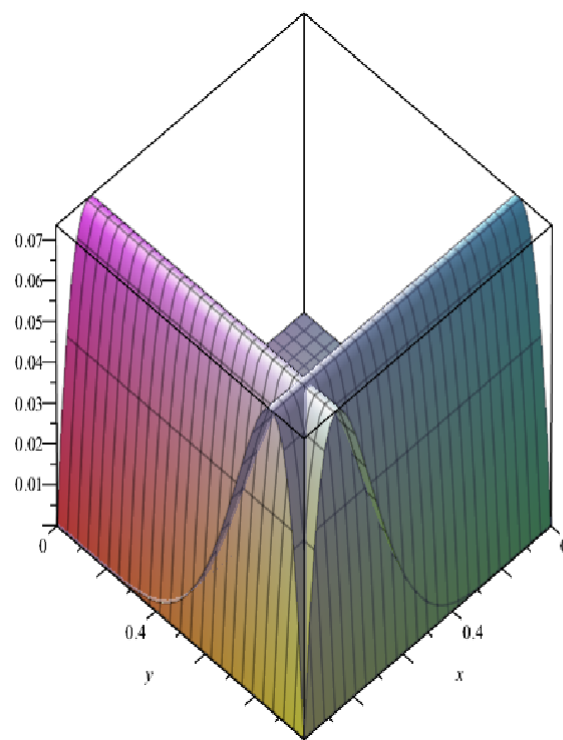


Figure 4. Exact Solutions and Approximate Solutions of Example 2 for $\alpha = 0.3$.

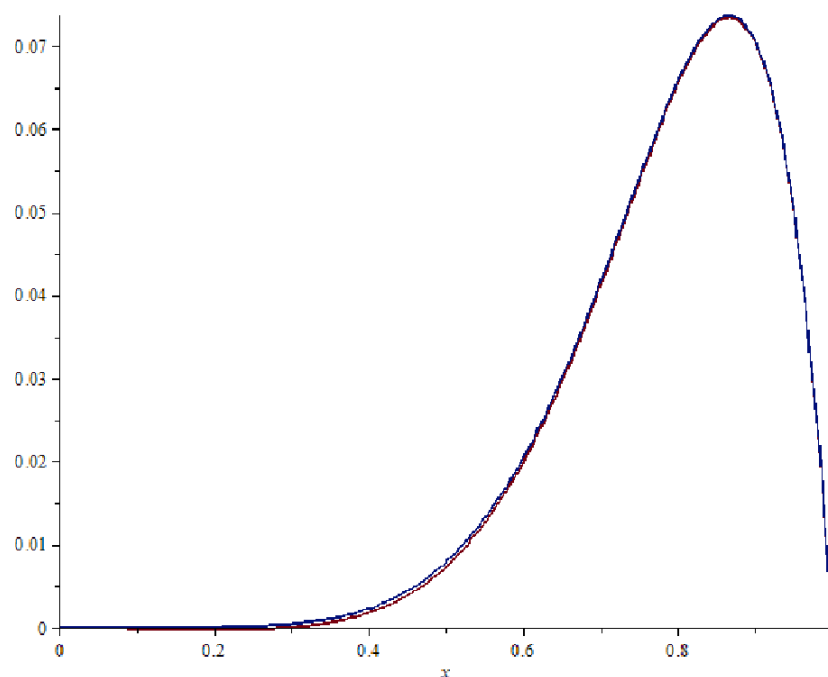


Figure 5. Exact Solutions and Approximate Solutions of Example 2 for $\gamma = 0.3$.

5. Conclusions

We studied the approximate solution of the fourth order fractional boundary value problems in this paper. We applied the reproducing kernel Hilbert space method to our problem. We demonstrated our results by tables and figures. We proved the accuracy of the reproducing kernel Hilbert space method for solutions of fourth order fractional differential equations.

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